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Décembre 2017

IHES/M/17/17

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December 5, 2017

Abstract

The hierarchy of equations belonging to two different but related integrable systems, the Nonlinear Schrödinger and its derivative variant, DNLS are subjected to two distinct deformation procedures, viz. quasi-integrable deformation (QID) that generally do not preserve the integrability, only asymptotically integrable, and non-holonomic deformation (NHD) that does. QID is carried out generically for the NLS hierarchy while for the DNLS hierarchy, it is first done on the Kaup-Newell system followed by other members of the family. No QI anomaly is observed at the level of EOMs which suggests that at that level the QID may be identified as some integrable deformation. NHD is applied to the NLS hierarchy generally as well as with the specific focus on the NLS equation itself and the coupled KdV type NLS equation. For the DNLS hierarchy, the Kaup-Newell(KN) and Chen-Lee-Liu (CLL) equations are deformed non-holonomically and subsequently, different aspects of the results are discussed.

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Mathematics Subject Classifications (2010): 35Q55, 37K10, 37K30

Keywords and Keyphrases. Nonlinear Schrödinger equation, Derivative nonlinear Schrödinger equation, Quasi-integrable deformation, Non-holonomic deformation, hierarchy.

1 Introduction

Completely integrable systems have many important and diverse physical applications such as in water waves, plasma physics, field theory and nonlinear optics [1]. The standard procedure to study the integrable models is by using the Lax pair, exploiting the zero curvature condition. Systems are considered to be integrable if they contain infinitely many conserved quantities that give rise to the stability of the soliton solutions. These constants of motion delineate the system dynamics, allowing them to be solved by the method of Inverse Scattering Transform (IST) with appropriate variables [2, 3, 4]. Another interesting feature of integrable hierarchies is the fact that they possess a local bi-hamiltonian structure [5, 6]. It is well-known that starting from a suitably chosen spectral problem, one can set up a hierarchy of nonlinear evolution equations. One of the important challenges in the study of integrable systems is to determine new such system s which are associated with nonlinear evolution equations of physical significance.

The Nonlinear Schrödinger (NLS) equation, in one space and one time $(1+1)$ dimensions, is a very well known integrable PDE. It also incorporates semi-classical solitonic solutions, that are physically realizable, and reflect a high degree of symmetry. The latter property corresponds to the infinitely many conserved quantities. There are different variants of the NLS equation such as the coupled KdV type NLS, the generalized NLS, the Kundu-Eckhaus equation, the dimensionless vector NLS etc [7, 8, 9, 10]. The derivative NLS (DNLS) equation is another celebrated system, its different examples being the Kaup-Newell (KN)[11], the Chen-Lee-Liu (CLL) [12] and Gerdjikov-Ivanov (GI) equations [13].

The concept of complete integrability is difficult to establish in case of field-theoretical models because of their infinite number of degrees of freedom. Real physical systems are definitely non-integrable; however, the importance of integrable models in the purview of such systems stems from the fact that the study of continuous physical systems as slightly deformed integrable models is of significant interest. It was recently shown that the sine-Gordon model can be deformed as an approximate system, giving rise to a finite number of conserved quantities [14, 15]. Some non-integrable models have been shown to possess soliton-like configurations and display properties not significantly different from that of solitons in integrable models, examples being the Ward modified chiral models and the baby Skyrme models with many potentials [16].

The preceding discussion suggests that we may extend our reasoning beyond integrability and introduce the concept of quasi (*almost*)-integrability. This was precisely the approach of Ferreira *et. al.* [16, 17] who considered the modified NLS potential of the form $V\ (|\psi|^2)^{2+\varepsilon}$, with ε being a perturbation parameter, and proved that such models possess an infinite number of quasi-conserved charges. Exact dark and bright soliton configurations of QI NLS system [18, 19] have also been obtained, the latter possessing infinite towers of exactly conserved charges, bringing the system back closer to integrability. QI deformation has also been studied in supersymmetric SG models [20].

Another situation of current interest is the non-holonomic deformation of integrable systems in which the system is perturbed in such a way that under suitable differential constraints on the perturbing function, the system maintains its integrability [21]. It was shown by Karasu-Kalkani *et. al.* [22] that the integrable sixth order KdV equation represented the non-holonomic deformation of the KdV equation preserving its integrability and generating an integrable hierarchy. The terminology *non-holonomic deformation* (NHD) was used by Kuperschmidt [23]. In Ref. [24] a matrix Lax pair, the N-soliton solution using IST as well as a two-fold integrable hierarchy were obtained by for the non-holonomic deformation of the KdV equation. The work was extended in [25] to include the NHD of both KdV and mKdV equations as well as their symmetries, hierarchies and integrability. While studies on the non-holonomic deformation of DNLS and Lenells-Fokas equations were carried out in [26], NHD of generalized KdV type equations was discussed in [27], wherein a geometric angle was provided into the KdV6 equation. In this work, Kirrilov's theory of co-adjoint representation of the Virasoro algebra was used to generate a large class of KdV6 type equations equivalent to the original equation. It was further shown that the Adler-Kostant-Symes approach provided a geometric formalism to obtain non-holonomic deformed integrable systems. NHD for the coupled KdV system was thereby generated. In [28] the author extended Kupershmidt's infinite-dimensional construction to generate NHD of a wide class of coupled KdV systems, all of which follow from the Euler-Poincaré-Suslov flows.

The purpose of the present work is to study the behavior of equations in the NLS and DNLS hierarchies when subject to nonholonomic as well as quasi-integrable deformations narrated above. The prior preserves integrability whereas the latter preserves the same in a loose sense; having an infinite number of charges which are asymptotically conserved in the scattering of soliton-like solutions. These 'quasi-integrable charges' are not conserved in time and they do vary considerably during the scattering process. However the values coincide with the scattering with the values they had before. Conservation properties of these QI systems are demonstrated mostly via numerical methods [14, 15, 16, 17, 18, 19, 29] for the lower order hierarchical equations. On the other hand nonholonomically deformed systems remain completely integrable [23, 24, 25, 26, 27, 28]. The fact that the deformation is applied to the temporal Lax component automatically preserves the scattering data of the undeformed system [24, 25, 26]. The corresponding deformation functions are exclusively position-dependent, making the final system conservative given the original one being integrable subjected to higher order constraints. This fact automatically identifies these nonholonomic deformations as semiholonomic, which are affine in velocities [21]. The stress in the present work is on the detailed analysis of these two class of deformations in case of NLS hierarchies. There is no attempt of comparison between the two given their distinct integrability properties (asymptotic vs higher-order constraints). However, the study of these two deformations of a particular class of systems in a way extends their integrability structure, leading to even higher order derivative systems with particular asymptotic behaviors. We adopt the NLS and DNLS hierarchies for this purpose as their integrability structures are well-documented. Further, the explicit demonstration of QID and NHD are commonly realized for the lower-order members of these two hierarchies [14, 17, 18, 19, 26, 30]. We try to realize the QID and NHD genealogies of these two hierarchies which, to the best of our knowledge, has not been done before.

The paper is organized as follows. Section 2 introduces the NLS hierarchy and points to some specific equations therein. This is followed by a thorough analysis of the quasi-integrable deformation of the equations in the NLS hierarchy. NHD is considered next, first in a generalized format, followed by referencing particular equations of the hierarchy. Section 3 repeats this exercise in respect of the DNLS hierarchy. Section 4 lists some general conclusions and indicates how the work may be extended in future.

2 The NLS hierarchy

In this hierarchy, the space (L) and time (M) parts f the Lax pair are respectively given by,

$$L = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}$$

$$\text{and} \qquad M = \sum_{m=0}^{n} \lambda^{n-m} \begin{pmatrix} a_m & b_m \\ c_m & -a_m \end{pmatrix}. \tag{1}$$

In the above, a_m, b_m and c_m are connected through the recurrence relations:

$$\begin{aligned} a_{mx} &= qc_m - rb_m \\ b_{mx} &= -2ib_{m+1} - 2qa_m \ , \\ c_{mx} &= 2ic_{m+1} + 2ra_m \end{aligned} \tag{2}$$

with equations of motion at $\mathcal{O}\left(\lambda^0\right)$ of spectral space spanned by λ as:

$$q_t = b_{n,x} + 2qa_n (\equiv -2ib_{n+1})^1 \quad \text{and} \quad r_t = c_{n,x} - 2ra_n (\equiv 2ic_{n+1}) \tag{3}$$

Equations (2) are obtained by solving the adjoint representation of the spectral problem or the "stationary" equation $M_x = [L, M]$ while equations (3) derive from the zero curvature condition $L_t - M_x + [L, M] = 0$

Some specific values of a_m, b_m and c_m are as follows:

[1] $m \leq n$.

$$a_0 = \alpha \text{ a constant}, \quad b_0 = 0, \quad c_0 = 0; \tag{4}$$

$$b_1 = i\alpha q, \quad c_1 = i\alpha r, \quad a_1 = 0; \tag{5}$$

$$b_2 = -\tfrac{\alpha}{2} q_x, \quad c_2 = \tfrac{\alpha}{2} r_x, \quad a_2 = \tfrac{\alpha}{2} qr; \tag{6}$$

$$\begin{array}{r} b_3 = \frac{i\alpha}{2}(-\frac{1}{2}q_{xx} + q^2 r), \\ c_3 = \frac{i\alpha}{2}(-\frac{1}{2}r_{xx} + qr^2), \\ a_3 = \frac{i\alpha}{4}(rq_x - qr_x); \end{array} \tag{7}$$

$$\begin{array}{r} b_4 = \frac{\alpha}{8}(q_{xxx} - 6qq_x r), \\ c_4 = \frac{\alpha}{8}(-r_{xxx} + 6qrr_x), \\ a_4 = \frac{\alpha}{8}(q^2 r^2 + r_x q_x - qr_{xx} - rq_{xx}); \end{array} \tag{8}$$

and so on.

