Examples of pre-CY structures, associated operads and cohomologies

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Abstract

We consider the notion of pre-CY algebra introduced in [3], and in [4], [5], for finite dimensional space. Basic cases of those structures are classified. Namely, it is shown that there are two different nontrivial 1-pre-CY structures of signature (1,1), up to the gauge group action. For the signature (1,n) case, all 1-pre-CY structures are described as solutions of a functional equation in $R \otimes R \otimes R \otimes V^*$, where $R = K\langle\langle X \rangle\rangle$ is a ring of formal power series in noncommuting variables $X = x_1, ..., x_n$, and $V = R_1$ is a degree 1 component of R.

In some cases of signature (1,n), for example, for the direct sum of (1,1)-signature solutions:$\gamma = x_1 \cdot x_1 \otimes \frac{\partial}{\partial x_1} + \ldots + x_k \cdot x_k \otimes \frac{\partial}{\partial x_k} + 1\ cdot 1 \otimes \frac{\partial}{\partial x_{k+1}} + 0$, zero cohomology group H^0 of the complex associated to γ is calculated. It turns out to be spanned on elements $e^{-D}(w) \otimes \frac{\partial}{\partial x_p} : k+1 \leqslant p \leqslant n,\ w \in X \setminus \{x_{k+1}\}$, for the derivation $D = \sum D_i$ on R, where $D_i(x_i) = x_i x_{k+1} x_i,$ and $D_j(x_i) = 0,\ i \neq j$.

The operad $\mathcal{A}_{p,1}$ controlling the structure which is induced on A_1 in the case of (1,n)-signature 1-pre-CY structure is described precisely in terms of generators and relations.

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1 Introduction

We consider here the notion of pre-CY algebra introduced in [3] in general, and in [4], [5], for finite dimensional case. It turns out, that even in the simplest situations, corresponding to the one-loop quiver there is quite reach variety of pre-CY structures, which we are aiming to classify and explain here. Comparing to (cyclic or Ginzburg) CY structures, where the main task usually was (see, for example, [2, 1]) to characterize properties of algebras, which follow from the existence of the structure, the pre-CY structures always exist, and the goal is to describe/classify them.

Definition 1.1. d-pre-CY structure on A_∞-algebra A is an A_∞ structure on $A \oplus A^*[1-d]$, cyclically invariant w.r.t. natural non-degenerate pairing on $A \oplus A^*[1-d]$, such that A is A_∞-subalgebra in $A \oplus A^*[1-d]$.

Note that the *natural pairing* on $A \oplus A^*$ of degree N is given by $\langle (a, f), (b, g) \rangle = f(b) + (-1)^{\deg a} g(a)$, for $a, b \in A$, $f, g \in A^*$, and $\deg a + \deg b = N$.

Considering cyclic d-CY-structures on graded algebras, one can see that it forces the restriction on the number of nontrivial graded components of the algebra: $A_n = 0, n > d$ (if $A_{<0} = 0$). This suggests the idea to consider d-pre-CY structures with only $d+1$ nontrivial graded components. In particular, the basic case which we study here is 1-pre-CY structures on graded algebras with two non-trivial graded components, namely $A = A_0 \oplus A_1$. The pair of dimensions of those components $(\dim A_0, \dim A_1)$ we call a *signature* of the 1-pre-CY structure.

We start with classification of 1-pre-CY structures of signature $(1,1)$, which perhaps is as much the 'simplest' case, as the case of the field with one element.

2 Classification of 1-pre-CY structures of signature $(1,1)$ on A_∞-algebras

As an underlying object for 1-pre-CY structures we consider a graded algebra $A = A_0 \oplus A_1$. In this section we fix signature $(1,1)$, which means that $\dim A_0 = 1, \dim A_1 = 1$. In other words, we consider graded algebras of the quiver with one single loop.

In spite of what intuition raised by working with associative (Lie) algebras should say, in this simple situation (of two-dimensional underlying algebra) there is an infinite-dimensional space of 1-pre-CY structures on A. It looks like here it is a situation, where seemingly trivial object carries a lot of information, as it is the case with the field of one element, or with the whole set theory being constructed out of the empty set.

Actually, the infinite dimensional space of structures, we got on this simplest algebra, can be equivalently characterized as a space of structures, defined as a certain collection of operations on A_1: two unitary operations, three binary, etc., satisfying certain relations (which are consequences of the Maurer-Cartan equation on $A \oplus A^*$). In other words, we discover an operad, $\mathcal{A}_{p,1}$, such that all structures on A_1, which are in 1-1 correspondence with 1-pre-CY structures on $A = A_0 \oplus A_1$ of signature $(1,n)$ are algebras over the operad $\mathcal{A}_p$. Description of this operad by generators and relations is given in section 4.

Here we classify 1-pre-CY structures on $A = A_0 \oplus A_1$ of signature $(1,1)$, and it turns out that there are essentially two big classes of them. One, Σ_0 corresponds to the case of trivial (zero) differential in A_∞ structure on $A \oplus A^*$, another, Σ_1 - to a non-zero differential. These classes formed by orbits under the gauge group action.

We express those structures (solutions of the Maurer-Cartan) as a formal power series in two commuting variables - elements of $k[[x,y]]$.

There is a dense subset $\Omega \subset k[[x,y]]$, consisting of series, which could be expressed, as a rational function on x, y. We call those solutions (structures) *rational*.

