On the Ramanujan conjecture for automorphic forms over function fields I. Geometry

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Abstract. Let G be a split semisimple group over a function field. We prove the temperedness at unramified places of automorphic representations of G, subject to a local assumption at one place, stronger than supercuspidality, and assuming the existence of cyclic base change with good properties. Our method relies on the geometry of Bun_G. It is independent of the work of Lafforgue on the global Langlands correspondence.

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1. Main result

Let F be the function field of a smooth projective curve over a finite field k. The Ramanujan conjecture that every cuspidal automorphic representation of $\mathrm{GL}(n)$ is tempered is established by L. Lafforgue [33]. This is in sharp contrast with the situation for general reductive groups, where the only concrete examples of automorphic representations that are known to be tempered, not derived in a straightforward way from $\mathrm{GL}(n)$, are in the recent works of Heinloth-Ngô-Yun [26] and Yun [54, 53].

For a reductive group G, it is well-known that the cuspidality condition is not sufficient to imply temperedness, which led to the formulation of Arthur's conjectures [2]. For example, there are two classical constructions of cuspidal non-tempered automorphic representations for GSp_4 by Saito-Kurokawa and Howe–Piatetskii-Shapiro [28].

Thus we need a condition on π stronger than cuspidality. We shall impose that π_u is supercuspidal for one place u. This is still not sufficient as the above examples [28] show, and Arthur's

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conjecture points towards the condition that π_u belongs to a supercuspidal L-packet. We shall introduce a further condition that π_u is *monomial geometric supercuspidal*, and establish the Ramanujan conjecture in this case. The concept will be discussed in details below. In brief it means that π_u is compactly induced from a character on a "nice enough" open subgroup of $G(F_u)$. We also need another Condition BC from Section 5 below, on the existence of an automorphic base change for constant field extensions.

Theorem 1.1. *Assume that G is split semisimple, and that* $\mathrm{char}(F) > 2$. *Suppose that*

- *for at least one place u, the representation π_u is monomial geometric supercuspidal;*
- *π is base-changeable in the sense of Condition BC.*

Then π is tempered at every unramified place.

Langlands's theorem on the analytic continuation of Eisenstein series implies that CAP representations are non-tempered at every unramified place. Combined with Theorem 1.1, it follows that π is not CAP.

Remark 1.2. Recently, V. Lafforgue [35] constructed global parameters using shtukas and excursion operators. An automorphic consequence is that π is tempered at one unramified place if and only if it is tempered at every unramified place, which was [8, Conj. 4].

The present paper focuses on establishing a Ramanujan bound on average, see (1.1) below, and deducing Theorem 1.1. It is part of a series of two articles, and the next [45] will focus on providing examples of representations that satisfy Condition BC, and on establishing the functorial image between inner-forms which will enable us to reduce cases of the Ramanujan bounds for general reductive groups to the split semisimple case.

1.1. Monomial geometric supercuspidal representations.

The definition of monomial geometric supecuspidal is motivated by features of the problem and our method to attack it.

We rely on studying families defined by local conditions. If we can show temperedness for one member of the family, the same argument applies to every member of the family. So we must impose a strong enough local condition. At minimum, we should avoid Eisenstein series, and a supercuspidal representation is the easiest way to achieve this.

Our method is geometric, and requires a geometric way to check the local condition. Monomial local conditions (the condition that a representation contains a vector which transforms according to a one-dimensional character $\chi : J \to \mathbb{C}^\times$ under the action of a subgroup J) can be represented geometrically in a natural way as long as J is the group of rational points of an algebraic subgroup and χ is the trace function of a character sheaf. This is certainly not the most general possible way to construct a geometric object that defines a local condition – in fact the geometric Langlands program suggests that there should be geometric objects corresponding to all automorphic representations, in a suitable sense – but it is easy to work with and contains many important examples. A general formalism of monomial local conditions for automorphic representations was already used by Yun [52, 2.6.2]. Our setup (Section 6) is essentially Yun's formalism restricted to a special case for both geometric and notational simplicity (and for this reason we use somewhat different notation).

Geometric objects behave similarly over different fields. In our case, the relevant geometric objects are defined over the constant field, and so it is possible to base change them along a constant field extension. If we use any geometric property to prove temperedness, this property will be maintained over constant field extensions, and so temperedness must hold not only for all members of the family, but also for all members of the analogous family after extension of the constant field. In particular, these representations must not be Eisenstein series. Again, the easiest way to ensure this is to ensure that our character (J, χ) still prescribes a supercuspidal representation after a constant field extension. This yields the notion of monomial geometric supercuspidal datum (Definition 3.5).

Another advantage of adding the monomial and geometric modifiers to the supercuspidal local condition is that it allows us to sidestep the unipotent supercuspidal representations. The usual construction of these is not by a monomial representation but rather from representations of finite groups of Lie type. We expect that no monomial geometric construction of unipotent representations exists. For example in Deligne–Lusztig theory, irreducible representations are induced from characters on elliptic tori, but this fails to work uniformly after finite fields extensions, since every torus eventually splits.

The local conditions we define are "geometric" in precisely the sense of the geometric Langlands program. However, there is one major difference in our approach. Progress in the geometric Langlands program has mainly focused on first studying automorphic forms that are everywhere unramified, and then generalizing to unipotent or more general ramification, before beginning to tackle the general case. In our problem, we find it is convenient to study highly ramified automorphic forms - in particular, including local factors with wildly ramified Langlands parameters - which necessitates working in a more general setup. We do this because when one of the local factors is supercuspidal, the Hecke kernels in the family will correspond to pure perverse sheaves (Theorem 7.33), although we also believe the more general setup is interesting on its own terms.

More formally, let $\mathbf{G}$ be a quasi-split reductive group over a field κ. We start with the data of an algebraic subgroup H of $G[\![t]\!]$ containing the subgroup of elements congruent to 1 modulo t^m for some m, and a character sheaf $\mathcal{L}$ on H which is trivial on that subgroup. We say this data is geometrically supercuspidal if for every parabolic subgroup $P \subset \mathbf{G}$ with radical N, and every $g \in \mathbf{G}[\![t]\!]$, the restriction of $\mathcal{L}$ to $gNg^{-1} \cap H$ is not geometrically isomorphic to a constant sheaf. If $\kappa = \mathbb{F}_q$ is a finite field, this occurs if and only if $\text{c-ind}_{J_n}^{\mathbf{G}(\mathbb{F}_{q^n}(\!(t)\!))} \chi_n$ is admissible supercuspidal for every integer $n \geq 1$, where $J_n := H(\mathbb{F}_q^n)$ and χ_n is the trace function of H over $\mathbb{F}_{q^n}$.

1.2. Ramanujan bound for GL(n).

For the general linear group, the Ramanujan bound is the statement that a cuspidal automorphic representation of $\mathrm{GL}(n)$ over a function field F is tempered at every place. One can distinguish two main approaches:

- Laumon [40] under a cohomological condition at one place, extending Drinfeld's first proof [15] for GL(2), using elliptic modules.
- L. Lafforgue [33] in general, extending Drinfeld's second proof [17] for GL(2), using shtukas.

Our approach is yet different, even in the case of $\mathrm{GL}(n)$, under the mgs condition. Rather than using moduli spaces of elliptic modules or shtukas, we study moduli spaces $\mathrm{Bun}_{\mathrm{GL}(n)}$ of vector bundles, as in the geometric Langlands program. Functions on these moduli spaces give rise to families of automorphic forms satisfying certain local conditions. We will prove temperedness using estimates for an entire family at once, rather than working with individual automorphic forms in the family.

1.3. Outline of the proof.

We embed π in a suitable automorphic family $\mathcal{V}$. It consists of automorphic representations Π of $G(\mathbb{A}_F)$, such that Π_u has a non-zero (J, χ)-invariant vector, with bounded ramification at a finite set of places, and unramified elsewhere. More generally, for every integer $n \geq 1$, we define $F_n = F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $\mathcal{V}_n$ as a multi-set of automorphic representations of $G(\mathbb{A}_{F_n})$, counted with multiplicity, and with similar prescribed behavior. All $\Pi \in \mathcal{V}_n$ are cuspidal.

Let v be an unramified place. To study the temperedness of π_v, we shall consider the local components Π_v for $\Pi \in \mathcal{V}_n$. More precisely, for a coweight λ, we shall consider the collection of all Hecke eigenvalues $\operatorname{tr}_\lambda(\Pi_v)$.

We express the kernel of this Hecke operator as the trace function of a complex of sheaves, which we will show, under our mgs local hypothesis, is a pure perverse sheaf (Theorem 7.33). This will imply, by standard estimates for the trace functions of perverse sheaves, a bound for the trace of a Hecke operator in the family (Theorem 10.2), which takes the form

$$\sum_{\Pi \in \mathcal{V}_n} |\operatorname{tr}_\lambda(\Pi_v)|^2 \leq C_\lambda \cdot q^{nd} \tag{1.1}$$

Here d depends on the underlying group and level conditions, and C_λ is the dimension of some cohomology groups and it is essential for us that it is independent of n. In examining this formula it is helpful to observe that, in the case $\lambda = 0$, it implies that the total number of automorphic forms in the family is $O(q^{nd})$. By comparing the definition of d to the relevant adelic volume we can see that this is the correct order of magnitude. Furthermore, under the Ramanujan conjecture, the estimate we obtain for the individual terms $|\operatorname{tr}_\lambda(\Pi_v)|$ is $\dim(V_\lambda)$. So this bound is exactly the size one would hope for, except for the constant C_λ.

Because C_λ is constant in n while every other term is exponential in n, the quality of this estimate improves as n goes to infinity. To take advantage of this, we will use automorphic base change for constant field extensions to amplify the estimate, and deduce $|\operatorname{tr}_\lambda(\pi_v)| \leq \dim(V_\lambda) \cdot q^{\frac{d}{2}}$ for our original representation π. Varying λ, we can further bootstrap this estimate to $|\operatorname{tr}_\lambda(\pi_v)| \leq \dim(V_\lambda)$ which is the temperedness of π_v.

Remark 1.3. Recall from [8] the following conjecture: π should be tempered at every unramified place as soon as π_u is the Steinberg representation for some place u. Compared to this, our situation consists in replacing the Steinberg condition by a more ramified condition. Our method of proof doesn't extend to the case of the Steinberg representation because the Euler-Poincaré function is an alternating sum, which we do not know how to geometrize globally to a pure sheaf on Bun_G.

1.4. **Constrasting Drinfeld's modular varieties and** Bun_G**.** The moduli spaces of shtukas and Bun_G are both stacks whose geometry carries information about automorphic forms over function fields, but they carry it in different ways and have different properties.

Each moduli space of shtukas can be related to a particular family of automorphic forms with a particular set of Hecke operators acting it. The geometry of the moduli space casts light on this family. More precisely, the cohomology of the moduli space relative to the base is expected to be a sum over automorphic forms with fixed level structure of local systems constructed from their Langlands parameters. The arithmetic structure on the moduli space carries additional information about the automorphic forms in this family. For instance, the Galois action on the cohomology of a moduli space of GL_r-shtukas with level structure determines the Galois action on the Langlands parameters of the cusp forms of that level [33, Lemma VI.26 and Theorem VI.27].

On the other hand, Bun_G, and its variants with level structure, are each related to a sequence of spaces of automorphic forms, one over each finite field extension $\mathbb{F}_{q^n}$ of the base field $\mathbb{F}_q$. The spaces of automorphic forms arise as the spaces of functions on the $\mathbb{F}_{q^n}$-points of Bun_G, i.e., they are defined arithmetically in terms of rational points (Remark 6.8). Because geometry is insensitive to base change, the geometry of Bun_G is only related to asymptotic information about this sequence of spaces as $q^n \to \infty$ (or possibly other subtler sorts of information that are invariant on passing to subsequences). For instance, by the Lefschetz fixed point formula, the dimension of the space of automorphic forms equals the number of $\mathbb{F}_{q^n}$-points which equals the supertrace of Frobenius on the cohomology (Lemma 9.9 and Proposition 10.1), so the cohomology of Bun_G gives information about how complex a single formula to describe the dimension of all the spaces of automorphic forms in the sequence must be. (Although for any nonabelian G these spaces will have infinitely many $\mathbb{F}_{q^n}$-points, and so something must be done to remove Eisenstein series before this can be made precise – in our paper, supercuspidal local conditions, defined in Section 3, are used.)

This fundamental difference can explain many of the more basic difference between the geometry of the moduli space of shtukas and Bun_G. For instance, their dimensions. Because the cohomology of the compactified moduli space of shtukas should be a sum of contributions associated to different automorphic forms, with each contribution the tensor product of the representations associated to the Hecke operator at legs composed with the Langlands parameter, the dimension, which appears as the degree and the size of the Weil numbers in the cohomology, should be determined only by the Hecke operators at the legs. In particular, raising the level does not raise the dimension, and corresponds to taking a finite étale cover. On the other hand, the top cohomology of Bun_G should be the leading term in the dimension of the space of automorphic forms, and therefore the size of the Weil numbers, and thus the dimension, should be determined by the asymptotic growth rate of the dimension of the sequence of families of automorphic forms as $q^n \to \infty$. Thus, adding level structure, which multiplies the expected main term in the trace formula by a polynomial in q^n, should raise the dimension, and instead corresponds to a fibration of stacks by a linear algebraic group. In fact, the dimension of Bun_G is $(\dim G)(g-1)$, and the dimension with full level D structure for a divisor D is $(\dim G)(g+|D|-1)$, which matches the size of the adelic volume contribution to the number of cusp forms of G on $\mathbb{F}_{q^n}(X)$ in the trace formula. Further, the number of forms with a nonzero (J,χ)-equivariant vector has leading exponent $(\dim G)(g+|D|-1) - \dim H$ (see §10.3).

This also suggests differences in their potential arithmetic applications. The moduli spaces of shtukas are well-suited to prove the automorphic-to-Galois direction of the Langlands correspondence because each automorphic form, and its associated Langlands parameter, appears in their cohomology. Of course this is exactly why Drinfeld [15] introduced them and how L. Lafforgue [33] and V. Lafforgue [35] used them, and it seems likely that researches will continue to deduce information about the Langlands correspondence from study of these moduli spaces in the future. But Bun_G is not well-suited for this purpose, as with the number of automorphic forms going to infinity as $q^n \to \infty$, it is harder to pick out a single one. Though an analogue of the automorphic-to-Galois direction of the Langlands correspondence is part of the geometric Langlands program over the complex numbers, it is not clear what, if any, the finite field analogue might be. On the other hand, Bun_G does seem well-suited to answer asymptotic questions about how analytic quantities, such as averages of Hecke operators, behave when $q^n \to \infty$, as we demonstrate in the present paper. The Ramanujan conjecture seems to lie in the intersection of these two domains – it can be attacked using Langlands parameters, but also can be viewed as a question of the $q^n \to \infty$ limit. Thus there is potential to use both approaches to prove new cases of the Ramanujan conjecture.

1.5. **Results on families.** Because our method to prove the main theorem relies on families of automorphic forms defined by geometric monomial local conditions, along the way we obtain some new results about these families. We expect further results can be obtained this way using our work in the future. For this reason we discuss the strengths and weaknesses of restricting to monomial representations from the point of view of families (rather than with regards to proving the Ramanujan conjecture for individual automorphic forms). Given a family of automorphic forms unramified away from some finite set of places, and defined by some local conditions at the remaining places, questions such as the following have considered:

(1) Can the number of forms in the family be expressed as a finite sum of Weil numbers?
(2) What about the trace of a Hecke operator on this space of forms?
(3) Can the Weil numbers that appear in these sums be calculated explicitly?
(4) Can these sums be approximated, or can the largest Weil numbers appearing in them be estimated?

Question (1) and question (3) were answered affirmatively by Drinfeld [16] in the case of everywhere unramified automorphic forms on GL_2, by Flicker for forms that are Steinberg at one place and unramified everywhere else, and by Deligne and Flicker [14] for forms on GL_n that are Steinberg at at least two places, and unramified everywhere else. Of course answering (3) is sufficient to answer question (4).

In this paper we answer question (1) in the case of monomial geometric conditions, supercuspidal at at least one place, and unramified elsewhere (Proposition 10.4). And most importantly we answer question (2), in the form that $\sum_{\Pi\in\mathcal{V}_n} q^{n\langle\lambda,\rho\rangle}|\operatorname{tr}_\lambda(\Pi_v)|^2$ is a signed sum of length C_λ of nth powers of q-Weil integers of weight $\leq 2d+\langle\lambda,2\rho\rangle$. This is actually how we establish the main estimate (1.1). See Theorem 9.15 and §10.3 for details.

1.6. **Local conditions.** There are many different kinds of local conditions that appear in the theory of automorphic forms. As mentioned before, we work with local conditions that demand the representation contain an eigenvector of a compact open subgroup J with eigenvalue χ, where J and χ arise from geometric objects - an algebraic subgroup of $G(\kappa[t]/t^m)$ for some m and a character sheaf on that algebraic subgroup. The theory of inertial types produces many examples where this condition, for a suitable choice of J,χ, characterizes the representation up to an unramified twist (e.g. the twist-minimal supercuspidal representations of GL_2 with conductor not congruent to 2 modulo 4). However, not all representations can be characterized up to an unramified twist this way (e.g. the twist-minimal supercuspidal representations of GL_2 with conductor congruent to 2 mod 4). But it may still be possible to characterize the representation up to a tamely ramified twist or other mild variant.

Choosing J,χ whose associated local condition uniquely picks out a given representation is very similar to the problem of constructing the representation as an induced representation (but slightly easier as one is allowed to produce the representation with multiplicity). Yu has shown how to construct a wide class of supercuspidal representations using Deligne-Lusztig representations of algebraic groups over finite fields and Heisenberg-Weil representations. (For instance, in the GL_2 twist-minimal case with conductor congruent to 2 mod 4, Deligne-Lusztig theory is needed for conductor 2 and Heisenberg-Weil representations are needed for higher conductor).

The matrix coefficients of the Weil representation were expressed as the trace function of a perverse sheaf in a 1982 letter of Deligne, and the same was done in [24] to the coefficients in a basis consisting of the matrices appearing in the Heisenberg representation. It is likely that much of what we do can be generalized using this geometrization. Sheaves whose trace functions are the traces of discrete series representations were constructed [41] but we do not know if there is any way to do the same for matrix coefficients (it is not clear what basis to use). It could also be possible to replicate our methods using just the trace and not all the matrix coefficients, but we are less certain of it.

Using these tools to make these representations geometric would follow the strategy of [11]. Note, however, some differences with their work. Their goal was to geometrize the trace of the automorphic representation, while our construction has the effect of geometrizing a test function, and they handled p-adic groups while we work in the equal characteristic case.

For our problem, new difficulties appear when adding these new representations and their more complicated sheaves. Because restricting to one-dimensional characters, and their associated character sheaves, will simplify things at several points, we leave the full theory to a later date.

2. Preliminaries

2.1. **Unramified groups.** Let k be a finite field. A connected reductive group G over $k(\!(t)\!)$ is unramified if it is quasi-split and splits over $\overline{k}(\!(t)\!)$. The following is well-known. Since we couldn't locate the result in the literature, we provide a quick proof.

Lemma 2.1. *An unramified group over $k(\!(t)\!)$ is the base change $\mathbf{G}_{k(\!(t)\!)}$ of a quasi-split reductive group $\mathbf{G}$ over k.*

Proof. Recall from Bruhat-Tits [5, 4.6.10], and [37, Chap. II], the existence of smooth affine group schemes over $\mathfrak{o}_k$, with reductive special fiber, and special fiber G. According to [9, Remark 7.2.4], the classification of forms of a reductive group over a Henselian local field with finite residue field is the same as the classification over the residue field. Indeed let $\mathcal{G}$, and $\mathcal{G}'$ be two connected reductive group schemes over $k[\![t]\!]$. Suppose their special fiber over k are isomorphic. The scheme of isomorphisms from $\mathcal{G}$ to $\mathcal{G}'$ is smooth, and has a point over k, so has a section over $k[\![t]\!]$. In particular if we take $\mathcal{G}'$ to be a constant group scheme, we get that $\mathcal{G}$ is constant as well. □

Remark 2.2. The same notion of unramified group G arises in mixed characteristic, that is over a finite extension K of $\mathbb{Q}_p$. In that context, it is standard that there is a smooth model $\mathcal{G}$ over the local ring $\mathfrak{o}_K$, and that $\mathcal{G}(\mathfrak{o}_K)$ is a hyperspecial maximal subgroup. This is analogous to Lemma 2.1, where the model is given by $\mathbf{G}_{k[\![t]\!]}$, and the hyperspecial maximal subgroup by $\mathbf{G}(k[\![t]\!])$, only that in equal characteristic the statement is simpler, and it is not necessary to introduce the group scheme $\mathcal{G}$. In mixed characteristic, the lifting argument still works, but there is no notion of constant group scheme over $\mathfrak{o}_K$ (though an analogue could likely be constructed using Witt vectors).

Lemma 2.3. *Let G be a reductive group over a finite field k. Let X be a smooth connected algebraic curve over k. Then every G-torsor on X admits a trivialization over the generic point.*

Proof. Let $F = \mathbb{F}_q(X)$. By [42, Lemma 1.1], it is sufficient to check that the kernel $\ker^1(F, G)$ of the natural map from $H^1(F, G)$ to the product over all places x of $H^1(F_x, G)$ is trivial. By [50, Theorem 2.6(1)], the kernel $\ker^1(F, G) = \ker^1(F, Z(\widehat{G}))$. Then $Z(\widehat{G})$ is a finite abelian group with an action of Frob_q. Every torsor can be described as an action of the Galois group of F on a finite set. We must check that, if it is a nontrivial torsor, then it remains nontrivial upon restriction to the Frobenius element at some place. If the Galois action factors through the Galois group of k, then the Frobenius element at any place of degree prime to the order of the Galois group generates the whole group, and so it is nontrivial if and only if it is nontrivial at one of these places. If it does not factor through the Galois group of k, we may pass to a finite extension of k where the Galois action on G is trivial, and then because the Galois action remains nontrivial upon restriction, by Chebotarev some Frobenius element must act nontrivially, which implies it is a nontrivial torsor for a group with trivial Frobenius action. $\square$

2.2. **Satake isomorphism.** In this subsection, let G be a split connected reductive group over a finite field k. Let $F = k(\!(t)\!)$, $\mathfrak{o} = k[\![t]\!]$, $K = G(\mathfrak{o})$, and consider the unramified Hecke algebra

$$\mathcal{H}(G) = \mathcal{H}(G(F), K) = \mathcal{C}_c(K\backslash G(F)/K, \mathbb{C}).$$

The below results hold more generally over the base ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ rather than $\mathbb{C}$, see e.g. [23]. Let $T \subset G$ be a maximal torus. There is an identification $\mathbb{C}[X_*(T)] \simeq \mathcal{C}_c(T(F)/T(\mathfrak{o}))$, where a cocharacter $\lambda : \mathbb{G}_m \to T$ corresponds to the characteristic function $1_{\lambda(t)T(\mathfrak{o})}$. The Weyl group $W = N_G(T)/Z_G(T)$ acts on both sides of the above isomorphism, in particular we can form the subalgebras

$$\mathbb{C}[X_*(T)]^W \simeq \mathcal{C}_c(T(F)/T(\mathfrak{o}))^W.$$

Choose a Borel subgroup $B = TU$. Let $\delta : B(F) \to q^{\mathbb{Z}}$ be the modulus character, where q is the size of k. Denote by $\delta^{\frac{1}{2}} : B(F) \to q^{\frac{1}{2}\mathbb{Z}}$, the positive square-root. For every $\lambda \in X_*(T)$, we have $\delta^{\frac{1}{2}}(\lambda(t)) = q^{-\langle \rho, \lambda \rangle}$, where $\rho \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ is the half-sum of the positive roots. The Satake transform $\mathcal{S}(f)$ of a function $f \in \mathcal{H}(G) = \mathcal{H}(G(F), K)$ is defined by

$$\mathcal{S}(f)(s) := \delta^{\frac{1}{2}}(s) \int_{U(F)} f(su)du, \quad s \in T(F),$$

where du is the Haar measure on $U(F)$ that gives $U(F) \cap K = U(\mathfrak{o})$ volume one. The value of the integral depends only on s modulo $T(\mathfrak{o})$. It induces a $\mathbb{C}$-algebra isomorphism $\mathcal{S} : \mathcal{H}(G) \to \mathcal{C}_c(T(F)/T(\mathfrak{o}))^W$.

Lemma 2.4. *There is a bijection between isomorphism classes of K-unramified representations π, algebra homomorphisms $\mathrm{tr}(\pi) : \mathcal{H}(G) \to \mathbb{C}$, W-conjugacy classes of unramified characters $\chi : T(F)/T(\mathfrak{o}) \to \mathbb{C}^\times$, and semisimple conjugacy classes t_π in $\widehat{G}(\mathbb{C})$.*

Proof. The character χ is such that π is the K-unramified representation π_χ that is the irreducible quotient of the principal series induced from χ. The relationship with the Satake isomorphism is that we have

$$\mathrm{tr}(\pi_\chi)(f) = \int_{T(F)/T(\mathfrak{o})} \chi(s)\mathcal{S}(f)(s)ds, \quad f \in \mathcal{H}(G).$$

The relationship between χ and t_π is via the identification

$$\mathrm{Hom}(X_*(T), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times) = \widehat{T}(\mathbb{C}).$$

Let $\chi_\lambda \in \mathbb{C}[X_*(\widehat{T})]^W$ be the trace of the representation V_λ of $\widehat{G}(\mathbb{C})$ with highest weight $\lambda \in \Lambda^+ \subset X^*(T) = X_*(\widehat{T})$. Here Λ^+ denotes the positive Weyl chamber. Let $a_\lambda \in \mathcal{H}(G) \simeq \mathbb{C}[X_*(\widehat{T})]^W$ be the corresponding element in the Hecke algebra. The relationship between $\mathrm{tr}(\pi)$ and t_π is that for every $\lambda \in \Lambda^+$,

$$\mathrm{tr}(\pi)(a_\lambda) = \chi_\lambda(t_\pi) = \mathrm{tr}(t_\pi|V_\lambda).$$

$\square$

Definition 2.5. For a coweight $\lambda \in \Lambda$, and a K-unramified irreducible representation π, define

$$\mathrm{tr}_\lambda(\pi) := \mathrm{tr}(\pi)(a_\lambda) = \mathrm{tr}(t_\pi|V_\lambda).$$

We have that π is tempered if and only if χ is unitary, if and only if t_π is a compact element, if and only if for every λ,

$$|\operatorname{tr}_\lambda(\pi)| \leq \dim V_\lambda.$$

In fact, it is sufficient to have $|\operatorname{tr}_\lambda(\pi)| \leq C \cdot \dim V_\lambda$, for some constant C that is independent of λ, or more generally $|\operatorname{tr}_\lambda(\pi)| \leq C_\epsilon \cdot q^\epsilon$ for every $\epsilon > 0$, where C_ϵ is independent of λ.

We have been using consistently the unitary normalization of the of the Satake transform, of the Satake parameter t_π, and of parabolic induction. There is also an algebraic normalization, which is that $q^{\langle\lambda,\rho\rangle} a_\lambda$ corresponds to the trace function of the IC sheaf of $\overline{\mathrm{Gr}_\lambda}$. This will be used in Lemma 9.5 below, and yields to integrality properties of the Weil numbers that appear in the trace formula.

Example 2.6. For the trivial representation $\mathbf{1}$, we have $\operatorname{tr}_\lambda(\mathbf{1}) = \operatorname{tr}(\mathbf{1})(a_\lambda)$. The Satake parameter $t_{\mathbf{1}}$ is equal to the principal semisimple element $\rho(q) \in \widehat{T}(\mathbb{C})$, where ρ is seen as a cocharacter $X_*(\widehat{T})_{\mathbb{C}}$. In particular, we obtain

$$\sum_{x \in K\backslash G/K} a_\lambda(x) = \operatorname{tr}(\mathbf{1})(a_\lambda) = \operatorname{tr}(\rho(q)|V_\lambda) = q^{\langle\lambda,\rho\rangle}\left(1 + O(q^{-\frac{1}{2}})\right).$$

We conclude that $\operatorname{tr}_\lambda(\mathbf{1}) = q^{\frac{d(\lambda)}{2}}$, where

$$d(\lambda) := \langle\lambda, 2\rho\rangle = \dim \mathrm{Gr}_\lambda \in \mathbb{Z}_{\geq 0},$$

which we interpret as the degree of the Hecke operator of coweight λ.

2.3. **Base change.** Notation is as in the previous subsection, and we consider the degree n extension $k' = \mathbb{F}_{q^n}$ of $k = \mathbb{F}_q$. There is a base change algebra homomorphism $b : \mathcal{H}(G_{k'}) \to \mathcal{H}(G)$, see e.g. [31]. For any K-unramified irreducible representation π of $G(k(\!(t)\!))$, there corresponds a unique K'-unramified irreducible representation Π of $G(k'(\!(t)\!))$ such that $\operatorname{tr}(\Pi)(f) = \operatorname{tr}(\pi)(b(f))$ for every $f \in \mathcal{H}(G_{k'})$. Indeed the corresponding Satake parameters satisfy the relation $t_\Pi = t_\pi^n$. In particular the representation π is tempered if and only if the base change representation Π is tempered. We can identify the positive Weyl chaber $\Lambda \subset X_*(T)$ for the groups G and $G_{k'}$. We have then the relation,

$$\operatorname{tr}_\lambda(\Pi) = \operatorname{tr}(t_\pi^n|V_\lambda),$$

which will be used often in relation to taking the limit as $n \to \infty$.