The stationary equation $M_x = [L, M]$ can be rewritten as ,

$$(M)_x - [L, M] = \sum_{m=0}^{n} \lambda^{n-m} \begin{pmatrix} a_{mx} - qc_m + rb_m & b_{mx} + 2i\lambda b_m + 2qa_m \\ c_{mx} - 2i\lambda c_m - 2ra_m & -(a_{mx} - qc_m + rb_m) \end{pmatrix},$$

which upon using the recurrence relations and simplifying reduces to,

$$\begin{pmatrix} 0 & -2ib_{n+1} \\ 2ic_{n+1} & 0 \end{pmatrix}. \tag{9}$$

This leads to the NLS hierarchy of equations,

$$\begin{array}{l} q_t = -2ib_{n+1} \\ r_t = 2ic_{n+1} \end{array} . \tag{10}$$

Successive equations of the hierarchy can be generated by putting $n = 1, 2, 3$ etc. Putting $n = 2$, we obtain,

$$\begin{array}{l} q_t = -2ib_3 \\ r_t = 2ic_3 \end{array} . \tag{11}$$

Using the values of b_3 and c_3 from equation (6) we get,

$$\begin{array}{l} q_t = \alpha(-\frac{1}{2}q_{xx} + q^2 r) \\ r_t = \alpha(\frac{1}{2}r_{xx} - qr^2), \end{array} \tag{12}$$

which constitute a system of NLS equations.
Setting $n = 3$ leads to,

$$\begin{array}{l} q_t = i\alpha(-\frac{1}{4}q_{xxx} + \frac{3}{2}qq_x r) \\ r_t = i\alpha(-\frac{1}{4}r_{xxx} + \frac{3}{2}qrr_x) \end{array} , \tag{13}$$

which are a pair of coupled KdV type NLS equations.

2.1 Quasi-integrable deformation of NLS hierarchy

The Hamiltonians corresponding to the NLS hierarchy are,

$$\begin{aligned}
\mathcal{H}_1 &= \int_x qr, \\
\mathcal{H}_2 &= \int_x (rq_x - qr_x), \\
\mathcal{H}_3 &= \frac{1}{2}\int_x \left(q_x r_x + q^2 r^2\right), \\
\mathcal{H}_4 &= \int_x \left(qr_{xxx} - 3q^2 r r_x\right), \\
&\vdots
\end{aligned} \tag{14}$$

The corresponding Lax pair, which can be re-expressed as,

$$L = -i\lambda\sigma_3 + q\sigma_+ + r\sigma_-, \qquad M = \sum_{m=0}^{n} \lambda^{n-m}\left(a_m\sigma_3 + b_m\sigma_+ + c_m\sigma_-\right), \tag{15}$$

leads to the zero-curvature condition:

$$\begin{aligned}
&\left[q_t - \sum_m \lambda^{n-m}\left(b_{m,x} + 2a_m q + 2i\lambda b_m\right)\right]\sigma_+ + \left[r_t - \sum_m \lambda^{n-m}\left(c_{m,x} - 2ra_m - 2i\lambda c_m\right)\right]\sigma_- \\
&= \sum_m \lambda^{n-m}\left(a_{m,x} + rb_m - qc_m\right)\sigma_3,
\end{aligned} \tag{16}$$

from where the EOM results at $\mathcal{O}\left(\lambda^0\right)$ and consistency conditions of Eq. 2 at $\mathcal{O}\left(\lambda^m\right),\ m \leq n$. It is to be noted that the RHS above vanishes by virtue of the first of Eq.s 2, so does the $\mathcal{O}\left(\lambda^{n\neq 0}\right)$ contributions of the coefficients on the LHS from the second and third ones of the same set of equations, leaving out the EOMs at $\mathcal{O}\left(\lambda^0\right)$. On comparing the expressions for the coefficients in Eq.s 5, 6, 7 and 8 to the expressions for the Hamiltonians in Eq. 14, it is easy to see that,

$$b_m = \beta_m \frac{\delta H_m}{\delta r} \quad \text{and} \quad c_m = \gamma_m \frac{\delta H_m}{\delta q}, \tag{17}$$

where β_m and γ_m are suitable constants with no sum intended over m.

Incorporating deformations of the systems in terms of the Hamiltonians, the curvature is expressed as,

$$F_{tx} = \left[q_t - \sum_m \lambda^{n-m}\left\{\left(\beta_m \frac{\delta H_m}{\delta r}\right)_x + 2a_m q + 2i\lambda\beta_m \frac{\delta H_m}{\delta r}\right\}\right]\sigma_+$$

$$
+\left[r_t-\sum_m \lambda^{n-m}\left\{\left(\gamma_m\frac{\delta H_m}{\delta q}\right)_x-2ra_m-2i\lambda\gamma_m\frac{\delta H_m}{\delta q}\right\}\right]\sigma_-
$$

$$
-\sum_m \lambda^{n-m}\left(a_{m,x}+r\beta_m\frac{\delta H_m}{\delta r}-q\gamma_m\frac{\delta H_m}{\delta q}\right)\sigma_3. \tag{18}
$$

In general, due to the deformations of Eq.s 17, this curvature does not vanish. However, one can always consider the system, with first two coefficients vanishing, that leads to two deformed EOMs with time-evolution. This is allowed as one can consider suitable q and r dependence of the Hamiltonians H_ms for a given set of a_ms. However, once that is done, it is no longer necessary that the third coefficient in above vanishes, which is aptly identified as the *anomaly*. This particular system is the quasi-deformed NLS hierarchy. Of course a different choice could have been made with the last expression vanishing, but that would necessarily have meant sacrificing the time-evolution equations for at least one variable between q and r. Therefore, from this Quasi-Integrable (QI) mechanism, the set of equations are,

$$
\begin{aligned}
q_t &= \sum_m \lambda^{n-m}\left\{\left(\beta_m\frac{\delta H_m}{\delta r}\right)_x+2a_mq+2i\lambda\beta_m\frac{\delta H_m}{\delta r}\right\},\\
r_t &= \sum_m \lambda^{n-m}\left\{\left(\gamma_m\frac{\delta H_m}{\delta q}\right)_x-2ra_m-2i\lambda\gamma_m\frac{\delta H_m}{\delta q}\right\}\\
&\text{with non}-\text{vanishing anomalies}:\\
\mathcal{X}_m &:= q\gamma_m\frac{\delta H_m}{\delta q}-a_{m,x}-r\beta_m\frac{\delta H_m}{\delta r}.
\end{aligned} \tag{19}
$$

The $\mathcal{O}\left(\lambda^0\right)$ anomaly contribution, consistent with time evolution of the system, is,

$$
\mathcal{X}_n=q\gamma_n\frac{\delta H_n}{\delta q}-a_{n,x}-r\beta_n\frac{\delta H_n}{\delta r}, \tag{20}
$$

with higher order contributions accommodated by corresponding deformed versions of Eq.s 2.