Note, that inside the set of rational solutions there is a change of variables $x \mapsto x + const, y \mapsto y + const$ which allows to transform any solution from one orbit, Σ_1 (with a nontrivial differential) to the one of another form, Σ_0 (without a differential).

Remark 2.1. Taking into account two remarks above, we can say, that in case of signature $(1,1)$ all solutions could be described in terms of solutions without differential Σ_0.

2.1 Preliminary remark on d-pre-CY structure on associative algebra

Here we introduce the reader to the taste of A_∞ structures on $A \oplus A^*$, by proving the following quite easy fact. Consider for now a subclass of A_∞-algebras consisting of $\mathbb{Z}$-graded associative algebras, namely put $A = (A, m^{(1)})$, with $m^{(1)} = m_2^{(1)}$.

Let us repeat the definition from the introduction, to separate clearly its three parts.

Definition 2.2. A d-pre-CY structure on A_∞-algebra A is

(I). an A_∞ structure on $A \oplus A^*[1-d]$,

(II). cyclically invariant w.r.t. natural non-degenerate pairing on $A \oplus A^*[1-d]$, meaning:

$$\langle m_n(\alpha_1, ..., \alpha_n), \alpha_{n+1} \rangle = (-1)^\sigma \langle m_n(\alpha_{n+1}, \alpha_1, ..., \alpha_{n-1}), \alpha_n) \rangle$$

where $\langle (a,f),(b,g) \rangle = f(b) + g(a)$ for $a, b \in A, f, g \in A^*$ and $(-1)^\sigma$ stands for the sign assigned according to the Koszul rule.

(III) and such that A is A_∞-subalgebra in $A \oplus A^*[1-d]$.

Theorem 2.3. *Any $\mathbb{Z}$-graded associative finite dimensional algebra $A = (A, m), m = m_2$, $A = \oplus A_k$ has a structure of d-pre-CY algebra.*

Proof. We prove, that any associative multiplication on $A = \oplus A_k$ can be extended to an associative multiplication on $A \oplus A^*[1-d]$, so that the natural form on $A \oplus A^*[1-d]$ is cyclic with respect to the this multiplication.

Indeed, we write 8 conditions with triples from A and A^* involved, coming from the cyclicity condition of the form: $\langle xy, z \rangle = \langle z, xy \rangle$, $x, y, z \in A \oplus A^*$.

Let $a, b, c \in A,\ f, g, h \in A^*$, and denote for any product $u \times v,\ u, v \in A \oplus A^*,\ u \times v = uv + u \star v, uv \in A, u \star v \in A^*$, so uv stands for A component of $u \times v$ and $u \star v$ for the A^* component of $u \times v$.

The cyclicity

$$\langle (a+f) \times (b+g), c+h \rangle = \langle (c+h) \times (a+f), (b+g) \rangle$$

gives 7 conditions on A, A^* components:

(the condition $\langle ab, c \rangle = \langle ca, c \rangle$ is trivially satisfied, since the natural form is zero on $A \times A$)

(1).$(f \star b)(c) = (c \star f)(b)$

(2).$h(ab) = (h \star a)(b)$

(3).$(a \star g)(c) = g(ca)$

(4).$(f \star g)(c) = g(cf)$

(5).$h(fb) = (h \star f)(b)$

(6).$h(ag) = g(ha)$

(7).$h(fg) = g(hf)$

First, it is easy to see that this system splits into 3 independent groups of equations: 1-3,4-6, and 7. Only the equations 2,3 in the first group of equations involves multiplication on A, they are related by means of 1. But note, that 2 and 3 together imply 1:

$(f \star b)(c) = f(bc)$ by (2), and $(c \star f)(b) = f(bc)$ by 3, so we get 1.

Hence from 2 and 3 we just define $(f \star a)(b)$ and $(a \star g)(c)$ via multiplication on A. We get the following A-bimodule multiplication on A:

$$a \star f \star b(c) = f(bca).$$

Since conditions 4-7 does not involve multiplication on A, we can choose all these four expressions to be equal to zero, which would mean zero multiplication on A^*. Using associativity of multiplication on A we check, that this indeed provides an associative operation on $A \oplus A^*$, indeed:

$(a \times b) \times f = a \times (b \times f)$ since

$(a \times b) \times f = (ab) \times f$

and

$a \times (b \times f) = a \times (bf + b \star f)$

$(ab) \star f(c) = f(c(ab))$

$a \star (b \star f)(c) = b \star f(ca) = f((ca)b)$.

Also,$(a \star f) \star b = a \star (f \star b)$ due to associativity of A.

In fact the existence of such extension of associative structure can be generally formulated as follows:

Lemma 2.4. *Let A be an associative algebra, M an A-bimodule, then the following multiplication on $A \oplus M$ makes it into an associative algebra:*

$$(a+f)(b+g) = ab + af + gb$$

Proof. Indeed,

$$((a+b)(b+g))(c+h) = (ab + ag + fb)(c+h) = abc + agc + fbc + abh$$

and

$$(a+f)((b+g)(c+h)) = (a+f)(bc + bh + gc) = abc + abh + agc + fbc.$$

□

We need also to check that the multiplication we got works well with the graded structure. Indeed,

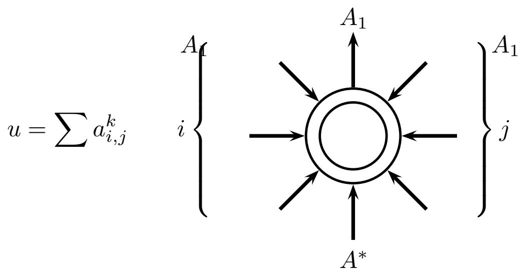
$(A \oplus A^*[1-d])_n = A_n \oplus (A_{1-d-n})^*$, so if we take $a, b, c \in A_m,\ f.g.h \in (A_{1-d-k})^* = (A^*[1-d])_k$, then $f \times a$ and $a \times f$ should be in $(A_{1-d-k-m})^* = (A^*[1-d])_{k+m}$.