2.4. **Character sheaves.**

Definition 2.7. For a connected algebraic group H, say a *character sheaf* on H is a rank one lisse sheaf $\mathcal{L}$ with an isomorphism between $\mathcal{L} \boxtimes \mathcal{L}$ and the pullback of $\mathcal{L}$ along the multiplication map $H \times H \to H$.

Remark 2.8. Given a character sheaf $\mathcal{L}$, we have an isomorphism $\mathcal{L}_e = \mathcal{L}_e \otimes \mathcal{L}_e$, hence an isomorphism $\overline{\mathbb{Q}}_\ell = \mathcal{L}_e$. Using the isomorphism between $\mathcal{L} \boxtimes \mathcal{L}$ and the pullback of $\mathcal{L}$, and associativity, we can define two isomorphism between $\mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{L}$ and the pullback of $\mathcal{L}$ to $H \times H \times H$. These two isomorphisms are necessarily equal, because they are maps between lisse sheaves on a connected scheme and are equal on the identity point.

For convenience, we give here many important facts about character sheaves, almost all of which are surely well-known.

Lemma 2.9. *Let H be an algebraic group over a finite field $\mathbb{F}_q$. The trace function of a character sheaf is a one-dimensional character of $H(\mathbb{F}_q)$.*

Proof. Let $\mathcal{L}$ be a character sheaf and let χ be the trace function of $\mathcal{L}$ on $H(\mathbb{F}_q)$. Then by the definition of a character sheaf, for $x, y \in H(\mathbb{F}_q)$, $\chi(xy) = \chi(x)\chi(y)$. Moreover because $\mathcal{L}$ is a rank one lisse sheaf, χ is nonzero. Hence it is an homomorphism to $\overline{\mathbb{Q}}_\ell^\times$ and thus a character. □

Remark 2.10. Not every character of $H(\mathbb{F}_q)$ necessarily arises from a character sheaf. Consider the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^p \\ 0 & 0 & 1 \end{pmatrix}$$

under matrix multiplication. Any character sheaf, restricted to the subgroup when $a = 0$, is a lisse character sheaf on $\mathbb{A}^1$. By evaluating the character sheaf on a commutator, one can see that this sheaf is necessarily trivial when pulled back along the map $(x, y) \to (x^p y - xy^p)$ whose generic fiber is geometrically irreducible, and hence the sheaf is trivial when restricted to this subgroup. However, not all characters of $H(\mathbb{F}_p)$ are trivial on this subgroup.

Let σ be the Frobenius automorphism of $H(\overline{\mathbb{F}}_q)$. The Lang isogeny is the covering $H \to H$ sending g to $\sigma(g)g^{-1}$, which is finite étale Galois with automorphism group $H(\mathbb{F}_q)$.

Lemma 2.11. *H be an algebraic group over a finite field $\mathbb{F}_q$, $\mathcal{L}$ a character sheaf on H, and χ its character. Then the pullback of $\mathcal{L}$ along the Lang isogeny is trivial, and as a representation of the fundamental group, L is equal to the composition of the map $\pi_1(H_{\mathbb{F}_q}) \to H(\mathbb{F}_q)$ with the character $\chi^{-1} : H \to \overline{\mathbb{Q}}_\ell^\times$.*

Proof. for the first fact, observe that the pullback of $\mathcal{L}$ along the Lang isogeny is $\sigma^*\mathcal{L} \otimes \mathcal{L}^{-1} = \overline{\mathbb{Q}}_\ell$ as $\mathcal{L}$ is defined over $\mathbb{F}_q$ and hence invariant under σ. It follows that the monodromy representation of $\mathcal{L}$ factors through $H(\mathbb{F}_q)$. By examining the Frobenius elements at points of $H(\mathbb{F}_q)$, we obtain χ - the inverse is obtained because of the difference between arithmetic and geometric Frobenius. □

Lemma 2.12. *Let H be an algebraic group over a finite field $\mathbb{F}_q$. Every one-dimensional character of $H(\mathbb{F}_q)$ arises from at most one character sheaf.*

The order of the arithmetic monodromy group of the character sheaf, the geometric monodromy group of the character sheaf, and the character all agree.

Proof. These statements follow immediately from Lemma 2.11. For the second, it is sufficient to observe that the image of the geometric fundamental group inside $H(\mathbb{F}_q)$ is also $H(\mathbb{F}_q)$, because the total space H of the Lang isogeny is geometrically connected. □

To check that a character arises from a character sheaf, we will mainly use the following lemma:

Lemma 2.13. *(i) Let H be an abelian algebraic group over $\mathbb{F}_q$. Every one-dimensional character of $H(\mathbb{F}_q)$ arises from a unique character sheaf. The trace function over $H(\mathbb{F}_{q^n})$ over this sheaf is the composition of the original character with the norm map.*

(ii) Let $f : H_1 \to H_2$ is an algebraic group homomorphism and let $\mathcal{L}$ be a character sheaf on H_2. Then $f^*\mathcal{L}$ is a character sheaf on H_1 whose trace function is the composition of the trace function of $\mathcal{L}$ with h.

Hence every character of the $\mathbb{F}_q$-points of an algebraic group that factors through a homomorphism to an abelian algebraic group arises from a unique character sheaf.

Proof. For assertion (i), one uses the construction of Lemma 2.11 to construct a sheaf from a character, and then checks immediately the necessary isomorphism to make it a character sheaf.

Assertion (ii) is a direct calculation. □

When performing harmonic analysis calculations with character sheaves, it is helpful to have a description of character sheaves directly in terms of points. This is provided, based on central extensions, with the following lemmas:

Lemma 2.14. *Let $\tilde{H}$ be a central extension of $1 \to \overline{\mathbb{Q}}_\ell^\times \to \tilde{H}(\overline{\mathbb{F}}_q) \to H(\overline{\mathbb{F}}_q) \to 1$ with an action of σ such that both maps involved are equivariant.*

Then there exists a unique character sheaf $\mathcal{L}$ on H whose trace function over $\mathbb{F}_q^n$ is given by $g \mapsto \sigma^n(\tilde{g})\tilde{g}^{-1}$ for $\tilde{g}$ any lift of g from $H(\overline{\mathbb{F}}_q)$ to $\tilde{H}(\overline{\mathbb{F}}_q)$.

Furthermore, every character sheaf arises from a central extension in this way.

Proof. For the purposes of this proof, it is simpler to define the trace function using the geometric Frobenius, and then we invert to get the true trace function.

Given a central extension $\tilde{H}$, we form the associated character of $\chi : H(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, $g \mapsto \sigma(\tilde{g})g^{-1}$. It is easy to check that this is actually a group homomorphism. Take the cover $H \to H$ that sends h to $\sigma(h)h^{-1}$, which is a finite étale Galois cover with Galois group the right action of $H(\mathbb{F}_q)$, giving a homomorphism $\pi_1(H) \to H(\mathbb{F}_q)$, and compose with χ to produce a homomorphism $\pi_1(H) \to \overline{\mathbb{Q}}_\ell^\times$ and hence a rank one sheaf.

Let us check that the trace function of this sheaf over $\mathbb{F}_{q^n}$ is given by $g \mapsto \sigma^n(\tilde{g})\tilde{g}^-1$. Let g be an element of $H(\mathbb{F}_{q^n})$ and let $\sigma(h)h^{-1} = g$. Then the trace function is given by viewing $h' = \sigma^n(h)$, which also satisfies $\sigma(h')h'^{-1} = g$, as a right translate of h by an element of $H(\mathbb{F}_q)$ and then applying χ. In other words, the trace function is $\chi(h^{-1}\sigma^n(h))$. Choose $\tilde{h}$ a lift of h, let $\tilde{g} = \sigma(\tilde{h})\tilde{h}^{-1}$, so that

$$\chi(h^{-1}\sigma^n(h)) = \sigma(\tilde{h}^{-1}\sigma^n(\tilde{h}))\left(\tilde{h}^{-1}\sigma^n(\tilde{h})\right)^{-1} = \sigma(\tilde{h})^{-1}\sigma^{n+1}(\tilde{h})\sigma^n(\tilde{h})^{-1}\tilde{h} = \tilde{h}\sigma(\tilde{h})^{-1}\sigma^{n+1}(\tilde{h})\sigma^n(\tilde{h})^{-1}$$
$$= \tilde{g}^{-1}\sigma^n(\tilde{g}) = \sigma^n(\tilde{g})\tilde{g}^{-1}.$$

In this equation we use the fact that we are working with an element of the center and hence may freely conjugate it by any element.

Second, let us check that the trace function over $\mathbb{F}_{q^n}$ is actually a character. This follows because

$$\sigma^n(\tilde{g}_1\tilde{g}_2)(\tilde{g}_1\tilde{g}_2)^{-1} = \sigma^n(\tilde{g}_1)\sigma^n(\tilde{g}_2)\tilde{g}_2^{-1}\tilde{g}_1^{-1} = \sigma^n(\tilde{g}_1)\tilde{g}_1^{-1}\sigma^n(\tilde{g}_2)\tilde{g}_2^{-1}$$

where we use the fact that $\sigma^n(\tilde{g}_2)\tilde{g}_2^{-1}$ is central.

It now follows by the Chebotarev density theorem that the sheaf admits an isomorphism $\mathcal{L} \boxtimes \mathcal{L} = m^*\mathcal{L}$ because these two sheaves have the same trace function over every finite field. The uniqueness follows from Lemma 2.12.

Given a character sheaf $\mathcal{L}$, define $\tilde{H}(\overline{\mathbb{F}}_q)$ to be the set of pairs of a point $x \in H(\overline{\mathbb{F}}_q)$ and a nonzero section of $\mathcal{L}_x$. Multiplication is given by $(x, s_x)(y, s_y) = (xy, s_x \otimes s_y)$ where we use the isomorphism $\mathcal{L}_x \otimes \mathcal{L}_y = \mathcal{L}_{xy}$ induced by taking stalks in the isomorphism $\mathcal{L} \boxtimes \mathcal{L} = m^*\mathcal{L}$ that is part of the definition of a character sheaf. Associativity for this multiplication follows from associativity for the isomorphism. To find units and inverses, it is sufficient to find them in the stalk over the identity of H, where they are obvious.

By definition, the trace function is the trace of Frobenius on the stalk, which because the stalk is one-dimensional is the eigenvalue of Frobenius on the stalk, which can be calculated as

$\sigma^n(s_x)s_x^{-1}$ for s_x a section of the stalk, which is equal to $\sigma^n(x,s_x)(x,s_x)^{-1}$ for (x,s_x) a lift of x. □

Lemma 2.15. *(1) For H_1, H_2 two algebraic groups, any character sheaf on $H_1\times H_2$ is $\mathcal{L}_1\boxtimes\mathcal{L}_2$ for $\mathcal{L}_1$ and $\mathcal{L}_2$ character sheaves on H_1 and H_2.*
(2) For H an algebraic group over $\mathbb{F}_{q^n}$, any character sheaf on $\mathrm{Res}^{\mathbb{F}_q}_{\mathbb{F}_{q^n}} H$ is the Weil restriction of a character sheaf on H.

Proof. (1) Let $\mathcal{L}$ be the character sheaf, let $\mathcal{L}_1$ be its pullback to H_1, and let $\mathcal{L}_2$ be its pullback to H_2. Then $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ and $\mathcal{L}$ have the same trace function, hence are equal.
(2) Let $\mathcal{L}$ be the chacter sheaf, let $\mathcal{L}'$ be its pullback to $(\mathrm{Res}^{\mathbb{F}_q}_{\mathbb{F}_{q^n}} H)_{\mathbb{F}_{q^n}}$ and then to H, embedded diagonally. Then $\mathcal{L}$ and $\mathrm{Res}^{\mathbb{F}_q}_{\mathbb{F}_{q^n}} \mathcal{L}'$ have the same trace function and thus are equal. □

2.5. **Weil Restrictions.**

Notation 2.16. We work with the convention that, for an algebraic group G over k and a finite-dimensional ring R over k, $G\langle R\rangle$ is the algebraic group whose S-points for a ring S over k are the $R\otimes_k S$ points of G. Equivalently, $G\langle R\rangle$ is the Weil restriction $\mathrm{Res}_k G_R$ from R to k of the base-change G_R.

Example 2.17. If we view k^n as a ring by pointwise multiplication, then $G\langle k^n\rangle = G^n$. More generally, $G\langle R_1\times R_2\rangle = G\langle R_1\rangle\times G\langle R_2\rangle$. For another generalization, if k' is a separable k-algebra of degree n, and $\overline{k}$ is the algebraic closure of k, then $(G\langle k'\rangle)_{\overline{k}} = G^n_{\overline{k}}$.

Example 2.18. $G\langle k[t]/t^2\rangle$ is an extension of G by the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{g}$ is viewed as an additive group scheme. More generally, $G\langle k[t]/t^n\rangle$ is an $n-1$-fold iterated extension of G by $\mathfrak{g}$.

By definition, we have $G\langle R\rangle(k) = G(R)$, which we will use several times. This is the "correct" way of constructing a scheme whose k-points are $G(R)$ in our situation because it is stable under base extension, i.e. for any field k' over k, $G_{k'}\langle R\otimes_k k'\rangle = (G\langle R\rangle)_{k'}$.

2.6. **Sheaves on Stacks.**

Lemma 2.19. *Let Y be a stack of finite type over an algebraically closed field and let K_1 and K_2 be bounded complexes of ℓ-adic sheaves on Y.*
(1) $H^i_c(Y, DK_1\otimes K_2)$ is naturally dual to $\mathrm{Ext}^{-i}_Y(K_2,K_1)$.
(2) If K_1 and K_2 are perverse, then $H^i_c(Y,DK_1\otimes K_2)$ vanishes for $i>0$.
(3) If K_1 and K_2 are perverse and semisimple, then $H^0_c(Y,DK_1\otimes K_2) = \mathrm{Hom}(K_1,K_2)$

Proof. For part 1, by the definition of cohomology with compact supports [38, 9.1],

$$H^i_c(Y,DK_1\otimes K_2) = \left(H^{-i}(Y, D(DK_1\otimes K_2))\right)^\vee.$$

By [38, Proposition 6.0.12 and Theorem 7.3.1],

$$H^{-i}(Y,D(DK_1\otimes K_2)) = H^{-i}(y,\mathcal{H}om(K_2,K_1)),$$

which in turn is equal to $\mathrm{Ext}^{-i}_Y(K_2,K_1)$, by definition of Ext, see [38, Remark 5.0.11].

Part 2 follows because perverse sheaves are the heart of a t-structure by [39, Theorem 5.1] and so their Ext^{-i} vanishes for $i>0$.

Part 3 follows because for semisimple perverse sheaves $\mathrm{Ext}^0(K_2,K_1) = \mathrm{Hom}(K_2,K_1)$ is dual to $\mathrm{Hom}(K_1,K_2)$, verifying the first claim. □

Lemma 2.20. *Let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an embedding. Let Y be an Artin stack of finite type over $\mathbb{F}_q$ with affine stabilizers and let K_1 and K_2 be bounded complexes of ℓ-adic sheaves on Y, ι-pure of weights w_1 and w_2. Then for any $j \in \mathbb{Z}$,*

$$\sum_{i=-\infty}^{j} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_{q^e}, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right) = O\left((q^e)^{\frac{j+w_2-w_1}{2}}\right),$$

where the constant in the big O is independent of e but may depend on (Y, K_1, K_2).

Proof. In proving this lemma we will use ι to view $\overline{\mathbb{Q}}_\ell$ as a subfield of $\mathbb{C}$, and thus avoid writing ι.

The tensor product $DK_1 \otimes K_2$ is necessarily mixed of weight $\leq w_2 - w_1$ so by [49, Theorem 1.4], $H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2)$ is mixed of weight $\leq i + w_2 - w_1$.

Let j' be an integer satisfying $j' \leq j - 1$ and $j' + w_2 - w_2 \leq 0$. By mixedness, all eigenvalues of Frob_q acting on the cohomology groups $H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))$ are $\leq q^{\frac{j'+w_2-w_1}{2}}$ and in particular are ≤ 1.

Let $|\,\mathrm{Frob}_q\,|$ be the operator that acts on generalized eigenspaces of Frob_q with eigenvalue the absolute value of the corresponding eigenvalue of $|\,\mathrm{Frob}_q\,|$. Then we have

$$\left|\sum_{i=-\infty}^{j'} (-1)^i \operatorname{tr}(\mathrm{Frob}_{q^e}, H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right|$$
$$\leq \sum_{i=-\infty}^{j'} \operatorname{tr}(|\,\mathrm{Frob}_q\,|^e, H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2).$$

Then because all eigenvalues of Frob_q are $\leq q^{\frac{j'+w_2-w_1}{2}}$, for any $0 < s \leq e$, we have

$$\sum_{i=-\infty}^{j'} \operatorname{tr}(|\,\mathrm{Frob}_q\,|^e, H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))$$
$$\leq q^{(e-s)\frac{j'+w_2-w_1}{2}} \left(\sum_{i=-\infty}^{j'} \operatorname{tr}(|\,\mathrm{Frob}_q\,|^s, H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right)$$

(by [49, Theorem 4.2(i)])

$$\leq q^{(e-s)\frac{j'+w_2-w_1}{2}} O(1).$$

Now because $j' \leq j - 1$, we have $q^{e\frac{j'+w_2-w_1}{2}} < q^{e\frac{j+w_2-w_1}{2}}$ so, for s sufficiently small we have

$$q^{(e-s)\frac{j'+w_2-w_1}{2}} < q^{e\frac{j+w_2-w_1}{2}} = (q^e)^{\frac{j+w_2-w_1}{2}}$$

and so this term is $O\left((q^e)^{\frac{j+w_2-w_1}{2}}\right)$, as desired.

Any remaining terms satisfy

$$(-1)^i \operatorname{tr}(\mathrm{Frob}_{q^e}, H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2)) = O\left((q^e)^{\frac{i+w_2-w_1}{2}}\right) = O\left((q^e)^{\frac{j+d(W_2)-d(W_1)}{2}}\right)$$

where the constant in the big O is the dimension of that cohomology group. ☐

2.7. **Linear recursive sequences and tensor power trick.** The following is a variant of Gelfand's formula $\lim_{n\to\infty} ||t^n||^{\frac{1}{n}}$ for the spectral radius of an endomorphism.

Lemma 2.21 ([12, §3], [4])**.** *Let V be a complex vector space, and $t \in \mathrm{End}(V)$. Then*

$$\rho := \limsup_{n\to\infty} |\operatorname{tr}(t^n|V)|^{\frac{1}{n}}$$

is the spectral radius of t, and

$$|\operatorname{tr}(t^n|V)| \le \dim V \cdot \rho^n, \quad \textit{for every } n \ge 0.$$

Proof. Let $\lambda_1, \dots, \lambda_{\dim(V)}$ denote the eigenvalues of t, so that $\operatorname{tr}(t^n|V) = \sum_i \lambda_i^n$. The power series

$$\sum_{n=0}^{\infty} \operatorname{tr}(t^n|V) z^n = \det(1 - zt|V)^{-1} = \prod_i (1 - z\lambda_i)^{-1}$$

has radius of convergence equal to ρ^{-1} by the Cauchy–Hadamard theorem. Since it is a rational fraction with poles at $z = \lambda_i^{-1}$, we deduce that ρ is equal to $\max_i |\lambda_i|$, the spectral radius of t. This establishes the first assertion, and then the inequality follows. □

3. Compactly induced representations

3.1. **Vanishing of Jacquet modules.** Let G be a reductive group over a non-archimedean field local F. Let P be a parabolic subgroup with Levi decomposition $P = MN$. The Jacquet module (π_N, V_N) of a smooth representation (π, V) of G is the M-module of the N-coinvariants of V. This is an exact functor.

Lemma 3.1. *Let χ be a character on an open-compact subgroup J. The following assertions are equivalent:*

(i) The N-Jacquet module of the induced representation $\text{c-ind}_J^G \chi$ vanishes;
(ii) for every $g \in G$, the restriction of χ to $gNg^{-1} \cap J$ is non-trivial;
(iii) for every $g_1, g_2 \in G$, $\int_N f_\chi(g_1 n g_2) dn = 0$, where

$$f_\chi(g) := \begin{cases} \chi(g), & \textit{if } g \in J, \\ 0, & \textit{if } g \notin J. \end{cases}$$

Proof. We first show the direction (i) → (ii). We view $\text{c-ind}_J^G \chi$ as the space of smooth compactly supported functions f on G satisfying $f(gh) = f(g)\chi(h)$ for $h \in J$. Take f in this space to be the function supported on $g^{-1}J$ such that $f(g^{-1}h) = \chi(h)$ for $h \in J$. Then $f \mapsto \int_{n\in N} f(ng^{-1})$ factors through the Jacquet module of $\text{c-ind}_J^G \chi$, hence vanishes, so

$$0 = \int_{n\in N} f(ng^{-1}) = \int_{n \in N \cap g^{-1}Jg} \chi(gng^{-1}) = \int_{h \in gNg^{-1}\cap J} \chi(h),$$

where the integrations are with respect to Haar measures. This implies that the restriction of χ to the subgroup $gNg^{-1} \cap J$ is non-trivial.

For the direction (ii) → (i), observe that a basis of $\text{c-ind}_J^G \chi$ consists of, for each coset gJ of J in G with chosen representative g, the function $f_g : gh \mapsto \chi(h)$ supported on gJ. For $h \in gNg^{-1} \cap J$, the right translation of f_g by h is equal to $\chi(h) f_g$, and the right translation of f_g by h is equal to the left translation of f_g by an element of N, which implies that the image of f_g and $\chi(h) f_g$ in the module of N-coinvariants is equal. Assuming that χ is nontrivial when restricted to $gNg^{-1} \cap J$, this implies that the image of f_g in the module of N-coinvariants is zero.

Making this assumption for all g, the image of all basis vectors in the modulo of N-coinvariants is zero, and so the module vanishes.

For the implication (ii) $\Rightarrow$ (iii), suppose that $g_1 n_0 g_2 \in J$ for some $n_0 \in N$. Then the condition $g_1 n g_2 \in J$ is equivalent to $g_2^{-1} n_0^{-1} n g_2 \in J$. Therefore

$$\int_{n\in N} f_\chi(g_1 n g_2) = \chi(g_1 n_0 g_2) \int_{h\in g_2^{-1} N g_2 \cap J} \chi(h) = 0.$$

The implication (iii) $\Rightarrow$ (ii) follows by taking $g_1 = g$ and $g_2 = g^{-1}$. □

Lemma 3.2. *The following assertions on the smooth irreducible representation π are equivalent: there is a non-zero (J,χ)-invariant vector; it is a quotient of* $\text{c-ind}_J^G \chi$. *If one of these conditions holds and the N-Jacquet module of* $\text{c-ind}_J^G \chi$ *vanishes, then $\pi_N = 0$.*

Proof. The equivalence of $\mathrm{Hom}_J(\chi,\pi) \neq 0$ and $\mathrm{Hom}(\text{c-ind}_J^G \chi, \pi) \neq 0$ is a form of Frobenius reciprocity [7, Thm. 3.2.4]. The second assertion is consequence of the exactness of the Jacquet functor. □

An admissible representation π is supercuspidal if $\pi_N = 0$ for every parabolic subgroup $P = MN$. It is equivalent [7, Thm. 5.3.1] that all the matrix coefficients have compact support mod center. If (π, V) is irreducible, then it is sufficient to verify that one nonzero matrix coefficient has compact support mod center. We deduce from Lemma 3.1 and Lemma 3.2 the following which will be used often.

Corollary 3.3. *Suppose that the restriction of χ to $N \cap J$ is non-trivial for every parabolic subgroup $P = MN$, or equivalently that* $\text{c-ind}_J^G \chi$ *has vanishing Jacquet modules. Then every smooth irreducible representation with a non-zero (J,χ)-invariant vector is supercuspidal, and the function f_χ is a cuspidal function.*

Remark 3.4. It is proved in [6] that the following assertions on the induced representation $\text{c-ind}_J^G \chi$ are equivalent: it is admissible, it is supercuspidal, it is a finite direct sum of irreducible supercuspidals. These imply that $\text{c-ind}_J^G \chi$ has vanishing Jacquet modules, but the converse doesn't hold because $\text{c-ind}_J^G \chi$ is never admissible if the center of G is non-compact (for example all unramified characters of $\mathbb{F}_q(\!(t)\!)^\times$ appear as quotient of $\text{c-ind}_{\mathbb{F}_q[\![t]\!]^\times}^{\mathbb{F}_q(\!(t)\!)^\times} 1$). If the center of G is compact, and under some additional assumptions, the vanishing of the Jacquet modules of $\text{c-ind}_J^G \chi$ imply that it is admissible, see [25, § III.2].

3.2. **Geometric version.** Let $\mathbf{G}$ be a reductive group over a finite field κ, m a natural number, H a subgroup of $\mathbf{G}\langle \kappa[t]/t^m \rangle$, and $\mathcal{L}$ a character sheaf on H. We call the quadruple $(\mathbf{G}, m, H, \mathcal{L})$ a *monomial datum.*

Let J be the inverse image of $H(\kappa)$ in $\mathbf{G}(\kappa[\![t]\!])$ and let χ be the character induced by $\mathcal{L}$ on $H(\kappa)$ (see Lemma 2.9), pulled back to J. The situation is described by the commutative diagram

$$\begin{array}{ccccc} U_m(\mathbf{G}(\kappa[\![t]\!])) & \hookrightarrow & J & \twoheadrightarrow & H(\kappa) \\ & & \downarrow & & \downarrow \\ & & \mathbf{G}(\kappa[\![t]\!]) & \twoheadrightarrow & \mathbf{G}(\kappa[t]/t^{m_x}) \end{array} \tag{3.1}$$

where $U_m(\mathbf{G}(\kappa[\![t]\!]))$ is the subgroup of $\mathbf{G}(\kappa[\![t]\!])$ consisting of elements congruent to 1 modulo t^m. In this diagram, the square is Cartesian and the sequence $U_m(\mathbf{G}(\kappa[\![t]\!])) \to J \to H(\kappa)$ is short exact.

This datum defines a monomial representation $\text{c-Ind}_{J_\kappa}^{\mathbf{G}(\kappa(\!(t)\!))} \chi_\kappa$. The following definition gives the geometric version of the condition that the Jacquet module of $\text{c-Ind}_{J_\kappa}^{\mathbf{G}(\kappa(\!(t)\!))} \chi_\kappa$ vanishes:

Definition 3.5. We say that the datum $(\mathbf{G}, m, H, \mathcal{L})$ is *geometrically supercuspidal* if for any parabolic subgroup P of $\mathbf{G}_{\overline{\kappa}}$ with radical N, and any $g \in \mathbf{G}(\overline{\kappa}[t]/t^m)$, the restriction of $\mathcal{L}_{\overline{\kappa}}$ to the intersection $gN\langle\overline{\kappa}[t]/t^m\rangle g^{-1} \cap H_{\overline{\kappa}}$ is non-trivial.

There is a close relationship between this geometric condition and the original vanishing of the Jacquet module. Indeed, the next lemma shows that, to establish that $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal, it suffices to verify that the associated Jacquet module vanishes, and, in addition, that the corresponding Jacquet module over each finite field extension vanishes as well. This will enable us to apply standard techniques from representation theory over local fields to verify geometric supercuspidality.

For any finite field extension κ' of κ, the datum $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal if and only $(\mathbf{G}_{\kappa'}, m, H_{\kappa'}, \mathcal{L}_{\kappa'})$ is geometrically supercuspidal. Let $J_{\kappa'}$ be the inverse image of $H(\kappa')$ in $\mathbf{G}(\kappa'[\![t]\!])$ as in the diagram (3.1). Let $\chi_{\kappa'}$ be the character induced by $\mathcal{L}$ on $H(\kappa')$, pulled back to $J_{\kappa'}$.

Lemma 3.6. *The following assertions are equivalent:*

(i) for every finite extension κ' of κ, the induction $\text{c-Ind}_{J_{\kappa'}}^{\mathbf{G}(\kappa'(\!(t)\!))} \chi_{\kappa'}$ has vanishing Jacquet modules;

(ii) for every finite extension κ'/κ, every parabolic subgroup $P = MN$ of $\mathbf{G}_{\kappa'(\!(t)\!)}$, the restriction of $\chi_{\kappa'}$ to $N(\kappa'(\!(t)\!)) \cap J_{\kappa'}$ is non-trivial;

(iii) for every field extension κ' of κ, any parabolic subgroup P of $\mathbf{G}_{\kappa'}$ with radical N, and any $g \in \mathbf{G}(\kappa'[t]/t^m)$, the restriction of $\mathcal{L}_{\kappa'}$ to the intersection of $gN\langle\kappa'[t]/t^m\rangle g^{-1}$ with $H_{\kappa'}$ is not geometrically isomorphic to a constant sheaf;

(iv) $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal.

Proof. The equivalence between (i) and (ii) follows from Lemma 3.1. The implication (iii) $\implies$ (iv) follows by taking $\kappa' = \overline{\kappa}$. The converse (iv) $\implies$ (iii) is because geometric non-triviality can be verified on the algebraic closure, and is preserved by composition of fields.

The direction (iii) $\implies$ (ii) is straightforward. First note that P is $\mathbf{G}(\kappa'(\!(t)\!))$-conjugate to the loop group $\mathbf{P}(\!(t)\!)$ of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}_{\kappa'}$, because $\mathbf{G}$ has a Borel on which Frobenius acts by a fixed automorphism of the Dynkin diagram [9, Example 7.2.3], so parabolic subgroups are classified by Frobenius-invariant subsets of the roots, and the same classification holds for $\mathbf{G}_{\kappa'(\!(t)\!)}$. Because of the Iwasawa decomposition $\mathbf{G}(\kappa'(\!(t)\!)) = P(\kappa'(\!(t)\!))\mathbf{G}(\kappa'[\![t]\!])$, we have that P is $\mathbf{G}(\kappa'[\![t]\!])$-conjugate to $\mathbf{P}(\!(t)\!)$. The restriction of $\chi_{\kappa'}$ to $gN(\kappa'(\!(t)\!))g^{-1} \cap J_{\kappa'}$ is the trace function over κ' of the restriction of $\mathcal{L}$ to $g\mathbf{N}\langle\kappa'[t]/t^m\rangle g^{-1} \cap H$. If this sheaf is geometrically nontrivial, then it remains nontrivial when restricted to the identity component, hence by Lemma 2.12 its trace function is nontrivial.