Abelianization can be implemented at this point through suitable $sl(2,c)$ rotation, essentially yielding the on-shell *non*-zero-curvature condition [17],

$$
\tilde{F}_{tx}\equiv\mathcal{X}_n\sigma_3. \tag{21}
$$

To this end, we choose the gauge operator,

$$
\tilde{g}=\exp\left(\frac{i}{2}\varphi b^0\right), \tag{22}
$$

over the following representation of $sl(2,c)$ algebra:

$$
\begin{aligned}
&b^j=\lambda^j\sigma_3, \qquad F_1^j=\frac{\lambda^j}{2}\left(\kappa\sigma_+-\sigma_-\right), \qquad F_2^j=\frac{\lambda^j}{2}\left(\kappa\sigma_++\sigma_-\right), \quad \kappa\in\mathbb{R};\\
&\left[b^j,b^k\right]=0, \qquad \left[b^j,F_{1,2}^k\right]=F_{2,1}^{j+k}, \qquad \left[F_1^j,F_2^k\right]=\frac{\kappa}{2}b^{j+k}.
\end{aligned} \tag{23}
$$

With the identification,

$$\exp(2i\varphi) = -\kappa\frac{r}{q}, \tag{24}$$

this leads to,

$$\begin{aligned}
\tilde{L} &= \tilde{g}L\tilde{g}^{-1} + \tilde{g}_x\tilde{g}^{-1} = -ib^1 + \frac{i}{2}\varphi_x b^0 + 2i\sqrt{\frac{qr}{\kappa}}F_1^0, \\
\tilde{M} &= \frac{i}{2}\varphi_t b^0 + \sum_{m=0}^{n}\Big\{a_m b^{n-m} + \left(\frac{1}{\kappa}e^{i\varphi}b_m - e^{-i\varphi}c_m\right)F_1^{n-m} \\
&\quad + \left(\frac{1}{\kappa}e^{i\varphi}b_m + e^{-i\varphi}c_m\right)F_2^{n-m}\Big\}.
\end{aligned} \tag{25}$$

As EOMs are utilized, coefficients of $\sigma_\pm$ in Eq. 18 can be equated to zero as the deformed EOMs, validating Eq. 22, as the $\sigma_3 \equiv b^0$ component remains the same for $\tilde{F}_{tx} \to \tilde{g}F_{tx}\tilde{g}^{-1}$ for $\tilde{g}$ of Eq. 22. Therefore:

$$\tilde{F}_{tx} \equiv \sum_m \lambda^{n-m}\mathcal{X}_m b^0. \tag{26}$$

Through *another* gauge transformation with respect to,

$$\bar{g} = \exp\left(\sum_{j=1}^{\infty}\mathcal{F}^{-j}\right), \qquad \mathcal{F}^{-j} = \xi_1^{-j}F_1^{-j} + \xi_2^{-j}F_2^{-j} \tag{27}$$

an Abelian sub-algebra representation,

$$\bar{L} = -ib_1 + \sum_{j=0}^{\infty}\mathcal{L}_0^{-j}b^{-j}, \tag{28}$$

by choosing $\xi_{1,2}^{-j}$s judiciously such that $\bar{L}$ does not depend on $F_{1,2}^{-j}$. The other Lax component acquires the general expression:

$$\bar{M} \equiv \sum_{m=0}^{n}\lambda^{n-m}a_m\sigma_3 + \sum_{j=0}^{\infty}\left[\mathcal{M}_0^{-j}b^{-j} + \mathcal{M}_1^{-j}F_1^{-j} + \mathcal{M}_2^{-j}F_2^{-j}\right], \tag{29}$$

where $\mathcal{M}_{1,2}^{-j}$s contain sums containing b_ms and c_ms. However, their exact forms are not relevant for calculating the QI anomalies.

From Eq. 22, the final curvature takes the form,

$$\bar{F}_{tx} := \bar{L}_t - \bar{M}_x + \left[\bar{L}, \bar{M}\right] \equiv \mathcal{X}_n\bar{g}b^0\bar{g}^{-1} := \mathcal{X}_n\sum_{j=0}^{\infty}\left[\alpha_0^{-j}b^{-j} + \alpha_1^{-j}F_1^{-j} + \alpha_2^{-j}F_2^{-j}\right], \tag{30}$$

at the lowest spectral order,

$$\mathcal{L}_{0,t}^{-j} - \mathcal{M}_{0,x}^{-j} = \mathcal{X}_n \alpha_0^{-j}, \tag{31}$$

leading to the anomalous charge conservation law,

$$\frac{d}{dt} Q^j = \Gamma^j; \quad \text{where} \quad Q^j = \int_x \mathcal{L}^{-j} \quad \text{and} \quad \Gamma^j = \int_x \mathcal{X}_n \alpha_0^{-j}. \tag{32}$$

Following Ferreira *et. al.*'s treatment [17] of utilizing the two $\mathbb{Z}_2$ transformations available in the system, namely the $sl(2,c)$ automorphism and space-time parity, the possible reduction of the parent algebra into Image and Kernel subspaces, allows to show that α_0^{-j}s are parity-even. This additionally requires that q and r are so chosen that φ is odd under parity. For those members of NLS hierarchy for which the *same* choice of q and r yields parity-odd $\mathcal{X}_n$ (for example, the standard NLSE), Γ^j vanishes asymptotically, ensuring quasi-integrability.

As a short summary of definite parity evaluation of α_0^{-j}s, let us introduce the $Sl(2,c)$ automorphism operator $\mathfrak{A}$ and space-time parity operator $\mathfrak{P}$ with actions:

$$\begin{aligned}
&\mathfrak{A}(b^n) = -b^n, \qquad \mathfrak{A}(F_1^n) = -F_1^n, \qquad \mathfrak{A}(F_2^n) = F_2^n; \\
&\mathfrak{P}: \quad (\tilde{x}, \tilde{t}) \to (-\tilde{x}, -\tilde{t}), \qquad \tilde{x} = x - x_0, \quad \tilde{t} = t - t_0.
\end{aligned} \tag{33}$$

Here (x_0, t_0) is any arbitrary origin. b^ns define the Kernel subspace, separating it from the Image one:

$$\mathcal{G} = \mathrm{Im} + \mathrm{Ker}; \qquad [b^n, \mathrm{Ker}] = 0, \quad [b^n, \mathcal{G}] = \mathrm{Im}. \tag{34}$$

Thus $\bar{L}$ in Eq. 28 lies in the Kernel subspace. As $\bar{L}$ is effected neither by the hierarchy order, nor by the QID, the constraints satisfied by $\xi_{1,2}^{-j}$s are exactly like those evaluated by Ferreira *et. al.* [17]. On considering different spectral order contributions to $\bar{L}$, we obtain:

$$\begin{aligned}
&\bar{L}^{(1)} = -ib^1, \\
&\bar{L}^{(0)} = i\left[b_1, \mathcal{F}^{-1}\right] + \tilde{L}^{(0)}, \quad \tilde{L}^{(0)} := \frac{i}{2}\varphi_x b^0 + 2i\sqrt{\frac{qr}{\kappa}} F_1^0, \\
&\bar{L}^{(-1)} = i\left[b^1, \mathcal{F}^{-2}\right] + \left[\mathcal{F}^{-1}, \tilde{L}^{(0)}\right] - \frac{i}{2!}\left[\mathcal{F}^{-1}, \left[\mathcal{F}^{-1}, b^1\right]\right] + \mathcal{F}_x^{-1}, \\
&\vdots
\end{aligned} \tag{35}$$

Now considering definite and same parity for q and r, as φ is parity-odd, it is easy to see from the first of Eq.s 25 that,

$$\Omega \tilde{L} \equiv -\tilde{L}, \qquad \Omega := \mathfrak{A}\mathfrak{P}, \tag{36}$$

true for *each* part of $\tilde{L}$. Then, from Eq.s 35,

$$
\begin{aligned}
&(1+\Omega)\bar{L}^{(1)} = 0,\\
&(1+\Omega)\bar{L}^{(0)} \equiv i\left[b_1,(1-\Omega)\mathcal{F}^{-1}\right],\\
&\Omega\bar{L}^{(-1)} \equiv -i\left[b_1,\Omega\mathcal{F}^{-2}\right]-\left[\Omega\mathcal{F}^{-1},\tilde{L}^{(0)}\right]+\frac{i}{2!}\left[\Omega\mathcal{F}^{-1},\left[\Omega\mathcal{F}^{-1},b^1\right]\right]-\mathcal{F}_x^{-1},\\
&\vdots
\end{aligned}
\tag{37}
$$

From the second equation above, as all spectral order contribution to $\bar{L}$ are odd under Ω-operation (Eq. 28), the LHS vanishes. So,

$$
\text{either}\quad (1-\Omega)\mathcal{F}^{-1}\in \text{Ker}\quad \text{or}\quad (1-\Omega)\mathcal{F}^{-1}=0,
\tag{38}
$$

where the second one conclusion is true by definition. Then, from the third of Eq.s 37,

$$
\begin{aligned}
&(1+\Omega)\bar{L}^{(-1)} \equiv i\left[b_1,(1-\Omega)\mathcal{F}^{-2}\right],\\
or,\quad &(1-\Omega)\mathcal{F}^{-2}=0.
\end{aligned}
\tag{39}
$$