Indeed, according to 2 $f \times a(b) = f(ab)$. If $f \in (A_{1-d-k})^*$ and $deg\, a + deg\, b = 1 - d - k$, then $deg\, b = 1 - d - k - m$, so $f \times a \in (A_{1-d-k-m})^*$. The same for $a \times f(b)$.

So, we got an extension of associative structure from A to $A \oplus A^*[1-d]$, which respects grading, and cyclically symmetric with respect to the natural form on $A \oplus A^*$. (Note also that construction does not depend on d). □

2.2 Functional equation for the 1-pre-CY structures of signature $(1,1)$

Taking into account that only two graded components of our algebra are non-zero, and the fact that n-ary operation should be of degree $-n+2$, from the Maurer-Cartan equation, written on the (cyclically invariant) inputs we get quadratic equations $[u,u]_{MC}=0$, and equations $[u,v]_{MC}=0$, where u consists of operations with the following structure: one input from A^*, one output (from A_1) , and $i+j$ inputs from A_1, where i and j is the number of inputs before the input from A^* and after, respectively. Graphically these cyclically invariant operations could be depicted as follows.



The variable v also is composed of cyclically invariant operations with certain type of inputs/outputs. Our main concern here will be solution of the equation

$(*)$ $[u,u]_{MC}=0$.

After this is done, we get a linear system of equations on v for any u, which will give a natural vector bundle structure on the space of solutions.

To any element u, which is the linear combination of operations of mentioned above structure, parameterized by two indexes i,j, we can associate a generating function in two variables x,y:

$$f_u(x,y) = \sum_{i+j=n-1} a^k_{ij} x^i y^j$$

The Maurer-Cartan equation (*) on u, in terms of generating function will have a form of the following functional equation.

$$(**) \quad f(x,y)\frac{f(x,z)-f(y,z)}{x-y} - f(y,z)\frac{f(x,y)-f(x,z)}{y-z}$$

First, let us make the following nice observation about the symmetric property of the solutions of this equation.

For any $f(x,y) \in k\langle\langle x,y\rangle\rangle$, denote by $a(x,y,z)$ the following element from $k\langle\langle x,y,z\rangle\rangle$:

$$a(x,y,z) = f(x,y)\frac{f(x,z)-f(y,z)}{x-y} - f(y,z)\frac{f(x,y)-f(x,z)}{y-z}$$

Theorem 2.5. *$f \in k\langle\langle x,y\rangle\rangle$ is a solution of the above functional equation if and only if $a(x,y,z) \in k\langle\langle x,y,z\rangle\rangle$ is a symmetric series, meaning stable with respect to S_3 action.*

Proof.

Lemma 2.6. *If $f(x,y) \in k\langle\langle x,y\rangle\rangle$ is a solution of the above functional equation then $f(y,x) \in k\langle\langle x,y\rangle\rangle$ is.*

Proof. (of lemma) Any function $f(x,y) \in k\langle\langle x,y\rangle\rangle$ can be presented as

$$f(x,y) = \frac{f(x,y)+f(y,x)}{2} + \frac{f(x,y)-f(y,x)}{2}$$

Denote $f_s = \frac{f(x,y)+f(y,x)}{2}$, $f_a = \frac{f(x,y)-f(y,x)}{2}$

The following holds.

1. if $f(x,y)$ is a solution, then f_a and f_s are solutions.
2. if $f(x,y)$ is a solution and $f(x,y) = -f(y,x)$, then $f = 0$

So, all solutions are symmetric. □

Now we can see that $a(x, y, z)$ stable under S_3:
i. $a(x, y, z) = a(y, z, x)$ - equation says this, for symmetric f
ii. $a(x, y, z) = a(y, x, z)$, since $f(x, y) = f(y, x)$ by lemma.
Being stable with respect to generators of S_3, $a(x, y, z)$ is stable under S_3. □

2.3 Solution of the functional equation for the 1-pre-CY structures of signature (1, 1)

Now we are going to describe all solutions of the functional equation (**) obtained in the previous section.

Suppose f is not a zero function. We can multiply the equation (**) by $(x - y)(y - z)$, and obtain

$$(y - z)f(x, y)(f(x, z) - f(y, z)) - (x - y)f(y, z)(f(x, y) - f(x, z)) = 0$$

We divide this by $(x - y)(x - z)(y - z)$, and get

$$\frac{f(x, y)}{(x - y)}\frac{f(x, z)}{(x - z)} + \frac{f(x, z)}{(x - z)}\frac{f(y, z)}{(y - z)} = \frac{f(x, y}{(x - y)}\frac{f(y, z)}{(y - z)}$$

Denote by $g(x, y) := f(x, y)/(x - y)$

Thus for $g(x, y)$ we have the equation:

$$g(x, y)g(x, z) + g(x, z)g(y, z) = g(x, y)g(y, z)$$

And hence, $h(x, y) := 1/g(x, y)$ satisfies the equation

$$h(y, z) + h(x, y) = h(x, z)$$

Now we are going to prove the following theorem.