So it remains to prove the converse (ii) $\implies$ (iv) $\implies$ (iii).

To check (iv) $\implies$ (iii), let us first check that, given a morphism $f : Y \to X$ of schemes of finite type over a field and a lisse sheaf $\mathcal{F}$ on Y, the property that $\mathcal{F}$ restricted to a fiber of f is constant defines a constructible subset of X. By Noetherian induction, it is sufficient to solve the problem after restricting to any open subset of X. By [13, Th. Finitude, Théorème 1.9(2)], there exists an open subset of X such that for each point x in that subset,

$$(f_*\mathcal{F})_x = H^0(Y_x, \mathcal{F}).$$

Restrict to that open subset. Because the image of each irreducible component of Y is constructible, we can choose a smaller open subset of X which is contained in the image of each irreducible component of Y with dense image and does not intersect any irreducible component of Y without dense image. After base-changing to this open subset, each irreducible component of Y maps surjectively onto X (because the irreducible components without dense image no longer exist). Now we prove the result in this case. At any point x, if there is a section of $H^0(Y_x, \mathcal{F})$ that gives an isomorphism between $\mathcal{F}$ and the constant sheaf, then the corresponding section of $f_*\mathcal{F}$ extends to some neighborhood, which gives an extension of the section of $\mathcal{F}$ to the inverse image of that neighborhood, where because $\mathcal{F}$ is lisse it must be an isomorphism on every connected component of Y that intersects that fiber, which is everywhere. So the set is open, hence constructible, verifying the claim.

Consider the family of schemes $gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H$ parameterized by $g \in \mathbf{G}\langle\kappa[t]/t^m\rangle$. Let $\mathcal{F}$ be the pullback of $\mathcal{L}$ to this family. The set in $\mathbf{G}\langle\kappa[t]/t^m\rangle$ where $\mathcal{F}$ is geometrically trivial on the fiber is constructible. Geometric supercuspidality is equivalent to the claim that this set does not contain any point defined over any field extension of κ. Because this set is constructible, it is sufficient to check this for every point defined over $\overline{\kappa}$. This establishes the direction (iv) $\Longrightarrow$ (iii).

We now establish (iii) $\Longrightarrow$ (ii). Fix a point $g \in \mathbf{G}(\overline{\kappa}[t]/t^m)$. There exist some finite field extension κ^* of κ such that g is defined over κ^* and every connected component of $gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H$ is defined over κ^*. If the character sheaf $\mathcal{L}$ is geometrically trivial on $gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H$, then its trace function is necessarily constant on each connected component of $gN\langle\kappa^*[t]/t^m\rangle g^{-1} \cap H$, and hence it corresponds to a character of the component group $\pi_0\left(gN\langle\kappa^*[t]/t^m\rangle g^{-1} \cap H\right)$. Thus the eigenvalue of Frobenius at each point is a root of unity of order dividing the order of the component group. We can pass to a further finite field extension κ'/κ^* that trivializes the eigenvalues of Frobenius at each point. Over this field extension, the corresponding character $\chi_{\kappa'}$ must be trivial when restricted to

$$\left(gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H\right)(\kappa') = gN(\kappa'[t]/t^m)g^{-1} \cap H(\kappa').$$

This contradicts assumption (ii). □

We will see some examples of $(G, m, H, \mathcal{L})$ satisfying these conditions later in this section.

Lemma 3.7. *If $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal, then H is unipotent mod the center of G.*

Proof. We will prove the contrapositive. Assume that H is not unipotent, we will show that $(\mathbf{G}, m, H, \mathcal{L})$ is not geometrically supercuspidal. A smooth connected algebraic group fails to be unipotent if and only if admits a nontrivial homomorphism from $\mathbb{G}_m$. Thus H admits a homomorphism from $\mathbb{G}_m$ that is nontrivial modulo the center. Let$\alpha : \mathbb{G}_m \to H$ be one such.

The group $\mathbf{G}\langle\kappa[t]/t^m\rangle$ is a semidirect product of the the reductive group $\mathbf{G}$ by a unipotent group, where the section $\mathbf{G} \to \mathbf{G}\langle\kappa[t]/t^m\rangle$ is induced by the tautological map $\kappa \to \kappa[t]^m$. Every nontrivial reductive subgroup of $\mathbf{G}\langle\kappa[t]/t^m\rangle$ has nontrivial image in $\mathbf{G}$, and two such subgroups are conjugate if and only if their images in $\mathbf{G}$ are conjugate, so every such subgroup is conjugate to a subgroup contained in $\mathbf{G}$. Hence the image of α is conjugate to a one-dimensional torus in $\mathbf{G}$. Because the definition of geometric supercuspidal is invariant under conjugacy, we may assume that the image α lies in the image of this section, that is it factors through a nontrivial homomorphism still denoted $\alpha : \mathbb{G}_m \to \mathbf{G}$.

Let T be a maximal split torus of $\mathbf{G}$ containing α. Let P be the parabolic subgroup containing T and every root subgroup of T on which α acts by conjugating with eigenvalue a nonnegative power of the parameter. Let N be the maximal unipotent subgroup of P. Then α acts on each root of N with eigenvalue a positive power. Let $H' = H \cap N\langle \kappa[t]/t^m\rangle$. Then H' an iterated extension of copies of $\mathbb{G}_a$, on each of which α acts by conjugation by a nonzero power. In other words, H' admits a α-invariant filtration $\{1\} = H'_0 \subseteq H'_1 \dots . H'_m = H'$. Let i be the largest natural number such that $\mathcal{L}$ is geometrically trivial on H'_i. Then $\mathcal{L}$ descends to H'/H'_i and is nontrivial on H'_{i+1}/H'_i. Because $\mathcal{L}$ is a character sheaf on H, it is conjugacy-invariant. Hence it is invariant by the conjugacy action of α. Hence its restriction to H'_{i+1} and descent to H'_{i+1}/H'_i is invariant to the action of α, which is scaling by some nonzero power. But there is no nontrivial lisse sheaf on $\mathbb{G}_a$ which is invariant to scaling by an arbitrary constant. So in fact $i = m$, and $\mathcal{L}$ is trivial on H', so $(\mathbf{G}, m, H, \mathcal{L})$ is not geometrically supercuspidal. □

3.3. **Intertwining.** Let $\mathbf{G}$ be a reductive group over a finite field κ, m a natural number, H a subgroup of $G\langle \kappa[t]/t^m\rangle$ containing the center, and $\mathcal{L}$ a character sheaf on H. We can check that $(G, m, H, \mathcal{L})$ is geometrically supercuspidal using a geometric analogue of the standard method, based on intertwining sets.

The intertwining of $\mathcal{L}$ is the set of $g \in G(\overline{\kappa}((t)))$ such that $\mathcal{L} = \mathcal{L}^g$ on $H_{\overline{\kappa}} \cap H^g_{\overline{\kappa}}$.

Lemma 3.8. *If the intertwining set is equal to $H_{\overline{\kappa}}$, then $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal.*

Proof. We apply Lemma 3.6. So it suffices to verify the vanishing of the Jacquet module for every finite extension κ'. This follows from irreducibility of the induced representation. See also [6, Prop. (2.4)]. □

Lemma 3.9. *Suppose that there is another subgroup K containing H as a normal subgroup, and that the intertwining is equal to the set of $g \in K$ such that $\mathcal{L} = \mathcal{L}^g$. Then $(\mathbf{G}, m, H, \mathcal{L})$ is geometric monomial.*

Proof. We again apply Lemma 3.6, and verify the vanishing of the Jacquet module for every finite extension κ'. This follows from [43, Lemma 2.2]. □

3.4. **Monomial geometric supercuspidal representations.** Let $\mathbf{G}$ be a reductive group over a finite field κ. We say that an irreducible smooth representation of $\mathbf{G}(\kappa((t)))$ is *mgs* if there exists some natural number m, connected subgroup H of $\mathbf{G}\langle \kappa[t]/t^m\rangle$, and character sheaf $\mathcal{L}$ on H, such that

(1) $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal
(2) π is a quotient of $\text{c-Ind}_J^{\mathbf{G}(\kappa((t)))} \chi$ where J is the inverse image of $H(\kappa)$ in $\mathbf{G}(\kappa[[t]])$ and χ is the trace function of $\mathcal{L}$ on $H(\kappa)$, pulled back to J.

Furthermore, in this setting, we say that $(\mathbf{G}, m, H, \mathcal{L})$ are *mgs data* for π.

Lemma 3.10. *Let π be an mgs representation of $\mathbf{G}(\kappa((t)))$. Then the pullback of π by any automorphism of the field $\kappa((t))$ is an mgs representation.*

Proof. Any such automorphism is a composition of an automorphism of κ with a change of variables that sends t to a power series with leading term a constant multiple of t. Automorphisms of κ act in a natural way on the mgs data $(\mathbf{G}, m, H, \mathcal{L})$. Changes of variables in t act in a natural way on $\mathbf{G}\langle \kappa[t]/t^m\rangle$ and hence act in a natural way on H and $\mathcal{L}$. Both of these automorphisms agree with the action of the field automorphism on the induced representation, hence preserve the Jacquet module vanishing condition, and also therefore agree with the pullback of χ. □

Lemma 3.11. *Let π be an mgs representation of $\mathbf{G}(\kappa((t)))$. Then the pullback of π by any automorphism of the group $\mathbf{G}$ defined over $\kappa((t))$ is an mgs representation.*

Proof. Let $(\mathbf{G}, m, H, \mathcal{L})$ be mgs data for π. By Lemma 3.7, H is unipotent. In particular, its image inside $\mathbf{G}$ is solvable, and hence contained in a Borel subgroup B. The inverse image of B in $\mathbf{G}(\kappa[[t]])$ is a minimal parahoric subgroup of $\mathbf{G}(\kappa((t)))$. (This follows from the explicit description of the parahoric subgroup in terms of roots. If we take an apartment corresponding to the inverse image of a torus of $\mathbf{G}$ and and perturb the hyperspecial point associated to $\mathbf{G}(\kappa[[t]])$ in a generic direction, producing a point in the interior of a chamber whose associated subgroup is a minimal parahoric, we see that the parahoric subgroup is the inverse image of some Borel, and because all Borels are conjugate all such subgroups are minimal parahoric.) Because all minimal parahoric subgroups are conjugate [37, Section 9], every automorphism of $\mathbf{G}_{\kappa((t))}$ can be expressed as an inner automorphism composed with an automorphism that sends this minimal parahoric subgroup to itself. Conjugation by an element of $\mathbf{G}(\kappa((t)))$ produces a representation isomomorphic to π, so we may assume that the automorphism σ fixes this minimal parahoric.

Expressing σ in the coordinates of $\mathbf{G}$, let δ be the highest power of t^{-1} that appears. Then for any g in the minimal parahoric, $\sigma(g)$ is in the minimal parahoric, and $\sigma(g)$ modulo t^m depends only on g modulo $t^{m+\delta}$. Hence σ defines a map from the subset of $\mathbf{G}\langle\kappa[t]/t^{m+\delta}\rangle$ congruent to B modulo t to the subset of $\mathbf{G}\langle\kappa[t]/t^m\rangle$ congruent to B modulo t. Because $\mathbf{G}$ acts as an automorphism of the minimal parahoric, this map is surjective.

Consider the data $(\mathbf{G}, m+\delta, \sigma^{-1}(H), \sigma^*\mathcal{L})$. Let J' be the inverse image of $\sigma^{-1}(H)(\kappa)$ in $\mathbf{G}(\kappa[[t]])$ and let χ' be the pullback of the trace function of $\sigma^*\mathcal{L}$ to J'. Then $J' = \sigma^{-1}(J)$ and $\chi' = \chi \circ \sigma$, so c-Ind$_{J'}^{\mathbf{G}(\kappa((t)))} \chi'$ is the pullback of c-Ind$_{J}^{\mathbf{G}(\kappa((t)))} \chi$ by σ and hence $\pi \circ \sigma$ is a quotient of it.

Similarly, over any finite field extension κ' of κ, c-Ind$_{J'_{\kappa'}}^{\mathbf{G}(\kappa'((t)))} \chi'_{\kappa'}$ is the pullback of c-Ind$_{J'_{\kappa'}}^{\mathbf{G}(\kappa'((t)))} \chi_{\kappa'}$, hence has vanishing Jacquet module for every parabolic subgroup, so by Lemma 3.6 it is geometrically supercuspidal.

Hence $\pi \circ \sigma$ is an mgs representation. □

Any unramified reductive group G over $\kappa((t))$ necessarily descends to a reductive group $\mathbf{G}$ over κ (Lemma 2.1). Combined with the previous two lemmas, that allows us to give an intrinsic definition of mgs representations of G as follows.

Definition 3.12. Let G be an unramified reductive group over an equal characteristic local field F. We say that an irreducible smooth representation π of $G(F)$ is mgs if for some (equivalently any) isomorphism $F \cong \kappa((t))$, where κ is the residue field of F, and for some (equivalently any) reductive group $\mathbf{G}$ over κ and isomorphism $\mathbf{G}_{\kappa((t))} = G$, π is a quotient of c-ind$_J^{G(F)} \chi$ for some geometrically supercuspidal datam $(\mathbf{G}, m, H, \mathcal{L})$.

We can make a similar definition over a mixed characteristic local field as follows: Let F be a mixed characteristic local field, $\mathfrak{o}_F$ its ring of integers, and κ its residue field. Let G be an unramified group over F and let $\mathcal{G}$ be a group scheme over $\mathfrak{o}_F$ whose generic fiber is isomorphic to G. Let $\mathcal{G}_m$ be the group scheme over κ whose R points for a ring R over κ are the $W_m(R) \otimes_{W(\mathbb{F}_q)} \mathfrak{o}_F$ points of G, where W is the Witt vectors functor and $W_m(R)$ is the ring of Witt vectors modulo p^m. (Here the Witt vectors are defined using universal polynomials over an imperfect ring). Let H be connected closed subgroup of $\mathcal{G}_m$ and let $\mathcal{L}$ be a character sheaf on H such that for every parabolic subgroup of P with maximal unipotent N, and every $g \in cG_m$, the restriction of $\mathcal{L}$ to $H \cap gN_m g^{-1}$ is nontrivial. Let J be the inverse image of $H(\kappa) \subseteq \mathcal{G}_m(\kappa) = \mathcal{G}(\mathfrak{o}_F/p^m)$ in $\mathcal{G}(\mathfrak{o}_F)$

and let χ be the pullback of the trace function of $\mathcal{L}$ from $H(\kappa)$ to J. Then we say that every irreducible smooth representation of $G(F)$ that appears as a quotient of $\text{c-Ind}_J^{G(F)} \chi$ is mgs, and we say that any irreducible smooth representation that does not appear this way for any $m, H, \mathcal{L}$ is not mgs.

3.5. **Moy-Prasad types and epipelagic representations.** Let G be an unramified semisimple group over $F = \kappa((t))$. Let x be a point in the Bruhats-Tit building of G and let $r > 0$ be a number. Then $G(F)_{x,r}/G(F)_{x,r+}$ is a vector space over $\mathbb{F}_q$. Let χ be a character of $G(F)_{x,r}$ that factors through this vector space.

Lemma 3.13. *$G_{x,r}$ is conjugate to a subgroup of $G(\kappa[[t]])$.*

Proof. It is contained in a minimal parahoric subgroup (e.g. the one associated to any adjacent chamber of the Bruhat-Tits building), and we may conjugate it to a minimal parahoric subgroup inside $G(\kappa[[t]])$ [37, Section 9]. □

Lemma 3.14. *Let x and r be such that $G_{x,r} \subseteq G(\kappa[[t]])$. Then there exists a natural number m, $H \subseteq G\langle \kappa[t]/t^m \rangle$, $H' \subseteq H$ such that the inverse image of $H(\kappa)$ in $G(\kappa[[t]])$ is $G(F)_{x,r}$ and the inverse image of $H'(\kappa)$ is $G(F)_{x,r+}$. Furthermore, for any finite field extension κ' of κ, the inverse image of $H(\kappa)'$ is $G(\kappa'((t)))_{x,r}$ and the inverse image of $H'(\kappa')$ is $G(\kappa'((t)))_{x,r+}$.*

Finally, H/H' is isomorphic to a vector space as an algebraic group.

The conditions on the rational points uniquely characterize the groups H and H'.

Proof. For some m, G_{x,r^+} contains the subgroup of elements congruent to the identity modulo t^m, so that $G_{x,r}$ and G_{x,r^+} are the inverse images of their projections to $G(\kappa[t]/t^m)$.

It is clear from the definition of $G_{x,r}$ and G_{x,r^+} that these projections are algebraic subgroups - the Moy-Prasad subgroups are defined as the subgroups generated by certain additive and multiplicative groups, and we can simply take the algebraic subgroup generated by these groups.

Furthermore, because $r > 0$, all the involved subgroups are additive, and their commutators lie in $G_{x,r+}$, so the H/H' is a vector space. □

Any character χ of $G(F)_{x,r}$ trivial on $G(F)_{x,r+}$ defines a character of $G(F)_{x,r}/G(F)_{x,r+} = H(\mathbb{F}_q)/H'(\mathbb{F}_q) = H/H'(\mathbb{F}_q)$ and hence by Lemma 2.13, a character sheaf $\mathcal{L}$ on H. By construction, this data $(G, m, H, \mathcal{L})$ satisfies $J = G(F)_{x,r}$ and $\chi = \chi$. Hence if $(G, m, H, \mathcal{L})$ is geometrically supercuspidal, any representation containing a vector on which $G(F)_{x,r}$ acts through the character χ is mgs.

A concrete description of when this occurs is provided by Lemma 3.6.

We give here a different condition, inspired by the construction of epipelagic representations of Reeder and Yu [43].

Lemma 3.15. *Let H and H' be the subgroups of Lemma 3.14. Let $\lambda : H/H' \to \mathbb{G}_a$ be a linear map, let $pr : H \to H/H'$ be the projection, let ψ be an additive character of $\mathbb{F}_q$, and let $\chi = \psi \circ \lambda \circ pr$ be the character of the character sheaf $pr^* \lambda^* \mathcal{L}_\psi$.*

Then $(G, m, H, pr^*\lambda^*\mathcal{L}_\psi)$ is geometrically supercuspidal if and only if λ is GIT-semistable for the action of $G(F)_{x,0}/G(F)_{x,0+}$ on $(H/H')^\vee$.

Proof. By conjugation, we may assume that $G(F)_{x,0}$ contains the standard minimal parahoric subgroup (the inverse image in $G(\kappa[[t]])$ of a fixed Borel subgroup of the quasi-split group $G(\kappa)$) and hence that x lies in the apartment of the standard maximal torus. Let P be a standard parabolic, and consider a conjugate gPg^{-1}. By the affine Bruhat decomposition, we may write

g as an element of the affine Weyl group composed on the left and the right with elements of the standard minimal parahoric. Because the standard minimal parahoric and maximal torus are contained in P, we may write g up to the right action of P as an element g_0 of the standard parahoric composed with an element w of the Weyl group.

Because P is a standard parabolic subgroup, there is some cocharacter $\alpha : \mathbb{G}_m \to T$ of the standard maximal torus T such that the unipotent subgroup N of P consists of those roots which have positive eigenvalue under α. Then wNw^{-1} consists of those roots which have a positive eigenvalue under $w\alpha w^{-1}$. Hence $wNw^{-1} \cap G_{x,r}/(wNw^{-1} \cap G_{x,r+}) = wNw^{-1} \cap H/(wNw^{-1} \cap H')$ is generated by the elements of H/H' which have a positive eigenvalue under $w\alpha w^{-1}$, as H/H' has a basis consisting of roots. Hence the projection onto H/H' of gNg^{-1} is generated by the elements which have a positive eigenvalue under $g_0 w\alpha w^{-1} g_0^{-1}$, which is a cocharacter of $G(F)_{x,0}$. Hence the set of linear forms on H/H' that vanish on that projection is the subspace generated by the elements which have a nonnegative eigenvalue under $g\alpha g^{-1}$. If λ is GIT-semistable, then by the Hilbert-Mumford criterion it does not lie in this space, so it is nontrivial on the image, hence the pullback of $\mathcal{L}_\psi$ under λ is nontrivial on this image, as desired.

For the converse, if λ is not semistable, we have a cocharacter of $G(F)_{x,0}$ such that λ is a sum of linear forms on H/H' that are eigenvectors of this cocharacter with nonnegative eigenvalue. Hence λ vanishes on all elements of H/H' that have positive eigenvalue under the cocharacter. Now let P be the parabolic subgroup of G generated by the maximal torus and all the roots that have nonnegative eigenvalue under this character, then all elements of N have positive eigenvalue, so λ vanishes on $H \cap N$ and therefore $(G, m, H, pr^*\lambda^*\mathcal{L}_\psi)$ is not mgs. □

Corollary 3.16. *If G is unramifed semisimple, then the epipelagic supercuspidal representations constructed in* [43] *are mgs.*

Proof. They are by definition summands of $\text{c-Ind}_{G(F)_{x,r}}^{G(F)} \chi$ for r the minimum positive value and χ a GIT-semistable character of $G(F)_{x,r}/G(F)_{x,r}^+$. □

Example 3.17. We review the simplest example of an epipelagic representation, which is also the simplest example of an mgs representation. Let $G = SL_2$ and let x be the midpoint of an edge between two vertices of the Bruhat-Tits tree. Then $G(F)_{x,1/2}$ is the subgroup of matrices of the form $\begin{pmatrix} 1 + t\kappa[[t]] & \kappa[[t]] \\ t\kappa[[t]] & \kappa[[t]] \end{pmatrix}$ and $G(F)_{x,1/2+}$ is the subgroup of matrices of the form $\begin{pmatrix} 1 + t\kappa[[t]] & t\kappa[[t]] \\ t^2\kappa[[t]] & \kappa[[t]] \end{pmatrix}$, so the quotient is isomorphic to κ^2, given by extracting the leading terms of the top-right and bottom-left matrix entries. The conjugation action of a matrix congruent to $\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}$ mod t is by multiplication by ab^{-1} on the top-right entry and $(ab^{-1})^{-1}$ on the bottom-left entry, so the semistable characters are exactly the characters nontrivial on the top-left and bottom-right entries.

The associated mgs data has $m = 2$, H the four-dimensional subgroup of matrices in $SL_2\langle\kappa[t]/t^2\rangle$ congruent mod 2 to an upper-triangular unipotent matrix, and $\mathcal{L}$ the unique character sheaf on H whose trace function is this character.

3.6. **Adler datum and toral representations.** We now describe a special case of the construction of [1] that produces mgs representations. To that end, we borrow some notation from [1]. Let G be an unramified semsimple group over $F = \kappa((t))$ satisfying [1, Hypothesis 2.1.1]. This allows us to take a G-equivariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G such that the induced isomorphism between $\mathfrak{g}$ and its dual .

Let T be a maximal F-torus of G that splits over a tamely ramified extension E of F but such that $T/Z(G)$ has no nontrivial map to $\mathbb{G}_m$ defined over any unramified extension of F. Let X be an element of the Lie algebra of T. Assume that there is a positive rational number r such that the valuation of $d\alpha(X)$ for every root α of T defined over E is equal to r.

Let x be the unique point of the Bruhat-Tits building of G that belongs to the apartment of T inside the Bruhat-Tits building of $G(E)$. Let $G_{x,r}, G_{x,r^+}, \mathfrak{g}_{x,r}, \mathfrak{g}_{x,r^+}$ be the corresponding Moy-Prasad subgroups of G and $\mathfrak{g}$. Then because $r > 0$, we may identify $G_{x,r}/G_{x,r^+} = \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ [1, (1.5.2)]. Using the bilinear form, we may view X as a character of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$, defining a character χ of $G_{x,r}$.

Proposition 3.18. *Any irreducible representation π of $G(F)$ that contains $(G_{x,r}, \chi)$ is mgs.*

Proof. We use the data (G, m, H, ψ) constructed in the previous section. It remains to check that this data is geometrically supercuspidal, which we do using Lemma 3.6.

It is sufficient to show that, after base-changing to a finite extension of κ, the Jacquet module of this induced representation vanishes. Because all our assumptions are stable under base change of κ, it in fact suffices to show that, for all F, G, T, X satisfying these assumptions, the Jacquet module of $\text{c-ind}_{G_{x,r}}^{G(F)} \chi$ vanishes.

Adler defines M to be the centralizer of X in G. In our case, that is simply $T(F)$, because our assumptions imply that $d\alpha(X) \neq 0$ for every root α of T. Because T is anisotropic, M is compact. Adler defines J as $\phi_x(\mathfrak{m}_{x,r} \oplus \mathfrak{m}^{\perp}_{x,(r/2)})$, where $\mathfrak{m}_{x,r}$ and $\mathfrak{m}^{\perp}_{x,(r/2)}$ are the Moy-Prasad subspaces of the Lie algebra of T and its orthogonal complement in the Lie algebra of G respectively, and ϕ_x is an approximate exponential map. For our purposes, it is most significant that J is compact and contains $G_{x,r}$, and is normalized by M, so MJ is compact and contains $G_{x,r}$ as an open subgroup.

Thus $\text{c-ind}_{G_{x,r}}^{MJ} \chi$ is a sum of irreducible representations σ of MJ, each containing $(G_{x,r}, \chi)$ by Frobenius reciprocity. The induced representations $\text{c-ind}_{G_{x,r}}^{G(F)} \chi = \text{c-ind}_{MJ}^{G(F)} \text{c-ind}_{G_{x,r}}^{MJ} \chi$ is the sum of $\text{c-ind}_{G_{x,r}}^{G(F)} \sigma$, and by the discussion at the beginning of [1, 2.5], these are supercuspidal, so it is a sum of supercuspidal representations, hence has vanishing Jacquet module. □

This construction shows that all the representations produced by the construction of Adler in the case where G is unramified and semisimple and the centralizer M of X is not just anisotropic over the base field but over all unramified extensions. (To see this, we must observe that M anisotropic over unramified extensions implies that M is a torus, as all groups become quasi-split over some unramified extensions, and hence equals T. If $M = T$, then $d\alpha(X) \neq 0$ for any root α of T. This condition, plus the stronger anisotropic condition for T, are our only points of departure from the setup of Adler.)

3.7. **Non-examples.** We discuss some examples of data $(G, m, H, \mathcal{L})$ that are not geometrically supercuspidal and so do not lead to mgs representations.

Example 3.19. First, if $\mathcal{L}$ is trivial then $(G, m, H, \mathcal{L})$ cannot be mgs. In particular, we can simply take H trivial.

Example 3.20. Second, if the order of the monodromy group of $\mathcal{L}$ (which is equal to the order of the associated character by Lemma 2.12) is prime to p, then its pullback to the intersection with any unipotent subgroup will have order prime to p, but the order of the unipotent subgroup is a power of p, so the character sheaf is trivial on that intersection.

For instance, we can take $m = 1$, H a Borel subgroup of G, and $\mathcal{L}$ the pullback of a character sheaf on the maximal torus. It is possible in this case for c-ind$_J^{G(\kappa[[t]])} \chi$ to be irreducible (the inflation of an irreducible principle series representation of $G(\kappa)$ but the Jacquet module of the induced representation is nonvanishing.

Example 3.21. We provide an example of a $(G, m, H, \mathcal{L})$ which is not mgs even though the Jacquet module of the induced representation is nontrivial. Let $G = GL_2$, $m = 2$, and H be the subgroup of elements congruent to 1 mod t, which is isomorphic to the Lie algebra of G, i.e. the vector space of 2×2 matrices. We take a linear function on H which sends a matrix A to $\mathrm{tr}(AB)$ where B is an element of a non-split Cartan of $M_2(\kappa)$, and pull back an Artin-Schreier sheaf $\mathcal{L}_\psi$ under this map. Then for any parabolic subgroup P, $gNg^{-1} \cap H$ is a one-dimensional vector space of nilpotent matrices, so the character is trivial when pulled back to that subgroup if and only if the trace of B times the nilpotent matrix vanishes, which happens if and only if B is contained in the associated Borel. Over κ, this is impossible, so the Jacquet module vanishes, and the induced representation is a sum of supercuspidals. However, over κ^2, there are two Borels containing this matrix, so $(G, m, H, \mathcal{L})$ is not geometrically supercuspidal.

3.8. **Preservation of mgs.** We note some properties that show mgs representations are preserved under some natural operations on algebraic groups.

Lemma 3.22. *Let $f : G_1 \to G_2$ be a homomorphism of unramified reductive groups over an equal characteristic local field F whose kernel is a torus and whose image is a normal subgroup with quotient a torus. Let π_2 be an mgs representation of $G_2(F)$. Then any irreducible quotient π_1 of $\pi_2 \circ f$ is mgs.*

Proof. Let $F = \kappa((t))$. We may choose descents $\mathbf{G}_1$ and $\mathbf{G}_2$ of G_1 and G_2 to κ such that f is defined over κ, because $\mathbf{G}_1$ and $\mathbf{G}_2$ have the same Bruhat-Tits tree and the same hyperspecial points.