Recursively in this way, one finds $(1-\Omega)\mathcal{F}^{-j}=0$ and therefore,

$$
(1-\Omega)g=0.
\tag{40}
$$

Now on considering the *Killing form* of the $sl(2)$ loop algebra:

$$
T_r\left(b^n b^m\right)=\frac{1}{2}\delta_{n,-m},\qquad T_r\left(b^n F^m_{1,2}\right)=0,\qquad T_r\left(\star\right):=-\frac{i}{2\pi}\oint\frac{d\lambda}{\lambda}t_r\left(\star\right),
\tag{41}
$$

with the second trace over matrices, from Eq. 30,

$$
\alpha_0^{-j}=2T_r\left(\bar{g}b^0\bar{g}^{-1}b^n\right)=2T_r\left(\mathfrak{A}(\bar{g})b^0\mathfrak{A}(\bar{g}^{-1})b^n\right),
\tag{42}
$$

as the Killing form is invariant under automorphism and b^ns are odd under the same. Therefore,

$$
\mathfrak{P}(\alpha_0^{-j})\equiv 2T_r\left(\Omega(\bar{g})b^0\Omega(\bar{g}^{-1})b^n\right)=2T_r\left(\bar{g}b^0\bar{g}^{-1}b^n\right)\equiv\alpha_0^{-j}.
\tag{43}
$$

Thus, following the last of Eq.s 32, for the anomaly $\mathcal{X}_n$ being *parity-odd*, we will have quasi-conservation: $\Gamma^j=0$.

The particular anomalies: From the expressions of the Hamiltonian in Eq.s 14 and the definition of anomaly in the last of Eq.s 19, we have,

$$
\begin{aligned}
&\mathcal{X}_1=(\gamma_1-\beta_1)qr,\\
&\mathcal{X}_2=-\frac{\alpha}{2}(qr)_x-2(\gamma_2qr_x+\beta_2rq_x),\\
&\mathcal{X}_3=2(\gamma_3+\beta_3)q^2r^2+\beta_3rq_{xx}-\gamma_3qr_{xx}-i\frac{\alpha}{4}(rq_{xx}-qr_{xx}),\\
&\mathcal{X}_4=q\gamma_4(r_{xxx}-6qrr_x)+r\beta_4(q_{xxx}-6rqq_x)-\frac{\alpha}{8}(q^2r^2+r_xq_x-qr_{xx}-rq_{xx}),\\
&\vdots,
\end{aligned}
\tag{44}
$$

where the expressions from Eq.s 5-8 have been utilized. As $n = 2$ gives the usual NLSE, with q and r being parity-even, for $\gamma_2 = \beta_2$, we have $\mathcal{X}_2 \propto (qr)_x$ which is parity-odd, therefore serving the purpose of quasi-integrability (Ref. [17]). A similar result is obtained for $n = 4$. However, for $n = 1, 3$, the corresponding anomaly is parity-even. By simple observation of power of the variables in the NLS hierarchy, it is clear that only *even* ordered systems can be quasi-integrable.

Explicit coefficients: For the sake of completion, we express the relations satisfied by the $\bar{g}$ coefficients in order to have $\bar{L}$ in the Kernel subspace as follows:

$$\begin{aligned}
\mathcal{O}(0):&\quad \xi_1^{-1} = 0, \quad \xi_2^{-1} = -2\sqrt{qr/\kappa},\\
\mathcal{O}(1):&\quad \xi_1^{-2} = -2i\left(\sqrt{qr/\kappa}\right)_x, \quad \xi_2^{-2} = -\varphi_x\sqrt{qr/\kappa},\\
\mathcal{O}(2):&\quad \xi_1^{-3} = -i\varphi_{xx}\sqrt{qr/\kappa} - 2i\varphi_x\left(\sqrt{qr/\kappa}\right)_x,\\
&\quad \xi_2^{-3} = 2\left(\sqrt{qr/\kappa}\right)_{xx} - \frac{1}{2}(\varphi_x)^2\sqrt{qr/\kappa} - \frac{4}{3}\kappa\,(qr/\kappa)^{3/2},\\
\mathcal{O}(3):&\quad \xi_1^{-4} = 2i\left(\sqrt{qr/\kappa}\right)_{xxx} - 4i\kappa\sqrt{qr/\kappa}\left(\sqrt{qr/\kappa}\right)_x + \frac{3}{2}\varphi_x\varphi_{xx}\sqrt{qr/\kappa}\\
&\qquad + \frac{3}{2}(\varphi_x)^2\left(\sqrt{qr/\kappa}\right)_x,\\
&\quad \xi_2^{-4} = \varphi_{xxx}\sqrt{qr/\kappa} + 3\varphi_{xx}\left(\sqrt{qr/\kappa}\right)_x + 3\varphi_x\left(\sqrt{qr/\kappa}\right)_{xx} - \frac{1}{4}(\varphi_x)^3\sqrt{qr/\kappa}\\
&\qquad - \frac{10}{3}\kappa\varphi_x(qr/\kappa)^{3/2},\\
&\vdots
\end{aligned} \tag{45}$$

Thus, given the solution is known, the coefficients can be evaluated in principle. In accord with these constraints, the 'anomaly coefficients' $\alpha_0^{n-m,-j}$, for a given set of (n, m) are,

$$\begin{aligned}
\alpha_0^{n-m,0} &= 1, \quad \alpha_0^{n-m,-1} = 0, \quad \alpha_0^{n-m,-2} = \frac{\kappa}{4}\left\{\left(\xi_2^{-1}\right)^2 - \left(\xi_1^{-1}\right)^2\right\} \equiv qr,\\
\alpha_0^{n-m,-3} &= \frac{\kappa}{2}\left(\xi_2^{-1}\xi_2^{-2} - \xi_1^{-1}\xi_1^{-2}\right) \equiv \varphi_x qr,\\
\alpha_0^{n-m,-4} &= \frac{\kappa}{4}\left\{2\xi_2^{-1}\xi_2^{-3} - 2\xi_1^{-1}\xi_1^{-3} + \left(\xi_2^{-2}\right)^2 - \left(\xi_1^{-2}\right)^2\right\} + \frac{\kappa^2}{96}\left\{\left(\xi_1^{-1}\right)^2 - \left(\xi_2^{-1}\right)^2\right\}^2\\
&\equiv \frac{3}{2}(qr)^2 + \frac{3}{4}(\varphi_x)^2 qr + \left\{\left(\sqrt{qr}\right)_x\right\}^2 - 2\sqrt{qr}\left(\sqrt{qr}\right)_{xx},\\
&\vdots
\end{aligned} \tag{46}$$

From Eq.s 32, $\Gamma^1 = 0$, thereby imposing quasi-integrability for given (n, m). For the anomaly $\mathcal{X}_n$ being a total derivative, $\Gamma^0 = 0$ too. The charges are given as space-integral of the coefficients of $\bar{L}$:

$$\mathcal{L}_0^1 = -i, \quad \mathcal{L}_0^0 = \frac{i}{2}\varphi_x, \quad \mathcal{L}_0^{-1} = iqr, \quad \mathcal{L}_0^{-2} = \frac{i}{2}\varphi_x qr,$$

$$\mathcal{L}_0^{-3} = -i\left\{(\sqrt{qr})_x\right\}^2 + \frac{i}{4}(\varphi_x)^2 qr + \frac{i}{2}(qr)^2, \cdots \tag{47}$$

Among them, following Eq.s 46, the particular ones are conserved, namely, $\mathcal{L}_0^{-1}$.

2.2 Non-holonomic deformation of NLS hierarchy

The non-holonomic deformation of the NLS hierarchy starts with the deformation of the temporal Lax component in Eq. 1 by an amount,

$$\delta M = \sum_l \lambda^l \mathcal{G}_l, \qquad \mathcal{G}_l = \mathfrak{a}_l\sigma_3 + \mathfrak{b}_l\sigma_+ + \mathfrak{c}_l\sigma_-. \tag{48}$$

This effectively amounts to the modifications:

$$\begin{aligned} &a_m \to a_m^d = a_m + \mathfrak{a}_{n-m}, \quad b_m \to b_m^d = b_m + \mathfrak{b}_{n-m}, \quad c_m \to c_m^d = c_m + \mathfrak{c}_{n-m}, \\ &\text{for} \quad l = n - m, \end{aligned} \tag{49}$$

with other values of l contributing through the remainder of δM, leading to the characteristic non-holonomic constraints. Eq.s 2-8 are altered accordingly. From Eq. 16 the dual set of equations appear as,

$$\begin{aligned} q_t &= \sum_m \lambda^{n-m}\left(b^d_{m,x} + 2a^d_m q + 2i\lambda b^d_m\right), \\ r_t &= \sum_m \lambda^{n-m}\left(c^d_{m,x} - 2ra^d_m - 2i\lambda c^d_m\right), \end{aligned} \tag{50}$$

accompanied by the set of non-holonomic constraints:

$$\begin{aligned} &\mathfrak{a}_{l,x} = a\mathfrak{c}_l - r\mathfrak{b}_l, \\ &\mathfrak{b}_{l,x} + 2i\mathfrak{b}_{l-1} + 2q\mathfrak{a}_l = 0, \\ &\mathfrak{c}_{l,x} = 2i\mathfrak{c}_{l-1} + 2r\mathfrak{a}_l, \qquad \text{where,} \quad l \neq n - m. \end{aligned} \tag{51}$$

Finally, Eq.s 10-13 are extended by the deformation parameters.