Theorem 2.7. *If $h(x, y) \in k\langle\langle x\rangle\rangle_{k\langle\langle y\rangle\rangle}$ is a solution of the equation*

$$h(y, z) + h(x, y) = h(x, z),$$

then $h(x, y) = P(x) - P(y)$, for some $P(x) \in k\langle\langle x\rangle\rangle$.

Here by $k\langle\langle x\rangle\rangle = \sum_{i=-N}^{\infty} \alpha_i x^i, \alpha_i \in k$ we denote the field of formal Laurent series on x (finite from one side).

Proof. Denote by $Q(A(x, y))$ the field of fractions of the domain $A(x, y) = \{\sum_{i,j=0}^{\infty} x^i y^j\}$ - formal power series of positive degrees.

Obviously, $Q(A(x)) = k[[x]] \subset k((x))$ and

$$Q(A(x, y)) \subset k\langle\langle x\rangle\rangle_{k\langle\langle y\rangle\rangle}$$

$$Q(A(x, y)) \subset k\langle\langle y\rangle\rangle_{k\langle\langle x\rangle\rangle}$$

So, we consider solutions of the equation in the bigger extension then $Q(A(x, y))$.

Write the equation by degrees of y:

$$\sum_{k=a}^{\infty} p_k(x)y^k + \sum_{k=b}^{\infty} g_k(z)y^k = h(x, z).$$

Obviously,

$$Q(A(x, y)) = Q(A(y, x)) \subset k\langle\langle x\rangle\rangle_{k\langle\langle y\rangle\rangle} \cap k\langle\langle y\rangle\rangle_{k\langle\langle x\rangle\rangle}$$

Collect terms near y^k:

$$\sum_{k=min(a,b)}^{\infty}(p_k(x)+q_k(z))y^k = h(x,z).$$

$h(x,z)$ does not depend on y. Thus, p_0 and q_0 are arbitrary, $p_k(x)+q_k(z) = 0$ for all $k \neq 0$, so $p_k(x) = c_k \in \mathbb{C}, q_k(x) = -c_k \in \mathbb{C}$. Write h in the form

$$h(x,y) = p_0(x) + \sum_{k\neq 0, k>a} c_k y^k = p_0(x) + r(x).$$

Substituting it into the equation we get $r(y) = p_0(y)$, hence $h(x,y) = p_0(x) - p_0(y)$, for $p_0(x) \in k\langle\langle x\rangle\rangle$. □

3 Generalization of the functional equation for the 1-pre-CY of signature $(1,n)$

Here we consider the case $A = A_0 \oplus A_1$ and $\dim A_0 = 1$, $\dim A_1 = n$ (signature $(1,n)$). Let X be the set of free variables $X = \{x_1,\dots,x_n\}$, the linear space $V = \operatorname{span} X$ and the algebra of formal power series on these non-commuting variables $R = \langle\langle x_1,\dots,x_n\rangle\rangle$.

Let $\mathbb{A} = R\otimes R\otimes V^*$ and $\mathbb{B} = R\otimes R\otimes R\otimes V^*$. Denote generators of V^* by $\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$. We encode the solutions of the Maurer-Cartan equation as a set of elements from $\mathbb{A}$ of the form:

$$F = \sum c_{m_1,m_2,u} m_1\otimes m_2\otimes u$$

(where $m_1, m_2 \in \langle X\rangle$, $u \in X^* = \{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$). The Maurer-Cartan equation itself is defined via the operation $\star : \mathbb{A}\times\mathbb{A}\to\mathbb{B}$, and constitutes a condition that certain element in $\mathbb{B}$ is zero. Namely, $F\in\mathbb{A}$ is a solution of Maurer-Cartan equation if and only if

$$F\star F = 0.$$

The operation $\star$ is defined on monomials as follows and then extended by linearity. For arbitrary $\mu_1,\mu_2,m_1,m_2\in\langle X\rangle$ and $u,v\in X$, we define $\star$ as

$$(\mu_1\otimes\mu_2\otimes v)\star(m_1\otimes m_2\otimes u) = \sum_{m_1=avb} a\mu_1\otimes\mu_2 b\otimes m_2\otimes u - \sum_{m_2=cvd} m_1\otimes c\mu_1\otimes\mu_2 d\otimes u.$$

The sums are over all occurrences of $v\in X$ in the monomials m_1 and m_2 respectively (one with the '+' sign and another with the '–' sign), which naturally splits there in two parts, $a,b\in\langle X\rangle$ and $c,d\in\langle X\rangle$.

The above presentation of the Maurer-Cartan equation as a 'functional equation' $F\star F = 0$ on elements of $\mathbb{A}$ admits the following reformulation as an infinite system of quadratic equations on the structural constants of the operations, which are precisely the coefficients of F. Let us write explicitly monomials $m_1,m_2,m_3\in\langle X\rangle$ as

$$m_1 = x_{i_1}\dots x_{i_k},\ m_2 = x_{j_1}\dots x_{j_m},\ m_3 = x_{s_1}\dots x_{s_q}.$$

An element $F\in\mathbb{A}$ has the shape

$$F = \sum c_{i_1,\dots,i_k,j_1,\dots,j_m,r} x_{i_1}\dots x_{i_k}\otimes x_{j_1}\dots x_{j_m}\otimes\frac{\partial}{\partial x_r}.$$