Let $(\mathbf{G}_2, m, H, \mathcal{L})$ be mgs data for π_2. Let J_2 be the subgroup defined by this data and χ the character. We know that π_2 contains a vector which transforms under the subgroup J_2 by the character χ_2, some conjugate of which must be nonzero in π_1, so it contains a vector which transforms under the subgroup $f^{-1}(gJ_2g^{-1})$ by $\chi_2 \circ f$. Now conjugation by g is an outer automorphism of G_1, so $f^{-1}(gJ_2g^{-1})$ is this conjugation applied to $f^{-1}(J_2)$. Because geometric supercuspidality is preserved by automorphisms (Lemma 3.11), we may assume π_1 is contains a vector that transforms under the subgroup $f^{-1}(J_2)$ by the character $\chi_2 \circ f$.

We have a map $f : \mathbf{G}_1\langle \kappa[t]/t^m \rangle \to \mathbf{G}_2\langle \kappa[t]/t^m \rangle$. It suffices to show that $(G_1, m, f^{-1}(H), f^*\mathcal{L})$ is mgs data for π_1. Let J_1 be the subgroup defined by this data and let χ_1 be the character. We have $J_1 = f^{-1}(J)$ and $\chi_1 = \chi_2 \circ f$, so it remains to show that $(\mathbf{G}_1, m, f^{-1}(H), f^*\mathcal{L})$ is geometrically supercuspidal. Let P_1 be a parabolic subgroup of $\mathbf{G}_1$. Then P_1 is the inverse image under f of a parabolic subgroup P_2 of $\mathbf{G}_2$. Moreover, for N_1 and N_2 their maximal unipotent subgroups, $f : N_1\langle \kappa[t]/t^m \rangle \to N_2\langle \kappa[t]/t^m \rangle$ is an isomorphism, because the kernel of f is a torus and does not intersect the unipotent subgroups, while the cokernel of f is a torus and so the image of the unipotent subgroup in it is trivial. So for any g in $\mathbf{G}_1\langle \kappa[t]/t^m \rangle$, $f : gN_1g^{-1} \cap f^{-1}(H) \to f(g)N_2f(g^{-1}) \cap H$ is an isomorphism, and since the pullback of $\mathcal{L}$ to $f(g)N_2f(g^{-1}) \cap H$, the pullback of $\mathcal{L}$ to $gN_1g^{-1} \cap f^{-1}(H)$ is nontrivial. ☐

Lemma 3.23. *Let G_1 and G_2 be unramified reductive groups over an equal characteristic local field F. Let $\pi = \pi_1 \boxtimes \pi_2$ be a mgs representation of $G_1(F) \times G_2(F)$, where π_1 is a representation of $G_1(F)$ and π_2 is a representation of $G_2(F)$. Then π_1 and π_2 are mgs representations of $G_1(F)$ and $G_2(F)$ respectively.*

Proof. Let $F = \kappa((t))$. Choose descents $\mathbf{G}_1$ and $\mathbf{G}_2$ and isomorphisms $\mathbf{G}_{1,F} = G_1, \mathbf{G}_{2,F} = G_2$. Let $G = G_1 \times G_2$, and $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$.

Choose mgs data $(\mathbf{G}, m, H, \mathcal{L})$ for π. Let $H_1 = H \cap \mathbf{G}_1\langle \kappa[t]/t^m\rangle$ and let $H_2 = H \cap \mathbf{G}_2\langle \kappa[t]/t^m\rangle$. Let $\mathcal{L}_1$ be the pullback of $\mathcal{L}$ to H_1 and let $\mathcal{L}_2$ be the pullback of $\mathcal{L}$ to H_2.

To show that $(\mathbf{G}_1, m, H_1, \mathcal{L}_1)$ and $(\mathbf{G}_2, m, H_2, \mathcal{L}_2)$ are geometrically supercuspidal, observe that for any parabolic subgroup P_1 of $\mathbf{G}_1$ with maximal unipotent subgroup N_1, $P_1 \times \mathbf{G}_2$ is a parabolic subgroup of $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ with maximal unipotent subgroup $N_1 \times e$, and for any $(g_1, g_2) \in \mathbf{G}_1\langle \kappa[t]/t^m\rangle \times \mathbf{G}_2\langle \kappa[t]/t^m\rangle$,

$$H \cap (g_1, g_2)(N_1 \times e)(g_1, g_2)^{-1} = H_1 \cap g_1 N_1 g_1^{-1}$$

so the pullback of $\mathcal{L}_1$ to $g_1 N_1 g_1^{-1}$ is geometrically nontrivial. The same argument works symmetrically for $\mathbf{G}_2$.

Letting $J, J_1, J_2, \chi, \chi_1, \chi_2$ be the subgroups and characters associated to the various data, we have $J_1 \times J_2 \subseteq J$ and $\chi_1 \times \chi_2$ is the restriction of χ to $J_1 \times J_2$, so there is a surjection

$$\text{c-Ind}_{J_1}^{G_1(F)} \chi_1 \boxtimes \text{c-Ind}_{J_2}^{G_2(F)} \chi_2 = \text{c-Ind}_{J_1 \times J_2}^{G_1(F)\times G_2(F)} (\chi_1 \times \chi_2) \to \text{c-Ind}_{J}^{G_1(F)\times G_2(F)} \chi \to \pi = \pi_1 \boxtimes \pi_2$$

and thus surjections $\text{c-Ind}_{J_1}^{G_1(F)} \to \pi_1$ and $\text{c-Ind}_{J_2}^{G_2(F)} \chi_2 \to \pi_2$, as desired. □

Lemma 3.24. *Let E/F be an unramified extension of local fields, let G be an unramified reductive group over E. Let π be a representation of $G(E)$. Then π is an mgs representation of G over E if π is an mgs representation of the F-points of the Weil restriction of G from E to F.*

Proof. We may take $F = \kappa((t))$ and let $E = \kappa'((t))$. Let $\mathbf{G}$ be a group over κ' with $\mathbf{G}_E = G$. Let $\mathbf{G}'$ be the Weil restriction of $\mathbf{G}$ from κ' to κ. Then $\mathbf{G}'_F$ is the Weil restriction of G from E to F. Let $(\mathbf{G}', m, H', \mathcal{L};)$ be mgs data for π.

There is a natural map $\mathbf{G} \to \mathbf{G}'_{\kappa'}$. Let H' be the inverse image of H under this map and let $\mathcal{L}$ be the restriction of $\mathcal{L}'$ to H. Then to check that $(\mathbf{G}, m, H, \mathcal{L})$ is geometrically supercuspidal, observe that $\mathbf{G}'_{\kappa'} = \mathbf{G}^{[\kappa':\kappa]}$, and the natural embedding $\mathbf{G} \to \mathbf{G}'_{\kappa'}$ is the inclusion of one of these factors. For P a parabolic subgroup of $\mathbf{G}'$, let P' be the product of P on one factor with $\mathbf{G}_\kappa$ on all the other factors, so that N' is the image of N under this embedding, and thus for any $g \in G\langle \kappa[t]/t^m\rangle$, $gN'g^{-1}$ is the image of gNg^{-1}. Hence because $\mathcal{L}$ is nontrivial on $gN'g^{-1} \cap H'$, the restriction of $\mathcal{L}$ is nontrivial on $gNg^{-1} \cap H$.

Then J is a subgroup of J' and χ is the restriction of χ' to J, so since π contains a vector transforming under the character χ' of the subgroup J', it contains a vector transforming under the character χ of the subgroup J. □

3.9. **Admissibility.** In §3.1 we discussed the vanishing of Jacquet modules of certain induced representations $\text{c-ind}_J^G \chi$, but did not otherwise describe the structure of these representations $\text{c-ind}_J^G \chi$. We now present a lemma giving a condition for these induced representations (and slightly more general ones) to be finite direct sums of supercuspidals, which follows quickly from that they are admissible.

Lemma 3.25. *Let $F = \kappa((t))$, G a semisimple group over κ, J a compact open subgroup of $G(\kappa[[t]])$, and σ a smooth finite-dimensional representation of J. Suppose for any proper parabolic subgroup P of $G(F)$, with unipotent radical N, the restricted representation $\sigma|_{J\cap N}$ does not contain the trivial representation. Then* $\text{c-ind}_J^{G(F)}(\sigma)$ *is a finite direct sum of supercuspidal representations.*

The same assertion holds for an unramified group G over a local field F of characteristic zero. The semisimplicity condition is necessary because we take J to be compact; for G reductive, one would require that J is compact-modulo-center, see [6].

Proof. By [6, Theorem 1, (ii) $\implies$ (iv)], the assertion follows if we prove that $\text{c-ind}_J^G(\sigma)$ is admissible.

Let U_m be the principal congruence subgroup of $G(\kappa[\![t]\!])$ consisting of elements congruent to 1 mod t^m. To prove that $\text{c-ind}_J^G(\sigma)$ is admissible, it is sufficient to prove that the subspace of U_m-invariant vectors is finite-dimensional for every integer m. It suffices to prove that there are finitely many double cosets $U_m gJ$ such that

(C) σ restricted to $g^{-1}U_m g \cap J$ contains the trivial representation.

By the Cartan decomposition, we write $g = k'\mu(t)k$ with $k, k' \in G(\kappa[\![t]\!])$ with μ a cocharacter of G. We have $g^{-1}U_m g = k^{-1}\mu^{-1}(t)U_m\mu(t)k$ (because K_m is normalized by k').

It is sufficient to prove that there are only finitely many possibilities for μ such that there is g satisfying the condition (C), as U_m and J are finite index in $G(\kappa[\![t]\!])$.

We shall show that (C) implies that $\langle \mu, \alpha \rangle < m$ for any simple root α. This defines a finite subset of the cocharacter lattice.

Suppose for contradiction that $\langle \mu, \alpha \rangle < m$ for some simple root α. Let N be the maximal unipotent of the maximal parabolic associated to α.

Then $\mu^{-1}(t)U_m\mu(t)$ contains $N \cap G(\kappa[\![t]\!])$. To check this, it is sufficient to check that for any element $u \in N \cap G(\kappa[\![t]\!])$, the matrix coefficients of $\mu(t)u\mu(t)^{-1}$ in any representation are congruent to the identity matrix mod t^m. Expressed in a basis of eigenvectors for the maximal torus T, the matrix coefficients of u that do not match the matrix coefficients of the identity have T-eigenvalue a nonempty product of roots of N, hence μ-eigenvalue a nonempty sum of pairings of roots of N with the cocharacter t. This exponent is at least the pairing $\langle \alpha, t \rangle$ of the simple root α of P with t. Hence the $\mu(t)$-eigenvalues of these matrix coefficients are divisible by t^m.

So after conjugation by k, we obtain that $g^{-1}U_m g$ contains $k^{-1}Nk \cap G(\kappa[\![t]\!])$, thus $k^{-1}Nk \cap J$. By assumption, the restriction of σ to $k^{-1}Nk \cap J$ does not contain the trivial representation. A fortiori, the restriction of σ to $g^{-1}U_m g \cap J$ does not contain the trivial representation, hence (C) is not satisfied. □

4. The base change transfer for mgs matrix coefficients

In [31], Kottwitz proves the base change fundamental lemma for unramified extensions at not just the unit elements of Hecke algebras but the characteristic functions of quite general compact open subgroups. In this section, we prove the analogous statement for one-dimensional characters of these compact open subgroups.

There is no direct way to base change the data of a compact open subgroup J and a one-dimensional character χ of it from a field to an unramified extension. On the other hand it is easy to base change the monomial datum $(G, m, H, \mathcal{L})$ mentioned earlier, and this datum can be used to define a subgroup J and a character χ. The fact that the fundamental lemma holds in this setting can be motivated by the geometric Langlands philosophy: because the induced representations defined over two different fields from the data $(G, m, H, \mathcal{L})$ correspond to the same geometric object, i.e., the category of $(H, \mathcal{L})$-equivariant sheaves on the loop group $G((t))$, they should have the same geometric Langlands parameter, so automorphic base change should take one to the other, which suggests that the fundamental lemma should hold.

However, in the proof of the fundamental lemma, the geometric description is not necessary. We have isolated the data needed for a compact open subgroup of a group over a local field and a character to both have well-defined base changes to an arbitrary unramified extension. Our results hold in this setting, and work equally well over equal characteristic and mixed characteristic local fields. They may be of general interest.

4.1. **Character datum.** Let F be a non-archimedean local field, let L be the completion of its maximal unramified extension, let σ be the Frobenius of F acting on L, and let G be a connected reductive group over F.

Definition 4.1. A *character datum* on $G(F)$ consists of a bounded open σ-invariant subgroup J_L of $G(L)$ and a central extension of topological groups with an action of σ

$$1 \to \mathbb{C}^\times \to \tilde{J}_L \to J_L \to 1.$$

We take the discrete topology and the trivial σ action on $\mathbb{C}^\times$.

For $E \subset L$ a degree l unramified extension of F, the subgroup $G(E)$ consists in the σ^l-invariant elements of of $G(L)$. Given character data on $G(F)$, define the subgroup J_E to be the σ^l-invariant subset of J_L and define $\chi_E : J_E \to \mathbb{C}^\times$ to take a σ^l-invariant element g to $\sigma^l(\tilde{g})\tilde{g}^{-1}$, where $\tilde{g}$ is a lift of g from J_L to $\tilde{J}_L$. In particular, in the $l = 1$ case, the character χ_F sends $g \in J_F$ to $\sigma(\tilde{g})\tilde{g}^{-1}$. Note that J_E and χ_E are invariant under σ and hence independent of the choice of isomorphism of E with the σ^l-invariant subfield of L.

Definition 4.2. Given an integer $l \geq 1$, we say that the character datum satisfies the axiom Lang_l if the map $g \mapsto \sigma^l(g)g^{-1}$ from J_L to itself is surjective.

Let $(\mathbf{G}, m, H, \mathcal{L})$ be a monomial datum, that is a group $\mathbf{G}$ over a finite field κ, a natural number m, a connected algebraic subgroup H of $\mathbf{G}\langle\kappa[t]/t^m\rangle$, and a character sheaf $\mathcal{L}$ on H, we can define a character data on $G(\kappa((t)))$. Take J_L to be the elements of $\mathbf{G}(\overline{\kappa}[[t]])$ congruent mod t^m to elements of $H(\overline{\kappa})$. Lemma 2.14 defines a central extension of $H(\overline{\kappa})$ by $\overline{\mathbb{Q}}_\ell^\times$ with an action of σ associated to $\mathcal{L}$. By applying an embedding ι of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, and pulling back from $H(\overline{\kappa})$ to J_L, we obtain a central extension $1 \to \mathbb{C}^\times \to \tilde{J}_L \to J_L \to 1$.

Lemma 4.3. *When we obtain a character datum from $(\mathbf{G}, m, H, \mathcal{L})$ in this way, the following holds:*

1. *The axiom Lang_l is satisfied for every integer $l \geq 1$.*
2. *For κ' a finite extension of κ, and $E = \kappa'((t))$ the character χ_E is equal to $\iota \circ \chi_{\kappa'}$, the trace function of $\mathcal{L}_{\kappa'}$, pulled-back from $H(\kappa')$ to $J_E = J_{\kappa'}$.*

Proof. (1) By Lang's theorem [47, 4.4.17], the map $g \to \sigma^l(g)g^{-1}$ from $H(\overline{\kappa})$ to itself is surjective for all l, and by iteratively lifting solutions to the equation $\sigma^l(g)g^{-1} = h$, the same map is surjective on J_L, so the axiom Lang_l is satisfied for all l.

(2) This follows by comparing the definition with Lemma 2.14. ☐

Remark 4.4. Character data have many of the nice geometric properties of monomial data, in particular those needed to prove the base change fundamental lemma below, without bringing any geometry into the definition. A character datum does not necessarily come from an algebraic subgroup, even if one assumes the axioms Lang_1 and Lang_l. For instance, consider the group of all matrices in $\mathrm{SL}_2(\mathbb{F}_q[[t]])$ that are unipotent upper triangular $\pmod t$ and whose upper-right entry $\pmod t$ lies in an extension of $\mathbb{F}_{q^l}$ of degree a power of p. Then for any a in $\mathbb{F}_{q^{lp^r}}$, the action of $\mathrm{Frob}_{q^{lp^r}}$ on solutions of $x^q - x = a$ and $x^{q^l} - x = a$ is by translation, hence has order at most p, so both these equations have solutions in $\mathbb{F}_{q^{lp^{r+1}}}$.

4.2. **Matching of orbital integrals.** Assume that G_{der} is simply connected. Let $l \geq 1$, and $\tilde{J}_L \to J_L$ be character datum on $G(F)$ satisfying Lang_1 and Lang_l. We keep the other notation from the definition of character data.

Let E be an unramified extension of F of degree l, embedded as the fixed points of σ^l in L. Let θ be an automorphism of E, with $E^\theta = F$.

Let f on $G(F)$ be equal to χ on J_F and 0 elsewhere. Let f_E on $G(E)$ equal χ_E on J_E and 0 elsewhere. We have the orbital integral

$$O_\gamma(f) = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)dg/dt$$

for G_γ the centralizer of γ in G, dg the Haar measure on g that gives J_F measure one, and dt any fixed Haar measure on $G_\gamma(F)$.

Similarly, we define

$$O_{\delta\theta}(f_E) = \int_{I_{\delta\theta}(F)\backslash G(E)} f_E(g^{-1}\delta\theta(g))dg_E/d\mu$$

where $I_{\delta\theta} \subset \mathrm{Res}_F^E G$ is the subgroup of fixed points of conjugation by δ composed with θ, g_E is the Haar measure on G such that J_E has total mass one, and $d\mu$ is a Haar measure on $I_{\delta\theta}(F)$. We shall assume that these integrals converge absolutely.

Let j be an integer such that $\theta = \sigma^j$ as automorphisms of E, and let a, b be integers with $al - bj = 1$.

Kottwitz's argument [31] relies on the system in (γ, δ, c) of two equations

$$\begin{cases} c\gamma^a\sigma^l c^{-1} = \sigma^l, \\ c\gamma^b\sigma^j c^{-1} = \delta\sigma^j, \end{cases} \tag{4.1}$$

valued in the semidirect product of $G(L)$ with the free abelian group on σ.

Lemma 4.5. *Suppose that $\gamma \in J_F$, $\delta \in J_E$, $c \in J_L$ satisfy the system* (4.1). *Then* $\chi(\gamma) = \chi_E(\delta)$.

Proof. Choose lifts $\tilde{\gamma}$ and $\tilde{\delta}$ to $\tilde{G}_L$. We will perform calculations in the semidirect product of $\tilde{J}_L$ with the free abelian group on σ. We have

$$\chi(\gamma) = \sigma(\tilde{\gamma})\tilde{\gamma}^{-1} = [\tilde{\gamma}, \sigma].$$

Because γ and σ commute, $\tilde{\gamma}$ and σ commute modulo center, so because $bl - aj = 1$,

$$[\tilde{\gamma}, \sigma] = [\tilde{\gamma}^a\sigma^l, \tilde{\gamma}^b\sigma^j].$$

Then because this commutator is central, it commutes with c, and thus

$$[\tilde{\gamma}^a\sigma^l, \tilde{\gamma}^b\sigma^j] = c[\tilde{\gamma}^a\sigma^l, \tilde{\gamma}^b\sigma^j]c^{-1} = [c\tilde{\gamma}^a\sigma^l c^{-1}, c\tilde{\gamma}^b\sigma^j c^{-1}].$$

Finally, because this commutator is independent of the choice of lift to a central extension,

$$[c\tilde{\gamma}^a\sigma^l c^{-1}, c\tilde{\gamma}^b\sigma^j c^{-1}] = [\sigma^l, \tilde{\delta}\sigma^j] = [\sigma^l, \tilde{\delta}] = \sigma^l(\tilde{\delta})\tilde{\delta}^{-1} = \chi_E(\delta).$$

□

Lemma 4.6. *Suppose that $\gamma \in G(F)$, $\delta \in G(E)$, $c \in G(L)$ satisfy* (4.1), *and also satisfy $x^{-1}\gamma x \in J_F$, $y^{-1}\delta\theta(y) \in J_E$, $y^{-1}cx \in J_L$.*

Then $\chi_E(y^{-1}\delta\theta(y)) = \chi(x^{-1}\gamma x)$.

Proof. This follows by applying Lemma 4.5 to $x^{-1}\gamma x$, $y^{-1}\delta\theta(y)$, $y^{-1}cx$, which can be immediately seen to satisfy the system of equations (4.1). □

The remainder of the argument closely follows [31]. We repeat the arguments in our setting for clarity, and because Kottwitz works in mixed characteristic only and we need equal characteristic.

Lemma 4.7. *Suppose that $\gamma \in G(F)$, $\delta \in G(E)$, $c \in G(L)$ satisfy (4.1) Conjugation by c defines an isomorphism from G_γ to $I_{\delta\theta}$, and we have*

$$O_{\delta\theta}(f_E) = O_\gamma(f),$$

where we use this isomorphism to match the Haar measures on G_γ and $I_{\delta\theta}$.

Proof. We break the integral $\int_{I_{\delta\theta}(F)\backslash G(E)} f_E(g^{-1}\delta\theta(g))dg_E/d\mu$ into a sum over double cosets $x \in I_{\delta\theta}(F)\backslash G(E)/J_E$. For each double coset, we claim that f_E is constant. This is because f_E vanishes outside J_E, a set which is invariant under twisted J_E-conjuation, and is a θ-invariant character on J_E which is also invariant under twisted J_E conjugation. This follows from the fact that for $\tilde{k}$ a lift of k, and $g \in J_E$ $\sigma^j(\tilde{g})\tilde{k}\tilde{g}^{-1}$ is a lift of $\theta(g)\theta(k)g^{-1}$, and we have

$$\sigma^l(\sigma^j(\tilde{g})\tilde{k}\tilde{g}^{-1}) = \sigma^j(\sigma^l(\tilde{g}))\sigma^l(\tilde{k})\sigma^l(\tilde{g})^{-1} = \sigma^j(\tilde{g}\chi_E(g))\tilde{k}\chi_E(k)\tilde{g}^{-1}\chi_E(g)^{-1} = \sigma^j(\tilde{g})\tilde{k}\tilde{g}^{-1}\chi_E(k).$$

Hence we can express the integral as a sum over $y \in I_{\delta\theta}(F)\backslash G(E)/J_E$ such that $y^{-1}\delta\theta(y) \in J_E$ of $\chi_E(y^{-1}\delta\theta(y))$ times the measure of $I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)yJ_E$.

Similarly, in the $l = 1$ case, the integral is the sum over $x \in G_\gamma(F)\backslash G(F)/J_F$ such that $x^{-1}\gamma x \in J_F$ of $\chi(x^{-1}\gamma x)$ times the measure of $G_\gamma(F)\backslash G_\gamma(F)x$.

Using the axiom Lang_l, one can view $G(E)/J_E$ as the σ^l-fixed points in $G(L)/J_L$, and the set with $y^{-1}\delta\theta(y) \in J_E$ as the $\delta\sigma^j$ -fixed points among those. Similarly, by Lang_1, $G(F)/J_F$ is the set of σ-fixed points in $G(L)/J_L$, and the subset of x with $x^{-1}\gamma x \in J_F$ is the γ-fixed points. Now (4.1) implies precisely that the map that sends x to $y = cx$ gives a bijection between the points fixed by γ and σ and the points fixed by σ^l and $\delta\sigma^j$. Furthermore, the points of $G(L)$ fixed by conjugation by γ and σ are precisely $G_\gamma(F)$, and the points fixed by $\delta\sigma^j$ and σ^l are precisely $I_{\delta\theta}(F)$, so this gives a bijection between the double cosets $I_{\delta\theta}(F)\backslash G(E)/J_E$ and $G_\gamma(F)\backslash G(F)/J_F$.

By construction, for x and y paired by this bijection, we have $y = cx \in G(L)/J_L$, so $y^{-1}cx \in J_L$, thus by Lemma 4.6, $\chi_E(y^{-1}\delta\theta(y)) = \chi(x^{-1}\gamma x)$.

It remains to check that, for x and y paired by this bijection, the measure of $I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)yJ_E$ equals the measure of $G_\gamma(F)\backslash G_\gamma(F)xJ_F$. To do this, observe that we have fixed measures so that J_E and J_F, so that the measure of $I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)yJ_E$ is equal to the inverse of the measure of the stabilizer of yJ_E in $I_{\delta\theta}$, and $G_\gamma(F)\backslash G_\gamma(F)xJ_F$ is equal to the inverse of the measure of the stabilizer of xJ_F in $G_\gamma(F)$. We can equivalently view these stabilizers as the stabilizers of the poitns x and y in $G(L)/J_L$. Thus, because $y = cx$, these stabilizers are sent to each other by the isomorphism between $G_\gamma(F)$ and $I_{\delta\theta}(F)$ defined by conjugation by c, which by assumption is a measure-preserving isomorphsm, so these measures are equal.

Hence the sums are equal and the orbital integrals are equal. □

4.3. **Stable orbital integrals.** An inner twisting between two algebraic groups is an isomorphism defined over the separable closure of the base field, which is Galois-invariant up to compositions with inner automorphisms, and where we take two inner twistings to be equivalent if they are equal up to composition with an inner automorphisms [40, p. 68]. Given an inner twisting between two groups, there is a natural transfer, explained in *loc. cit*, of Haar measures from one group to Haar measures on the other via the Lie algebras.

In particular, if γ and γ' are stably conjugate, then there is a canonical inner twisting (i.e. canonical isomorphism over the separable closure of the base field, up to conjugacy) between their centralizers G_γ and $G_{\gamma'}$. This enables us to define, after fixing a Haar measure on G_γ, the stable orbital integral

$$SO_\gamma(f) = \sum_{\gamma'} e(G_{\gamma'}) O_{\gamma'}(f)$$

where γ' is a system of conjugacy classes of elements stably conjugate to γ, and $e(G_{\gamma'})$ is the sign defined by Kottwitz.

Less obviously, for $\delta \in G_E$ let $\mathcal{N}\delta = \delta\theta(\delta)\theta^2(\delta)\dots\theta^{l-1}(\delta)$ be the norm of δ. If $\mathcal{N}\delta$ is stably conjugate to γ then there is a canonical inner twisting $I_{\delta\theta} \to G_\gamma$. Indeed,

Lemma 4.8. *Let p be the projection $I_E \to G_E$ defined using the fact that R-points of I are $R \otimes_F E$-points of G for any ring R by the map $G(R \otimes_F E) \to G(R)$ for an E-algebra R induced by the multiplication map $R \otimes_F E \to R$.*

For $d \in G(F^s)$ such that $d^{-1}\mathcal{N}\delta d = \gamma$, the map $g \mapsto d^{-1}p(g)d$ from $I_{\delta\theta,F^s} \to G_{\gamma,F^s}$ is an isomorphism.

This defines an inner twisting $I_{\delta\theta} \to G_\gamma$ which depends only on γ, δ.

The proof is the same as [30, Lemma 5.8] and [40, I, p. 115], though neither reference is in the exact context we work in. The idea is to construct an explicit isomorphism $I_{\delta\theta} \to G_\gamma$ over the separable closure of F.

Proof. We use the fact that $I_E \cong G^{\mathrm{Gal}(E/F)}$, where $I = \mathrm{Res}_F^E G$. Under this isomorphism, the action of δ is by translation, and the map p is projection onto one of the factors. (This follows from the fact that $E \otimes_F E = E^{\mathrm{Gal}(E/F)}$, with the action of δ by translation, and the multiplication map to E is projection onto one of the factors).

Thus the action of $\delta\theta$ on I is by conjugation by δ and then translating by $\theta \in \mathrm{Gal}(E/F)$. So a fixed point of this action is determined a tuple of l elements of G, each of which when conjugated by δ becomes equal to the next one. Such a tuple is determined by its value in one copy of G, and an element of G extends to a tuple if and only if it returns to itself when conjugated and translated l times, which is equivalent to commuting with $\mathcal{N}\delta$. This shows that the projection p defines an isomorphism $I_{\delta\theta} \cong G_{\mathcal{N}\delta}$ over L, and then conjugating by d gives a further isomorphism onto G_γ.

This is a canonical inner twisting because any d' satisfying the same equation as d, for instance a Galois conjugate of d, is equal to d times an element of G_γ, so this map depends only on δ, γ up to conjugation by G_γ. ☐

Using this canonical inner twisting to transfer a fixed Haar measure on G_γ, we can define the stable twisted orbital integral

$$SO_{\delta\theta}(f_E) = \sum_{\delta'} e(I_{\delta'\theta}) O_{\delta'\theta}(f_E)$$

where δ' are a system of representatives for the twisted conjugacy classes inside the stable twisted conjugacy class of δ.

We will now show an identity of stable twisted orbital integrals, continuing to follow [31].

Lemma 4.9. *For each $\delta \in G(E)$, there is at most one $\gamma \in G(F)$ up to conjugacy satisfying* (4.1)*, and always at least one if $O_{\delta\theta}(f_E) \neq 0$. Similarly, for each $\gamma \in G(F)$, there is at most one $\delta \in G(E)$ up to θ-conjugacy satisfying* (4.1)*, and always at least one if $O_\gamma(f) \neq 0$.*

Finally, δ and γ satisfying (4.1) *have $\mathcal{N}\delta = c\gamma c^{-1}$, where $\mathcal{N}\delta$ is the norm of δ.*

Proof. Fix γ. The identity $c\gamma^a\sigma^l c^{-1} = \sigma^l$ implies

$$c^{-1}\sigma^l(c) = \gamma^a,$$

which uniquely determines c up to left multiplication by something σ^l-invariant. In other words, this determines c up to left-multiplication by an element of $G(E)$. For any choice of c, the identity $c\gamma^b\sigma^j c^{-1} = \delta\sigma^j$ determines δ, and multiplying c on the left by $G(F)$ is equivalent to conjugating $\delta\sigma^j$ by an element of $G(E)$ and thus is equivalent to θ-conjugating δ by an element of $G(E)$. So for each γ, there is at most one δ up to θ-conjugacy.