We illustrate the non-holonomic deformation by considering the specific example of NLSE given by equation (11) with the choice $\alpha = -i$. The spatial and temporal components of the Lax pair of this equation are given by,

$$\begin{aligned} &L = -i\lambda\sigma_3 + q\sigma_+ + r\sigma_-, \\ &M_{\text{Original}} = -i\lambda^2\sigma_3 + \lambda(q\sigma_+ + r\sigma_-) + -(\tfrac{i}{2})qr\sigma_3 + (\tfrac{i}{2}q_x)\sigma_+ - (\tfrac{i}{2}r_x)\sigma_-, \end{aligned} \tag{52}$$

where the subscript indicates the time component prior to deformation. To obtain the deformation, let us introduce,

$$M_{\text{Deformed}} = \tfrac{i}{2}\textstyle\sum_{n=1}\lambda^{-n}G^{(n)}, \tag{53}$$

where,

$$G^{(n)} = a^n\sigma_3 + g_+^n\sigma_+ + g_-^n\sigma_-. \tag{54}$$

The spectral order n can take any integer value which directly or indirectly determines the order of the constraint equations, and thereby, that of the resultant hierarchy itself, as will be seen. The time part of the Lax pair takes the form

$$\tilde{M} = M_{\text{Original}} + M_{\text{Deformed}} \tag{55}$$

The zero curvature condition used with L and $\tilde{M}$ shows that while the positive powers of λ are trivially satisfied, the zeroth power (or the λ free term) leads to the perturbed dynamical systems (equations), while the negative powers of λ give rise to the differential constraints. The deformed pair of the NLS equations are given by,

$$q_t - \tfrac{i}{2}q_{xx} + iq^2r = -g_1, \quad g_1 = g_+^1, \tag{56}$$

$$r_t + \tfrac{i}{2}r_{xx} - iqr^2 = g_2, \quad g_2 = g_-^1. \tag{57}$$

Crucially, only the λ^{-1} term from M_{Deformed} contributes in deforming the dynamical equation, whereas all the other values of $n > 1$ contribute only to the constraint conditions, and thereby to the hierarchy itself. Therefore, there is a clear sectioning in the spectral space.
In order to elucidate this, we consider the simplest case of the perturbation $M_{\text{Deformed}} \equiv \lambda^{-1}G^{(1)}$. Then, on equating the λ^{-1} order coefficients of the generators σ_3, σ_+, σ_- from the zero curvature condition successively, we obtain the following individual constraint conditions on the functions a, g_1 and g_2 as,

$$a_x = qg_2 - rg_1, \tag{58}$$

$$g_{1x} + 2aq = 0, \tag{59}$$

$$g_{2x} - 2ar = 0. \tag{60}$$

The foregoing equations can be shown to give rise to the differential constraint:

$$\hat{L}(g_1, g_2) = rg_{1xx} + q_x g_{2x} + 2qr(qg_2 - rg_1) = 0. \tag{61}$$

On eliminating the deforming functions g_1 and g_2, we can derive a new higher order equation as,

$$\begin{aligned} &-r(q_t - \tfrac{i}{2}q_{xx} + iq^2r)_{xx} + q_x(r_t + \tfrac{i}{2}r_{xx} - iqr^2)_x \\ &+2qr[q(r_t + \tfrac{i}{2}r_{xx} - iqr^2) + r(q_t - \tfrac{i}{2}q_{xx} + iq^2r)] = 0, \end{aligned} \tag{62}$$

which is a *fourth order* equation.

Next we consider the contribution up to the second order ($n = 2$) deformation of the NLS equation:

$$M_{\text{Deformed}}(\lambda) = \tfrac{i}{2}(\lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)}), \tag{63}$$

where the function $G^{(2)}$ is given by,

$$G^{(2)} = b\sigma_3 + f_1\sigma_+ + f_2\sigma_-, \tag{64}$$

and $G^{(1)}$ is already defined in equation (54). The zero-curvature condition is now applied with L as before but M_{Deformed} as defined in (63). The following results arise:

(i) No change occurs in the deformed NLS equations, as inferred above. From this, it can immediately be concluded that no contribution from M_{Deformed} with $n > 1$ can effect the deformed NLS equation further, as their corresponding contribution will not occur at the same spectral order (λ^0) as that equation.

(ii) Picking up the terms in λ^{-1} from the zero curvature equation and equating the coefficients of the generators σ_3, σ_+ and σ_- successively, we are led to the following individual constraints:

$$a_x = qg_2 - rg_1, \tag{65}$$

$$g_{1x} + 2if_1 + 2aq = 0, \tag{66}$$

$$g_{2x} - 2if_2 - 2ar = 0. \tag{67}$$

The preceding set of equations finally lead to the following differential constraint:

$$\hat{L}(g_1, g_2) + 2i(rf_{1x} - q_x f_2) = 0, \tag{68}$$

with,

$$\hat{L}(g_1, g_2) = rg_{1xx} + g_{2x}q_x + 2qr(qg_2 - rg_1). \tag{69}$$

On comparison with Eq.s $58 - 61$, however, it is observed that the constraint conditions do change due to the $n = 2$ contribution, yielding a more elaborate structure.

(iii) The terms in λ^{-2} give rise to a second constraint,

$$\hat{L}(f_1, f_2) = 0, \tag{70}$$

where the functional form of the above expression is already given by (69) while f_1, f_2 make up the argument in (70). However, from Eq.s $66-67$, $f_{1,2}$ are first order in derivatives in $g_{1,2}$. Therefore, the elimination of $f_{1,2}$ from Eq. 70 in terms of q and r will now lead to a *fifth order* differential equation, unlike the case with only $n = 1$ perturbation (Eq. 62).

Thus, this is an example where the perturbed equations are kept the same, but the order of the differential constraint is increased recursively, thereby creating a new integrable hierarchy for the NLS equation. We may also consider NHD of systems other than the NLS system, with order of NHD focused on highlighting the complete spectral sector that deforms the dynamics. Any different order extension to M_{deformed} will only build-up the constraint-induced hierarchy.

For the sake of completeness, we now discuss the coupled KdV type NLSE, for which the space and time components of the Lax pair are given by:

$$\begin{aligned} &L = -i\lambda\sigma_3 + q\sigma_+ + r\sigma_- \quad \text{and} \\ &M_{\text{Original}} = -i\lambda^3\sigma_3 + \lambda^2(q\sigma_+ + r\sigma_-) + \lambda[(-\tfrac{i}{2}qr)\sigma_3 + \tfrac{i}{2}q_x\sigma_+ - \tfrac{i}{2}r_x\sigma_-] \\ &\qquad\qquad +[\tfrac{1}{4}(rq_x - qr_x)\sigma_3 + (\tfrac{1}{2}q^2r - \tfrac{1}{4}q_{xx})\sigma_+ + (\tfrac{1}{2}qr^2 - \tfrac{1}{4}r_{xx})\sigma_-]. \end{aligned} \tag{71}$$

M_{Original} now includes a term in λ^3 as compared to λ^2 in the previous example of the NLS equation. This would lead to a higher order dispersion term. We take $M_{\text{Deformed}} = \frac{i}{2}\lambda^{-1}G^{(1)}$ and therefore, $\tilde{M} = M_{\text{Original}} + M_{\text{Deformed}}$. Then, on using the zero-curvature condition, we arrive at the following deformed equations:

$$q_t + \tfrac{1}{4}q_{xxx} - \tfrac{3}{2}qq_x r = -g_1, \tag{72}$$

and

$$r_t + \tfrac{1}{4}r_{xxx} - \tfrac{3}{2}rr_x q = g_2, \tag{73}$$

along with the differential constraint:

$$\hat{L}(g_1, g_2) = 0. \tag{74}$$

In this example, the constraint is held fixed at its lowest level, but the order of the NLS equation is increased (terms enter with higher order dispersion) and thus a new integrable hierarchy is formed.

