# Pre-Calabi-Yau algebras as noncommutative Poisson structures

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**Abstract**

We show how the double Poisson algebra introduced in [2] appear as a particular part of a pre-Calabi-Yau structure, i.e. cyclically invariant, with respect to the natural inner form, solution of the Maurer-Cartan equation on $A \oplus A^*$. Specific part of this solution is described, which is in one-to-one correspondence with the double Poisson algebra structure. As a consequence we have that appropriate pre-Calabi-Yau structures induce a Poisson bracket on representation spaces $(\mathrm{Rep}_n A)^{Gl_n}$ for any associative algebra $A$.

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# 1 Introduction

We consider the structures introduced in [6], [7], [11], [13], which are cyclically invariant with respect to the natural inner form solutions of the Maurer-Cartan equation on the algebra $A \oplus A^*$, for any graded associative algebra $A$. This structure is called pre-Calabi-Yau algebra. We show, how the double Poisson algebra [2] appear as a particular part of a pre-Calabi-Yau structure.

It was suggested in [5] that cyclic structure on $A_\infty$-algebra with respect to certain non-degenerate inner form should be considered as a symplectic form on the formal noncommutative manifold. Here we demonstrate that specific case of the form on $A \oplus A^*$, namely pre-Calabi-Yau structure serves as a noncommutative version of Poisson bracket.

Indeed, we check that pre-Calabi-Yau structure of appropriate kind on $A$ induce a $Gl_n$- invariant Poisson bracket on the representation spaces $(\mathrm{Rep}_n A)$ of $A$. More precisely, we found the way to associate to a pre-Calabi-Yau structure of appropriate arities $A = (A, m = m_2^{(1)} + m_3^{(1)})$ a double Poisson bracket, which satisfies all axioms of the double Poisson algebra [2]. This allows to consider pre-Calabi-Yau structures as a noncommutative version of Poisson structure according to ideology introduced in [8], saying that noncommutative structure should manifest as a corresponding commutative structure on representation spaces.

The way we establish the correspondence between the two structures is the following. First, we associate to a solution of the Maurer-Cartan equation of type B the Poisson algebra structure. Most subtle point here is the choice of the definition of the bracket via the pre-Calabi-Yau structure.

**Theorem 1.1.** *Let the bracket is defined by the formula*

$$\langle g \otimes f, \{\{b, a\}\} \rangle := \langle m_3(a, f, b), g \rangle,$$

*where $a, b \in A$, $f, g \in A^*$ and $m_3(a, f, b) = c \in A$ corresponds to the component of type B of the solution to the Maurer-Cartan $m_3$: $A \times A^* \times A \to A$. Then this bracket does satisfy all axioms of the double Poisson algebra.*

*Thus, pre-Calabi-Yau structures $(A \oplus A^*, m = m_2^{(1)} + m_3^{(1)})$ of type B (i.e. corresponding to the tensor of the type $A \otimes A^* \otimes A \otimes A^*$ or $A^* \otimes A \otimes A^* \otimes A$) are in one-to-one correspondence with double Poisson brackets $\{\{\cdot, \cdot\}\} : A \otimes A \to A \otimes A$ for an arbitrary associative algebra $A$.*

Here we concentrate on the non-graded version of the double Poisson structure, and show how it could be obtained as a part of the solution of the Maurer-Cartan equation on the algebra $A \oplus A^*$, which is already a graded object - otherwise the Maurer-Cartan equation would be trivial. Namely, we consider the grading on $R = A \oplus A^*$, where $R_0 = A$, $R_1 = A^*$, and find it quite amazing how graded continuation of an arbitrary non-graded associative algebra can induce an interesting non-graded structure on $A$ itself. In order the non-graded version of double Poisson algebra to be induced on associative algebra $A$ (sitting in degree zero) it have to be 2-pre-Calabi-Yau structure in case of this grading. Analogous results for an arbitrary graded associative algebra will be treated elsewhere.

The structure of the paper is the following. We explain the notions of strong homotopy associative algebra ($A_\infty$-algebra) and of pre-Calabi-Yau structure on $A_\infty$ or graded associative algebra in Section 1.

Then in Section 2 we investigate the general structure of equations coming from the Maurer-Cartan of appropriate arities (4 and 5) to be compared with the double Poisson bracket. We spot some peculiarities of this system of equations, where, for example, equations obtained from $MC_5$ never contain both $XX$ and $XY$ terms. Here $XX$ means the composition of two operations of type B, while $XY$ means the composition of operations of different kinds (of type $A$ and type $B$).

In Section 3 we perform main computations to show that parts of the solutions to the Maurer-Cartan equation corresponding to exclusively operations of the type $A \otimes A^* \otimes A \otimes A^*$ are in one-to-one correspondence with double Poisson brackets.

In Section 4 we finally discuss how appropriate kind of pre-Calabi-Yau structures via double Poisson bracket induce Poisson structures on representation spaces of an arbitrary associative algebra.

# 2 Finite and infinite dimensional pre-Calabi-Yau algebras

We deal here with the definition of a d-pre-Calabi-Yau structure on $A_\infty$-algebra. Further in the text we consider mainly pre-Calabi-Yau structures on an associative algebra $A$. Since in the definition of pre-Calabi-Yau structure the main ingredient is $A_\infty$-structure on $A \oplus A^*$ we start with the definition of $A_\infty$-algebra, or *strong homotopy associative algebra* introduced by Stasheff [12].

Let $A$ be a $\mathbb{Z}$ graded vector space $A = \bigoplus_{n \in \mathbb{Z}} A_n$. Let $C^l(A, A)$ be Hochschild cochains $C^l(A, A) = \underline{Hom}(A[1]^{\otimes l}, A[1])$, for $l \geqslant 0$, $C^\bullet(A, A) = \prod_{k \geqslant 1} C^l(A, A)$. With respect to the Gerstenhaber bracket $[-, -]_G$ we have the Maurer-Cartan equation

$$[m^{(1)}, m^{(1)}]_G = \sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^{\varepsilon} m_p(x_1, \ldots, x_{i-1}, m_q(x_j, \ldots, x_{i+q-1}), \ldots, x_k) = 0, \tag{2.1}$$

where

$$\varepsilon = |x_1|' + \ldots + |x_{i-1}|', \qquad |x_i|' = |x_i| - 1 = \mathrm{deg} x_i - 1$$

In fact, there are two accepted conventions of grading of an $A_\infty$-algebra. They differ by a shift in numeration of graded components. In one convention, we call it *Conv.1*, each operation has degree 1, while the other is determined by making the binary operation to be of degree 0, and the degree of operation of arity $n$, $m_n$ to be $2-n$. This second convention will be called *Conv.0*. If the degree of element $x$ in *Conv.0* is $\mathrm{deg} x = |x|$, then shifted degree in $A^{sh} = A[1]$, which fall into *Conv.1*, will be $\mathrm{deg}^{\mathrm{sh}} x = |x|'$, where $|x|' = |x| - 1$, since $x \in A^i = A[1]^{i+1}$.