Let us write down the coefficient of $F\star F$ near the monomial

$$(*)\qquad x_{i_1}\dots x_{i_k}\otimes x_{j_1}\dots x_{j_m}\otimes x_{s_1}\dots x_{s_q}\otimes\frac{\partial}{\partial x_r} = m_1\otimes m_2\otimes m_3\otimes\frac{\partial}{\partial x_r}.$$

Such a monomial can be obtained either by substitution to a monomial in $\mathbb{A}$ with the 'output' x_r of another monomial from $\mathbb{A}$ (with an arbitrary output x_p) from the left of 'zero', or from the right of 'zero'. More precisely, the monomial $(*)$ can be obtained by the following substitution corresponding to a splitting $m_1 = ab$, $m_2 = cd$

$$\begin{array}{ccccc} & & & x_r & \\ & & & \uparrow & \\ a & x_p & d & \otimes & m_3 \\ & \uparrow & & & \\ b & \otimes & c & & \end{array}$$

or by the following substitution corresponding to a splitting $m_2 = a'b'$, $m_3 = c'd'$:

$$\begin{array}{ccccc} & x_r & & & \\ & \uparrow & & & \\ m_1 & \otimes & a' & x_p & d' \\ & & & \uparrow & \\ & & b' & \otimes & c' \end{array}$$

These two possibilities for any fixed monomial $(*)$ $m_1 \otimes m_2 \otimes m_3 \otimes \frac{\partial}{\partial x_r} \in \mathbb{B}$ give us the terms with sign '+' and sign '–' respectively of the equation

$$\begin{aligned} &\textstyle\sum_{m_1=ab,\, m_2=cd,\, p} c_{ax_pd, m_3, \frac{\partial}{\partial x_r}} \cdot c_{b,c,\frac{\partial}{\partial x_p}} \\ &\quad = \textstyle\sum_{m_2=a'b',\, m_3=c'd',\, p} c_{m_1, a'x_pd', \frac{\partial}{\partial x_r}} \cdot c_{b',c',\frac{\partial}{\partial x_p}}. \end{aligned}$$

The sums are over $1 \leqslant p \leqslant n$ and all possible splittings of m_1, m_2 as products $m_1 = ab$, $m_2 = cd$ of monomials a, b, c, d and m_2, m_3 as products $m_2 = a'b'$, $m_3 = c'd'$ of monomials a', b', c', d' respectively.

We have such a quadratic equation for every monomial $m_1 \otimes m_2 \otimes m_3 \otimes \frac{\partial}{\partial x_r} \in \mathbb{B}$.

4 Operad $\mathcal{A}_{p,1}$

Any 1-pre-CY structure on $A = A_0 \oplus A_1$ of signature $(1, n)$ induces on A_1 a structure of an algebra over the operad $\mathcal{A}_{p,1}$. This operad is more complicated than the A_∞ operad and perhaps than most of others known so far.

For each N, it has $N + 1$ N-ary operations, which we denote π_j^N with $0 \leqslant j \leqslant N$. These operations correspond to the operations from the 1-pre-CY structure with the output in A_1, exactly one input from A_0^* preceded by j inputs from A_1 and followed by $N - j$ inputs from A_1.

The precise relations on the generators of this operad are as follows:

$$\sum_{\substack{s+1-l=a,\, m-l+j-s=b \\ k-j=c,\, s\leqslant j}} \pi_j^k \circ_s \pi_l^m = \sum_{\substack{j=a,\, s-1-j+l=b \\ m-l+k-s=c,\, s>j}} \pi_j^k \circ_s \pi_l^m$$

for every non-negative integers a, b, c satisfying $a + b + c > 0$. The terms with different signs correspond to substitution 'from different sides of 0'.

This gives a presentation of the operad $\mathcal{A}_{p,1}$ by generators and relations.

5 Structural theory

Bearing in mind the Remark 1, we restrict ourself meantime by the case of solutions with zero differential on $A \oplus A^*$.

Suppose for now, we are in the following situation, A_0 is one-dimensional, A_1 is two-dimensional, the extended differential $A_0^* \to A_1$ is zero, the 'unary' operations $\delta_0^1, \delta_1^1 : A_1 \to A_1$ of type $0-1$ and $1-0$ are NOT both zero (Upper index stands for arity of operation on A_1, lower index - means the number of entries from A_1 before the A_0 entry).

In this situation we perform reduction of solutions of signature (1,2), to already obtained solutions of signature (1,1). More precisely, we show that

Lemma 5.1. *There is an A_∞ subalgebra in any solution of signature (1,2), which is a solution of signature (1,1).*

Proof. By the Maurer-Cartan $(\delta_0^1)^2 = (\delta_1^1)^2 = \delta_0^1\delta_1^1 - \delta_1^1\delta_0^1$, which implies the one-dimensional space

$Im\delta_0^1 = Im\delta_1^1 = ker\delta_0^1 \ = ker\delta_1^1 = B.$

(to include the case when one of deltas is zero: $Im\delta_0^1 + Im\delta_1^1 = ker\delta_0^1 \cap ker\delta_1^1 = B.$)

Claim: B is invariant under all our operations in the following sense. If we take the $(k+m)$-ary operation on A_1 δ_k^{k+m} (corresponds to the operation with k entries from A_1 followed by the entry from A_0^* and then by m entries from A_1 with the output from A_1 treated as an $(k+m)$-ary operation on A_1), then plugging in all entries from B yields the output from B.

Without loss of generality $\delta_0^1 \neq 0$.