For there to exist at least one δ satisfying (4.1), it suffices that the equation $c^{-1}\sigma^l(c) = \gamma^a$ has a solution, for which by the axiom Lang_l it suffices that γ is conjugate to an element of J_F, which is implied by the nonvanishing of $O_\gamma(f)$. Moreover, any δ satisfying (4.1) lies in $G(E)$ because the two equations together imply that δ commutes with σ^l.

For the opposite direction, we change the equations slightly. Because γ and σ commute with each other, and σ^l and $\delta\sigma^j$ commute with each other, we can invert the two-by-two matrix to obtain the equivalent equations

$$(\delta\sigma^j)^l\sigma^{-jl} = c\gamma c^{-1}$$
$$(\delta\sigma^j)^{-a}\sigma^{bl} = c\sigma c^{-1}$$

Fixing δ, the second equation determines $c\sigma c^{-1}$, hence determines c up to right multiplication by an element of $G(F)$. Examining the first equation, we see it determines γ after fixing δ, c, and right multiplying c by an element of $G(F)$ has the effect of conjugating γ by an element of $G(F)$.

For γ to exist, it suffices that there exists a c with $c\sigma(c)^{-1} = (\delta\sigma^j)^{-a}\sigma^{bl-1}$, for which by the axiom Lang_1 it suffices that $\delta\sigma^j$ is θ-conjugate to an element of J_F, which is implied by the nonvanishing of $O_{\delta\theta}(f)$. Furthermore this implies $\gamma \in G(F)$, because it implies γ commutes with σ.

Finally, observe that

$$c\gamma c^{-1} = (\delta\sigma^j)^l\sigma^{-jl} = \delta\theta(\delta)\theta^2(\delta)\dots\theta^{l-1}(\delta) = \mathcal{N}\delta.$$

□

Lemma 4.10. *For any δ, γ, c satisfying (4.1), the map from $I_{\delta\theta}(F)$ to $G_\gamma(F)$ defined by conjugation by c in fact arises from an isomorphism of group schemes over F, which is equivalent to the isomorphism of Lemma 4.8 in the case $d = c$.*

In particular, the transfer of the Haar measure from $G_\gamma(F)$ to $I_{\delta\theta}(F)$ under this map matches the transfer via the canonical inner twisting.

Proof. The isomorphism $g \mapsto c^{-1}p(g)c$ of Lemma 4.8 is, by construction, defined over L.

To show it descends from L to F, we use the fact that G_γ and $I_{\delta\theta}$ are reductive, so there exists a scheme parameterizing isomorphisms between them. To check that an L-point of this scheme is defined over F, it suffices to check that it is stable under the Frobenius σ. In other words we must check that it commutes with σ. It suffices to check it commutes with σ^l and σ^j.

Observe that σ^l commutes with p, and that

$$\sigma^l(c^{-1}gc) = \sigma^l(c)^{-1}\sigma^l(g)\sigma^l(c) = \gamma^{-a}c^{-1}\sigma^l(g)c\gamma^a = c^{-1}\sigma^l(g)c$$

using (4.1) and the fact that γ commutes with $c^{-1}\sigma^l(g)c \in G_\gamma$.

Next observe that

$$\sigma^j(c^{-1}p(g)c) = \sigma^j c^{-1}p(g)c\sigma^{-j} = \gamma^{-b}c^{-1}\delta\sigma^j p(g)\sigma^{-j}\delta^{-1}c\gamma^b$$
$$= \gamma^{-b}c^{-1}p(\delta\theta\sigma^j g\sigma^{-j}\theta\delta^{-1})c\gamma^b = \gamma^{-b}c^{-1}p(\sigma^j(c))c^{-1} = c^{-1}p(\sigma^j(c))c^{-1}$$

using (4.1), the fact that $\sigma^j(g) \in I_{\delta\theta}$ commutes with $\delta\theta$, and the fact that $c^{-1}p(\sigma^j(c))c^{-1} \in G_\gamma$ commutes with γ. □

Theorem 4.11. *For every semisimple $\gamma \in G(F)$, the stable orbital integral $SO_\gamma(f_E)$ vanishes unless the stable conjugacy class of γ is equal to the norm $\mathcal{N}\delta$ for some $\delta \in G(E)$, in which case it is given by $SO_\gamma(f) = SO_{\delta\theta}(f_E)$.*

Here we define both stable orbital integrals using the same Haar measure on G_γ.

Proof. For each stable conjugate γ' of γ, if the associated orbital integral is nonvanishing, then γ' is conjugate to an element of K. Hence by Lemma 4.9 there exists a δ' satisfying Kottwitz's equations, and the norm of δ' is stably conjugate to γ.

So we may assume that γ is stably conjugate to the norm of δ. Now for each γ' for which the orbital integral is nonvanishing there exists a unique δ' up to θ-conjuacy satisfying (4.1) by Lemma 4.9, and because the norm of δ' is stably conjugate to the norm of δ, δ' are stably θ-conjugate to δ. (To see, this, base change to E, so that $I = G^l$ and θ acts by permutation. Then if two elements of G^l have conjugate norms, we can θ-conjugate one to the other by adjusting each element of the l-tuple step-by-step.) By Lemma 4.7 and Lemma 4.10, the orbital integrals and signs of γ' and δ' agree. (The signs agree because they depend only on the isomorphism class, and we have an isomorphism between the two groups.) Because each γ' corresponds to a unique δ' up to stable θ-conjugacy, and by Lemma 4.9 each δ' with nonvanishing orbital integral corresponds to a unique γ' up to stable conjugacy, the signed sums of orbital integrals over conjugacy classes and θ-conjugacy classes agree, so the orbital integrals agree. □

The analogue for κ-orbital integrals should also be possible, by an argument analogous to that in [31].

5. Automorphic base change

For every place x of every constant field extension F_n of F of degree $n \geq 1$, we will always take the standard hyperspecial maximal compact $G(\mathfrak{o}_x)$ defined by the globally split structure of G. We say that a representation is unramified when it is $G(\mathfrak{o}_x)$-unramified. Let π be an automorphic representation, and $u \in |X|$ a place such that π_u is mgs. In this context, we say that π is base-changeable if the following holds.

Condition (BC). *There exists a finite set of mgs data at u, such that for every constant field extension F_n of F, there exists a base change representation Π_n of $G(\mathbb{A}_{F_n})$, which at places lying over u is mgs with one of the given mgs datum, over the unramified places of π is unramified and compatible under the Satake isomorphism, and at all other places has depth bounded independent of n.*

We make the following conjecture.

Conjecture 5.1. *Every automorphic representation of $G(\mathbb{A}_F)$ that is mgs at a place u satisfies Condition BC.*

This is a standard conjecture on the existence of cyclic base change, analogous to results that have been proved over number fields by Labesse [32, Thm 4.6.2], except for the compatibility condition at places lying over u, and for the boundenness of depth [21]. Our main evidence that a cyclic base change compatible at u should exist is Theorem 4.11, which gives the local transfer identities needed to compare twisted orbital integrals involving a test function which detects the mgs condition with usual orbital integrals for an analogous test function. Hence cases of the conjecture is accessible by endoscopically stabilizing the trace formula and twisted trace formula and proving a comparison result between them, and special cases are accessible either by establishing stability of a finite set of mgs data at u, or by inserting stabilizing test functions at an additional place. We shall do this in the sequel [45].

6. Geometric setup

We now discuss geometric models for a family of automorphic forms with local conditions.

Let k be a field, let X be a curve over k, and let $F = k(X)$. When we connect to analysis we will assume k finite, but for the purely geometric parts we will not need that assumption. Let G be a split reductive algebraic group over k. Let D be a divisor on X, which we will often view as a closed subscheme in X. We can write $D = \sum_{x\in D} m_x[x]$ where m_x is the multiplicity of x in D.

Definition 6.1. Let $\mathrm{Bun}_{G(D)}$ be the moduli space of G-bundles on X with a trivialization along D (notation is in analogy with that of principal congruence subgroups).

We write $|X|$ for the set of closed points of X and $|X - D|$ for the points outside the support of D. For $x \in |X|$, let κ_x be the residue field at x. We fix a local coordinate t of X at each closed point x, so that $\mathfrak{o}_x = \kappa_x[[t]]$ is the complete local ring at x, but our constructions will be independent of the choice of coordinate and so this is really just a notational convenience. With this convention, $F_x = \kappa_x((t))$. The adele ring $\mathbb{A}_F$ is the restricted product $\prod'_{x\in|X|} F_x$.

Notation 6.2. Let

$$\mathbf{K}(D) = \prod_{x\in|X-D|} G(\mathfrak{o}_x) \times \prod_{x\in D} U_{m_x}(G(\mathfrak{o}_x))$$

where $U_{m_x}(G(\kappa_x[[t]]))$ is the subgroup of $G(\kappa_x[[t]])$ consisting of elements congruent to 1 modulo t^{m_x}. Then Weil's parameterization lets us write $\mathrm{Bun}_{G(D)}(k)$ as the adelic double quotient $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$.

Let $\mathcal{O}_D$ be the ring of global sections of the structure sheaf on the scheme D, so that $G\langle\mathcal{O}_D\rangle$ is the group of automorphisms of the trivial G-bundle on D.

Lemma 6.3. *We have isomorphisms*

$$\mathcal{O}_D \simeq \prod_{x\in D} \kappa_x[t]/t^{m_x}, \quad G\langle\mathcal{O}_D\rangle \simeq \prod_{x\in D} G\langle\kappa_x[t]/t^{m_x}\rangle.$$

Proof. The first isomorphism follows from viewing D as a disjoint union of schemes $m_i x_i$, and choosing local coordinates for each x_i, and the second isomorphism follows from the first. □

Definition 6.4. Say that an algebraic subgroup $H \subseteq G\langle\mathcal{O}_D\rangle$ is *factorizable* if it is equal to a product $\prod_{x\in D} \mathrm{Res}^k_{\kappa_x} H_x$ where H_x is an algebraic subgroup of $G_{\kappa_x}\langle\kappa_x[t]/t^{m_x}\rangle$ and $\mathrm{Res}^k_{\kappa_x} H_x$ is its Weil restriction from κ_x to k, making it a subgroup of $G\langle\kappa_{x_x}[t]/t^{m_x}\rangle$.

Lemma 6.5. *If $H \subseteq G\langle\mathcal{O}_D\rangle$ is factorizable, then for any separable field extension k' of k, the base change $H_{k'}$ of H from k to k' remains factorizable as a subgroup of $G_{k'}\langle\mathcal{O}_D \otimes k'\rangle$.*

Proof. This happens because the base change from k to k' of the Weil restriction from κ_x to k of a subgroup equals the Weil restriction from $\kappa_x \otimes k'$ to k' of the base change from κ_x to $\kappa_x \otimes k'$ of the same subgroup. If $\kappa_x \otimes k'$ is a product of fields, then the Weil restriction is the product of the Weil restrictions over each field, and hence the resulting subgroup is factorizable. □

Fix a smooth connected factorizable subgroup $H \subseteq G\langle \mathcal{O}_D \rangle$ and a character sheaf $\mathcal{L}$ on H. By Lemma 2.15, H splits as a product $\boxtimes_{x\in D} \operatorname{Res}^k_{\kappa_x} \mathcal{L}_x$ for character sheaves $\mathcal{L}_x$ on H_x. We refer to this data as a set of monomial local conditions on an automorphic representation of $G(\mathbb{A}_F)$.

Notation 6.6. Let J_x be the inverse image of $H_x(\kappa_x)$ in $G(\kappa_x[\![t]\!])$, which maps to $G(\kappa_x[t]/t^{m_x}) = G\langle \kappa_x[t]/t^{m_x}\rangle(\kappa_x)$ by the natural projection.

Definition 6.7. Let χ_x be the character of $H_x(\kappa_x)$, and thus of J_x, induced by $\mathcal{L}_x$ and let χ be the character of $H(k)$ induced by $\mathcal{L}$.

Under these definitions, we have a commutative diagram

$$\begin{array}{ccccc}
\mathbf{K}(D) & \hookrightarrow & \prod_{x\in|X-D|} G(\mathfrak{o}_x) \times \prod_{x\in D} J_x & \twoheadrightarrow & H(\kappa) \\
 & & \big\downarrow & & \big\downarrow \\
 & & \prod_{x\in|X|} G(\mathfrak{o}_x) & \twoheadrightarrow & G(\mathcal{O}_D)
\end{array}$$

where the square is a Cartesian and the top row is a short exact sequence.

For clarity and concreteness, we mention the data $(G, m_x, H_x, \mathcal{L}_x)$ that will appear in the proof of the main theorem of the paper. At each place, we will either take H_x the trivial group and $\mathcal{L}$ the trivial sheaf, or we will take $(G, m_x, H_x, \mathcal{L}_x)$ to be geometrically supercuspidal. Examples of the second kind of data were provided in Lemma 3.5.

Remark 6.8. Consider the space of functions on $\mathrm{Bun}_{G(D)}(k) = G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ that are χ-equivariant for the natural right action of

$$H(k) \subseteq G\langle \mathcal{O}_D\rangle(k) = \prod_{x\in D} G(\kappa_x[t]/t^{m_x}) = \prod_{x\in D} G(\mathfrak{o}_x)/U_{m_x}(G(\mathfrak{o}_x)) = G(\mathfrak{o}_F)/\mathbf{K}(D)$$

on $\mathrm{Bun}_{G(D)}(k)$, where $\mathfrak{o}_F = \prod_{x\in X} \mathfrak{o}_x = \prod_{x\in X} \kappa_x[\![t]\!]$.

We view this as a space of automorphic forms.

We break this space into a sum of eigenspaces under Hecke operators, indexed by automorphic representations of $G(\mathbb{A}_F)$. All automorphic representations that appear in this sum are unramified away from D, and at a point $x \in D$ admit a nontrivial map from the compact induction $\text{c-Ind}_{J_x}^{G(\kappa_x(\!(t)\!))} \chi_x$.

The dimension of the space associated to an automorphic representation π of $G(\mathbb{A}_F)$ is equal to its global multiplicity in $L^2(G(F)\backslash G(\mathbb{A}_F))$ times the product over x of the dimension of the (J_x, χ_x) eigenspace in π_x. The dimension of this eigenspace for different groups J_x, χ_x is studied in the general theory of newforms as well as the theory of depth and Moy-Prasad types.

Remark 6.9. We compare our data $(G, D, H, \mathcal{L})$ defining a space of automorphic forms to the "geometric automorphic datum" defined by Yun in [52, 2.6.2]. Both are geometric versions of the notion of an automorphic representation defined by local conditions, but Yun's is somewhat more general, as we have made various restrictions for technical and notational simplicity.

We work with semisimple groups, while Yun fixes a central character. The group "$\mathbf{K}_S$" in [52] carries the same information as our H. It is a pro-algebraic subgroup of $\prod_{x\in S} G\langle \kappa_x[\![t]\!]\rangle$, whereas H is an algebraic subgroup of $G\langle \mathcal{O}_D\rangle$. This is only a technical difference - by truncating, we avoid working with pro-algebraic groups. More significantly, Yun allows the local subgroups to be contained in any parahoric subgroup, while we allow only the standard hyperspecial subgroup, and he allows them to be arbitrary subgroups of $G\langle \kappa_x[\![t]\!]\rangle$ and not just Weil restrictions from $G_{\kappa_x[\![t]\!]}$, which means that his definition is not stable under base field extension (this can be repaired by either specializing to subgroups that are Weil restrictions or generalizing to subgroups of the product of local groups at all places, rather than products of local subgroups). His "$\mathcal{K}_S$" are our $\mathcal{L}_x$, $x \in D$, and behaves similarly.

Remark 6.10. Most of our methods apply over an arbitrary base field, and it would not be surprising if they could be generalized to the derived category of D-modules. For instance, Theorem 7.33 could possibly be established for D-modules, in which case Lemma 8.3 would be the statement that a D-module pushforward is supported in a single degree. Similarly, the Ramanujan conjecture in a particular case established in [26] has been used in [36] to prove that certain character D-modules were concentrated in a single degree.

If this was done, it might have relevance to the characteristic zero geometric Langlands program. However, it is easy to see that the geometric supercuspidality condition cannot be satisfied by any tamely ramified character sheaf, and thus cannot be satisfied at all for sheaves, or D-modules with regular singularities, in characteristic zero. Hence using this technique requires dealing with irregular singularities.

Remark 6.11. We note that this geometric setup can also be used to motivate Condition BC. Let π is an automorphic representation generated by some automorphic function on $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ which is χ-equivariant for the right action of $H(k)$. Suppose that it is the trace function of a Hecke eigensheaf on $\mathrm{Bun}_{G(D)}$ that is $\mathcal{L}_\psi$-equivariant for the right action of H, then π satisfies Condition BC, except possibly for finitely many extensions. Indeed, over each finite field extension k' of k, we can take the trace function of the Hecke eigensheaf over k', which is a Hecke eigenfunction, and hence is a sum of functions lying in one or more automorphic representations with the same Satake parameters at unramified places. Because the Hecke eigenvalues come from the same geometric Langlands parameter as the Hecke eigensheaf associated to π, they have matching Satake parameters. Because the automorphic function lies on $\mathrm{Bun}_{G(D)}(k')$, the associated representations have bounded depth, and because it is $(H(k'), \chi_{k'})$-equivariant, the associated representations are compatible with the same mgs data at every mgs place. The only potential problem is if the trace function is identically zero, which can only happen for finitely many field extensions.

6.1. Moduli Spaces.

As in §2.2, let Λ^+ be a Weyl cone in the cocharacter lattice of G (which is naturally in bijection with a Weyl cone in the character lattice of $\widehat{G}$).

Let x be a point in X and let $U \subseteq X$ be a neighborhood of x. Let α_1 and α_2 be two G-bundles defined over U, and let $\varphi : \alpha_1 \to \alpha_2$ be an isomorphism over $U - \{x\}$. If we choose trivializations of α_1 and α_2 in a formal neighborhood of x, we can represent φ as an element of $G(\kappa_x(\!(t)\!))$. Changing the trivializations corresponds to the left and right action of $G(\kappa_x[\![t]\!])$ on this element, so the isomorphism φ defines a double coset in $G(\kappa_x[\![t]\!])\backslash G(\kappa_x(\!(t)\!))/G(\kappa_x[\![t]\!])$. These double cosets are naturally in bijection, under the Bruhat decomposition, with Λ^+. We can view this decomposition as coming from the affine Grassmannian $G(\!(t)\!)/G[\![t]\!]$, because each double coset in $G(\kappa_x[\![t]\!])\backslash G(\kappa_x(\!(t)\!))/G(\kappa_x[\![t]\!])$ is a $G(\kappa_x[\![t]\!])$-orbit in the κ_x-points $G(\kappa_x(\!(t)\!))/G(\kappa_x[\![t]\!])$ of the affine Grassmannian. These orbits are the Bruhat cells of the affine Grassmannian, which again are in bijection with Λ^+. This geometric description makes clear that, in any family of G-bundles α_1, α_2 and maps φ, the set of points where the double coset associated to φ is in a particular cell of the affine Grassmannian is locally closed and, moreover, the set of points where φ is in the closure of a particular Bruhat cell of the affine Grassmannian is closed. Using these closed cells, we will define a Hecke correspondence.

Let W be a function from $|X|$ to Λ^+, that sends all but finitely many points to the trivial cocharacter and sends all the points of D to the trivial cocharacter. Define the support of W to be the set of points that W sends to a nontrivial cocharacter (i.e. the usual definition of the support of a function, if we view the trivial cocharacter as the zero element of Λ^+). We will view the trivial cocharacter as the zero element of Λ^+, so that the support of W is the set of points sent to nontrivial characters.

Definition 6.12. Let $\mathcal{H}k_{G(D),W}$ be the moduli space of pairs of G-bundles with an isomorphism away from the support of W, and with a trivialization of the first bundle along D, such that near each point x of the support of W, when the isomorphism is viewed as a point in the formal loop space $G((t))$, it projects to a point in the affine Grassmannian that lies in the closed cell corresponding to W_x.

Definition 6.13. We define a map $\Delta^W : \mathcal{H}k_{G(D),W} \times H \to \mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$ where the left projection is taking the first G-bundle with trivialization and the right projection is taking the second G-bundle, using the isomorphism to carry over the trivialization, and then twisting the trivialization by the element of H.

We will work with the intersection cohomology complex $IC_{\mathcal{H}k_{G(D),W}}$ on $\mathcal{H}k_{G(D),W}$, which by definition is the unique irreducible perverse sheaf isomorphic to $\overline{\mathbb{Q}}_\ell[\dim \mathcal{H}k_{G(D),W}]$ on the open set where $\mathcal{H}k_{G(D),W}$ is smooth.

Remark 6.14. The trace function of $\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$, which is a function on $\mathrm{Bun}_{G(D)}(k) \times \mathrm{Bun}_{G(D)}(k)$, is the kernel for the composition of the Hecke operator associated to W by the Satake isomorphism with the averaging operator of χ associated to H (Lemma 9.9). Thus it acts as a Hecke operator on the space of automorphic forms described in Remark 6.8.

The aim of Section 7 will be to prove the following cleanness property of Δ^W. Namely we shall establish in Theorem 7.33 that if, for some $u \in D$, $(G_{\kappa_u}, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal, then the natural map $\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \to \Delta^W_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is an isomorphism. Using this, in Section 8, we will prove that $\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is a pure perverse sheaf, which we will use in Section 9 to derive concrete numerical consequences.

Remark 6.15. Let us explain some of the motivation for Theorem 7.33. As we mentioned before, the trace function of $R\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is a Hecke kernel on a particular space of automorphic forms. In particular, in the case when W is trivial, it is simply the idempotent projector onto this space of automorphic forms.

In the case where $G = SL_2$, D is empty, and W is trivial, the trace function of $R\Delta^W_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ was calculated by Schieder [46, Proposition 8.15]. Viewing the trace function as a kernel, the induced operator on the space of automorphic forms was calculated by Drinfeld and Wang, who found that it acts as the identity on cusp forms [18, Proposition 3.2.2(i), Theorem 1.3.4, and Equation 3.2], and a similar calculation was done by Wang for general groups in [51, Theorem C.7.2 and Theorem 1.4.3]. If this fact is true for the families of automorphic forms with more general families of local conditions, then the trace function of $R\Delta^W_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ should equal the trace function of $R\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ as soon as one of the local conditions ensures that the automorphic forms in the family are cuspidal by mandating that one of the local factors is supercuspidal. If we believe this, then we might conjecture that they should agree as sheaves and not just trace functions as long as the local condition also forces cuspidality over finite field extensions.

7. Cleanness of the Hecke complex

As before, let X be a smooth projective curve over a finite field k, G a split semisimple algebraic group over k, D a divisor on X, H a smooth factorizable subgroup of $G\langle\mathcal{O}_D\rangle$, and $\mathcal{L}$ a character sheaf on H.

7.1. A compactification of $\mathcal{H}k_{G(D),W} \times H$.

Let V be a faithful representation of G, which we also view as a functor $\alpha \mapsto V(\alpha)$ from G-bundles to vector bundles. Throughout this section, we will be working geometrically and so we can assume that k is algebraically closed. Assume that the pairing of any root of G with any weight of V is less than the characteristic p of k. We fix a maximal torus and a Borel $T \subset B$ inside G. As in the previous section, let W be a function from $|X|$ to Λ^+ with finite support disjoint from the effective divisor $D = \sum_{x\in|X|} m_x[x]$.

Definition 7.1. Let $\{W\} : |X| \to \mathbb{Z}$ be the divisor, whose multiplicity $\{W\}_x$ at a point $x \in |X|$ is equal to minus the lowest weight of the composition $\mathbb{G}_m \xrightarrow{W_x} G \to \mathrm{GL}(V)$ of the representation V with the cocharacter $W_x \in \Lambda^+$. (This composition is a representation of $\mathbb{G}_m$ so its weights are integers.)

The support of $\{W\}$ is a subset of the support of W. In particular we have that $\{W\}$ is disjoint from D.

Example 7.2. (i) If $G = Sp_{2n}$, V is the standard representation, and W_x is the cocharacter with eigenvalues $\lambda^{w_1}, \dots, \lambda^{w_n}, \lambda^{-w_n}, \dots, \lambda^{-w_1}$ where $w_1, \dots, w_n$ are integers with $w_1 \geq \cdots \geq w_n \geq 0$ then $\{W\}_x = w_1$.

(ii) If $G = SL_n$, V is the adjoint representation, and W_x is the cocharacter whose eigenvalues on the standard representation are $\lambda^{w_1}, \dots, \lambda^{w_n}$ for $w_1, \dots, w_n$ integers with $w_1 \geq \cdots \geq w_n$ and $\sum_{i=1}^n w_i = 0$, then its eigenvalues on the adjoint representation have the form $\lambda^{w_1 - w_n}$ so $\{W\}_x = w_1 - w_n$.

Before compactifying $\mathcal{H}k_{G(D),W} \times H$, we compactify G:

Notation 7.3. Let $\overline{G}$ be the closure of $G \subseteq \operatorname{End} V \subseteq \mathbb{P}(\operatorname{End} V \oplus k)$. (The map $G \to GL(V) \to \operatorname{End} V$ is an immersion because V is a faithful representation).

Given two pairs $(\alpha_1, t_1), (\alpha_2, t_2)$ of a G-bundle and a trivialization over D and a projective section $\varphi \in \mathbb{P}(\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$, because $\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k$ is the vector space of global sections of

$$\mathcal{H}om(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X,$$

we can view φ as a nonzero global section of $\mathcal{H}om(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X$, well-defined up to scaling. Locally over any open set, closed set, punctured formal neighborhood, that does not intersect the support of W and where we have at trivialization of α_1 and α_2, we obtain a section of $\operatorname{End} V \oplus k$ up to scaling.

Definition 7.4. Let $\overline{\mathcal{H}k}_{G(D),H,W,V}$ be the moduli space of five-tuples consisting of $\alpha_1, t_1, \alpha_2, t_2, \varphi$ where $(\alpha_1, t_1), (\alpha_2, t_2)$ are two pairs of a G-bundle and a trivialization over D and φ is a section of

$$\mathbb{P}(\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$$

such that

(1) Over any open set in the complement of the support of W, for any trivialization of α_1 and α_2 over that open set, the induced section of $\operatorname{End} V \oplus \mathcal{O}_X$ lies in the affine cone on $\overline{G}$. (Note that $\overline{G}$ is invariant under the left and right action of G, so this does not depend on the choice of trivialization.)
(2) In a punctured formal neighborhood of any point x in the support of W, for any trivialization of α_1 and α_2 over that punctured formal neighborhood, the induced section of $\operatorname{End} V \oplus \mathcal{O}_X$, when viewed as a point in the formal loop space $(\operatorname{End} V \oplus k)((t))$, is in the closure of the set of pairs $(\lambda V(g), \lambda)$ where $\lambda \in \mathbb{G}_m$ and $g \in G((t))$ is in the Bruhat cell associated to W_x.
(3) Over D, using the trivializations t_1 and t_2, the induced element of $(\operatorname{End} V \oplus k)\langle \mathcal{O}_D \rangle$ lies in the closure of the set of pairs $(\lambda h, \lambda)$ where $h \in H \subseteq G\langle \mathcal{O}_D \rangle \subseteq \operatorname{End} V \langle \mathcal{O}_D \rangle$. Equivalently, using an arbitrary trivialization over D, $t_2 \circ f|_D \circ t_1^{-1}$ lies in this closure.

For interpreting the last two conditions, remember that a global section of $\mathcal{O}_X$ is always constant over X so forcing the last coordinate to be locally constant over X is not any additional restriction. Recall from Definition 6.12 that $\mathcal{H}k_{G(D),W}$ is the moduli space of four-tuples consisting of a pair of G-bundles α_1, α_2, an isomorphism $f : \alpha_1 \to \alpha_2$ away from the support of W, that near each point in the support of W is in the closure of the cell of the affine Grassmannian associated to the corresponding representation, and a trivialization t_1 of α_1.

Lemma 7.5. *There is a well-defined map* $j : \mathcal{H}k_{G(D),W} \times H \to \overline{\mathcal{H}k}_{G(D),H,W,V}$ *that sends* $(\alpha_1, t_1, \alpha_2, f, h)$ *to* $((\alpha_1, t_1), (\alpha_2, h \circ t_1 \circ f^{-1}), \varphi)$ *where*

$$\varphi \in \operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \subseteq \mathbb{P}(\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$$

is $V(f) : V(\alpha_1) \to V(\alpha_2)$ *tensored with the natural map* $\mathcal{O} \to \mathcal{O}(\{W\})$.

Proof. First we show that φ is in fact a homomorphism from $V(\alpha_1)$ to $V(\alpha_2) \otimes \mathcal{O}_X(\{W\})$ defined everywhere on X. This is clear away from the support of W, where f is an isomorphism. In a formal neighborhood of each point x in the support of W, for f whose associated point of $G((t))$ is in the Bruhat cell corresponding to W_x, the order of the pole of $V(f)$ is at most $\{W\}_x$, by definition of $\{W\}$. For f whose associated point of $G((t))$ is in the closure of the Bruhat cell, because the pole order is a lower semicontinuous function, the order of the pole is also at most $\{W\}_x$, and so it becomes a homomorphism after we tensor with $\mathcal{O}(\{W\})$.

Next we show that φ satisfies the local conditions (1), (2), and (3) of the definition of $\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\}))$. It satisfies condition (1) because f is an isomorphism away from the support of W, condition (2) because f is in the closure of the correct cell of the affine Grassmannian near points in the support of W, and condition (3) because over D, we have $t_2 \circ f \circ t_1^{-1} = h \in H$. □

Let $\overline{\Delta}^W : \overline{\mathcal{H}k}_{G(D),H,W,V} \to \operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)}$ send $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ to $((\alpha_1, t_1), (\alpha_2, t_2))$.