The formulae for the Maurer-Cartan equations and cyclic invariance of the inner form are different in different conventions. Since we mainly will use the *Conv.1*, but in a way need *Conv.0* as well we present both of them.

The the Maurer-Cartan in *Conv.0* is:

$$[m^{(1)}, m^{(1)}] = \sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^{\varepsilon} m_p(x_1, \ldots, x_{i-1}, m_q(x_j, \ldots, x_{i+q-1}), \ldots, x_k) = 0, \tag{2.2}$$

where

$$\varepsilon = i(q+1) + q(|x_1| + \ldots + |x_{i-1}|,$$

**Definition 2.1.** An element $m^{(1)} \in C^\bullet(A, A)[1]$ which satisfies the Maurer-Cartan equation $[m^{(1)}, m^{(1)}]_G$ with respect to the Gerstenhaber bracket $[-,-]_G$ is called an $A_\infty$-structure on $A$.

Equivalently, it can be formulated in a more compact way as a coderivation on the coalgebra of the bar complex of $A$.

In particular, associative algebra with zero derivation $A = (A, m = m_2^{(1)})$ is an example of $A_\infty$-algebra. The component of the Maurer-Cartan equation of arity 3, $MC_3$ will say that the binary operation of this structure, the multiplication $m_2$ is associative:

$$(ab)c - a(bc) = dm_3(a,b,c) + (-1)^{\sigma} m_3(da,b,c) + (-1)^{\sigma} m_3(a,db,c) + (-1)^{\sigma} m_3(a,b,dc)$$

We can give now definition of pre-Calabi-Yau structure (in $Conv.1$).

**Definition 2.2.** A d-pre-Calabi-Yau structure on a finite dimensional $A_\infty$-algebra $A$ is

(I). an $A_\infty$ structure on $A \oplus A^*[1-d]$,

(II). cyclic invariant with respect to natural non-degenerate pairing on $A \oplus A^*[1-d]$, meaning:

$$\langle m_n(\alpha_1, ..., \alpha_n), \alpha_{n+1} \rangle = (-1)^{|\alpha_1|'(|\alpha_2|' + \ldots + |\alpha_{n+1}|')} \langle m_n(\alpha_2, ... \alpha_{n+1}), \alpha_1) \rangle$$

where the inner form $\rangle, \langle$ on $A + A^*$ is defined naturally as $\langle (a,f), (b,g) \rangle = f(b) + (-1)^{|g|'|a|'} g(a)$ for $a, b \in A, f, g \in A^*$

(III) and such that $A$ is $A_\infty$-subalgebra in $A \oplus A^*[1-d]$.

The signs in this definition written in *Conv.1*, are assigned according to the Koszul rule. It is not quite the case in *Conv.0*, where the cyclic invariance with respect to the natural non-degenerate pairing on $A \oplus A^*[1-d]$, from (II) sounds:

$$\langle m_n(\alpha_1, ..., \alpha_n), \alpha_{n+1}\rangle = (-1)^{n+|\alpha_1|'(|\alpha_2|'+...+|\alpha_{n+1}|')}\langle m_n(\alpha_2, ...\alpha_{n+1}), \alpha_1)\rangle$$

The appearance of the arity $n$, which influence the sign in this formula, does not really fit with the Koszul rule, this is the feature of the *Conv.0*, and this is why it is more convenient to work with the *Conv.1*.

As we will need to refer to these later, let us define separately the cyclic invariance condition and inner form symmetricity in *Conv.1*:

$$\langle m_n(\alpha_1, ..., \alpha_n), \alpha_{n+1}\rangle = (-1)^{|\alpha_1|'(|\alpha_2|'+...+|\alpha_{n+1}|')}\langle m_n(\alpha_2, ...\alpha_{n+1}), \alpha_1)\rangle \tag{2.3}$$

$$\langle x, y\rangle = -(-1)^{|x|'\,|y|'}\langle y, x\rangle \tag{2.4}$$

The notion of pre-Calabi-Yau algebra introduced in [7], [11], [13] seemingly use the fact that $A$ is finite dimensional, since there is no natural grading on the dual algebra $A^* = \mathrm{Hom}(A, \mathbb{K})$, induced form the grading on $A$ in infinite dimensional case. The general definition suitable for infinite dimensional algebra was given in [7], [6], and it is equivalent to the definition, where the $\mathrm{Hom}(A, \mathbb{K})$ is substituted with the graded version: $A^* = \oplus(A_n)^* = \underline{\mathrm{Hom}}(A, \mathbb{K})$. We will give this general definition and show the equivalence further in this section.

**Example.** The most simple example of pre-Calabi-Yau structure demonstrates that this structure does exist on any associative algebra. Namely, the structure of associative algebra on $A$ can be extended to the associative structure on $A \oplus A^*[1-d]$ in such a way, that the natural inner form is (graded)cyclic with respect to this multiplication. This amounts to the following fact: for any $A$-bimodule $M$ the associative multiplication on $A \oplus M$ is given by $(a+f)(b+g) = ab + af + gb$. In this simplest situation both structures on $A$ and on $A + A^*$ are in fact associative algebras. More examples one can find in [3], [10], [1].

One can reformulate the above definition without $A^*$, using the inner product, to change inputs and outputs of operations, and by this to substitute $A^*$ with $A$, as it was done in [7]. The $A_\infty$-structure on $A \oplus A^*$ means first of all the bunch of linear maps

$$m_N : (A \oplus A^*)^N \to A \oplus A^*.$$

Such a map splits as a collection of linear maps of the type

$$\xi = m^{q_1,...,q_l}_{p_1,...,p_l} : A^{\otimes p_1} \otimes A^{*\otimes q_1} \otimes ... \otimes A^{\otimes p_l} \otimes A^{*\otimes q_l} \to A \,(\text{or } A^*)$$

where $\sum p_i + q_i = N$, $0 \leqslant p_i \leqslant N$.

These could be interpreted, using the inner product, as tensors of the type, $A^{\otimes p_1} \otimes A^{*\otimes q_1} \otimes ... \otimes A^{\otimes p_l} \otimes A^{*\otimes q_l}$, $\sum p_i + q_i = N + 1$, and graphically depicted as operations where incoming edges correspond to elements of $A$, outgoing edges to elements of $A^*$ and the marked point correspond to the output of operation $m^{q_1,...,q_l}_{p_1,...,p_l}$. (Of course, due to cyclic invariance operations with different marked points are equal up to sign). This gave rise to the definition below.