3 The DNLS hierarchy

In this case the Lax operator L is taken as,

$$\mathcal{L} = \begin{pmatrix} -i\lambda^2 - is & \lambda q \\ \lambda r & i\lambda^2 + is \end{pmatrix}. \tag{75}$$

The M operator is slightly complicated given by,

$$M = \tilde{V}^{(n)} + \triangle_n, \tag{76}$$

where,

$$\tilde{V}^{(n)} = (\lambda^{2n+2}V)_+, \qquad V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

and the elements a, b and c are expanded in negative powers of λ. In the above $\triangle_n = 2\beta a_{2(n+1)}\sigma_3(0)$, where β is a constant. $\sigma_3(0)$ is the loop algebra generator corresponding to $\lambda = 0$. The coefficients of the elements a, b and c, when expanded in negative powers of λ, are governed by the following recurrence relations:

$$\begin{aligned} a_{mx} &= qc_{m+1} - rb_{m+1} \\ b_{mx} &= -2ib_{m+2} - 2isb_m - 2qa_{m+1} \\ c_{mx} &= 2ic_{m+2} + 2isc_m + 2ra_{m+1} \end{aligned}\ , \tag{77}$$

from which it can be shown that,

$$a_{(m+1)x} = (rsb_m - qsc_m) - \frac{i}{2}(qc_{mx} + rb_{mx}). \tag{78}$$

In general it is found that:

$$a_{2j+1} = 0, b_{2j} = 0, c_{2j} = 0, \tag{79}$$

for $j = 0, 1, 2, 3, ...$ The zero-curvature equation leads to the following dynamical systems,

$$\begin{array}{l} q_t = b_{(2n+1)x} + i(1+2\beta)qrb_{2n+1} + 2(1+2\beta)qa_{2(n+1)} \\ r_t = c_{(2n+1)x} - i(1+2\beta)qrc_{2n+1} - (1+2\beta)ra_{2(n+1)} \end{array} . \tag{80}$$

These represent a coupled system of hierarchy of equations. Here, we have substituted,

$$s = \frac{1}{2}(1+2\beta)qr, \tag{81}$$

following the equations of motion from zero-curvature condition, leading to the expression of $(qr)_t$, and also that at $\mathcal{O}\left(\lambda^0\right)$, $(1+2\beta)a_{2n+2,x} \equiv -is_t$.

Putting $n = 1$, we obtain,

$$\begin{array}{l} q_t = b_{3x} + i(1+2\beta)qrb_3 + 2(1+2\beta)qa_4 \\ r_t = c_{3x} - i(1+2\beta)\beta qrc_3 - 2(1+2\beta)ra_4. \end{array} \tag{82}$$

After using $2s = (1+2\beta)qr$, we obtain the following expressions for b_3, c_3 and a_4 viz.

$$\begin{array}{l} b_3 = iq_x - 2\beta q^2 r \\ c_3 = -ir_x - 2\beta qr^2 \\ a_4 = \frac{1}{2}(rq_x - qr_x) + (2\beta + \frac{1}{4})iq^2r^2. \end{array} \tag{83}$$

Hence Eq. 82 yields,

$$\begin{array}{l} q_t = iq_{xx} - (4\beta+1)q^2r_x - 4\beta qq_xr + \frac{i}{2}(1+2\beta)(4\beta+1)q^3r^2 \\ r_t = -ir_{xx} - (4\beta+1)r^2q_x - 4\beta rr_xq - \frac{i}{2}(1+2\beta)(4\beta+1)q^2r^3. \end{array} \tag{84}$$

Eq.s 84 represent coupled Kundu-type systems.

Several reductions of Eq. 84 are possible. On putting $\beta = -\frac{1}{2}$, we get,

$$\begin{array}{l} q_t = iq_{xx} + (q^2r)_x \\ r_t = -ir_{xx} + (qr^2)_x, \end{array} \tag{85}$$

which form a coupled Kaup-Newell (KN) system. $\beta = -\frac{1}{4}$ leads to,

$$\begin{array}{l} q_t = iq_{xx} + qq_xr \\ r_t = -ir_{xx} + rr_xq, \end{array} \tag{86}$$

which is the coupled Chen-Lee-Liu (CLL) system. Finally, $\beta = 0$ yields,

$$\begin{array}{l} q_t = iq_{xx} - q^2r_x + \frac{i}{2}q^3r^2 \\ r_t = -ir_{xx} - r^2q_x - \frac{i}{2}q^2r^3, \end{array} \tag{87}$$

which is a coupled GI system. Putting $r = q^*$ in the above system of equations leads to further reductions, *e. g.* setting $r = q^*$ in 84 leads to,

$$q_t = iq_{xx} - (4\beta+1)q^2q_x^* - 4\beta|q|^2q_x + i\beta(4\beta+1)|q|^4q, \tag{88}$$

which again represents Kundu type equation.

3.1 Quasi-integrable deformation of DNLS hierarchy

We consider the set of KN equations,

$$q_t = iq_{xx} + (q^2 r)_x \quad \text{and} \quad r_t = -ir_{xx} + (qr^2)_x, \tag{89}$$

obtained for $\beta = -1/2$, as the starting-point for the QI deformation of DNLS system. The specific Lax pair for this system is,

$$\begin{aligned} &L = -i\lambda^2\sigma_3 + \lambda q\sigma_+ + \lambda r\sigma_- \quad \text{and} \\ &M \equiv \left(\lambda^2 a_2 - 2i\lambda^4\right)\sigma_3 + \left(\lambda b_3 + \lambda^3 b_1\right)\sigma_+ + \left(\lambda c_3 + \lambda^3 c_1\right)\sigma_-, \end{aligned} \tag{90}$$

with,

$$a_2 = -iqr, \quad b_1 = 2q, \quad c_1 = 2r, \quad b_3 = iq_x + q^2 r \quad \text{and} \quad c_3 = -ir_x + r^2 q. \tag{91}$$

This system originates from a Hamiltonian,

$$H = \frac{1}{2}\int_x \left(iq_x r - ir_x q - q^2 r^2\right), \tag{92}$$

corresponding to the PB structure,

$$\{q(x), r(y)\} = \frac{1}{2}(\partial_x - \partial_y)\delta(x-y), \tag{93}$$

following the usual definition of time-evolution: $\alpha_t = \{\alpha, H\}$.

To attain the QI deformation of the system in Eq. 89, the following identifications are made:

$$E_q := q_t - iq_{xx} - \left(\frac{\delta H}{\delta r} + iq_{xx}\right)_x = 0 \quad \text{and} \quad E_r := r_t + ir_{xx} - \left(\frac{\delta H}{\delta q} + ir_{xx}\right)_x = 0, \tag{94}$$

corresponding to the coefficients,

$$a_2 = -\frac{i}{2}\frac{\delta^2 H}{\delta q \delta r}, \quad b_3 = iq_x + \frac{\delta H}{\delta r} + iq_{xx} \quad \text{and} \quad c_3 = -ir_x + \frac{\delta H}{\delta q} + ir_{xx}. \tag{95}$$

The corresponding curvature tensor takes the form,

$$\begin{aligned} F_{tx} \equiv & \left\{\lambda E_q + \lambda^3\left(\frac{\delta H}{\delta q} + ir_{xx} - \frac{1}{2}r\frac{\delta^2 H}{\delta q\delta r}\right)\right\}\sigma_+ + \left\{\lambda E_r + \lambda^3\left(\frac{\delta H}{\delta r} + iq_{xx} - \frac{1}{2}q\frac{\delta^2 H}{\delta q\delta r}\right)\right\}\sigma_- \\ & + \mathcal{X}\sigma_3; \quad \text{where,} \\ \mathcal{X} := & \lambda^2\left[\frac{i}{2}\left(\frac{\delta^2 H}{\delta q\delta r}\right)_x + q\left(-ir_x + \frac{\delta H}{\delta q} + ir_{xx}\right) - r\left(iq_x + \frac{\delta H}{\delta r} + iq_{xx}\right)\right]. \end{aligned} \tag{96}$$

Now, if the Hamiltonian H is deformed, Eq.s 94 will represent the deformed EOMs, whereas $\mathcal{X}$ will represent the QI anomaly at a *different* spectral order, vanishing identically for the undeformed system.