Lemma 7.6. *The map* $\overline{\Delta}^W$ *is projective and* $\overline{\Delta}^W \circ j = \Delta^W$.

Proof. The first claim follows immediately because the graph of $\overline{\Delta}^W$ is defined as a subset of a projective bundle consisting of triples satisfying three closed conditions, and thus is a closed subset, hence proper. The second claim follows because $\overline{\Delta}^W \circ j$ sends $(\alpha_1, \alpha_2, t_1, f, h)$ to $((\alpha_1, t_2), (\alpha_2, h \circ t_2))$ which is precisely the definition of Δ^W. □

Lemma 7.7. *j is an open immersion, and its image is the locus in* $\overline{\mathcal{H}k}_{G(D),H,W,V}$ *where* $\varphi \in \operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \subseteq \mathbb{P}(\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$.

Proof. By construction, a point in the image of j has φ contained in $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\}))$. Because affine subsets of projective spaces are open, the subset where $\varphi \in \mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\}))$ is an open subset of $\overline{\mathcal{H}k}_{G(D),H,W,V}$, and so to prove that j is an open immersion whose image is this subset, it suffices to find an inverse of j over this subset.

Given a point in this subset and an open set away from the support of W where α_1 and α_2 can be trivialized, so that $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) = \mathrm{End}\, V$, we have $\varphi \in \mathrm{End}\, V \cap \overline{G} \subseteq \mathrm{End}\, V$. Because $\mathrm{End}\, V \cap \overline{G} = G$, the induced section is locally obtained by functoriality from an isomorphism of G-bundles $\alpha_1 \to \alpha_2$.

Because V is faithful, this isomorphism is unique, and in particular extends to a global isomorphism away from W. Hence we obtain an isomorphism $f : \alpha_1 \to \alpha_2$ as G-bundles away from W. By assumption we know that φ, when viewed as a point in the formal loop space $(\mathrm{End}\, V \oplus k)(\!(t)\!)$, is in the closure of the set of pairs $(\lambda V(g), \lambda)$ where $\lambda \in \mathbb{G}_m$ and $g \in G(\!(t)\!)$ is in the Bruhat cell associated to W_x. Because $\varphi = (V(f), 1)$ and V is faithful, this implies that f, when viewed as a point in $G(\!(t)\!)$, it is in the closure of cell of the Bruhat cell associated to W_x, hence modulo $G[\![t]\!]$, it is in the closure of the cell of the affine Grassmannian associated to W_x.

Over D, $t_2 \circ f \circ t^{-1}$ lies in the closure of the set of points $(h\lambda, \lambda)$ for $h \in H$. Because the last coordinate is nonzero, we may fix it to equal 1, and thus take $\lambda = 1$, so it lies in the closure of H inside $\mathrm{End}\, V\langle \mathcal{O}_D \rangle$. Because H is a closed subgroup of $G\langle \mathcal{O}_D \rangle$, which is closed in $\mathrm{End}\, V\langle \mathcal{O}_D \rangle$, in fact $t_2 \circ f \circ t_1^{-1}$ lies in H, so we may take h to be $t_2 \circ f \circ t_1^{-1}$.

Verifying that this is an inverse is a routine calculation. $\square$

Lemma 7.8. *The map Δ^W is schematic and affine.*

Proof. By Lemma 7.6 and 7.7, this map is the composition of the open immersion j with the closed subset of a projective bundle $\overline{\Delta}^W$. Moreover, this open immersion is the complement of the hyperplane $\mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})))$ inside $\mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$. Thus because Δ^W is a hyperplane complement in a projective morphism, it is affine. $\square$

Remark 7.9. Throughout Section 7, we do not need the full formalism of étale cohomology on stacks. This is because the relevant morphism are schematic morphisms between Artin stacks, so we can define $\Delta^W_!$ and Δ^W_* smooth-locally as derived pushforwards with respect to morphisms of schemes.

If $G = SL_n$ and V is the standard representation, we can classify the points according to the rank of φ. For each rank, we can consider the maximal parabolic subgroup that preserves the kernel of φ, and its unipotent radical, elements of which fix φ when acting by composition on the right. Sections of this unipotent radical act as local automorphisms of $\overline{\mathcal{H}k}_{G,H,W,V}$. These automorphisms can be used to show the vanishing of $j_*(IC_{\mathcal{H}k_{G(D),H}} \boxtimes \mathcal{L})$ at these points. In the general group case, we will replace the study of the rank with the orbits in $\overline{G}$ of the joint left and right action of $G \times G$. We describe these orbits using the standard theory of reductive groups in the next subsection.

Remark 7.10. If the highest weight of V is a regular weight, i.e. not fixed by any element of the Weyl group, then $\overline{G}$ is equal to (or very similar to) the "wonderful compactification" of G. Similarly, in this case $\overline{\mathcal{H}k}_{G(0),H,1,V}$ is (or is very close to) the Drinfeld-Lafforgue-Vinberg compactification of Bun_G as defined by Schieder [46]. Our proof uses heavily the explicit representation V as a form of coordinates, but it seems plausible that a "coordinate-free" proof of the same result can be obtained using the abstract theory of the wonderful and Drinfeld-Lafforgue-Vinberg compactifications.

However, for our proof, there is no reason to choose V to be the representation associated to a regular weight. If we instead choose a representation like the standard representation (for G a classical group), the compactification we use, and other concepts involved like the height, admit particularly simple descriptions. The reader may wish to follow along with the case $G = Sp_{2g}$ in mind, say.

7.2. **Lemmas on semisimple groups.** Let G be a split semisimple group, V a faithful representation over k, and fix a split maximal torus T of G.

Lemma 7.11. *Any point in $\overline{G} - G \subseteq \mathbb{P}(\operatorname{End} V \oplus k)$ can be expressed as $(g_1 e g_2, 0)$ where $g_1, g_2 \in G$ and e is the idempotent projector onto the sum of eigenspaces of T whose weights lie in some proper face of the convex hull of the weights of V.*

Example 7.12. Let us provide some examples of what these idempotent projectors look like:

(i) Let $G = SL_n$ and let V be the standard representation. Then the weights of V are n linearly independent vectors, forming the vertices of an $(n-1)$-simplex. Hence any nonempty proper subset of the weights is a proper face of the convex hull. Thus any diagonal matrix with all diagonal entries 0 and 1, not all 1 and not all 0, is such an e.

(ii) Let $G = Sp_{2g}$ and let V be the standard representation. Then the weights of V are the vectors with one entry ± 1 and the rest 0 in $\mathbb{Z}^g$. The convex polytope this forms is a cross-polytope, whose proper faces are all simplices. The weights lying in a face form a subset S of these vectors, such that for any $v \in S$, $-v \notin S$. Thus e is an idempotent projector onto an isotropic subspace, whose kernel contains a maximal isotropic subspace.

(iii) Let $G = G_2$ and let V be the unique seven-dimensional irreducible representation. Then the weights of V form the six vertices and center of a hexagon. The proper faces consist of either one vertex or two adjacent vertices, so the sum of the eigenspaces is a subspace of dimension one or two. These subspaces are isotropic under the G_2-invariant quadratic form on V and the two-dimensional subspaces are sent to zero by the unique G_2-equivariant map $\wedge^2 V \to V$ (as the product of their eigenvalues under T is not a weight of V).

In all the above examples, the stabilizer of the sum of the eigenspaces of T whose weights lie in a proper face is always a maximal parabolic subgroup of G. We will prove that it is always a parabolic subgroup, but it need not be maximal - for instance when $G = SL_n$ and V is the adjoint representation, it need not be maximal for $n \geq 3$.

Proof of Lemma 7.11. By the valuative criterion of properness, any element of the closure of G is the limit as t goes to 0 of a $k'((t))$-valued point of G for some field k'. By Bruhat decomposition, any such point can be written as $g_1(t)\chi(t)g_2(t)$ where g_1, g_2 are $k'[[t]]$-valued points of G and χ is a cocharacter of t. Now $\chi(t)$ converges as t goes to 0 to a point $\chi(0) \in \mathbb{P}(\operatorname{End} V \oplus k)$, and because the left and right group actions are continuous, $g_1(t)\chi(t)g_2(t)$ converges as t goes to 0 to $g_1(0)\chi(0)g_2(0)$.

If χ is trivial, then $\chi(0)$ is the identity element and this limit is in G.

Otherwise, we can write $\chi(t)$ in $\operatorname{End} V$ as the sum over eigenspaces of T of the idempotent projector onto that eigenspace times an integer power of t, where the integer power appearing is a linear function of the weight. In projective coordinates, it is the same but with an additional 1 appended. Because χ is nontrivial, not all these exponents are 0, and because G is semisimple, the sum of the exponents vanishes, so some are negative and some are positive. We can change the projective coordinates by dividing by t to power of the minimal exponent appearance. Having done this, all the coefficients of idempotent projectors onto eigenspaces with the minimal exponent are fixed at 1, and all other coefficients are positive powers of t which converge to 0, so $\chi(0)$ is the idempotent projector e onto the sum of eigenspaces of T where some nontrivial linear function of the weights is minimized, i.e. proper some face of the convex hull of the weights. The last coefficient of $\chi(0)$ is 0, so multiplying on the left by $g_1(0)$ and the right by $g_2(0)$ we obtain the $(g_1 e g_2, 0)$. □

Fix a proper face of the convex hull of the weights of V, and take the idempotent projector e, so that $\mathrm{Im}(e)$ is the sum of the T-eigenspaces whose weights lie on that face and $\ker(e)$ is the sum of the T-eigenspaces whose weights do not lie on that face.

From now on, assume that V lifts to the Witt vectors of k and assume that the pairing of any weight of V with any coroot of G is less than p.

Lemma 7.13. *The stabilizer of* $\ker(e)$ *is a parabolic subgroup of* G*, and this stabilizer remains smooth after lifting* G *and* V *to the Witt vectors of* k*.*

Proof. We first check in characteristic zero that the stabilizer is a parabolic subgroup. It suffices to check that it is proper and contains a Borel subgroup. It is proper because the weights of $\mathrm{Im}(e)$ are the weights where some nontrivial linear function on the weight space is maximized, so the sum of that function over the weights of $\mathrm{Im}(e)$ is positive, and thus its sum over the weights of $\ker(e)$ is negative, which is impossible if $\ker(e)$ is a representation of G. Thus $\ker(e)$ is not G-stable and so its stabilizer is proper.

To show that the stabilizer contains a Borel, observe that the chosen face can be written as the locus where a linear form on the weight lattice takes its maximal value among the weights of V. The linear form is in some Weyl chamber of the dual space. With regards to the ordering induced by that Weyl chamber, the linear form takes nonnegative values on all the simple roots, hence takes nonnegative values on all the positive roots. Hence the set of weights of V where this linear form takes its maximal value is closed under addition of positive roots, and the complement of this set is closed under addition of negative roots. Therefore $\ker(e)$ is closed under the lowering operators and thus stable under the opposite Borel.

To show that the stabilizer of $\ker(e)$ is smooth over $\mathbb{Z}_p$, and thus remains parabolic in characteristic p, it suffices to check that the cotangent space of the schematic stabilizer at the identity is p-torsion free, in other words that every element of the Lie algebra of the stabilizer of $\ker(e)$ in characteristic p is the reduction mod p of an element in the Lie algebra of the stabilizer in characteristic zero. Because $\ker(e)$ is T-invariant, the Lie algebra of the stabilizer is a sum of T-eigenspaces, and so it is sufficient to check this for raising operators associated to roots. Let J_+ be the raising operator associated to a root and let J_- be the lowering operator associated to the opposite root. Suppose that J^+ does not stabilize $\ker(e)$ in characteristic zero but does in characteristic p. Because J^+ does not stabilize, it raises the linear function on the weight space of V which is maximized by $\mathrm{Im}(e)$, so $J^+ \mathrm{Im}(e) = 0$, and J^- lowers this linear function so $J^- \ker(e) \subseteq \ker(e)$. Thus in characteristic p, $J^+ J^- \mathrm{Im}(e) \subseteq J^+ \ker(e) \subseteq \ker(e)$, and $J^- J^+ \mathrm{Im}(e) \subseteq J^- 0 = 0$, so $[J^+, \ker(e)] \mathrm{Im}(e) \subseteq \ker(e)$. Now $[J^+, J^-]$ is an element of the Lie algebra of the maximal torus, the coroot corresponding to J^+, so $\mathrm{Im}(e)$ is a sum of eigenspaces of this coroot, and thus all the eigenvalues must be 0 mod p. Because the eigenvalues are pairings of the coroot corresponding to J^+ with weights of V, and hence are integers at most p, they must be zero. Because $J^+ \mathrm{Im}(e) = 0$, all eigenvalues of $[J^+, J^-]$ on $\mathrm{Im}(e)$ are highest weights of their corresponding representations, so all irreducible representations of the $\mathfrak{sl}_2$ generated by J^+, J^-, and $[J^+, J^-]$ other than those contained in $\mathrm{Im}(e)$ have highest weight zero, hence are trivial, hence have J^+ vanish on them, which contradicts the assumption that J^+ does not stabilize $\ker(e)$ in characteristic zero. $\square$

Lemma 7.14. *Let P be the stabilizer of* $\ker(e)$ *and let M be its Levi subgroup. The action of P on $V/\ker(e)$ factors through the projection $P \to M$.*

Proof. The set of weights in a proper face is the locus where some linear form on the weight lattice takes its maximal value among the weights of V.

Because the subspace $\ker(e)$ is stable under the maximal torus, P contains, and hence its normalized by, the maximal torus, so the Lie algebra of P generated by some subset of the raising and lowering operators corresponding to roots. The maximal unipotent subgroup of P is generated by the operators corresponding to some further subset of the roots.

If the raising and lowering operator corresponding to some root acts nontrivially on $V/\ker(e)$, then there must be two weights in the fixed face that differ by that root, so that root must be parallel to the face, and thus the operator corresponding to minus that root is also in the stabilizer P, and hence the corresponding unipotent element is in some SL_2-triple and thus is not in the maximal unipotent subgroup of P.

Because no generator of the maximal unipotent subgroup of P, the whole unipotent subgroup acts trivially, and so the action factors through M. $\square$

Lemma 7.15. *Let e be the idempotent projector on V onto the T-eigenspaces in some proper face of the convex hull of the weights of V. Let P be the parabolic subgroup of G consisting of elements stabilizing* $\ker(e)$.

1. *The natural map $\pi : G \to P\backslash G$ extends to a map π' from an open subset U of $\overline{G}$ to $P\backslash G$, such that $e \in U$ and $\pi'(e) = P \subseteq G/P$.*
2. *Let $\overline{P}$ be the projective closure of P inside $\mathbb{P}(\operatorname{End} V \oplus k)$. Any element of U sent to the identity under π' lies in $\overline{P}$.*

Proof. (1) Let $U \subseteq \overline{G}$ be the open subset consisting of $(x, \lambda) \in \overline{G}$ where $\operatorname{rank}(ex) = \operatorname{rank}(e)$. There is a map k from U to the Grassmannian $\operatorname{Gr}(\dim\ker(e), \dim V)$ that sends x to $\ker(ex)$. Such a map is invariant under the left action of P, which by definition preserves $\ker(e)$, so we have a commutative diagram.

$$\begin{array}{ccc} G & \xrightarrow{\quad\pi\quad} & P\backslash G \\ \downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{\;k\;} & \operatorname{Gr}(\dim\ker(\rho), \dim V) \end{array}$$

Because P is the schematic stabilizer of the kernel of e, i is an embedding, and because P is parabolic, $P\backslash G$ is proper, and so i is a closed immersion. Because G is dense in U, the image of k is contained in the closed image of this immersion, so we can factor $k = i \circ \pi'$ for a unique map $\pi' : U \to P\backslash G$. By commutativity, this extends π.

The open subset U where $\operatorname{rank}(ex) = \operatorname{rank}(e)$ includes e because $e^2 = e$ is idempotent. Furthermore $i \circ \pi'(e) = \ker(e^2) = \ker(e) = i(1)$ so because i is injective, $\pi'(e) = 1$.

(2) There is a map $m : \overline{P} \times G \to \overline{G}$ defined by the embedding $\overline{P} \subseteq \overline{G}$ and the right action of G on $\overline{G}$. Because m is stable under the action of $p \in P$ on $\overline{P} \times G$ that sends (x, g) to $(xp, p^{-1}g)$, m descends to a map map $\gamma : P\backslash(\overline{P} \times G) \to \overline{G}$. Now $P\backslash(\overline{P} \times G)$ is an $\overline{P}$-bundle on $P\backslash G$ and

both of these are proper so $P\backslash(\overline{P} \times G)$ is proper. Because the map to $\overline{G}$ is proper and has dense image it is surjective. Let U be the open subset of $\overline{G}$ on which the map $\pi' : G \to P\backslash G$ is defined. Then $\gamma^{-1}(U)$ admits two maps to $P\backslash G$, the first via π' and the second by projection to the second factor, which agree on the dense subset $P\backslash(P \times G) = G$, and hence are equal as $P\backslash G$ is separated. Hence every point that is sent to the identity must be an element of $(\overline{P} \times G)/P$ with the second factor in P, in other words an element of $\overline{P}$, as desired. □

Lemma 7.16. *Let G be a split semisimple algebraic group over a field k of characteristic p. If $p > 2$, then there exists a faithful representation V of G defined over $\mathbb{Z}$ such that the pairing of any weight of V with any coroot of G is less than p.*

If $p = 2$, there exists such a representation if each simple factor of G has nontrivial center in G.

Proof. In fact, we will construct V where all the pairings are at most 2. We will construct it over $\mathbb{Z}$ as a sum of highest weight representations, and reduce modulo p. It is sufficient to show that, for each character of the center of G, there exists such a representation whose central character is that character and whose kernel is contained in the center. Then summing, the kernel will be the intersection in the center of all characters of the center, and hence be trivial.

For this statement, we can assume that G is simply-connected, as any representation of the universal cover with a central character pulled back from G is in fact a representation of G with the same central character. Because a tensor product of two representations satisfying the condition on weights is a representation of the product group satisfying the pairing condition, and the same is true for the kernel condition, we may assume that G is simple. We can then check the existence of such representations for each possible central character of each simple group explicitly by the classification.

For the trivial character, the adjoint representation satisfies the pairing condition if and only if the Dynkin diagram has no edges of multiplicity greater than 2, so we can use the adjoint representation for any group except G_2. Because the center of G_2 is trivial, we can use the seven-dimensional standard representation for G_2.

It remains to handle the nontrivial characters. For any simple group, there exists a unique minuscule representation for each central character, and for any nontrivial character, the minuscule representation satisfies both conditions. Indeed, because it is not the trivial representation, its kernel is contained in the center, and because the Weyl group acts transitively on the weights (the definition of minuscule) the weights lie on a sphere, and so no three are colinear. But any weight whose pairing with a coroot is k lies in a $k+1$-dimensional representation of the SL_2 containing the dual root, hence lies in a series of $k+1$ weights in a line, so we must have $k \leq 1$.

In the $p = 2$ case, we can take the sum of all minuscule representation of G, which necessarily have all pairings ≤ 1. There exists one such representation of each central character, so the kernel of V intersects the center of G only at the identity. Hence kernel of V is connected and every simple factor of the kernel has trivial center in G. □

7.3. **Vanishing near the cusp.** We continue to assume that V lifts to the Witt vectors of k and assume that the pairing of any weight of V with any coroot of G is less than p.

Let $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ be a point of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ not in the image of j. Then viewing φ as a section of $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X$ up to scaling, and evaluating this section at each point of X where it doesn't vanish, we obtain a point of $\overline{G}$, well-defined up to the left and right action of G. By Lemma 7.7, the last coordinate of this point vanishes, and so it lies in $\overline{G} - G$, hence, by Lemma 7.11, takes the form $(g_1 e g_2, 0)$ for an idempotent projector e onto the space of T-eigenvalues of some proper face of the convex hull of the weights of V. Because there are finitely many such $(G \times G)$-orbits, and each is constructible, one must contain an open subset $X_0 \subseteq X$.

We focus attention on this orbit, and its associated idempotent projector e. Let P be the stabilizer of the kernel of e, which by Lemma 7.13 is a parabolic subgroup, and let N be its unipotent radical.

The fibration $P \backslash \alpha_1$ is a locally trivial fibration with fibers isomorphic to $P \backslash G$. Because the stabilizer of the kernel of e is P, $P \backslash \alpha_1$ is equal to the bundle over X consisting of subspaces of $V(\alpha_1)$ that are conjugate under G to the kernel of e. On the open subset X_0, $P \backslash \alpha_1$ admits a section given by the kernel of φ. Because this fibration is proper, this bundle admits a global section. Let $\mathcal{P}_{\alpha_1,\varphi} \subseteq \mathrm{Aut}(\alpha_1)$ be the group scheme over X of automorphisms preserving this section, which is locally conjugate to P. Let $\mathcal{N}_{\alpha_1,\varphi}$ be the unipotent radical of $\mathcal{P}_{\alpha_1,\varphi}$, which is locally conjugate to N.

Lemma 7.17. *Let σ be a section of $\mathcal{N}_{\alpha_1,\varphi}$, viewed as an automorphism of $V(\alpha_1)$. Then*

$$\varphi \circ \sigma = \varphi.$$

Proof. Because this equation is a closed condition, it suffices to check this over X_0, and to work locally. In particular, we may trivialize α_1 and α_2. Using that trivialization, from Lemma 7.11, φ can be expressed as $g_1 e g_2$. From the definition of $\mathcal{P}_{\alpha_1,\varphi}$ and $\mathcal{N}_{\alpha_1,\varphi}$, we see that $\mathcal{P}_{\alpha_1,\varphi} = g_2^{-1} P g_2$ and $\mathcal{N}_{\alpha_1,\varphi} = g_2^{-1} N g_2$. So it suffices to check that for $\sigma \in N$, $e\sigma = e$. Elements of N certainly lie in P and thus preserve the kernel of e, so to check $e\sigma = e$ it suffices to check that they act trivially on the quotient by this kernel, which is done in Lemma 7.14. ☐

Because G is split, we may assume that P is defined over $\mathbb{Z}$.

Definition 7.18. Let $N_{\mathbb{Q}} = N_{0,\mathbb{Q}} \supseteq N_{1,\mathbb{Q}} \supseteq N_{2,\mathbb{Q}} \supseteq \cdots \supseteq N_{r,\mathbb{Q}} = 1$ be the derived series of $N_{\mathbb{Q}}$. Let $N_{0,\mathbb{Z}} \supseteq N_{1,\mathbb{Z}} \supseteq N_{2,\mathbb{Z}} \supseteq \cdots \supseteq N_{r,\mathbb{Z}} = 1$ be their schematic closure in $N_{\mathbb{Z}}$, and let $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_r = 1$ be their reductions mod p.

Lemma 7.19. *For all i, N_i is a smooth connected P-invariant subgroup of N, and N_i/N_{i+1} is isomorphic to a vector space (i.e. a power of $\mathbb{G}_a$), where the action of P on N_i/N_{i+1} is by vector space automorphisms.*

Proof. We can verify all these facts by the theory of root groups.

Let U be a maximal unipotent subgroup of G, defined over $\mathbb{Z}$, containing N. For each root α of U, there is a root group U_α, a subgroup isomorphic to $\mathbb{G}_a$ over $\mathbb{Z}$, which in characteristic zero is the exponential of that root [9, Theorem 4.1.4 and Definition 4.2.3]. (In general the root group may be a line bundle, but over $\mathbb{Z}$ the only line bundle is $\mathbb{G}_a$.) Moreover, U is isomorphic as a scheme to the product of these root groups, with the isomorphism given by multiplication in the group law, for any fixed ordering of the roots [9, Theorem 5.1.13].

Choose an ordering where the roots not in $N_{\mathbb{Q}}$ are first, then the roots in $N_{0,\mathbb{Q}}$ but not in $N_{1,\mathbb{Q}}$, and so on, and use the induced isomorphism to a product of copies of $\mathbb{G}_a$ as coordinates on U. In this ordering, each of the closed subsets $N_{i,\mathbb{Q}}$ is defined by the vanishing of an initial segment of the coordinates. Hence their schematic closures, and the reductions mod p, are defined by the same equations. In particular, they are smooth and connected. The fact that these closed subsets are P-invariant, and are subgroups, can be expressed by algebraic equations and hence holds in the reduction mod p because it holds over $\mathbb{Q}$.

Because the commutator of two roots in $N_{i,\mathbb{Q}}$ necessarily lies in $N_{i+1,\mathbb{Q}}$, the group law on $N_{i,\mathbb{Q}}/N_{i+1,\mathbb{Q}}$ is simply given by addition in our fixed coordinates, and thus the action of P is linear in these coordinates. Because these are both closed conditions, they also hold modulo p. □

Definition 7.20. Let $\mathcal{N}_{\alpha_1,\varphi,i}$ be the subgroup of $\mathcal{N}_{\alpha_1,\varphi}$ defined by the fact that $\mathcal{N}_{\alpha_1,\varphi}$ is the conjugate of N by a P-torsor and N_i is a P-invariant subgroup of N.

Lemma 7.21. *The quotient $\mathcal{N}_{\alpha_1,\varphi,i}/\mathcal{N}_{\alpha_1,\varphi,i+1}$ is a vector bundle on X.*

Proof. This follows from the fact that N_i/N_{i+1} is a vector space and P acts by vector space automorphisms. □

Definition 7.22. Let the height of $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ be minus the smallest degree of a line bundle which occurs as a quotient of any of the vector bundles $\mathcal{N}_{\alpha_1,\varphi}$.

Lemma 7.23. *Fix $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ in $\overline{\mathcal{H}k}_{G(D),H,W,V}$. Consider the map $c : H \to \overline{\mathcal{H}k}_{G(D),H,W,V}$ that sends $h \in H$ to $(V_1, V_2, h \circ t_1, t_2, \varphi)$. The pullback $c^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is isomorphic to the tensor product of the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ at $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ with $\mathcal{L}^{-1}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{H}k_{G(D),W} \times H & \xrightarrow{\ j\ } & \overline{\mathcal{H}k}_{G(D),H,W,V} & & \\
\uparrow a & & \uparrow b & \nwarrow c & \\
\mathcal{H}k_{G(D),W} \times H \times H, & \xrightarrow{\ j\ } & \overline{\mathcal{H}k}_{G(D),H,W,V} \times H & \xleftarrow[d]{} & H
\end{array}$$

where the vertical map b sends $((\alpha_1, \alpha_2, t_1, t_2, \varphi), h)$ to $(\alpha_1, \alpha_2, h \circ t_1, t_2, \varphi)$, the vertical map a preserves the $\mathcal{H}k_{G(D),W}$ part and sends h_1, h_2 to $h_2 h_1^{-1}$, the arrow d sends $h \in H$ to $((V_1, V_2, h \circ t_1, t_2, \varphi), h)$, and so c sends $h \in H$ to $(V_1, V_2, h \circ t_1, t_2, \varphi)$.

We have

$$c^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = d^* b^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}).$$

We have

$$b^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = j_* a^*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \boxtimes \mathcal{L}^{-1}) = j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \boxtimes \mathcal{L}^{-1}$$

with the first identity by smooth base change, because the left square is Cartesian, the second by the character sheaf property of $\mathcal{L}$, and the last by the Künneth formula.

Hence pulling back to a particular copy of H in $\overline{\mathcal{H}k}_{G(D),H,W,V} \times H$, we obtain the stalk of $j_*(IC_{\mathcal{M}} \otimes s^* \mathcal{L}^*)$ at a particular point tensored with $\mathcal{L}$, as desired. □

Lemma 7.24. *Let β be a P-bundle on X. Let $\mathcal{P}_\beta$ be the associated twisted form of P and $\mathcal{N}_\beta$ its unipotent radical. Assume that all vector bundles in the canonical filtration of $\mathcal{N}_\beta$ have no nontrivial quotients of degree at most $2g - 2 + |D|$. Then there is a section of $\mathcal{N}_\beta$ over $\mathrm{Res}_k^D(\mathcal{N}_\beta|D) \times X$, whose restriction to $\mathrm{Res}_k^D(\mathcal{N}_\beta|D) \times D$ is the canonical section.*

Proof. Let $i : D \to X$ be the immersion, so that $\Gamma(D, i^*\mathcal{N}_\beta) = \Gamma(X, i_* i^* \mathcal{N}_\beta)$. First we will show that the map $\Gamma(X, \mathcal{N}_\beta) \to \Gamma(X, i_* i^* \mathcal{N}_\beta)$ is surjective. The cokernel is contained in the H^1 of X with coefficients in the kernel of the natural map $\mathcal{N}_\beta \to i_* i^* \mathcal{N}_\beta$. The kernel of the natural map $\mathcal{N}_\beta \to i_* i^* \mathcal{N}_\beta$ has a filtration, induced by pulling back the filtration of $\mathcal{N}_\beta$, whose associated graded objects are $(\mathcal{N}_{i,\beta}/\mathcal{N}_{i+1,\beta})\otimes\mathcal{O}(-D)$. By the assumption on height, $(\mathcal{N}_{i,\beta}/\mathcal{N}_{i+1,\beta})\otimes\mathcal{O}(-D)$ has no line bundle quotients of degree $2g-2$, thus admits no nontrivial maps to the canonical bundle, hence has vanishing H^1, so the kernel has vanishing H^1 as well, and the map is surjective.