But first, we should define *higher Hochschild cochains* and *generalised necklace bracket.*

**Definition 2.3.** For $k \geqslant 1$ the space of *k-higher Hochschild chains* is defined as

$$C^{(k)}(A) := \prod_{r\geqslant 0} \underline{Hom}(A[1]^{\otimes r}, A^{\otimes k})$$

$$= \prod_{r_1,...,r_k\geqslant 0} \underline{Hom}(\overset{k}{\underset{i=1}{\otimes}} A[1]^{\otimes r_i}, A^{\otimes k})$$

Note, that $C^{(1)}(A) = C^{\bullet}(A, A)$ is the space of usual Hochschild cochains.

We can see that element of the higher Hochschild cochain can be interpreted as depicted above operation with $r$ incoming edges and $k$ outgoing edges. There is a marked point in the picture as well, but because of the cyclic invariance condition one can move this marked point with the change of the sign. Indeed, suppose that the last position is marked, then it can be moved to the one but last using the formula:

$$< x_1, m(x_2, ..., x_n) >= (-1)^{\sigma} < x_n, m(x_1, ..., x_{n-1}) >$$

where $x_i \in A$ or $A^*$. Thus, just higher Hochschild cochains, without a specified point will appear in the definition of pre-Calabi-Yau structure.

The composition of two operations of this kind translates according to definition2.2 to the explained above picture via the notion of generalised necklace bracket:

**Definition 2.4.** The *generalised necklace bracket* between two elements $f, g \in C^{(k)}(A)$ is given as $[f, g]_{gen.neckl} = f \circ g - (-1)^{\sigma} g \circ f$, where composition $f \circ g$ consists of inserting all outputs of $f$ to all inputs from $g$ with signs corresponding to the Koszul rule.

**Definition 2.5.** Let $A$ be a $\mathbb{Z}$-graded space $A = \oplus A_n$. The pre-Calabi-Yau structure on $A$ is a solution $m = \sum_{k\geqslant 0} m^{(k)}$, $m^{(k)} \in C^{(k)}(A)$ of the Maurer-Cartan equation $[m, m]_{gen.neckl} = 0$ with respect to. generalised necklace bracket.

**Definition 2.6.** The pre-Calabi-Yau structure on a $\mathbb{Z}$-graded space $A = \oplus A_n$ is a cyclically invariant $A_\infty$ structure on $A \oplus A^*[1-d]$, where $A^*$ is understood as $A^* = \oplus(A_n)^* = \underline{\mathrm{Hom}}(A, \mathbb{K})$.

**Proposition 2.7.** *The definitions of pre-Calabi-Yau structures 2.5 and 2.6 are equivalent.*

*Proof.* To demonstrate this we will start with an element $m = \sum_{k\geqslant 0} m^{(k)}$, $m^{(k)} \in C^{(k)}(A)$, $m^{(k)} = A^{\otimes r} \to A^{\otimes k}$ depicted as an operation with r incoming arrows, k outgoing arrows, and one marked point.

From this data we construct a collection of operations $m_n : (A \oplus A^*)^{\otimes n} \to A \oplus A^*$, to form an $A_\infty$-structure on $A \oplus A^*$.

So let us have an element $\xi$ in the tensor product (in some order) of $r$ copies of $A$ and $k$ copies of $A^*$, where the last position is specified. Thus we have an operation $E : A^{\otimes r} \to A^{\otimes k}$ with one fixed entry. This defines an element $\widehat{\xi} \in (A^*)^{\otimes r} \otimes A^{\otimes k}$ (by means of the natural pairing) such that

$$\langle E(a_1 \otimes ... \otimes a_r), f_1 \otimes ... \otimes f_k\rangle = \langle \widehat{\xi}, a_1 \otimes ... \otimes a_r \otimes f_1 \otimes ... \otimes f_k\rangle$$

Note that here we use the equality $A^{**} = A$, which is true only for finite dimensional spaces. We should make sure that we use duals satisfying $A^{**} = A$, as it is done in definition2.6.

Now we can define an operation from the $A_\infty$-structure on $A \oplus A^*$ corresponding to the above operation $E$,

$$m_{n-1}(a_1, f_1, \ldots, \widehat{f_k})$$

if the marked point have an outgoing edge and

$$m_{n-1}(a_1, f_1, \dots, \widehat{a_r})$$

if the marked point has an incoming edge. Here $n = k + r$ and the order of entries of elements from $A$ and from $A^*$ is dictated by the order in $\xi$. In these two cases we define $m_{n-1}$ as follows:

$$\langle f_k, m_{n-1}(a_1, f_1, \dots, \widehat{f_k})\rangle = \langle\widehat{\xi}, a_1 \otimes \dots \otimes a_r \otimes f_1 \otimes \dots \otimes f_k\rangle;$$
$$\langle a_r, m_{n-1}(a_1, f_1, \dots, \widehat{a_r})\rangle = \langle\widehat{\xi}, a_1 \otimes \dots \otimes a_r \otimes f_1 \otimes \dots \otimes f_k\rangle.$$

□

In spite definition 2.5 looks more beautiful and reveals nice graphically presented connection with A infinity structure, we will use definition 2.6, since we find it easier to work with and make sure all details are correct.

# 3 Structure of the Maurer-Cartan equations

The general Maurer-Cartan equations on $C = A \oplus A^*$ for the operations $m_n : C[1]^n \to C[1]$ have the shape

$$\sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^{\varepsilon} m_p(x_1, \dots, x_{i-1}, m_q(x_j, \dots, x_{i+q-1}), \dots, x_k),$$

where

$$\varepsilon = |x_1|' + \dots + |x_{i-1}|', \qquad |x_s|' = \mathrm{deg} x_s - 1$$

Now the equations we get from the Maurer-Cartan in arities four and five will look as follows. In arity 4, $MC_4$ reads:

$$(-1)^0 m_3(x_1x_2, x_3, x_4) + (-1)^{|x_1|'} m_3(x_1, x_2x_3, x_4) + (-1)^{|x_1|'+|x_2|'} m_3(x_1, x_2, x_3x_4) +$$
$$(-1)^{|x_1|'} m_2(x_1, m_3(x_2, x_3, x_4)) + (-1)^0 m_2(m_3(x_1, x_2, x_3), x_4)$$

In arity 5, $MC_4$ reads:

$$(-1)^0 m_3(m_3(x_1, x_2, x_3), x_4, x_5) + (-1)^{|x_1|'} m_3(x_1, m_3(x_2, x_3, x_4), x_5) +$$
$$(-1)^{|x_1|'+|x_2|'} m_3(x_1, x_2, m_3(x_3, x_4, x_5)) = 0$$

Operations of arity 4 or higher are absent since we are looking for the structure of the form $m = m_2^{(1)} + m_3^{(1)}$.