We use induction by $k+m$. If $k+m=1$, the statement is true since both δ_0^1 and δ_1^1 take values in B. Now assume that $k+m=n>1$ and the statement holds in the case of $< n$-ary operations. We start with $k=0$, $m=n$. Write the Maurer-Cartan equation for total entries of signature 001...1 and plug in the 1-entries from A_1, B,..., B. All the outputs fall into B by the induction hypothesis except for possibly $\delta_0^n(\delta_0^1\cdot,\cdot,...,\cdot)$. Since the image of δ_0^1 is B, we have that δ_0^n sends B^n to B, as required.

Now use this and repeat the procedure for Maurer-Cartan equation for total entries of signature 0101...1. Only two terms of Maurer-Cartan will involve operation of arity n-1, one of which is the same as above: $\delta_0^n(\delta_0^1\cdot,\cdot,...,\cdot)$, already shown to have image in B.

In the same way, δ_1^n sends B^n to B and so on, until we get all δ_k^n.

□

6 Examples of 1-pre-CY structures of signature $(1, n)$ and their zero cohomology

Here we consider particular solutions of Maurer-Cartan of signature $(1, n)$ which are obtained as a direct sum of solutions of signature $(1, 1)$ described above. We calculate cohomology H^0 of the complex associated to this solution. It terns out to be formulated in terms of explicitly given derivation on the ring R of formal power series on noncommutative variables $x_1, ..., x_n$

Consider particular solution γ of signature $(1, n)$ which is a direct sum of solutions of signature $(1, 1)$ of the type $x \cdot x \otimes \frac{\partial}{\partial x}$, $1 \cdot 1 \otimes \frac{\partial}{\partial x}$ and 0.

Theorem 6.1. *Let $\gamma = x_1 \cdot x_1 \otimes \frac{\partial}{\partial x_1} + \ldots + x_k \cdot x_k \otimes \frac{\partial}{\partial x_k} + 1 \cdot 1 \otimes \frac{\partial}{\partial x_{k+1}} + 0$, $X = \{x_1, \ldots, x_n\}$ and $X' = X \setminus \{x_{k+1}\}$. Then*

$$H^0(\gamma) = \mathop{\oplus}_{p=k+1}^{n} \mathrm{Sp}\,\{\widetilde{\Phi}(w) \otimes \frac{\partial}{\partial x_p} : k+1 \leqslant p \leqslant n,\ w \in \langle X' \rangle\},$$

where $\widetilde{\Phi}$ is an automorphism of R, $\widetilde{\Phi} = e^{-D}$ for the derivation $D = \sum D_i$, where

$$D_i(x_i) = x_i x_{k+1} x_i,\ D_j(x_i) = 0,\ i \neq j,$$

(and extended by the Leibniz rule to the monomials).

Let us describe $\widetilde{\Phi}$ more constructively and give few examples. Denote by Φ the operation of replacing one occurrence of an x_i with $1 \leqslant i \leqslant k$ in w by $x_i x_{k+1} x_i$, while

$$\widetilde{\Phi}(w) = \sum_u (-1)^s u,$$

where the sum is taken over all monomials u, which can be obtained from w by applying operations Φ, while $s = s(u)$ is the number of operations required.

For example, if $w \in \langle x_{k+2}, \ldots, x_n \rangle$, then $\widetilde{\Phi}(w) = w$, for $1 \leqslant i \leqslant k$,

$$\widetilde{\Phi}(x_i) = x_i - x_i x_{k+1} x_i + x_i x_{k+1} x_i x_{k+1} x_i - \ldots = x_i \sum_{m=0}^{\infty} (-1)^m (x_{k+1} x_i)^m,$$

while for distinct i, j satisfying $1 \leqslant i, j \leqslant k$,

$$\widetilde{\Phi}(x_i x_j) = x_i x_j - x_i x_{k+1} x_i x_j - x_i x_j x_{k+1} x_j + \ldots = x_i \left(\sum_{m,r=0}^{\infty} (-1)^{m+r} (x_{k+1} x_i)^m (x_j x_{k+1})^r \right) x_j.$$

From these formulas one can directly see that

$$\widetilde{\Phi}(x_i x_j) = \widetilde{\Phi}(x_i) \widetilde{\Phi}(x_j).$$

Proof of Theorem 6.1. Let us write down the action of γ, where

$$\gamma = x_1 \cdot x_1 \otimes \frac{\partial}{\partial x_1} + \ldots + x_k \cdot x_k \otimes \frac{\partial}{\partial x_k} + 1 \cdot 1 \otimes \frac{\partial}{\partial x_{k+1}} + 0$$

on the monomial $w \otimes \frac{\partial}{\partial x_p} = x_{i_1} \ldots x_{i_N} \otimes \frac{\partial}{\partial x_p}$:

$$[x_{i_1} \ldots x_{i_N} \otimes \frac{\partial}{\partial x_p}, \gamma] = \begin{cases} 0 & \text{if } p \geqslant k+1; \\ (x_p \cdot w - w \cdot x_p) \otimes \frac{\partial}{\partial x_p} & \text{if } p \leqslant k \end{cases}$$
$$+ \sum_{s: i_s = k+1} x_{i_1} \ldots x_{i_{s-1}} \cdot x_{i_{s+1}} \ldots x_{i_N} \otimes \frac{\partial}{\partial x_p}$$
$$+ \sum_{s: i_s \leqslant k} x_{i_1} \ldots x_{i_s} \cdot x_{i_s} \ldots x_{i_N} \otimes \frac{\partial}{\partial x_p}$$

For convenience, we call the first group of terms in this equality (they appear from inserting into γ) — group I, the terms in the second group are called group II, while the terms in the last sum are called group III.