Moreover, the H^1 of the kernel will still vanished when base changed by any affine scheme, as these are flat over the base field, and so the natural map $\Gamma(X\times Y,\mathcal{N}_\beta)\to\Gamma(X\times Y, i_*i^*\mathcal{N}_\beta)$ is flat for any affine Y. We take Y to be the Weil restriction $\mathrm{Res}_k^D(\mathcal{N}_\beta|D)$ of $\mathcal{N}_\beta$ from D to k, over which there is a canonical element of $\Gamma(D,\mathcal{N}_\beta)$. This gives a section of $\mathcal{N}_\beta$ over $\mathrm{Res}_k^D(\mathcal{N}_\beta|D)\times X$, whose restriction to $\mathrm{Res}_k^D(\mathcal{N}_\beta|D)\times D$ is the canonical section. □

Lemma 7.25. *Assume that some $(G,m_u,H_u,\mathcal{L}_u)$ is geometrically supercuspidal. Then the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}}\boxtimes\mathcal{L})$ vanishes at points whose height is greater than $2g-2+|D|$.*

Proof. Consider a point $(\alpha_1,\alpha_2,t_1,t_2,\varphi)$ in $\overline{\mathcal{H}k}_{G(D),H,W,V}$ of height greater than $2g-2+|D|$. Let β be the associated P-bundle, so that $\mathcal{P}_\beta=\mathcal{P}_{\alpha_1,\varphi}$. By Lemma 7.24, there is a section s of $\mathcal{N}_{\alpha_1,\varphi}$ over $\mathrm{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|D)\times X$, whose restriction to $\mathrm{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|D)\times D$ is the canonical section.

Now consider the map τ from $\mathrm{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|D)$ to $\overline{\mathcal{H}k}_{G(D),H,W,V}$ that sends $g\in\mathrm{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|D)$ to $(\alpha_1,\alpha_2,g\circ t_1,t_2,\varphi)$. This map is actually equal to the constant map by a diagram

$$\begin{array}{ccc} \alpha_1 & \xrightarrow{\varphi} & \alpha_2 \\ \big\downarrow{\scriptstyle s(g)} & & \big\downarrow{\scriptstyle e} \\ \alpha_1 & \xrightarrow{\varphi} & \alpha_2 \end{array}$$

which commutes by Lemma 7.17 because $s(g)\in\mathcal{N}_{\alpha_1,\varphi}$, and because $g\circ t_1=s(g)|_D\circ t_1$ by the definition of $s(g)$.

Hence the pullback of $j_*(IC_{\mathcal{H}k_{G(D),W}}\boxtimes\mathcal{L})$ along τ is the constant sheaf tensored with stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}}\boxtimes\mathcal{L})$ at $(\alpha_1,\alpha_2,t_1,t_2,\varphi)$.

Now $g\circ t_1=t_1\circ(t_1^{-1}gt_1)$. Because the $P\backslash G$-bundle defining $\mathcal{N}_{\alpha_1,\varphi}$ admits a section over D, $\mathcal{N}_{\alpha_1,\varphi}$ is conjugate over D to N, and so $\mathrm{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|D)$ is isomorphic to $N\langle\mathcal{O}_D\rangle$, in such a way that the embedding $g\mapsto(t_1^{-1}gt_1)$ into $G\langle\mathcal{O}_D\rangle$ is conjugate to the standard embedding.

Now consider the pullback of $j_*(IC_{\mathcal{H}k_{G(D),W}}\boxtimes\mathcal{L})$ along the map that sends h to $(\alpha_1,\alpha_2,t_1\circ h,t_2,\varphi)$ for h in the intersection of H with this conjugate copy of $N\langle\mathcal{O}_D\rangle$. This pullback is a constant sheaf tensored with the stalk of $j_*(IC_{\mathcal{M}}\otimes s^*\mathcal{L}^*)$ at $(V_1,V_2,t_1,t_2,L,\varphi)$. From Lemma 7.23, we know it is the same stalk tensored with the pullback of $\mathcal{L}^{-1}$. From the definition of geometric supercuspidal, we know that even restricting to a further intersection with H_x, the pullback of $\mathcal{L}^{-1}$ is not a geometrically constant sheaf, and so its tensor product with no nonzero vector space is geometrically constant, and hence the stalk vanishes, as desired. □

7.4. **Hecke Correspondences.** We continue to assume that V lifts to the Witt vectors of k and assume that the pairing of any weight of V with any coroot of G is less than p.

We will use the following space to compare the stalks of $j_*(IC_{\mathcal{H}k_{G(D),W}}\boxtimes\mathcal{L})$ at different points:

Definition 7.26. Fix a geometric point $Q\in X$ that is neither in D nor the support of W and a cocharacter μ in the Weyl cone of G. Let $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$ be the moduli space of quadruples consisting of two points $(\alpha_1,\alpha_2,t_1,t_2,\varphi)$ and $(\alpha_3,\alpha_4,t_1,t_2,\varphi')$ in $\overline{\mathcal{H}k}_{G(D),H,W,V}$ and isomorphisms $m_1:\alpha_3\to\alpha_1$ and $m_2:\alpha_4\to\alpha_2$ away from Q, such that $t_1\circ m_1=t_3$, $t_2\circ m_2=t_4$, $\varphi\circ V(m_1)=V(m_2)\circ\varphi'$, and such that m_1 and m_2, expressed as points in $G(\!(t)\!)$ via local coordinates at Q, are in $G[\![t]\!]\mu(t)G[\![t]\!]$. (Note that here we use a Bruhat cell and not its closure.)

Let pr_{12} and $pr_{34} : \mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right) \to \overline{\mathcal{H}k}_{G(D),H,W,V}$ be the maps induced by $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ and $(\alpha_3, \alpha_4, t_1, t_2, \varphi')$ respectively.

Let $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ be a point of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ not in the image of j. As we did at the beginning of the previous subsection, we can choose some open set X_0 where φ locally takes the form $g_1 e g_2$ for the idempotent projector e onto the space of T-eigenvalues of some proper face of the convex hull of the weights of V. Equivalently, we can trivialize α_1 and α_2 over X_0 so that φ in the induced coordinates is an idempotent projector e. Let Q be a point in X_0 that does not lie in D. Let P be the stabilizer of the kernel of e. Let $\mu : \mathbb{G}_m \to T$ be a cocharacter such that the eigenvalue of $g \to \mu(\lambda)^{-1} g \mu(\lambda)$ is a nonnegative power of λ on roots in P and is negative on roots not in P (which exists by [10, Proposition 2.2.9]).

In this subsection, we will show how to choose a point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$ whose image under pr_1 is $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$, whose image under pr_2 has greater height than $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$, and such that the stalks of the pullbacks of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ on its image under pr_1 and its image under pr_2 are isomorphic. This is precisely what we will need to inductively show that the stalk vanishes in the proof of Theorem 7.32 in the next subsection.

The key step in comparing the stalks is to show that the maps pr_{12} and pr_{34} are smooth, as it allows us to use the smooth base change theorem. This can be checked by comparing sections of the relevant stalks over the local ring, which can be reduced by a Beauville-Laszlo argument to a purely algebraic calculation, which we handle first:

Lemma 7.27. *Let R be a Henselian local ring, with maximal ideal $\mathfrak{m}$. Let $M \in \operatorname{End} V(R[[t]])$ be a matrix and $s \in R[[t]]$ an element such that (M, s) are the projective coordinates of an $R[[t]]$-point of $\overline{G}$. Assume that (M, s) is congruent to $(e, 0)$ modulo $\mathfrak{m}$. Let g_a and g_b be elements of $G(R[[t]])$ such that $g_a \mu(t)^{-1} g_b$ is congruent to $\mu(t)$ mod $\mathfrak{m}$.*

Then there exist g_c, g_d in $G(R[[t]])$, such that $g_c \mu(t) g_d$ is congruent to $\mu(t)$ mod $\mathfrak{m}$, and such that

$$\left(g_a \mu(t)^{-1} g_b\right) M \left(g_c \mu(t) g_d\right)$$

is in $\operatorname{End} V(R[[t]]) \subseteq \operatorname{End} V(R((t)))$.

Moreover the products $g_c \mu(t) g_d$ for all g_c, g_d satisfying these two conditions lie in a single orbit under the right action of $G(R[[t]])$.

Proof. We make a series of reductions.

First note that we may assume R is Noetherian. This is because the problem only depends on finitely many entries of M, g_a, g_b - those entries that are nonvanishing mod a power of t equal to the sum of the highest negative power of t appearing in entries of $\mu(t^{-1})$ and $\mu(t)$. Hence the problem is defined over a Henselization of a finitely generated subring of R, which is Noetherian. For the uniqueness statement, because

$$\left(G(R[[t]]) \mu(t) G(R[[t]])\right) / G(R[[t]])$$

is represented by a scheme of finite type - more specifically, a Bruhat cell of the affine Grassmannian - we may check uniqueness in the Henselization of another finitely generated subring of R, those generated by the finitely many entries of M, g_a, g_b plus the coordinates in this Bruhat cell of two different possible values of g_c, g_d.

Next we may assume that g_b is congruent to 1 mod $\mathfrak{m}$. This is because the map

$$G[[t]] \to G[[t]] \backslash \left(G[[t]] \mu(t)^{-1} G[[t]]\right)$$

that sends g to $G[[t]]\mu(t)^{-1}g$ (equivalently to $G[[t]]g_a\mu(t)^{-1}g$) is smooth at the identity, and so we can lift $G[[t]]g_a\mu(t)^{-1}g_b$, which is congruent to $\mu(t)$ mod $\mathfrak{m}$, to an R-point of $G[[t]]$ congruent to 1 mod $\mathfrak{m}$.

Now because $\overline{G}$ is stable under left-multiplication by G, we may replace M by g_bM and so assume $g_b = 1$. Because left-multiplication by g_a does not affect integrality, we may assume $g_a = 1$.

Now applying Lemma 7.15(1), from $(M, s) \in \overline{G}(R[[t]])$ we obtain a point in

$$\pi'(M, s) \in (P\backslash G)(R[[t]]) = P(R[[t]])\backslash G(R[[t]])$$

with the identity because the projection $G \to P\backslash G$ is smooth so we may lift points of $(P\backslash G)(R[[t]])$ to points of $G(R[[t]])$. Pick some element $\sigma \in G(R[[t]])$ in the left $P(R[[t]])$-coset $\pi'(M, s)$. We can multiply M on the right by σ^{-1} without affecting the existence of g_c, g_d or their uniqueness, because we can always multiply g_c on the left by σ to cancel it. So we may assume that $\pi'(M, s)$ is the identity, and hence by Lemma 7.15(2) that (M, s) lies in $\overline{P}(R[[t]])$.

This implies the existence of a solution. In fact we can take $g_c = g_d = 1$, so it suffices to check that $\mu(t)^{-1}M\mu(t)$ is integral. By construction, all the nonzero entries of elements of the Lie algebra of P are multiplied by a nonnegative power of t when conjugated by $\mu(t)$. In characteristic zero, this implies that all the nonzero entries of elements of P are multiplied by a nonnegative power of t when conjugated by $\mu(t)$, as these are exponentials of the Lie algebra elements. Because the representation lifts to characteristic zero, the same thing is true for the nonzero entries in the characteristic p representation, and thus the same thing is true for elements of the closure $\overline{P}$ of P, including (M, s). So indeed $\mu(t)^{-1}M\mu(t)$ is integral, as desired.

The argument for uniqueness is more subtle. It suffices to show that, for M, g_a, g_b in this special form, all solutions $g_c\mu(t)g_d$ map to the point $\mu(t)G(R[[t]])$ of the Bruhat cell

$$(G(R[[t]])\mu(t)G(R[[t]]))\,/G(R[[t]]).$$

Using the Noetherian hypothesis and induction, it is sufficient assume that the solution maps to this point modulo $\mathfrak{m}^n$ for some $n \geq 1$ and show that it also maps to this point modulo $\mathfrak{m}^{n+1}$. Because the map $G(R[[t]]) \to (G(R[[t]])\mu(t)G(R[[t]]))\,/G(R[[t]])$ sending g to $g\mu(t)$ is smooth, and because $g_c\mu(t)g_d$ is congruent to 1 modulo $\mathfrak{m}^n$, we may assume g_c is is congruent to 1 modulo $\mathfrak{m}^n$. Then modulo $\mathfrak{m}^{n+1}$, g_c is $1 + \tau$ for some $\tau \in \mathfrak{m}^n\mathfrak{g}(R[[t]])$, where $\mathfrak{g}$ is the Lie algebra of G. Then we can write

$$\mu(t^{-1})Mg_c\mu(t)g_d = \mu(t^{-1})M(1+\tau)\mu(t)g_d = \mu(t^{-1})M\mu(t)g_d + \mu(t^{-1})M\tau\mu(t)g_d.$$

We know that $\mu(t^{-1})M\mu(t)g_d$ is integral, so this implies that $\mu(t^{-1})M\tau\mu(t)g_d$ is integral, which, inverting g_d, implies that $\mu(t)^{-1}M\tau\mu(t)$ is integral. Because τ is divisible by $\mathfrak{m}^n$ and M is congruent to e modulo $\mathfrak{m}$, modulo $\mathfrak{m}^{n+1}$ we have

$$\mu(t)^{-1}M\tau\mu(t) = \mu(t)^{-1}e\tau\mu(t) = e\mu(t)^{-1}\tau\mu(t).$$

Hence $\mu(t)^{-1}\tau\mu(t)$ must be t-integral modulo the kernel of left multiplication by e. The kernel of left-multiplication by e consists of matrices whose image lies in the kernel of e, which includes those matrices that send the kernel of e to the kernel of e, which is the Lie algebra of the stabilizer of the kernel of e, which by definition is the Lie algebra of P. Hence $\mu(t)^{-1}\tau\mu(t)$ is t-integral modulo P. But $\mu(t)^{-1}\tau\mu(t)$ is also $t-$integral modulo a $\mu(t)$-invariant complement of P, as the eigenvalues of conjugation by $\mu(t)$ on the Lie algebra of P are nonnegative powers of t and hence conjugation by $\mu(t)$ on the Lie algebra of P preserves t-integrality. So $\mu(t)^{-1}\tau\mu(t)$ is integral, which because $g_c\mu(t)g_d \equiv 1\mu(t)(1 + \mu(t)^{-1}\tau\mu(t))g_d \mod m^{n+1}$, shows that $g_c\mu(t)g_d$

maps to the point $\mu(t)G(R[\![t]\!])$ of the Bruhat cell $(G(R[\![t]\!])\mu(t)G(R[\![t]\!]))\,/G(R[\![t]\!])$ modulo $\mathfrak{m}^{n+1}$, as desired. $\square$

We will define a special point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$ where the smoothness of pr_{12} and pr_{34} is as easy as possible to check. Recall that we have already fixed trivalizations of α_1 and α_2 on the open set X_0, and thus on a formal neighborhood of Q. Let $m_1 : \alpha_3 \to \alpha_1$ and $m_2 : \alpha_4 \to \alpha_2$ be the unique modifications of α_1 and α_2 respectively that are isomorphisms away from Q and that in a formal neighborhood of Q are locally isomorphic to the map $\mu(t)$. (This uniquely characterizes them by Beauville-Laszlo). Let $t_3 = t_1 \circ m_1$ and $t_4 = t_2 \circ m_2$ be the trivializations. Let $\varphi' : V(\alpha_3) \to V(\alpha_4)$ be the map that, away from Q, is φ, and in a formal neighborhood of Q, is e. Let $y = ((\alpha_1, \alpha_2, t_1, t_2, \varphi), (\alpha_3, \alpha_4, t_1, t_2, \varphi'), m_1, m_2)$. Because e commutes with $\mu(t)$, $\varphi \circ V(m_1) = V(m_2) \circ \varphi'$ and so y is a point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$.

We can translate Lemma 7.27 into a geometric lifting lemma:

Lemma 7.28. *Let R be a Henselian local ring with maximal ideal $\mathfrak{m}$. Let $(\alpha_1^*, \alpha_2^*, t_1^*, t_2^*, \varphi^*)$ be an R-point of of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ that modulo the maximal ideal of R is $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$. Let α_4^* be a G-bundle on X_R and let m_2^* be an isomorphism: $m_2^* : \alpha_4^* \to \alpha_2^*$ away from Q that expressed in local coordinates over a formal neighborhood of Q lies in $G[\![t]\!]\mu(t)G[\![t]\!]$ and such that (α_4^*, m_2^*) mod $\mathfrak{m}$ is isomorphic to (α_4, m_2).*

Then there exists a unique triple of a G-bundle α_3^* on X_R, isomorphism $m_1^* : \alpha_3^* \to \alpha_1^*$ away from Q that in a formal neighborhood of Q lies in $G[\![t]\!]\mu(t)G[\![t]\!]$, and $\varphi'^* \in \mathbb{P}(\mathrm{Hom}_X(V(\alpha_3), V(\alpha_4)+1)$ such that $\varphi^* \circ V(m_1^*) = V(m_2^*) \circ \varphi'^*$, that is congruent to $(\alpha_3, m_1, \varphi')$ modulo $\mathfrak{m}$ up to isomorphism.

Proof. Fix trivializations of $\alpha_1^*, \alpha_2^*, \alpha_4^*$ over the formal neighborhood of Q that agree modulo $\mathfrak{m}$ with the trivializations of α_1 and α_2 we have chosen and with the trivialization of α_4 in which m_2 is $\mu(t)$.

By Beauville-Laszlo, the data of α_3^* is equivalent to the data of a G-bundle over a formal neighborhood of Q, a G-bundle over the complement of Q, and an isomorphism between the two over the punctured formal neighborhood. Because m_1^* is an isomorphism over the complement of Q, we can take the G-bundle over the complement of Q to be α_1^*, so the data of (α_3^*, m_1^*) is simply a G-bundle over a formal neighborhood of Q with an isomorphism to α_1^* over the punctured formal neighborhood. Because we have a trivialization of α_1^*, this data is equivalent to an element of $G(R(\!(t)\!))$ modulo the right action of $G(R[\![t]\!])$. We can view this element as m_1^* because it is the isomorphism from α_3^* to α_1^* in formal coordinates.

The map $\varphi^{'*}$ is uniquely determined by the other data, as we must have $V(m_2^*)^{-1} \circ \varphi^* \circ V(m_1^*) = \varphi^{'*}$. However, this formula may not define any $\varphi^{'*}$, as it defines a section of $\mathcal{H}om(V(\alpha_3), V(\alpha_4)) + \mathcal{O}_X$ away from Q that may have a pole of Q.

If we express φ^* in our trivialization over the punctured formal neighborhood as (M, s), then by assumption M, s are the projective coordinates of an $R[\![t]\!]$-point of $\overline{G}$ and are congruent to $(e, 0)$ mod $\mathfrak{m}$.

If we view m_2^* over the punctured formal neighborhood of Q as an element of $G(R(\!(t)\!))$, by assumption on m_2, it can be expressed as $g_b^{-1}\mu(t)g_a^{-1}$ for $g_a, b \in G(R[\![t]\!])$ and it is congruent to $\mu(t)$ modulo $\mathfrak{m}$.

Then the possible values of (α_1^*, m_1^*) are parameterized by those elements of $G(R(\!(t)\!))$ that are of the form $g_c\mu(t)g_d$, that are congruent to $\mu(t)$ modulo $\mathfrak{m}$, and such that $g_a\mu(t)^{-1}g_b x g_c \mu(t) g_d$ is integral, up to the right action of elements of $G(R[\![t]\!])$ that are congruent to 1 modulo $\mathfrak{m}$. By Lemma 7.27, there is a unique such element up to equivalence. $\square$

We can now prove the desired smoothness statement:

Lemma 7.29. *Both pr_{12} and pr_{34} are smooth at y.*

Proof. We can factor pr_{12} as the composition of first, the map p' that projects onto a point $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ with a G-bundle α_4 and isomorphism $m_2 : \alpha_4 \to \alpha_2$ such that m_2 near Q is in the cell of the affine Grassmannian corresponding to μ with, second, the map that forgets α_4 and m_2. This second map is a locally trivial fibration by the cell of the affine Grassmannian associated to μ and hence is smooth.

Thus it is sufficient to show that the first projection p' is étale at y. To do this we may ignore the trivializations t_3, t_4 as these are uniquely determined by the other data. The projection p' is then defined by adding α_3, m_1, φ'. Then p' is schematic of finite type, since the data of the pair (α_3, m_1) is equivalent to a section of a locally trivial fibration by the cell of the affine Grassmannian associated to μ, and then φ' is a section of a projective bundle satisfying a closed condition, so the p' is represented by a closed subset of a projective bundle on a fibration by a variety. To check that p' is étale at the point y, we use the fact that each R-point of the base for a Henselian local ring R congruent mod $\mathfrak{m}$ to the image of y has a unique lift to an R-point of the total space congruent mod $\mathfrak{m}$ to y, which is Lemma 7.28. This implies that there is a section of p' over the étale local ring at the $p'(y)$, and that this section is equal over the étale local ring at y to the identity, which implies the natural map between from the étale local ring at $p'(y)$ to the étale local ring at y is an isomorphism and so the map is étale.

Finally, we can deduce the pr_{34} case from the pr_{12} case by symmetry, taking the dual of V and so reversing all the arrows. Note that the assumption on the weights of V is preserved by duality. □

Lemma 7.30. *The stalks of $pr_{12}^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ and $pr_{34}^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ at y are isomorphic.*

Proof. By Lemma 7.7, the image of j inside $\overline{\mathcal{H}k}_{G(D),H,W,V}$ consists of those $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ where the last coordinate of φ is nonzero. For a point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$, the equation $\varphi \circ V(m_1) = V(m_2) \circ \varphi'$ ensures that this property holds for φ if and only if it holds for φ'. Let $\mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right)$ be the open subset with this property, j' its inclusion into $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$, and pr'_{12} and pr'_{34} the projections onto $\mathcal{H}k_{G(D),W} \times H$. This gives a commutative diagram:

$$\begin{array}{ccccc}
\overline{\mathcal{H}k}_{G(D),H,W,V} & \xleftarrow{pr_{12}} & \mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right) & \xrightarrow{pr_{34}} & \overline{\mathcal{H}k}_{G(D),H,W,V} \\
{\scriptstyle j}\uparrow & & {\scriptstyle j'}\uparrow & & {\scriptstyle j}\uparrow \\
\mathcal{H}k_{G(D),W} \times H & \xleftarrow[pr'_{12}]{} & \mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right) & \xrightarrow{pr'_{34}} & \mathcal{H}k_{G(D),W} \times H
\end{array}$$

To show the isomorphism, observe that in a neighborhood of y, $pr_{12}^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = j'_* pr'^*_{12}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ by smooth base change and Lemma 7.29. So it suffices to show that $pr'^*_{12}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = pr'^*_{34}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$. Let $p_c : \mathcal{H}k_{G(D),W} \times H \to \mathcal{H}k_{G(D),W}$ and $p_h : \mathcal{H}k_{G(D),W \times H} \to H$ be the projections. We have $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} = p_c^* IC_{\mathcal{H}k_{G(D),W}} \otimes p_h^* \mathcal{L}$ so

$$pr'^*_{12}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = pr'^*_{12} p_c^* IC_{\mathcal{H}k_{G(D),W}} \otimes pr^{*'}_{12} p_h^* \mathcal{L},$$

and similarly for pr_{34}. Hence it suffices to show that

$$pr'^{*}_{12} p_c^* IC_{\mathcal{H}k_{G(D),W}} = pr'^{*}_{34} p_c^* IC_{\mathcal{H}k_{G(D),W}}$$

and

$$pr'^{*}_{12} p_h^* \mathcal{L} = pr'^{*}_{34} p_h^* \mathcal{L}.$$

Because $pr_{12'}$ is smooth by Lemma 7.29, and p_c is smooth because H is, $pr'^{*}_{12} p_c^* IC_{\mathcal{H}k_{G(D),W}}$ is simply a shift of $IC_{\mathcal{H}k_{Q,\mu}(\mathcal{H}k_{G(D),W} \times H)}$, and the same for pr_{34}, which gives the first desired identity.

The second desired identity follows from $p_c \circ pr'_{12} = p_c \circ pr'_{34}$, which can be expressed also as the commutativity of the extended diagram

$$\begin{array}{ccccc}
\overline{\mathcal{H}k}_{G(D),H,W,V} & \xleftarrow[pr_{12}]{} & \mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right) & \xrightarrow{pr_{34}} & \overline{\mathcal{H}k}_{G(D),H,W,V} \\
{\scriptstyle j}\uparrow & & {\scriptstyle j'}\uparrow & & {\scriptstyle j}\uparrow \\
\mathcal{H}k_{G(D),W} \times H & \xleftarrow[pr'_{12}]{} & \mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right) & \xrightarrow{pr'_{34}} & \mathcal{H}k_{G(D),W} \times H \\
& \searrow^{p_c} & & \swarrow_{p_c} & \\
& & H & &
\end{array}$$

If $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ is in the image under j of some point $(\alpha_1, t_1, \alpha_2, f), h) \in \mathcal{H}k_{G(D),W \times H}$, then $\varphi = V(f)$ for some isomorphism f of G-bundles $\alpha_1 \to \alpha_2$, and $t_2 = h \circ t_1 \circ f^{-1}$, so $h = t_2 \circ f \circ t_1^{-1}$. Similarly if $\varphi' = V(f')$ then we have $h' = t_4 \circ f' \circ t_3^{-1}$. To check that the diagram commutes, we must check $h = h'$. Because V is faithful, the identity $V(m_1) \circ \varphi' = \varphi \circ V(m_2)$ implies $m_2 \circ f' = f \circ m_1$. Thus we have

$$t_2 \circ f \circ t_1^{-1} = t_2 \circ f \circ m_1 \circ t_3^{-1} = t_2 \circ m_2 \circ f' \circ t_3^{-1} = t_4 \circ f' \circ t_3^{-1}$$

showing that the diagram commutes and completing the proof. □

Lemma 7.31. *For $y = ((\alpha_1, \alpha_2, t_1, t_2, \varphi), (\alpha_3, \alpha_4, t_1, t_2, \varphi'), m_1, m_2)$ defined as before, the height of $(\alpha_3, \alpha_4, t_1, t_2, \varphi')$ is strictly greater than the height of $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$.*

Proof. Consider the natural isomorphism $\mathcal{N}_{\alpha_1,\varphi} \to \mathcal{N}_{\alpha_3,\varphi'}$ away from Q that is induced by the isomorphism m_1. This isomorphism respects the canonical filtration of N by vector spaces. Hence it defines an isomorphism from the associated graded vector bundles of $\mathcal{N}_{\alpha_1,\varphi}$ to the associated graded vector bundles of $\mathcal{N}_{\alpha_3,\varphi'}$. We will show that each map of vector bundles appearing this way extends to a map of vector bundles over all of X that vanishes over the fiber of Q.

To do this, it is sufficient to calculate in a neighborhood of Q. Over that neighborhood, we can assume that φ and φ' are both simply the map e, so that $\mathcal{N}_{\alpha_1,\varphi}$ and $\mathcal{N}_{\alpha_3,\varphi'}$ are each N, and the induced map is the homomorphism $g \to m_1^{-1} \circ g \circ m_1 = \mu(t)^{-1} g \mu(t)$. So it is sufficient to show that the eigenvalues of $\mu(t)$ acting by conjugation on the associated graded module of the canonical filtration of N are all positive powers of t. Because the associated graded is also the associated graded of the Lie algebra of a filtration on the Lie algebra of N, it is sufficient to show that all the eigenvalues of $\mu(t)$ on the Lie algebra of N are positive powers of t. To do this, observe that for any root in the Lie algebra of N, its dual root is not in the Lie algebra of P, so the eigenvalue of $\mu(t)$ on it is a negative power of μ.

Given a map of vector bundles that vanishes at a point, any line bundle that appears as a quotient of the second vector bundle admits a nontrivial map from the first vector bundle that vanishes at the point, and so some line bundle of lower degree is a quotient of the first vector bundle. It follows that the height of $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ is less than the height of $(\alpha_3, \alpha_4, t_1, t_2, \varphi')$. □

7.5. **Conclusion.**

Theorem 7.32. *Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and* $\mathrm{char}(k) > 2$. *Then the natural map*

$$j_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \to j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$$

is an isomorphism.

Proof. By passing to an algebraically closed field, we may assume that G is split. By Lemma 7.16, there exists a suitable representation V, so we may apply the results from the previous subsections.

We check the isomorphism on stalks at each point. By Lemma 7.7, j is an open immersion, and thus the isomorphism holds for points in the image of j. At points outside the image of j, it is sufficient to prove that the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ vanishes. We do this by induction on the height. The base case when the height is greater than $2g-2+|D|$ is handled by Lemma 7.25.

For the induction step, we assume it is true for height $> h$ and prove it for height h. Given a point $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ of height h, we have defined a point y of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$. By Lemma 7.30, the stalk at $pr_{12}(y)$ is equal to the stalk at $pr_{34}(y)$. By Lemma 7.31, the height of $p_{34}(y)$ is greater than h, so by induction hypothesis the stalk vanishes, and then the stalk at $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ vanishes, completing the induction step. □

Theorem 7.33. *Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and* $\mathrm{char}(k) > 2$. *Then the natural map*

$$\Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) \to \Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)$$

is an isomorphism.

Proof. We have observed that $\overline{\Delta}^W \circ j = \Delta^W$ and that $\overline{\Delta}^W$ is proper. We thus have

$$\Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) = \overline{\Delta}^W_* j_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) = \overline{\Delta}^W_* j_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)$$
$$= \Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right).$$
□

In fact, this result also holds in characteristic 2 if G has no simple factor with trivial center in G Lemma 7.16.

8. Properties of the Hecke complex

Let X be a smooth projective curve over k, G a reductive group over k, D a divisor on X, H a smooth connected factorizable subgroup of $G\langle \mathcal{O}_D \rangle$, and $\mathcal{L}$ a character sheaf on H.