Since we have Maurer-Cartan equations on $A \oplus A^*$, it essentially means that any equation splits into the set of equations with various distributions of inputs/outputs from $A$ and $A^*$. Note that solutions of the Maurer-Cartan which are interesting for us correspond to operations $A \otimes A \to A \otimes A$ (which can serve as a double bracket). These are operations from tensors with exactly two $A$th and two $A^*$th.

Remind that an operation, say, $A^* \times A^* \times A \to A^*$ can be naturally interpreted as an element of the space $A \otimes A \otimes A^* \otimes A^*$ and this tensor due to cyclic invariance of the structure equals to its cyclic permutations up to sign, in this case $A^* \otimes A \otimes A \otimes A^*$, $A^* \otimes A^* \otimes A \otimes A$ and $A \otimes A^* \otimes A^* \otimes A$.

There is another type of tensor from $A \otimes A^* \otimes A \otimes A^*$ for which there is only one cyclic permutation $A^* \otimes A \otimes A^* \otimes A$. Due to cyclic invariance

$$\langle m_3(f, a, g), b\rangle = \pm\langle m_3(b, f, a), g\rangle$$

operation $A^* \times A \times A^* \to A^*$ corresponding to tensor $A \otimes A^* \otimes A \otimes A^*$ is the same as operation $A \times A^* \times A \to A$ corresponding to tensor $A^* \otimes A \otimes A^* \otimes A$. These tensors encode the second type of operations.

Two types of operations mentioned above which are different up to cyclic permutation on tensors will serve as variables in the equations we obtain from the Maurer-Cartan.

Let us list 6 tensors corresponding to 2 operations, corresponding to two types of *main variables* in our equations.

| Type A | | Type B | |
|---|---|---|---|
| $A^* \otimes A \otimes A \otimes A^*,$ | $A \times A^* \times A^* \to A^*,$ | | |
| $A^* \otimes A^* \otimes A \otimes A,$ | $A \times A \times A^* \to A,$ | $A \otimes A^* \otimes A \otimes A^*,$ | $A^* \times A \times A^* \to A^*,$ |
| $A \otimes A^* \otimes A^* \otimes A,$ | $A^* \times A \times A \to A,$ | $A^* \otimes A \otimes A^* \otimes A,$ | $A \times A^* \times A \to A,$ |
| $A \otimes A \otimes A^* \otimes A^*,$ | $A^* \times A^* \times A \to A^*.$ | | |

**Definition 3.1.** We say that operations corresponding to the tensor $A \otimes A \otimes A^* \otimes A^*$ (and its cyclic permutations) are operations of *type A*, and operations corresponding to the tensor $A \otimes A^* \otimes A \otimes A^*$ (and its cyclic permutations) are operations of *type B*.

The other operations which are also variables in the Maurer-Cartan equation, correspond to the tensors containing not exactly two $A$ and two $A^*$. We call them *secondary type* variables, as opposed to the main type, consisting of variables of type $A$ and $B$. These are the following.

| Secondary type | | | |
|---|---|---|---|
| $A \otimes A \otimes A \otimes A^*,$ | $A^* \times A^* \times A^* \to A^*,$ | $A^* \otimes A^* \otimes A^* \otimes A,$ | $A \times A \times A \to A,$ |
| $A^* \otimes A \otimes A \otimes A,$ | $A \times A^* \times A^* \to A,$ | $A \otimes A^* \otimes A^* \otimes A^*,$ | $A^* \times A \times A \to A^*,$ |
| $A \otimes A^* \otimes A \otimes A,$ | $A^* \times A \times A^* \to A,$ | $A^* \otimes A \otimes A^* \otimes A^*,$ | $A \times A^* \times A^* \to A^*,$ |
| $A \otimes A \otimes A^* \otimes A,$ | $A^* \times A^* \times A \to A,$ | $A^* \otimes A^* \otimes A \otimes A^*,$ | $A \times A \times A^* \to A^*,$ |
| $A \otimes A \otimes A \otimes A,$ | $A^* \times A^* \times A^* \to A,$ | $A^* \otimes A^* \otimes A^* \otimes A^*,$ | $A \times A \times A \to A^*.$ |

Let us look at what we can get from the Maurer-Cartan in arity 5.

First consider the input row containing 4 or more entries from $A$ (or $A^*$). It is easy to check that in this case all terms of equations we get contain secondary type variables. For example, consider the input $A, A, A^*, A, A$. The term $m_3(m_3(a, f, b), c, d)$ is zero if $m_3(a, b, f) \in A$, since $A$ is associative algebra and $m_3(a_1, a_2, a_3) = 0$ for all $a_1, a_2, a_3 \in A$. If $m_3(a, b, f) \in A^*$ then the operation is of secondary type, from tensor $A^* \otimes A^* \otimes A \otimes A^*$.

Another group of equations correspond to input containing three $A$ (or $A^*$). These are divided according to what is the output of the corresponding operation of arity 5. In case of operations with 3 inputs from $A$, two inputs from $A^*$ and output from $A^*$ as well as 3 inputs from $A^*$, two inputs from $A$ and output from $A$, all terms of the equations still contain at least one variable of secondary type.

This property of equations will allow us to restrict any solution of the Maurer-Cartan to the ones containing only main variables (take the projection of solution to the space of main variables, and ensure that we have a solution again).

In the cases of operations with 3 inputs from $A$, two inputs from $A^*$ and output from $A$ as well as 3 inputs from $A^*$, two inputs from $A$ and output from $A^*$, all terms of the equations contain only variables of the main type. Each of these cases corresponds to 10 (5 choose 2) equations on main variables. We consider their structure in more detail. These equations on main variables contain both variables of types $A$ and of $B$. Call variables of type $B$ by $X$'s and of type $A$ by $Y$'s. Then the system of equations again splits into those each term of which contains a $Y$ variable and those which are equations only on $X$'s.