Our goal is to describe

$$H^0 = \left\{ a = \sum_{\substack{w \in \langle X \rangle \\ 1 \leqslant p \leqslant n}} a_{w \otimes \frac{\partial}{\partial x_p}} w \otimes \frac{\partial}{\partial x_p} : [a, \gamma] = 0 \right\}.$$

For time being, we assume $a \in H^0$ and see how this inclusion reflects on the coefficients of a.

Proposition 6.2. *Let $1 \leqslant p \leqslant n$, $w = w_1 x_\alpha x_{k+1} x_\beta w_2$, $1 \leqslant \alpha, \beta \leqslant n$. If $\mathtt{deg}\, w_1 \geqslant 1$ and $\mathtt{deg}\, w_2 \geqslant 1$, then $a_{w \otimes \frac{\partial}{\partial x_p}} = 0$ except perhaps for the case $\alpha = \beta \leqslant k$.*

Proof. Note that

$$[w \otimes \frac{\partial}{\partial x_p}, \gamma] = ... \pm w_1 x_\alpha \cdot x_\beta w_2 \otimes \frac{\partial}{\partial x_p} + \ldots$$

and this term can only occur in the II-part of $[w \otimes \frac{\partial}{\partial x_p}, \gamma]$. Since the relation $\alpha = \beta \leqslant k$ fails, this term never appears in the III-part. It can not appear in the I-part either, because $\mathtt{deg}\, w_1 \geqslant 1$ and $\mathtt{deg}\, w_2 \geqslant 1$. Thus $[w \otimes \frac{\partial}{\partial x_p}, \gamma]$ contains a monomial, which does not feature in pairing of γ with anything else. Hence the equality $[a, \gamma] = 0$ yields $a_{w \otimes \frac{\partial}{\partial x_p}} = 0$. □

Proposition 6.3. *Assume that w starts with x_{k+1}. Then $a_{w \otimes \frac{\partial}{\partial x_p}} = 0$ for $1 \leqslant p \leqslant n$.*

Proof. By assumption $w = x_{k+1} w'$. Note that

$$[w \otimes \frac{\partial}{\partial x_p}, \gamma] = ... \pm 1 \cdot w' \otimes \frac{\partial}{\partial x_p} + \ldots$$

and this term can not occur in any other way. As in the proof of the previous proposition, it can not cancel and therefore the equality $[a, \gamma] = 0$ yields $a_{w \otimes \frac{\partial}{\partial x_p}} = 0$. □

Proposition 6.4. *Assume that w is a monomial of the form $w = x_p x_{k+1} x_i w'$, where $1 \leqslant p \leqslant k$ and $1 \leqslant i \leqslant n$. Then $a_{w\otimes\frac{\partial}{\partial x_p}} = -a_{x_i w'\otimes\frac{\partial}{\partial x_p}}$ if $i \neq p$ and $a_{w\otimes\frac{\partial}{\partial x_p}} = -2a_{x_i w'\otimes\frac{\partial}{\partial x_p}}$ if $i = p$.*

Proof. Indeed, the term $x_p \cdot x_i w' \otimes \frac{\partial}{\partial x_p}$ occurs (with + sign) in $[w \otimes \frac{\partial}{\partial x_p}, \gamma]$ once if $i \neq p$ (type I) and twice if $i = p$ (types I and III). The same term features once (type II) with plus sign in $[x_i w' \otimes \frac{\partial}{\partial x_p}, \gamma]$ and does not pop up in any other way. Hence the equality $[a, \gamma] = 0$ yields $a_{w\otimes\frac{\partial}{\partial x_p}} = -a_{x_i w'\otimes\frac{\partial}{\partial x_p}}$ if $i \neq p$ and $a_{w\otimes\frac{\partial}{\partial x_p}} = -2a_{x_i w'\otimes\frac{\partial}{\partial x_p}}$ if $i = p$. □

Proposition 6.5. *Assume that w is a monomial of the form $w = w_1 x_i x_{k+1} x_i w_2$, where $1 \leqslant i \leqslant k$, $\mathtt{deg}\, w_1 \geqslant 1$ and $\mathtt{deg}\, w_2 \geqslant 1$. Then for $1 \leqslant p \leqslant n$, $a_{w\otimes\frac{\partial}{\partial x_p}} = -a_{w_1 x_i w_2\otimes\frac{\partial}{\partial x_p}}$.*

Proof. Indeed, the term $w_1 x_i \cdot x_i w_2 \otimes \frac{\partial}{\partial x_p}$ occurs (with + sign) in $[w \otimes \frac{\partial}{\partial x_p}, \gamma]$ once (type III) and in $[w_1 x_i w_2 \otimes \frac{\partial}{\partial x_p}, \gamma]$ once (type II) and does not feature in any other way. Hence the equality $[a, \gamma] = 0$ yields $a_{w\otimes\frac{\partial}{\partial x_p}} = -a_{w_1 x_i w_2\otimes\frac{\partial}{\partial x_p}}$. □

We use the above information to prove the following fact.