Before embarking on quasi-deformation, a few points are to be taken note of:

1. The standard Abelianization will be different, as the anomaly $\mathcal{X}$ is of a different order than the EOMs; unlike the case of SG, SSG, NLS or KdV systems.
2. However upon Abelianization, for some α_0^{-j} being constant, only $\mathcal{X}$ will take part in the conservation expression:
$$\frac{dQ}{dt} = \int_x \mathcal{X}.$$
So having it as a total derivative or to be parity odd will do. Looking at its expression in Eq.s 96, this will amount to having the term:
$$q\frac{\delta H}{\delta q} - r\frac{\delta H}{\delta r}$$
as a total derivative. Albeit, this is subjected to finding *constant* terms as coefficient of a particular order in the $sl(2)$ gauge rotation of the F_{tx}.
3. Alternatively, to utilize *parity* and *automorphism* properties of the system to establish quasi-integrability, It should be possible in the same line as Ferreira *et. al.* as the spectral order of σ_3 term is different that those of $\sigma_\pm$ in the expression for L in Eq.s 90.

It is to be noticed that the spatial Lax component of the KN system in Eq. 90 is just λ times that for the NLS systems. Therefore, we perform a gauge-rotation by the *same* operator as that in Eq. 22, that leads to λ times the expression in the first of Eq.s 25, *i. e.*,

$$\tilde{L} \equiv -ib^2 + \frac{i}{2}\varphi_x b^1 + 2i\sqrt{\frac{qr}{\kappa}}F_1^1. \tag{97}$$

Here also, there is an underlying $sl(2,c)$ loop algebra, exactly similar to that explained in Eq.s 23. From the above equation, it is clear that all the generic structures regarding the $\mathbb{Z}_2$ symmetry treatment of the system will pass through, leading to a parity-even counterpart to α_0^{-j} of Eq.s 32 and 43. Therefore, the only job left is to determine the parity property of the current anomaly term $\mathcal{X}$, whose expression has already been determined. For the sake of completeness, we provide the expression for the temporal Lax component below:

$$\tilde{M} \equiv \left(\frac{i}{2}\varphi_t + \lambda^2 a_2 - 2i\lambda^4\right)\sigma_3 + e^{i\varphi}\left(\lambda b_3 + \lambda^3 b_1\right)\sigma_+ + e^{-i\varphi}\left(\lambda c_3 + \lambda^3 c_1\right)\sigma_-. \tag{98}$$

As both single and double space derivatives appear explicitly only with q and r in Eq.s 96, for definite parity solutions q and r, the anomaly cannot have definite parity and thus the corresponding Γ^js do not vanish in general. However for particular form of H, the overall $\mathcal{X}$ can still be odd. Additionally, if it is a *total derivative*, then for particular α_0^{-j}s with $\partial_x\alpha_0^{-j} = 0$, the corresponding charges will be conserved, yielding quasi-integrability.

In order to determine the coefficients α_0^{-j}s, we consider the rotated Lax component,

$$\bar{L} = \bar{g}\tilde{L}\bar{g}^{-1} + \bar{g}_x\bar{g}^{-1}, \tag{99}$$

and impose the condition that it is confined to the sub-algebra spanned by b^ns. The corresponding constraints are:

$$\begin{aligned}
&\mathcal{O}(1): \quad \xi_1^{-1} = 0, \quad \xi_2^{-1} = -2\sqrt{qr/\kappa}, \\
&\mathcal{O}(0): \quad \xi_1^{-2} = 0, \quad \xi_2^{-2} = -\varphi_x\sqrt{qr/\kappa} \\
&\mathcal{O}(-1): \quad \xi_1^{-3} = -2i\left(\sqrt{qr/\kappa}\right)_x, \quad \xi_2^{-3} = -\frac{4}{3}\kappa\,(qr/\kappa)^{3/2} - \frac{1}{2}(\varphi_x)^2\sqrt{qr/\kappa}, \\
&\mathcal{O}(-2): \quad \xi_1^{-4} = -i\varphi_{xx}\sqrt{qr/\kappa} - 2i\varphi_x\left(\sqrt{qr/\kappa}\right)_x, \\
&\qquad\qquad \xi_2^{-4} = -\frac{1}{4}(\varphi_x)^3\sqrt{qr/\kappa} - 3\kappa\varphi_x\,(qr/\kappa)^{3/2}, \\
&\vdots
\end{aligned} \tag{100}$$

The above relations are to be implied while evaluating the coefficients in the expression of,

$$\bar{F}_{tx} = \bar{g}\tilde{F}_{tx}\bar{g}^{-1}, \tag{101}$$

leading to the expressions:

$$\begin{aligned}
&\alpha_0^0 = 1, \quad \alpha_0^{-1} = 0, \quad \alpha_0^{-2} = \frac{\kappa}{4}\left\{\left(\xi_2^{-1}\right)^2 - \left(\xi_1^{-1}\right)^2\right\} \equiv qr, \\
&\alpha_0^{-3} = \frac{\kappa}{2}\left(\xi_2^{-1}\xi_2^{-2} - \xi_1^{-1}\xi_1^{-2}\right) \equiv \varphi_x qr, \\
&\alpha_0^{-4} = \frac{\kappa}{4}\left\{2\xi_2^{-1}\xi_2^{-3} - 2\xi_1^{-1}\xi_1^{-3} + \left(\xi_2^{-2}\right)^2 - \left(\xi_1^{-2}\right)^2\right\} + \frac{\kappa^2}{96}\left\{\left(\xi_1^{-1}\right)^2 - \left(\xi_2^{-1}\right)^2\right\}^2 \\
&\qquad \equiv \frac{3}{2}(qr)^2 + \frac{1}{4}(\varphi_x)^2 qr, \\
&\vdots
\end{aligned} \tag{102}$$

Therefore, from observation alone, for order $j = 1$, the corresponding charge is conserved and for $j = 0$, the same is true given $\mathcal{X}$ is a total derivative. These criteria may lead to quasi-integrability. The charges are given as space-integral of the coefficients of $\bar{L}$:

$$\mathcal{L}_0^2 = -i, \quad \mathcal{L}_0^1 = \frac{i}{2}\varphi_x, \quad \mathcal{L}_0^0 = iqr, \quad \mathcal{L}_0^{-1} = \frac{i}{2}\varphi_x qr, \quad \mathcal{L}_0^{-2} = \frac{i}{4}(\varphi_x)^2 qr + \frac{i}{2}(qr)^2, \cdots \tag{103}$$

Among them, following Eq.s 102, the particular ones are conserved, namely, $\mathcal{L}_0^{-1}$.

To study the quasi-integrable deformation for other members of the DNLS hierarchy, the form of the corresponding Hamiltonian is essential. From Ref. [31], the general form of the Hamiltonian for the DNLS hierarchy is given as,

$$\mathcal{H}_j^{\mathrm{D}} = \frac{1}{2j}\left\{4a_{2j+2} - rb_{2j+1} - qc_{2j+1}\right\}, \quad j \geq 1, \tag{104}$$

wherein Eq.s 79 has been effectively considered. This enables general definitions of the form: $\gamma_k = \gamma_k\left(\delta\mathcal{H}_k^{\mathrm{D}}/\delta q, \delta\mathcal{H}_k^{\mathrm{D}}/\delta r\right)$, where γ stands for (a,b,c), which explicitly are:

$$a_{2j+2} = -i\frac{j}{2}\left(\mathcal{H}_j^{\mathrm{D}} - r\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta q}\right), \quad b_{2j+1} = -2j\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta r}, \quad c_{2j+1} = -2j\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta q}. \tag{105}$$

This enables us to incorporate QIDs directly at the level of Lax pair coefficients. Then from Eq.s 80, the most general curvature expression for the deformed system can be expressed as,