**Lemma 3.2.** *Any equation on main variables coming from $MC_5$ either containing only terms $XX$, i.e. only variables of type $B$ or each term contains at least one variable $Y$ - variable of type $A$.*

*Proof.* Let us see from which inputs terms of type $XX$ can appear. There are two operations of type $B$: I. $A \times A^* \times A \to A$ and II. $A^* \times A \times A^* \to A^*$. Consider the case of composition of the type $m_3(x_1, m_3(x_2, x_3, x_4), x_5))$. In case I. to have a composition of two operations of type B we forced to start with input row $A^*(AA^*A)A^*$. In case II. to have a composition of two operations of type B we forced to start with input row $A(A^*AA^*)A$. The remaining types of compositions: $m_3(m_3(x_1, x_2, x_3), x_4, x_5))$ and $m_3(x_1, x_2, m_3(x_3, x_4, x_5))$ analogously give the same result. Thus the only rows of inputs from which $XX$ term can appear are those two rows. We see moreover that no compositions containing variable Y (operation of type A) appear from this row of input. Thus variables $X$ and $Y$ are separated in the above sense in this system of equations. □

This structure of the system of equations on operations which constitute an unwrapped Maurer-Cartan equation will be a key to relate any pre-Calabi-Yau structure concentrated in appropriate arities to the double Poisson bracket. Each equation that we get from $MC_5$ consists of 'quadratic' terms, meaning terms involving two operations. This system of equations has the feature that in no equation both terms containing two $X$ variables and $XY$ or $YY$ terms appear. These terms are separated. The above arguments allow us to see that

**Proposition 3.3.** *Projection of any $MC_5$ solution to the $B$-type component is also a solution of $MC_5$.*

# 4 Solutions to the Maurer-Cartan equations in arity four and five and double Poisson bracket

In this section we show that the pre-Calabi-Yau structures of type B, namely the ones which are solutions of type B (corresponding to the tensor $A \otimes A^* \otimes A \otimes A^*$ or $A^* \otimes A \otimes A^* \otimes A$) of the Maurer-Cartan equation on $A \oplus A^*$, are in one-to one correspondence with the non-graded double Poisson brackets.

We choose the main example of grading on $A + A^*$ in order to get correspondence with the non-graded double Poisson bracket. Namely, in order to have multiplication on $A$ to be of degree 0 (as it should be in *Conv.0* ), we have to have $A_0 = A$. Then, in order for the type $B$ operations (the most interesting part of the solution of the Maurer-Cartan, which is a ternary operation) to make sense, i.e. according to the *Conv.0*, to be of degree $-1$, we need $A^*$ to be in the component of degree 1. That is, $R = A \oplus A^*$ is graded by $R_0 = A$ and $R_1 = A^*$.

Now we shift this grading by one, to use more convenient formulae of *Conv.1.* Thus we get $inA^{sh} = A[1]$ $A^{sh}_{-1} = A$, and $R^{sh} = A^{sh} + A^{*sh}$ is graded by $R^{sh}_{-1} = A$, $R^{sh}_0 = A^*$, that is $A$ will have degree $-1$, and $A^*$, degree 0, when we are in shifted situation, and in *Conv.1.*

Let $A$ be an arbitrary associative algebra $A = (A, m = m_2^{(1)})$ with a pre-Calabi-Yau structure given as a cyclicly symmetric $A_\infty$-structure on $A \oplus A^*$: $(A \oplus A^*, m = m_2^{(1)} + m_3^{(1)})$. We define the double Poisson bracket via the pre-Calabi-Yau structure, more precisely its component corresponding to the tensor $A \otimes A^* \otimes A \otimes A^*$, as follows.

**Definition 4.1.** The double bracket is defined as:

$$\langle g \otimes f, \{\{b, a\}\}\rangle := \langle m_3(a, f, b), g\rangle,$$

where $a, b \in A$, $f, g \in A^*$ and $m_3(a, f, b) = c \in A$ corresponds to the component of $m_3$: $A \times A^* \times A \to A$.

By choosing this definition we set up a one-to-one correspondence between pre-Calabi-Yau structures of type B and double Poisson brackets from [2]. This choice have been done in such a way that it would be possible to show the double bracket defined above indeed satisfies all axioms of double Poisson bracket. Note, that it is most subtle point, since there are many possibilities for this choice, and only some choices give the required one-to-one correspondence.

We will check that double bracket defined in this way satisfies all axioms of the double Poisson bracket.

Anti-symmetry:

$$\{\{a, b\}\} = -\{\{b, a\}\}^{op} \tag{4.1}$$

Here $\{\{b, a\}\}^{op}$ means the twist in the tensor product, i.e. if $\{\{b, a\}\} = \sum_i b_i \otimes c_i$, then $\{\{b, a\}\}^{op} = \sum_i c_i \otimes b_i$.

Double Leibniz:

$$\{\{a, bc\}\} = b\{\{a, c\}\} + \{\{a, b\}\}c \tag{4.2}$$

and double Jacobi identity:

$$\{\{a, \{\{b, c\}\}\}\}_L + \tau_{(123)}\{\{b\{\{c, a\}\}\}\}_L + \tau_{(132)}\{\{c\{\{a, b\}\}\}\}_L \tag{4.3}$$

Here for $a \in A \otimes A \otimes A$, and $\sigma \in S_3$

$$\tau_\sigma(a) = a_{\sigma^{-1}}(1) \otimes a_{\sigma^{-1}}(2) \otimes a_{\sigma^{-1}}(3).$$

The $\{\{\ \}\}_L$ defined as

$$\{\{b, a_1 \otimes a_n\}\}_L = \{\{b, a_1\}\}_L \otimes a_1 \otimes ... \otimes a_n$$

**Theorem 4.2.** *Let the bracket is defined by the formula*

$$\langle g \otimes f, \{\{b, a\}\}\rangle := \langle m_3(a, f, b), g\rangle,$$

*where $a, b \in A$, $f, g \in A^*$ and $m_3(a, f, b) = c \in A$ corresponds to the component of type B of the solution to the Maurer-Cartan $m_3$: $A \times A^* \times A \to A$. Then this bracket does satisfy all axioms of the double Poisson algebra.*

*Thus, pre-Calabi-Yau structures $(A \oplus A^*, m = m_2^{(1)} + m_3^{(1)})$ of type B (i.e. corresponding to the tensor of the type $A \otimes A^* \otimes A \otimes A^*$ or $A^* \otimes A \otimes A^* \otimes A$) are in one-to-one correspondence with double Poisson brackets $\{\{\cdot, \cdot\}\} : A \otimes A \to A \otimes A$ for an arbitrary associative algebra $A$.*

*Proof.* Anti-symmetry of the double Poisson bracket reads in these notations:

$$\langle f\otimes g, \{\{b,a\}\}\rangle = -\langle g\otimes f, \{\{a,b\}\}\rangle,$$

So we need to check that

$$\langle m_3(b,g,a), f\rangle = -\langle m_3(a,f,b), g\rangle.$$

Indeed, using cyclic invariance, we have

$$\begin{aligned} \langle f\otimes g, \{\{b,a\}\}\rangle &= \langle m_3(a,f,b), g\rangle \\ &= (-1)^{|b|'(|g|'+|a|'+|f|')}\langle m_3(g,a,f), b\rangle = \\ &= (-1)^{|b|'(|g|'+|a|'+|f|')}(-1)^{|g|'(|a|'+|f|'+|b|')}\langle m_3(a,f,b), g\rangle \\ &= -\langle m_3(a,f,b), g\rangle = -\langle g\otimes f, \{\{a,b\}\}\rangle \end{aligned}$$

We used the fact that in our grading $\forall f\in A^*, |f|'=0$ and $\forall a\in A, |a|'=-1$.