Proposition 6.6. *Assume that $1 \leqslant p \leqslant k$. Then $a_{w\otimes\frac{\partial}{\partial x_p}} = 0$ for every monomial w.*

Proof. The case $w = 1$ is trivial. Assume now that w starts with x_i. By Proposition 6.4, $a_{w\otimes\frac{\partial}{\partial x_p}} = c a_{x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}}$, where $c = -1$ if $i \neq p$ and $c = -2$ if $i = p$. Applying Proposition 6.4 two more times, we get

$$a_{w\otimes\frac{\partial}{\partial x_p}} = c a_{x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}} = -2c a_{x_p x_{k+1} x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}} = 4c a_{x_p x_{k+1} x_p x_{k+1} x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}}.$$

Thus

$$a_{x_p x_{k+1} x_p x_{k+1} x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}} = -\frac{1}{2} a_{x_p x_{k+1} x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}}.$$

On the other hand, Proposition 6.5 applied with $w_1 = x_p x_{k+1}$ and $w_2 = x_{k+1} w$ implies that

$$a_{x_p x_{k+1} x_p x_{k+1} x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}} = -a_{x_p x_{k+1} x_p x_{k+1} w\otimes\frac{\partial}{\partial x_p}}$$

By the above 3 displays, $a_{w\otimes\frac{\partial}{\partial x_p}} = 0$. □

It remains to consider the case $p \geqslant k+1$. In this case terms of type I can not contribute. The only equations that occur are those provided by Propositions 6.4 and 6.5, which read $a_{w_1 x_i x_{k+1} x_i w_2\otimes\frac{\partial}{\partial x_p}} = -a_{w_1 x_i w_2\otimes\frac{\partial}{\partial x_p}}$ for $1 \leqslant i \leqslant k$ and every monomials w_1, w_2 (including the constants).

To complete the proof of theorem 6.1 we should show that the map $\widetilde{\Phi}$ we get can be obtained as e^{-D} for some D, and hence is an automorphism.

Consider the following derivation D on $R \otimes V^*$. Let

$$D_i(x_i) = x_i x_{k+1} x_i,\ D_j(x_i) = 0,\ i \neq j,$$

$D = \sum D_i$ and D is extended by the Leibnitz rule to the monomials, for example,

$$D(x_i x_j) = D(x_i) x_j + x_i D(x_j) = x_i x_{k+1} x_i x_j + x_i x_j x_{k+1} x_j.$$

Note, that introduced above $\widetilde{\Phi}(w) = D(w)$, when $s = 1$, i.e. when Φ applied once.

It is easy to see that

$$D(x_i) \circ D(x_j) = D(x_j) \circ D(x_i).$$

We can consider

$$e^{-D}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}D^n$$

and check, that $e^{-D}=\widetilde{\Phi}$ and it is an automorphism of R.

First, from the definitions one can see, that

$$\widetilde{\Phi}(u)=Id-D(u)+\frac{1}{2}D(u)-\frac{1}{6}D(u)+...$$

so $\widetilde{\Phi}(u)$ does indeed coincide with $e^{-D}(u)$.

Now we can see, that $\widetilde{\Phi}$ is a homomorphism, and after that it is obviously an automorphism, since inverse is given by $e^{D}(u)$.

Indeed, for any derivation, which increases the degree (so the series make sense)

$$e^D(a)e^D(b)=\left(\sum_{n=0}^{\infty}\frac{1}{n!}D^n(a)\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}D^m(b)\right)=\sum_{m,n\geqslant 0}\frac{1}{n!m!}D^n(a)D^m(b).$$

Using the iterated Leibnitz

$$D^k(ab)=\sum_{j=0}^{k}\binom{k}{j}D^j(a)D^{k-j}(b),$$

and for any derivation D we have

$$\begin{aligned}
e^D(ab)&=\sum_{k=0}^{\infty}\frac{1}{k!}D^k(ab)=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}D^j(a)D^{k-j}(b)\\
&=\sum_{m,n\geqslant 0}\binom{m+n}{n}\frac{D^n(a)D^m(b)}{k!}=\sum_{m,n\geqslant 0}\binom{m+n}{n}\frac{D^n(a)D^m(b)}{(m+n)!}\\
&=\sum_{m,n\geqslant 0}\frac{(m+n)!}{n!m!}\frac{D^n(a)D^m(b)}{(m+n)!}=\sum_{m,n\geqslant 0}\frac{1}{n!m!}D^n(a)D^m(b)=e^D(a)e^D(b)
\end{aligned}$$

□

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References

[1] Bocklandt, Raf; Graded Calabi Yau algebras of dimension 3. Journal of Pure and Applied Algebra 212 (2008), no. 1, 14–32.

[2] R. Bocklandt, T. Schedler and M. Wemyss, *Superpotentials and higher order derivations.* J. Pure Appl. Algebra **214** (2010), no. 9, 1501-1522.

[3] Kontsevich and Vlassopoulos, *Pre-Calabi-Yau algebras and topological quantum field theories,* [preprint]

[4] P. Seidel, *Fukaya A_∞ structures associated to Lefschetz fibrations,* ArXiv MathSG/0912.3932v2

[5] T. Tradler and M. Zeinalian, *Algebraic string operations,* K-Theory **38** (2007), 59–82

[6] Kontsevich and Soibelman, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry,* ArXiv MathRA/0606241 (2006)

[7] Loday and Valette, *Algebraic operads,* Springer 2013

[8] Kontsevich, *Formal non-commutative symplectic geometry,* in 'The Gelfand mathematical Seminars 1990–1992', Birkhäuser, (1993), 173-187.