$$\begin{aligned}
F_{tx}^{\mathrm{D}} \equiv \Big[&\lambda q_t + 2\lambda^{2n+2}\sum_j \lambda^{-(2j+1)} j\left(\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta r}\right)_x + 4i\lambda^{2n+2}\left\{\lambda^2 + \left(\frac{1}{2}+\beta\right)qr\right\} \\
&\times\sum_j \lambda^{-(2j+1)} j\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta r} + 2i\lambda q\Big\{\lambda^{2n+2}\sum_j \lambda^{-(2j+2)}\frac{j-1}{2}\left(\mathcal{H}_{j-1}^{\mathrm{D}} - r\frac{\delta\mathcal{H}_{j-1}^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_{j-1}^{\mathrm{D}}}{\delta q}\right) \\
&+\beta n\left(\mathcal{H}_n^{\mathrm{D}} - r\frac{\delta\mathcal{H}_n^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_n^{\mathrm{D}}}{\delta q}\right)\Big\}\Big]\sigma_+ \\
+\Big[&\lambda r_t + 2\lambda^{2n+2}\sum_j \lambda^{-(2j+1)} j\left(\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta q}\right)_x - 4i\lambda^{2n+2}\left\{\lambda^2 + \left(\frac{1}{2}+\beta\right)qr\right\} \\
&\times\sum_j \lambda^{-(2j+1)} j\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta q} - 2i\lambda r\Big\{\lambda^{2n+2}\sum_j \lambda^{-(2j+2)}\frac{j-1}{2}\left(\mathcal{H}_{j-1}^{\mathrm{D}} - r\frac{\delta\mathcal{H}_{j-1}^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_{j-1}^{\mathrm{D}}}{\delta q}\right) \\
&+\beta n\left(\mathcal{H}_n^{\mathrm{D}} - r\frac{\delta\mathcal{H}_n^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_n^{\mathrm{D}}}{\delta q}\right)\Big\}\Big]\sigma_- \\
+\Big[&-i\left(\frac{1}{2}+\beta\right)(qr)_t - \lambda^{2n+2}\sum_j \lambda^{-(2j+2)}\frac{j-1}{2}\left(\mathcal{H}_{j-1}^{\mathrm{D}} - r\frac{\delta\mathcal{H}_{j-1}^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_{j-1}^{\mathrm{D}}}{\delta q}\right)_x \\
&+in\beta\left(\mathcal{H}_n^{\mathrm{D}} - r\frac{\delta\mathcal{H}_n^{\mathrm{D}}}{\delta r} - q\frac{\delta\mathcal{H}_n^{\mathrm{D}}}{\delta q}\right)_x \\
&-2\lambda^{2n+3}\left\{q\sum_j \lambda^{-(2j+1)} j\left(\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta q}\right) - r\sum_j \lambda^{-(2j+1)} j\left(\frac{\delta\mathcal{H}_j^{\mathrm{D}}}{\delta r}\right)\right\}\Big]\sigma_3.
\end{aligned} \tag{106}$$

In the above, at $\mathcal{O}(\lambda^1)$, the coefficients of $\sigma_\pm$ represents deformed coupled equations of the hierarchy, while the coefficient of σ_3 is the anomaly term. It is to be noted that the constraints of Eq.s 77 that lead to Eq.s 79 are *still* valid as only the forms of he coefficients (a,b,c) have been generalized. This essentially ensures that there will be *no* anomaly contribution at $\mathcal{O}(\lambda^1)$ or at any odd order in general. This is a crucial fact as there will be no QI anomaly at the level of EOMs, and thus the QID should be identified as some *integrable* deformation, and thus can possibly be identified with some NHD. At other spectral orders, however, there will be anomaly contributions, which can directly be determined from Eq. 106. Then, quasi-conserved charges can be determined in accord with Eq.s 102.

3.2 Non-holonomic deformation of DNLS hierarchy

To discuss the non-holonomic deformation of the equations in the DNLS hierarchy, attention is first focused on the Kaup-Newell system with the Lax pair given in Eq.s 90. The modified temporal component of the Lax pair is given as $\tilde{M} = M_{original} + M_{deformed}$, with $M_{original}$ given by Eq.s 90 and

$$M_{deformed} = i(G^{(0)} + \lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)}), \tag{107}$$

where,

$$G^{(0)} = w\sigma_3 + m_1\sigma_+ + m_2\sigma_-, \tag{108}$$

$$G^{(1)} = a\sigma_3 + g_1\sigma_+ + g_2\sigma_-, \tag{109}$$

$$G^{(2)} = b\sigma_3 + f_1\sigma_+ + f_2\sigma_-. \tag{110}$$

Using the zero-curvature relation, we obtain the following deformed equations:

$$q_t - \frac{i}{2}q_{xx} - \frac{1}{2}(q^2r)_x + 2g_1 - 2iqw = 0, \tag{111}$$

$$r_t + \frac{i}{2}r_{xx} - \frac{1}{2}(qr^2)_x - 2g_2 + 2irw = 0. \tag{112}$$

Further, we obtain the following conditions on the different components of the deforming functions $G^{(i)}$: $m_1 = 0$, $m_2 = 0$, $a = 0$, $f_1 = 0$, $f_2 = 0$ and b_x

$$
\begin{aligned}
g_1 &= -2b(t)u, \\
g_2 &= 2b(t)v, \\
w &= 2b(t)uv + K(t),
\end{aligned} \tag{116}
$$

where K is again a function of t only. On eiminating g_1, g_2 and w from equations (111) and (112), we can rewrite the coupled perturbed $(deformed)$ DNLS equations in the following form:

$$
u_{xt} - \frac{i}{2}u_{xxx} - \frac{1}{2}(u_x^2 v_x)_x - 4ub(t) - 2iu_x(2b(t)uv + K(t)) = 0, \tag{117}
$$

$$
v_{xt} + \frac{i}{2}v_{xxx} - \frac{1}{2}(u_x v_x^2)_x - 4vb(t) + 2iv_x(2b(t)uv + K(t)) = 0. \tag{118}
$$

These are coupled evolution equations which are non-autonomous with arbitrary time-dependent coefficients $b(t)$ and $K(t)$. Clearly, no more constraints are left at this stage. Equations (117) and (118) generalize the coupled system of Lenells-Fokas equations by including a nonlinear derivative term as well as a higher order dispersion term. Also, it is to be noted that in this system both λ^0 and λ^{-1} ($i.\ e.\ n = 0, 1$) effects the dynamics, whereas $n > 2$ contributions only build the hierarchy up.

We now consider the Chen-Lee-Liu (CLL) system for which the Lax pair is given by,

$$
L = \lambda^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{2}qr \end{pmatrix}, \tag{119}
$$

and,

$$
\begin{aligned}
M = 2\lambda^4 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + 2\lambda^3 \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \lambda^2 qr \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \lambda \begin{pmatrix} 0 & iq_x + \frac{1}{2}q^2 r \\ -ir_x + \frac{1}{2}qr^2 & 0 \end{pmatrix} \\
+ \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}(rq_x - r_x q) + \frac{i}{4}r^2 q^2 \end{pmatrix}.
\end{aligned} \tag{120}
$$

We consider,

$$
M_{deformed} = i(G^{(0)} + \lambda^{-1}G^{(1)} + \lambda^{-2}G^{(2)}), \tag{121}
$$

with $G^{(0)} = w\sigma_3$, $G^{(1)} = g_1\sigma_+ + g_2\sigma_-$, $G^{(2)} = b\sigma_3$ where we have taken the cue from the preceding discussion in choosing the form of the matrices $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$. On taking $\tilde{M} = M + M_{deformed}$, and imposing the zero-curvature condition, we are led to the following deformed CLL equations:

$$
q_t = iq_{xx} + qq_x r - 2g_1 + 2iqw, \tag{122}
$$

and,

$$
r_t = -i - r_{xx} + rr_x q + 2g_2 - 2irw. \tag{123}
$$

The following differential constraints are also obtained:

$$
iw_x = qg_2 - rg_1, \tag{124}
$$

$$
g_{1x} + 2qb + \frac{i}{2}qrg_1 = 0, \tag{125}
$$

and,

$$
g_{2x} - 2rb - \frac{i}{2}qrg_2 = 0. \tag{126}
$$

We further get $b_x = 0$ which implies that b is a function of t only. However, it may be noted that it is not possible in this case to resolve the constraints and express the perturbing functions through the basic field variables by re-defining these variables. This is due to the presence of a nonlinear term in (125) and (126).

4 Discussion and conclusion

Two different deformation techniques are considered in this paper, both being applied to equations belonging to the NLS and DNLS families. For the NLS hierarchy, the QI deformation is done in detail yielding the particular anomalies, the explicit coefficients and the like. In case of DNLS, QID is first applied to the Kaup-Newell system and then for other members of the hierarchy leading to the significant observation that there cannot be any QI anomaly at the level of EOMs which means that in this case, the QID may be identified as some integrable deformation.

NHD is first generically applied to the NLS hierarchy followed by specific cases; first in case of the NLS equation itself and then for the coupled KdV type NLSE. In case of the DNLS hierarchy, NHD is carried out on two different systems, viz. the Kaup-Newell and Chen-Lee-Liu equations. The two different deformation procedures, one exactly preserving integrability (by construction) and the other only asymptotically, applied to two separate hierarchies, demonstrate an extended class of dynamical systems. These deformed systems, both (non-holonomically) integrable or/and quasi-integrable, adds to the known hierarchies as possible dynamical systems which could be of physical interest with possibly new aspects. Simultaneous application of these techniques to other families of integrable systems and/or their supersymmetric generalizations will be the topic for our future investigation.

Acknowledgement: The research of KA is supported by the TÜBITAK 2216 grant number 21514107-115.02-124285 of the Turkish government. PG would like to express his gratitude to Professor Maxim Kontsevich and other members of the IHES for their warm hospitality.

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