By this the anti-symmetry of obtained in this way from pre-Calabi-Yau structure (non-graded) bracket is proven.

Now we deduce the Leibnitz identity from the part of the arity 4 of the Maurer–Cartan equations with inputs from $A$,$A$, $A^*$ and $A$.

General Maurer–Cartan in arity 4 reads:

$$m_3(x_1x_2,x_3,x_4) + (-1)^{|x_1|'}m_3(x_1,x_2x_3,x_4) + (-1)^{|x_1|'+|x_2|'}m_3(x_1,x_2,x_3x_4) +$$
$$(-1)^{|x_1|'}x_1m_3(x_2,x_3,x_4) + m_3(x_1,x_2,x_3)x_4 = 0$$

Applying this to the input $a,b,f,c$ from $A$,$A$, $A^*$, $A$ we have

$$m_3(ab,f,c) + (-1)^{|a|'}m_3(a,bf,c) + (-1)^{|a|'+|b|'}m_3(a,b,fc) + (-1)^{|a|'}am_3(b,f,c) + m_3(a,b,f)c = 0.$$

Since we consider solutions containing only $B$-type components, two terms in the equation ($m_3(a,b,f)c$ and $m_3(a,b,fc)$) vanish, leaving us with

$$m_3(ab,f,c) - m_3(a,bf,c) - am_3(b,f,c) = 0. \tag{4.4}$$

after we applied our grading, where $|a|'=-1$ for all $a\in A$.

Now we pair the above equality obtained from $MC_4$ with $g$ (the equality holds if and only if it holds for any pairing with an arbitrary $g\in A^*$):

$$\langle m_3(ab,f,c), g\rangle - \langle m_3(a,bf,c), g\rangle - \langle am_3(b,f,c), g\rangle = 0.$$

and express the three terms appearing there via the double bracket.

For doing this we need the following lemma.

**Lemma 4.3.** *The following equalities hold:*

$$\text{(R)} \qquad \langle g\otimes af, \{\{b,c\}\}\rangle = \langle g\otimes f, \{\{b,c\}\}a\rangle$$

$$\text{(L)} \qquad \langle ga\otimes f, \{\{b,c\}\}\rangle = -\langle g\otimes f, a\{\{b,c\}\}\rangle$$

*Proof.* (R)

$$\langle g\otimes af,\{\{b,c\}\}\rangle=\sum\langle g\otimes af,b_i\otimes c_i\rangle=$$

$$\sum\langle g,b_i\rangle\langle af,c_i\rangle\overset{(4.5)}{=}\sum\langle g,b_i\rangle\langle f,c_ia\rangle=$$

$$\langle g\otimes f,\sum b_i\otimes c_ia\rangle=\langle g\otimes f,\{\{b,c\}\}a\rangle$$

We use here:

$$\langle af,c_i\rangle\overset{(2.3)}{=}(-1)^{|a|'(|f|'+|c_i|')}\langle fc_i,a\rangle\overset{(2.3)}{=}$$
$$(-1)^{|a|'(|f|'+|c_i|')}(-1)^{|f|'(|c_i|'+|a|')}\langle c_ia,f\rangle$$

$$\overset{(2.4)}{=}(-1)^{|a|'(|f|'+|c_i|')}(-1)^{|f|'(|c_i|'+|a|')}\cdot-(-1)^{|c_ia|'|f|'}\langle f,c_ia\rangle$$

and in our grading, where for all $a\in A, f\in A^*$, $|a|'=-1, |f|'=0$, we get

$$\langle af,c_i\rangle=\langle f,c_ia\rangle \tag{4.5}$$

(L)

$$\langle ga\otimes f,\{\{b,c\}\}\rangle=\sum\langle ga\otimes f\rangle\langle b_i\otimes c_i\rangle=$$

$$\sum\langle ga\otimes b_i\rangle\langle f\otimes c_i\rangle\overset{(4.6)}{=}\sum\langle ab_i,g\rangle\langle f,c_i\rangle$$

$$\overset{(4.7)}{=}-\sum\langle g,ab_i\rangle\langle f,c_i\rangle=-\sum\langle g\otimes f,ab_i\otimes c_i\rangle=-\langle g\otimes f,a\{\{b,c\}\}\rangle$$

We use here:

$$\langle ga,b_i\rangle\overset{(2.3)}{=}(-1)^{|g|'(|a|'+|b_i|')}\langle ab_i,g\rangle$$

$$\langle a,b_ig\rangle\overset{(2.4)}{=}(-1)^{|ab_i|'|g|'}\langle g,ab_i\rangle$$

and in our grading, where for all $a\in A, f\in A^*$, $|a|'=-1, |f|'=0$, we get

$$\langle ga,b_i\rangle=\langle ab_i,g\rangle \tag{4.6}$$

and

$$\langle ab_i,g\rangle=-\langle g,ab_i\rangle \tag{4.7}$$

respectively.

□

Now we are ready to express three terms of the Maurer-Cartan equation 4.4 via the bracket.

$$\langle m_3(ab,f,c),g\rangle\overset{def}{=}\langle g\otimes f,\{\{c,ab\}\}\rangle;$$
$$\langle m_3(a,bf,c),g\rangle\overset{def}{=}\langle g\otimes bf,\{\{c,a\}\}\rangle\overset{R}{=}\langle g\otimes f,\{\{c,a\}\}b\rangle;$$
$$\langle am_3(b,f,c),g\rangle\overset{cycl.m_2}{=}-\langle m_3(b,f,c),ga\rangle\overset{def}{=}-\langle ga\otimes f,\{\{c,b\}\}\rangle\overset{L}{=}\langle g\otimes f,a\{\{c,b\}\}\rangle$$

According to these the Maurer-Cartan can be rewritten as

$$\langle g\otimes f, \{\{c, ab\}\}\rangle - \langle g\otimes f, \{\{c, a\}\}b\rangle - \langle g\otimes f, a\{\{c, b\}\}\rangle$$

which is exactly the Leibniz identity:

$$\{\{c, ab\}\} = \{\{c, a\}\}b + a\{\{c, b\}\}$$

Now it remains to prove that the double bracket defined via the solution of the Maurer-Cartan (of type B) as

$$\langle g\otimes f, \{\{b, a\}\}\rangle = \langle m(a, f, b), g\rangle$$

for all $a, b \in A$, $f, g \in A^*$, does satisfy the Jacobi identity.

The appropriate part of the Maurer-Cartan equation to consider is the part of arity 5, with inputs from $A$,$A^*$, $A$, $A^*$ and $A$.

General Maurer–Cartan in arity 5 reads:

$$(-1)^0 m_3(m_3(x_1, x_2, x_3), x_4, x_5) + (-1)^{|x_1|'} m_3(x_1, m_3(x_2, x_3, x_4), x_5) + \\ (-1)^{|x_1|'+|x_2|'} m_3(x_1, x_2, m_3(x_3, x_4, x_5)) = 0$$

Applying this to the input $a, f, b, g, c$ from $A$,$A^*$, $A$, $A^*$, $A$ we get

$$m_3(m_3(a, f, b), g, c) + (-1)^{|a|'} m_3(a, m_3(f, b, g), c) + (-1)^{|a|'}(-1)^{|a|'+|f|'} m_3(a, f, m_3(b, g, c)) = 0.$$

Thus from the Maurer-Cartan we have.

$$m_3(m_3(a, f, b), g, c) - m_3(a, m_3(f, b, g), c) - m_3(a, f, m_3(b, g, c)) = 0. \tag{4.8}$$

Since we are going to prove the double Jacobi identity:

$$\{\{a, \{\{b, c\}\}\}\}_L + \tau_{123}\{\{b, \{\{c, a\}\}\}\}_L + \tau_{132}\{\{c, \{\{a, b\}\}\}\}_L = 0.$$

we need to express double commutators via the operations - solutions of the Maurer-Cartan equation.

**Lemma 4.4.** *For any $a, b, c \in A$ and $\alpha, \beta, \gamma \in A^*$ the from the definition 4.1 it follows:*

$$\langle \alpha\otimes\beta\otimes\gamma, \{\{a, \{\{b, c\}\}\}\}_L\rangle = \langle m_3(m_3(c, \gamma, b), \beta, a), \alpha\rangle$$

*Proof.*

$$\begin{aligned}\langle \alpha\otimes\beta\otimes\gamma, \{\{a, \{\{b, c\}\}\}\}_L\rangle &= \langle \alpha\otimes\beta, \{\{a, \langle \mathrm{id}\otimes\gamma, \{\{b, c\}\}\rangle\}\}\rangle \\ &= \langle m_3(\langle \mathrm{id}\otimes\gamma, \{\{b, c\}\}\rangle, \beta, a), \alpha\rangle \\ &= \langle m_3(m_3(c, \gamma, b), \beta, a), \alpha\rangle\end{aligned}$$

□

Clearly (4.8) is equivalent to

$$\langle m_3(m_3(a,f,b),g,c),h\rangle - \langle m_3(a,m_3(f,b,g),c),h\rangle - \langle m_3(a,f,m_3(b,g,c)),h\rangle = 0. \tag{4.9}$$

for any $a,b,c \in A$ and $f,g,h \in A^*$.

By Lemma (4.4), the first summand in (4.9) is given by

$$\langle m_3(m_3(a,f,b),g,c),h\rangle = \langle h\otimes g\otimes f, \{\{c,\{\{b,a\}\}\}\}_L\rangle. \tag{4.10}$$

We show now that the second term in (4.9) is expressed via double commutator as:

$$\langle m_3(a,m_3(f,b,g),c),h\rangle = -\langle g\otimes f\otimes h, \{\{b,\{\{a,c\}\}\}\}_L\rangle. \tag{4.11}$$

Indeed, using cyclic invariance and graded symmetry of the inner product, we see

$$\langle m_3(a,m_3(f,b,g),c),h\rangle = (-1)^{|a|'(|m(f,b,g)|'+|c|'+|h|')}\langle m_3(m_3(f,b,g),c),h),a\rangle =$$

$$(-1)^{|a|'(|m(f,b,g)|'+|c|'+|h|')}(-1)^{|m(f,b,g)|'(|c|'+|h|'+|a|')}\langle m_3(c,h,a),m_3(f,b,g))\rangle =$$

$$(-1)^{|a|'(|m(f,b,g)|'+|c|'+|h|')}(-1)^{|m(f,b,g)|'(|c|'+|h|'+|a|')}\cdot -(-1)^{|m(c,h,a)|'|m(f,b,g)|'}\langle m_3(f,b,g),m_3(c,h,a)\rangle =$$

$$(-1)^{|a|'(|m(f,b,g)|'+|c|'+|h|')}(-1)^{|m(f,b,g)|'(|c|'+|h|'+|a|')}\cdot -(-1)^{|m(c,h,a)|'|m(f,b,g)|'}$$
$$(-1)^{|f|'(|b|'+|g|'+|m(c,h,a)|')}\langle m_3(b,g,m_3(c,h,a)),f)\rangle =$$

$$(-1)^{|a|'(|m(f,b,g)|'+|c|'+|h|')}(-1)^{|m(f,b,g)|'(|c|'+|h|'+|a|')}\cdot -(-1)^{|m(c,h,a)|'|m(f,b,g)|'}$$
$$(-1)^{|f|'(|b|'+|g|'+|m(c,h,a)|')}(-1)^{|b|'(|g|'+|m(c,h,a)|'+|f|')}\langle m_3(g,m_3(c,h,a),f),b)\rangle =$$

$$(-1)^{|a|'(|m(f,b,g)|'+|c|'+|h|')}(-1)^{|m(f,b,g)|'(|c|'+|h|'+|a|')}\cdot -(-1)^{|m(c,h,a)|'|m(f,b,g)|'}$$
$$(-1)^{|f|'(|b|'+|g|'+|m(c,h,a)|')}(-1)^{|b|'(|g|'+|m(c,h,a)|'+|f|')}(-1)^{|g|'(|m(c,h,a)|'+|f|'+|b|')}\langle m_3(m_3(c,h,a)),f,b),g\rangle.$$

Taking into account that in our grading $|m(f,b,g)|' = |f|'+|b|'+|g|'+1 = 0$ and $|m(a,f,b)|' = |a|'+|f|'+|b|'+1 = -1$ for all $a,b \in A$, $f,g \in A^*$ we see that the latter sign is '-', hence we get the required:

$$\langle m_3(a,m_3(f,b,g),c),h\rangle = -\langle g\otimes f\otimes h, \{\{b,\{\{a,c\}\}\}\}_L\rangle,$$

since due to Lemma4.4

$$\langle m_3(m_3(c,h,a)),f,b),g\rangle = \langle g\otimes f\otimes h, \{\{b,\{\{a,c\}\}\}\}_L\rangle.$$

Now we consider the third term in (4.9) and show that it is expressed via double commutator as:

$$\langle m_3(a,f,m_3(b,g,c))),h\rangle = -\langle f\otimes h\otimes g, \{\{a,\{\{c,b\}\}\}\}_L\rangle. \tag{4.12}$$

Indeed, using cyclic invariance, we see

$$\langle m_3(a, f, m_3(b, g, c)), h\rangle = (-1)^{|a|'(|f|'+|m(b,g,c)|'+|h|')}\langle m_3(f, m_3(b, g, c), h), a\rangle =$$
$$(-1)^{|a|'(|f|'+|m(b,g,c)|'+|h|')}(-1)^{|f|'(|m(b,g,c)|'+|h|'+|a|')}\langle m_3(m_3(b, g, c), h, a), f\rangle$$

Taking into account signs in our grading, we see that the letter sign is "-", thus by 4.4

$$(-1)^{|a|'(|f|'+|m(b,g,c)|'+|h|')}(-1)^{|f|'(|m(b,g,c)|'+|h|'+|a|')}\langle m_3(m_3(b, g, c), h, a), f\rangle$$
$$-\langle m_3(m_3(b, g, c), h, a), f\rangle = -\langle f \otimes h \otimes g, \{\{a, \{\{c, b\}\}\}\}_L\rangle.$$

Thus 4.9 can be rewritten as:

$$\langle h \otimes g \otimes f, \{\{c, \{\{b, a\}\}\}\}_L\rangle + \langle g \otimes f \otimes h, \{\{b, \{\{a, c\}\}\}\}_L\rangle + \langle f \otimes h \otimes g, \{\{a, \{\{c, b\}\}\}\}_L\rangle = 0$$

We see that permutations of functionals $f, g, h \in A^*$ in our formulas match with the permutation on the images of the bracket in the double Jacobi identity:

$$\tau_{(123)}(h \otimes g \otimes f) = g \otimes f \otimes h, \tau_{(132)}(h \otimes g \otimes f) = f \otimes h \otimes g.$$

Thus we get the required identity4.3:

$$\{\{c\{\{b, a\}\}\}\}_L + \tau_{(123)}\{\{b\{\{a, c\}\}\}\}_L + \tau_{(132)}\{\{a\{\{c, b\}\}\}\}_L = 0.$$

□

# 5 Polyderivations from pre-Calabi-Yau structures induce a Poisson bracket on representation spaces

Consider polyderivations, that is maps $A_1 \otimes \cdots \otimes A_r \to A_1 \otimes \cdots \otimes A_k$, which satisfy kind of Leibniz identities.

**Definition 5.1.** Let $PolyDer(A^{\otimes n}, M)$, for any $A^{\otimes n}$-bimodule $M$, be the space of polyderivations, that is linear maps $\delta : A \otimes ... \otimes A \to M$ satisfying the Leibniz identity:

$$\delta(a_1 \otimes ... \otimes a_i' a_i'' \otimes ... \otimes a_n = (1 \otimes ... \otimes a_i' \otimes ... \otimes 1)\delta(a_1 \otimes ... \otimes a_i'' \otimes ... \otimes a_n) + \delta(a_1 \otimes ... \otimes a_i' \otimes ... \otimes a_n)(1 \otimes ... \otimes a_i'' \otimes ... \otimes 1)$$

**Definition 5.2.** We call a solution of the Maurer-Cartan on $A \oplus A$ (and a corresponding linear map $\delta : A \otimes A \to A \otimes A$) a *restricted polyderivation*, $\delta \in RPolyDer(A^{\otimes 2}, A^{\otimes 2})$, if its projection to $B$-component is a polyderivation.

Now we can see that any pre-Calabi-Yau structure, that is a cyclicly invariant $A_\infty$ structure on $(A \oplus A^*, m)$ concentrated in arities two and three $m = m_2^{(1)} + m_3^{(1)}$, which is a restricted polyderivation gives rise to the Poisson bracket on the space $\mathrm{Rep}(A, m)$ which is $\mathrm{GL_n}$ invariant.

This follows from

**Theorem 5.3.** *Pre-CY structure of appropriate arities $(A \oplus A^*, m = m_2^{(1)} + m_3^{(1)})$, which is additionally a restricted polyderivation $\delta : A \otimes A \to A \otimes A$, $\delta \in RPolyDer(A^{\otimes 2}, A^{\otimes 2})$ gives rise to the double Poisson bracket.*

*Proof.* As we have shown before in Theorem4.2, that there is one-to-one correspondence between pre-Calabi-Yau structures of type $B$ and double Poisson brackets. We now want to show that for any solution of $MC_5$, its projection to the space of solutions of type $B$ is also a solution. Thus this projection will create a double bracket.

This comes from the consideration of Section 2 on the structure of equations arising from $MC_5$. We showed that all equations but those which are entirely on operations of type $B$ contain in each term at least one operation of type $A$. Hence if we replace in a given solution of the Maurer-Cartan equation all $Y$s (corresponding to operations of type $A$) by zero, we get all equations with $Y$ in them automatically satisfied. System of equations arising from $MC_5$ turns into its restriction to those equations which are on operations of type $B$ only. The latter, as we know (Theorem4.2) under the assignment

$$\langle g \otimes f, \{\{a,b\}\}\rangle := \langle g, m_3(a,f,b)\rangle$$

(which automatically has antisymmetry) coincides with the double Jacobi identity.

Unfortunately, the analogous procedure of projection of any solution onto the components of type $B$ does not work in the same way for $MC_4$. This is why we additionally asked for our arbitrary pre-Calabi-Yau structure to be a restricted polyderivation. Now the type $B$ component of any Maurer-Cartan solution gives us a double Poisson bracket. □

**Corollary 5.4.** *Any pre-Calabi-Yau structure of an arbitrary associative algebra $A$ of signature $(A\oplus A^*, m = m_2^{(1)} + m_3^{(1)})$ which is a restricted polyderivation, $\rho : A\otimes A \to A\otimes A$ induces a Poisson structure on representation spaces* $\mathrm{Rep}(A,n)$*, which is $Gl_n$-invariant.*

*Proof.* It comes as a direct consequence of Van den Bergh's construction for double Poisson bracket [2], after the application of Theorem5.3. □

Note that there was a considerable freedom in the choice of definition for the double bracket via the pre-Calabi-Yau structure (in spite it is in many ways defined by various features of the double bracket), but only the one presented in definition4.1, together with appropriate choice of the grading on $A \oplus A^*$ allowed to deduce the axioms of double bracket. By this choice we thus found an embedding of a double Poisson structures into pre-Calabi-Yau structures.

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