For $W : |X| \to \Lambda^+$ a function with finite support, supported away from D, let

$$K_W = \Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) [\dim H].$$

We will use Theorem 7.33, and other tools, to show important properties of K_W. In subsection 8.1 we will show it is a pure perverse sheaf. In subsection 8.2 will describe its support. In subsection 9.1 we will calculate its trace function.

8.1. **Purity and Perversity.**

Notation 8.1. Let $d(W) = \sum_{x \in X} 2(\deg x)\langle W_x, \rho\rangle$ where ρ is half the sum of the positive roots of the maximal torus of G.

Lemma 8.2. *(1) The dimension of* $\mathrm{Bun}_{G(D)}$ *is* $(\dim G)(g + |D| - 1)$.
(2) The dimension of $\mathcal{H}k_{G(D),W}$ *is* $(\dim G)(g + |D| - 1) + d(W)$

Proof. (1) $\mathrm{Bun}_{G(D)}$ is a $G\langle \mathcal{O}_D\rangle$-torsor on Bun_G. The dimension of Bun_G is $(\dim G)(g-1)$ and the dimension of $G\langle\mathcal{O}_D\rangle$ is $(\dim G)|D|$.
(2) $\mathcal{H}k_{G(D),W}$ is a fiber bundle over $\mathrm{Bun}_{G(D)}$. The fiber over each point is the product over x in the support of W of Weil restriction from κ_x to k of the closure of the cell of the affine Grassmannian defined by W_x. The dimension of this fiber is the sum over x of the degree of x times the dimension of this cell. The dimension of the cell is $2\langle W_x, \rho\rangle$ so the sum is $d(W)$. □

Lemma 8.3. *Assume that* $(GL_n, m_x, H_x, \mathcal{L}_x)$ *is geometrically supercuspidal for some* $x \in D$ *and* $\mathrm{char}(k) > 2$. *Then the complex* K_W *is perverse, pure of weight* $\dim G(g+|D|-1)+d(W)+\dim H$, *and geometrically semisimple.*

Proof. By construction and Lemma 8.2, $IC_{\mathcal{H}k_{G(D),W}}$ is perverse and pure of weight $(\dim G)(g + |D|-1)+d(W)$. Because $\mathcal{L}$ is lisse on a smooth variety of dimension $\dim H$, $\mathcal{L}[\dim H]$ is perverse. By Lemma 7.8, Δ^W is schematic and affine. Thus by Artin's theorem, $\Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is semiperverse and $K_W = \Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is cosemiperverse. Because they are equal by Theorem 7.33, they are each perverse.

By Lemma 2.12, $\mathcal{L}$ has arithmetic monodromy of finite order, so every Frobenius eigenvalue of $\mathcal{L}$ has finite order, and hence has absolute value 1, so $\mathcal{L}$ is pure of weight 0. Thus its shift $\mathcal{L}[\dim H]$ is pure of weight $\dim H$, so the exterior product $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]$ is pure of weight $(\dim G)(g + |D| - 1) + d(W) + \dim H$. Hence by Deligne's theorem (which we may apply because Δ^W is schematic), $K_W = \Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is mixed of weight $\leq (\dim G)(g + |D| - 1) + d(W) + \dim H$ and $\Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is mixed of weight $\geq (\dim G)(g+|D|-1)+d(W)+\dim H$. Because they are equal by Theorem 7.33, they are each pure of weight $(\dim G)(g + |D| - 1) + d(W) + \dim H$.

The geometric semisimplicity of a pure perverse sheaf on an Artin stack with affine stabilizers follows from [48, Theorem 1.2] □

Lemma 8.4. *Assume that* $(G, m_u, H_u, \mathcal{L}_u)$ *is geometrically supercuspidal for some* $u \in D$ *and* $\mathrm{char}(k) > 2$. *Then the Verdier dual of* K_W *is the analogue of* K_W *defined with the dual character sheaf* $\mathcal{L}^\vee$, *twisted by* $\overline{\mathbb{Q}}_\ell(\dim G(g + |D| - 1) + d(W) + \dim H)$.

Proof. We have

$$DK_W = D\Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)[\dim H] = \Delta^W_* D \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)[\dim H]$$

$$= \Delta_*^W \left(DIC_{\mathcal{H}k_{G(D),W}} \boxtimes D\mathcal{L} \right) [\dim H].$$

Now $D\mathcal{L}[\dim H] = D\mathcal{L}^\vee(\dim H)[\dim H]$ and $DIC_{\mathcal{H}k_{G(D),W}} = IC_{\mathcal{H}k_{G(D),W}}(\dim \mathcal{H}k_{G(D),W}) = IC_{\mathcal{H}k_{G(D),W}}(\dim G(g+|D|-1)+d(W))$ so

$$DK_W = \Delta_*^W \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \right) (\dim G(g+|D|-1)+d(W)+\dim H))[\dim H]$$

$$= \Delta_!^W \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \right) (\dim G(g+|D|-1)+d(W)+\dim H))[\dim H]. \qquad \square$$

8.2. **Vanishing Properties.** The following definition is one way of generalizing to the ramified case the very unstable bundles of Frenkel-Gaitsgory-Villonen [20, 3.2].

Definition 8.5. Let P be a parabolic subgroup of G with maximal unipotent subgroup N. To a P-bundle on X, we attach a form of N twisted by the conjugation action of P on N, which admits a natural filtration into vector bundles (see Lemma 7.21). Say that a P-bundle is *very unstable* if none of these vector bundles admit a nontrivial map to $K_X(D)$. Say that a G-bundle is very unstable if it admits a reduction to a very unstable P-bundle for some maximal parabolic subgroup G.

This definition makes sense for G-bundles on X defined over any field, and in particular an algebraically closed field.

Lemma 8.6. *Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and* $\mathrm{char}(k) > 2$. *Then the stalk of K_W at a geometric point $((\alpha_1, t_1), (\alpha_2, t_2))$ of* $\mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{GL(D)}$ *vanishes if V_1 or V_2 is D-very unstable, as does the stalk of its dual.*

Proof. By Lemma 8.4, and because geometric supercuspidality is preserved by duality, we can reduce to the case of K_W. By switching α_1 and α_2 and replacing W by the conjugate of $-W$ under the longest element of the Weyl group, we can reduce to the case where α_1 is D-very unstable.

By proper base change, the stalk of K_W at $((\alpha_1, t_1), (\alpha_2, t_2))$ is the cohomology with compact supports of the fiber of Δ^W over $(\alpha_1, t_1), (\alpha_2, t_2)$ with coefficients in $IC_{\mathcal{H}kG(D)} \boxtimes \mathcal{L}$. This fiber consists of isomorphisms $\varphi : \alpha_1 \to \alpha_2$ away from the support of W, satisfying local conditions at points in the support of W, such that $t_2 \circ \varphi|_D \circ t_1 \in H$.

Let β be a reduction of α_1 to a very unstable P-bundle. By Lemma 7.24, there is a section over $\mathrm{res}_k^D(\mathcal{N}_\beta|D) \times X$ of $\mathcal{N}_\beta$, and therefore a section of the automorphism group of α_1, that restricted to D is the canonical section. Let S be the subgroup of $\sigma \in \mathrm{res}_k^D(\mathcal{N}_\beta|D) \times X$ such that $t_1^{-1} \circ \sigma \circ t_1 \in H$. Then S acts on this fiber by sending φ to $\varphi \circ s(\sigma)$, which satisfies

$$t_2^{-1} \circ \varphi|_D \circ s(\sigma)|_D \circ t_1 = t_2^{-1} \circ \varphi|_D \circ \sigma \circ t_1 \in H$$

by assumption. This action preserves $IC_{\mathcal{H}kG(D)}$, because it is canonical, but acts on $\mathcal{L}$ by tensoring with $\mathcal{L}(t_1^{-1}\sigma t_1)$. Hence the action of the automorphism on the cohomology is by tensoring with $\mathcal{L}(t_1^{-1}\sigma t_1)$, which is nontrivial by the geometrically supercuspidal assumption, so the cohomology is equal to itself tensored with a nontrivial local system, hence the cohomology vanishes, as desired. $\square$

It is necessary to prove a version of the main theorem of reduction theory that is uniform in q. In the work of [20], the role of this lemma is played by some calculations with the Harder-Narasimhan filtration. Recall that G is split.

Lemma 8.7. *Let B be a Borel subgroup of G and T a maximal torus. Then every G-bundle on X admits a reduction to a B-bundle whose induced T-bundle, composed with the cocharacter associated to any simple positive root to produce a line bundle, has degree $\geq -2g$.*

We use the convention that in the SL_2 triple where the upper-right nilpotent is the given positive root, the associated cocharacter is $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Proof. First we check that G admits a reduction to a B-bundle. To prove this, note that it admits a trivialization over the generic point, hence a B-reduction over the generic point, which extends to the whole curve because the associated G/B-bundle is proper.

Next we define a height on the set of B-reductions. Observe that the associated G/B-bundle (i.e. the G-bundle modulo the right action of B) is a projective scheme over X. Given a character of B, factoring through T, we can form the associated line bundle on this projective scheme. Fix a character of T that is in the interior of the Weyl chamber of B, so that it is positive on all the positive coroots. Then the associated line bundle is ample.

Any B-reduction defines a section of this G/B-bundle. The Weil height of this section according to this line bundle is defined to be the degree of its pullback along this section. This is manifestly an integer and is bounded below. Hence it takes a minimum value. The pullback of this line bundle along the section associated to a B-reduction is the inverse of the composition of the B-bundle with this character, so the height is minus the degree of that composition. Choose a B-reduction with the minimum value of height. We will show that, for this B-bundle, its composition with every simple root character has degree $\geq -2g$.

Fix a a simple root. Let χ be the associated character of B and let P be the associated parabolic. Then the quotient of the Levi subgroup of P by its center is a split adjoint-form group of rank one, hence is isomorphic to PGL_2. We have a Cartesian square.

$$\begin{array}{ccccc} \mathbb{G}_m & \xleftarrow{\lambda_1/\lambda_2} & B(PGL_2) & \longrightarrow & PGL_2 \\ & \nwarrow^{\chi} & \uparrow & & \uparrow \\ & & B & \longrightarrow & P \end{array}$$

Let α_1 be our choice of B-reduction, forming a B-bundle. Then α_1 defines, by functoriality, a P-bundle and hence a PGL_2-bundle, which we can view as a rank two vector bundle V on X, up to a twist by a line bundle. After twisting, we may assume that V has degree $2g-1$ or $2g$. By Riemann-Roch, $H^0(X,V)$ has dimension $\geq (2g-1)+2-2g = 1$, so it has a global section, and hence V can be written as the extension by a line bundle L_1 of degree ≥ 0 of another line bundle L_2, which necessarily has degree $\leq 2g$. This gives a reduction of the induced PGL_2-bundle of α_1 to $B(PGL_2)$. Let α_2 be the fiber product with α_1 over the induced PGL_2-bundle. Then because B is the fiber product of $B(PGL_2)$ with P over PGL_2, α_2 is a B-bundle that agrees with α_1 when projected to P, and hence is another B-reduction of G, and that agrees with the new reduction of the PGL_2-bundle when projected to $B(PGL_2)$.

We can express the fixed cocharacter in the interior of the Weyl chamber as a sum of some character that factors through P with a positive multiple of χ. This is because the characters that factor through P form a wall of the Weyl chamber, to which χ is perpendicular, and pointing towards the interior of the Weyl cone. Observe that the degree of $\chi(\alpha_2)$ is equal to the degree of L_1 minus the degree of L_2, which is at least $-2g$ by construction. So if $\chi(\alpha_1) < -2g$, then $\chi(\alpha_2) > \chi(\alpha_1)$, which contradicts the assumption that the height is minimized. $\square$

Let V be a faithful representation of G. Let r be the maximum number of simple roots that must be added to form a positive root of G and let k be the maximum ℓ^1-norm of weight of V, measured in a basis of simple roots of G. Let ϵ be 1 if $r = 1$ and D is empty and 0 otherwise. Let L be a line bundle on X of degree at least $k(2rg + \deg D + \epsilon) + 2g - 1$.

Definition 8.8. Let U be the set of $(\alpha, t) \in \mathrm{Bun}_{G(D)}$ such that $H^1(X, V(\alpha) \otimes L(-Q))$ vanishes for each point Q in X.

Lemma 8.9. *(1) U is an open subset of* $\mathrm{Bun}_{G(D)}$.

(2) U is quasicompact.

(3) U is the quotient of a smooth scheme of finite type by a reductive algebraic group of finite type.

(4) Every vector bundle in the complement of U inside $\mathrm{Bun}_{G(D)}$ *is D-very unstable.*

(5) The stalk of K_W vanishes on $\mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$ *outside $U \times U$.*

Proof. To prove assertion (1), observe that the set U is the complement of the projection from $\mathrm{Bun}_{G(D)} \times X$ to $\mathrm{Bun}_{G(D)}$ of the locus where $H^1(X, V(\alpha) \otimes L(-Q)) \neq 0$. By the semicontinuity theorem, this locus is closed, and X is proper, hence universally closed, so the projection is closed as well.

Assertion (2) follows from assertion (3). To prove assertion (3), observe that a G-bundle V satisfies this condition if and only if $V(\alpha) \otimes L$ is globally generated and satisfies $H^1(X, V(\alpha) \otimes L) = 0$. In this case, $H^0(X, V(\alpha) \otimes L)$ is a $(\dim V)(\deg L + 1 - g)$-dimensional vector space. Fixing a basis for this space, we obtain a map from X to the Grassmannian of rank $\dim V$ quotients of a fixed $(\dim V)(\deg L + 1 - g)$-dimensional vector space, where the map has degree $(\dim V)(\deg L)$. Moreover, in this case V is the pullback of the tautological bundle from the Grassmannian. The moduli space of such maps is finite-type, and the subspace where the pullback of the tautological bundle has no higher cohomology, and the global sections of the pullback of the tautological bundle map isomorphically to the fixed vector space, is an open subset. The morphism that assigns a reduction of structure group to G to the pullback of the tautological bundles is a schematic morphism of finite type [51, Corollary 3.2.4]. The moduli space of such objects with a trivialization over D of the pullback of the tautological bundle is a $G\langle\mathcal{O}_D\rangle$-bundle on it, hence also schematic of finite type. Then U^d is the quotient of this space by $GL_{(\dim V)(\deg L+1-g)}$.

To prove assertion (4), let α be a G-bundle outside U. Then for some point Q, we have $H^1(X, V(\alpha) \otimes L(-Q)) \neq 0$. Hence by Serre duality we have $H^0(X, K_X \otimes V(\alpha)^\vee \otimes L^\vee(Q)) \neq 0$, so $V(\alpha)$ admits a nontrivial map to the line bundle $K_X \otimes L^\vee(Q)$, which has degree at most $-k(2rg + \deg D + \epsilon)$. Choose a B-reduction of α as in Lemma 8.7, and let β be the induced T-bundle, where T is the maximal torus of T. As a representation of B, V admits a filtration by one-dimensional characters. The induced filtration of $V(\alpha)$ is a filtration by line bundles, each arising by β from a one-dimensional character of T. Because $V(\alpha)$ admits a nontrivial map to a line bundle of degree $\leq -k(2rg + \deg D + \epsilon)$ least one of these line bundles has degree $\leq -k(2rg + \deg D\epsilon)$. Because each weight of V is a combination of at most k simple roots, this implies that the absolute value of the degree of the composition of β with the character of B corresponding to one of the simple roots, must be at least $2rg + \deg D + \epsilon > 2g$. The property ensured by Lemma 8.7 is that all of these degrees must be $\geq -2g$, so in fact one of the compositions must be $\geq 2rg + \deg D + \epsilon$, as none can be $\leq -2rg - \deg D + \epsilon$.

Fix a simple root where this inequality holds. Let P be the parabolic subgroup defined by the set of all the other roots. Let N be its unipotent radical. Then N is an iterated extension, as an algebraic group, of one-dimensional representations of B, on the additive group, each a character of B corresponding to a negative root in the unipotent radical of P and thus to minus the sum of at most r positive roots, at least one of which is α. Because each of the other roots has degree $\geq -2g$ and α has degree $\geq 2rg + \deg D$, the product has degree at least $2g + \deg D$ and so does not admit a nontrivial map to $K_X(D)$. Hence none of the N_is do either, and the bundle is very unstable.

Assertion (5) follows from assertion (4) and Lemma 8.6. ☐

9. The trace function of the Hecke complex

9.1. **Calculation of the Trace Function.** Recall some of our earlier notation: $\mathbf{K}(D) = \prod_{x\in|X-D|} G(\mathfrak{o}_x) \times \prod_{x\in D} U_{m_x}(G(\mathfrak{o}_x))$, where $\mathfrak{o}_x \simeq \kappa_x[\![t]\!]$ and $U_{m_x}(G(\kappa_x[\![t]\!]))$ is the subgroup of $G(\kappa_x[\![t]\!])$ consisting of elements congruent to 1 modulo t^{m_x}.

Lemma 9.1. *There is a bijection between* $\mathrm{Bun}_{G(D)}(k)$ *and* $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$.

Moreover, this bijection arises from a bijection between the set of pairs (α, t) *of a* G*-bundle* α *and a tuple* t *consisting of a trivialization* $t_\eta : \alpha|_\eta \to G_\eta$ *of* α *over the generic point and a trivialization* $t_x : \alpha|_{\kappa_x[\![t]\!]} \to G_{\kappa_x[\![t]\!]}$ *for each closed point* x*, such that for* $x \in D$*,* t_x *agrees with* t *modulo* t^{m_x} *and* $G(\mathbb{A}_F)$*, where forgetting* t *corresponds to quotienting out by* $G(F)$ *on the left and* $\mathbf{K}(D)$ *on the right.*

Explicitly, the bijection sends (α, t) *to the tuple*

$$(t_\eta|_{\kappa_x(\!(t)\!)} \circ t_x^{-1}|_{\kappa_x(\!(t)\!)})_{x\in|X|} \in {\prod_{x\in|X|}}' G(\kappa_x(\!(t)\!)) = G(\mathbb{A}_F)$$

of transition maps defined over the punctured formal neighborhood of x*.*

Proof. This is the standard definition of the Weil parameterization. By Lemma 2.3, for any G-bundle there in fact exists a trivialization over the generic point, and because there are no nontrivial torsors of connected algebraic groups over finite fields, there exists a trivialization over a formal neighborhood of every closed point.

One then checks that this map sends the set of all possible trivializations to a double coset in $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ and that each double coset arises from a unique isomorphism class of G-bundles. ☐

Recall that J_x is the inverse image of $H_x(\kappa_x)$ under the map $G(\mathfrak{o}_x) \twoheadrightarrow G(\kappa_x)$.

Definition 9.2. For $g \in G(\mathbb{A}_F)$, let $\mathrm{Aut}_D(g)$ be the subgroup of $\gamma \in G(F)$ such that $g^{-1}\gamma g \in \mathbf{K}(D)$. Let $\mathrm{Aut}_{D,H}(g)$ be the subgroup of $\gamma \in G(F)$ such that

$$g^{-1}\gamma g \in {\prod_{x\in|X-D|}}' G(\mathfrak{o}_x) \times \prod_{x\in D} J_x.$$

We have that $\mathrm{Aut}_D(g)$ is a normal subgroup of $\mathrm{Aut}_{D,H}(g)$.

There is an action of $H(k)$ on $\mathrm{Bun}_{G(D)}(k)$, defined by compositing t with h. Precisely, write $h = (h_x)_{x\in D}$ under the identification $H(k) = \prod_{x\in D} H_x(\kappa_x)$, compose t_x with h_x, and leave t_η unchanged.

Lemma 9.3. *Let* g *be an element of* $G(\mathbb{A}_F)$*, and* (α, t) *be the point of* $\mathrm{Bun}_{G(D)}(k)$ *corresponding to the double coset of* g*. Then*

(1) *The automorphism group of* (α, t) *is* $\mathrm{Aut}_D(g)$.
(2) *Under the identification* $H(k) = \prod_{x\in D} H_x(\kappa_x) = \prod_{x\in D} J_x / \prod_{x\in D} U_{m_x}(G(\mathfrak{o}_x))$, *the action of* $H(k)$ *on* $\mathrm{Bun}_{G(D)}(k)$ *is intertwined with the action of* $\prod_{x\in D} J_x$ *by right multiplication on* $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$.
(3) *The subgroup of* $h \in H(k)$ *that fixes the point* (α, t) *is* $\mathrm{Aut}_{D,H}(g)/\mathrm{Aut}_D(g)$.

Proof. (1) Any automorphism of (α, t), when restricted to the generic point by the trivialization t_η, defines an element $\gamma \in G(F)$. Conversely, any element $\gamma \in G(F)$ defines an automorphism of α over the generic point. The condition that the automorphism extends to a place x is precisely the condition that $g_x^{-1}\gamma g_x$ is in $G(\mathfrak{o}_x)$. For $x \in D$, the condition that the automorphism commute with the trivialization t is the condition that $g_x^{-1}\gamma g_x \in U_{m_x}(G(\mathfrak{o}_x)$.
(2) The action of $H(k)$ is by adjusting the trivializations t. By the definition of the Weil parameterization (Lemma 9.1), we must adjust the trivializations t_x for $x \in D$ to agree with the element modulo m_x. This is exactly composing t_x with an element of J_x in the inverse image of the fixed element of $H(k)$, or equivalently multiplying g_x by that element.
(3) $\mathrm{Aut}_D(g)$ is the kernel of the natural map from $\mathrm{Aut}_{D,H}(g)$ to $H(k)$ given by projection $g_x^{-1}\gamma g_x$ from J_x to $H_x(\kappa_x)$. The elements in the image are exactly those elements that can be lifted from $H(k)$ to elements in $\prod_{x\in D} J_x$ whose action by right multiplication fixes the double coset of g, i.e. the stabilizer in $H(k)$ of (α, t). □

From now on, let q be the cardinality of k, so $k = \mathbb{F}_q$. We need a lemma about the compatibility of the geometric and classical Satake isomorphisms, which is well-known. This is implicit in the 1982 combinatorial formulas of Lusztig and Kato, whose relationship to the IC sheaf is the generalization to the affine Grassmanians of the calculations by Kazhdan–Lusztig of the trace of Frobenius on the IC sheaves of Schubert varieties in a complete flag variety. Our proof is an elaboration of a sketch by Richarz and Zhu [44, p. 449], and we provide some details since we were not able to find a more detailed exposition in the literature.

Lemma 9.4. *Let* $\lambda \in \Lambda^+$ *be a coweight of* G. *Let* IC_λ *be the IC-sheaf of the closure of the cell of the affine Grassmanian* $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$ *associated to* λ. *The trace of Frobenius on the stalk* $IC_{\lambda,x}$ *of* IC_λ *at a point* $x \in \mathrm{Gr}_G(\mathbb{F}_q) = G(\mathbb{F}_q(\!(t)\!))/G(\mathbb{F}_q[\![t]\!])$ *is equal to the integral over the* $N(\mathbb{F}_q(\!(t)\!))$*-orbit of* x *in* $\mathrm{Gr}_G(\mathbb{F}_q)$ *of the spherical function associated to the representation of* $\widehat{G}$ *with highest weight* λ *by the Satake isomorphism, times* $q^{\langle\lambda,\rho\rangle}$.

Proof. Consider the function on $\mathrm{Gr}_G(\mathbb{F}_q)$ defined by the stalks of IC_λ times $q^{-\langle\lambda,\rho\rangle}$, i.e., the stalks of the twist $IC_\lambda(\langle\lambda,\rho\rangle)$. Because the Bruhat cell is left $G(\mathbb{F}_q[\![t]\!])$-invariant, IC_λ is left $G(\mathbb{F}_q[[t]])$-invariant, and so this is a function on $G(\mathbb{F}_q[\![t]\!])\backslash G(\mathbb{F}_q(\!(t)\!))/G(\mathbb{F}_q[\![t]\!])$. Because the Satake transform is an isomorphism, it suffices to check that the Satake transform of this function is the character of the representation of $\widehat{G}$ with highest weight λ.

The Satake transform is a function on the cocharacter lattice of G whose value at a cocharacter μ is $q^{-\langle\mu,\rho\rangle}$ times the integral of the trace function of $IC_\lambda(\langle\lambda,\rho\rangle)$ over $N(\mathbb{F}_q[\![t]\!])\mu(t)$, where N is the unipotent radical of a Borel, and we take a Haar measure where $N \cap G(\mathbb{F}_q[\![t]\!])$ has measure one. It $g \in N\mu(t)$, then the total measure assigned to $g\mathbb{F}_q[\![t]\!]$ by this integral is the measure of $\mu(t)\,(N \cap G(\mathbb{F}_q[\![t]\!]))\,\mu(t)^{-1}$, which is $q^{2\langle\mu,\rho\rangle}$. So it is equivalent to sum the trace function of $IC_\lambda(\langle\lambda,\rho\rangle)$ over $N(\mathbb{F}_q(\!(t)\!))\mu(t)G(\mathbb{F}_q[\![t]\!])/G(\mathbb{F}_q[\![t]\!])$ and multiply by $q^{2\langle\mu,\rho\rangle-\langle\mu,\rho\rangle} = q^{\langle\mu,\rho\rangle}$.

The subset $N(\mathbb{F}_q[\![t]\!])\mu(t)G(\mathbb{F}_q[\![t]\!])/G(\mathbb{F}_q[\![t]\!]) \subseteq \mathrm{Gr}_G(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-points of the locally closed subscheme S_μ of the affine Grassmanian defined by Mirković-Villonen [55, 5.3.5], see also [3, §3.2]. Hence the sum of the trace function of $IC_\lambda(\langle\lambda,\rho\rangle)$ over this set is simply the trace of Frobenius on the cohomology of S_μ with coefficients in the pullback of $IC_\lambda(\langle\lambda,\rho\rangle)$. By [55, Theorem 5.3.9(2)] and [3, prop. 10.1], all eigenvalues of Frobenius on this cohomology group are equal to $q^{-\langle\mu,\rho\rangle}$ and occur in degree $\langle 2\rho,\mu\rangle$, so the trace of Frobenius is $q^{-\langle\mu,\rho\rangle}$ times the dimension of the cohomology group. In particular, the image under the Satake transform is the sum of cocharacters μ of G, or characters of the maximal torus of $\widehat{G}$ weighted by the dimension of this cohomology group. By [55, Theorem 5.3.9(3) and Lemma 5.3.17], this cohomology group is equal to the $\widehat{T}$-eigenspace with character μ in the representation of $\widehat{G}$ with heighest weight λ, which is exactly the multiplicity of the character μ of $\widehat{T}$ in the character of the representation $\widehat{G}$ with highest weight λ, as desired. □

Lemma 9.5. *Let g_1, g_2 be two elements of $G(\mathbb{A}_F)$, and let (α_1, t_1), (α_2, t_2) be the corresponding points of $\mathrm{Bun}_{G(D)}(k)$. There is a natural bijection between*

Isomorphisms $\varphi : \alpha_1 \to \alpha_2$ away from the support of W, that expressed as elements of $G((t))$ by local coordinates near each point x in the support of W are in the closed cell of the affine Grassmannian associated W_x, and such that $t_2 \circ \varphi|_D \circ t_1^{-1}$ is contained in H (i.e. points of $\mathcal{H}k_{G(D),W} \times H$)

and

elements $\gamma \in G(F)$ such that $g_2^{-1}\gamma g_1$ is in $G(\mathfrak{o}_x)$ at all points outside the support of W and the support of D, is in the closure of the cell of the Bruhat decomposition of $G(F_x)$ associated to W_x for each point x in the support of W, and lies in J_x for each point $x \in D$.

Furthermore, the composition of projection to $H(k)$ for the first set of objects with this bijection sends γ to, in $H(k)$, the product over i of the projection of the local component of $g_2\gamma g_1^{-1}$ from J_x to $H_x(\kappa_x)$.

Finally, the composition of this bijection with the trace function of the intersection cohomology complex on $\mathcal{H}k_{G(D),W}$ is $q^{d(W)/2}$ the product over places in the support of W of the function associated by the Satake isomorphism to the character of the representation of $\widehat{G}$ whose highest weight corresponds to W_x.

Proof. Let $t_{\eta,1}, t_{x,1}, t_{\eta,2}, t_{x_2}$ be the trivializations of α_1 and α_2 at the generic point and in formal neigborhoods respectively. Then because $t_{\eta,1}$ and $t_{\eta,2}$ are isomorphisms, there is a bijection between isomorphisms $\varphi : \alpha_1 \to \alpha_2$ over the generic points and the elements $t_{\eta,2} \circ \varphi \circ t_{\eta,1}^{-1}$ of $G(F)$. Let $\gamma = t_{\eta,2} \circ \varphi \circ t_{\eta,1}^{-1}$. Then restricted to the punctured formal neighborhood of x,

$$t_{x,2} \circ \varphi \circ t_{x,1}^{-1} = t_{x,2} \circ t_{\eta,2}^{-1} \circ \gamma \circ t_{\eta,1} \circ t_{x,1}^{-1} = g_{2,x}^{-1}\gamma g_{1,x}$$

is the local component of $g_2\gamma g_1^{-1}$ at x.

The condition that φ be an isomorphism away from the support of W is thus equivalent to $g_2^{-1}\gamma g_1$ lying in $G(\mathfrak{o}_x)$ for all closed points x away from the support of W (for the points of D, this is implied by the condition that it lie in J_x). The condition that, expressed in local coordinates at x φ is in the closure of the cell in the affine Grassmannian associated to W_x is equivalent to $g_2^{-1}\gamma g_1$ lying in the closure of the cell of the Bruhat decomposition of $G(F_x)$ associated to W_x. The fact that $t_2 \circ \varphi|_D \circ t_1^{-1}$ lies in H is equivalent to the condition that $g_2^{-1}\gamma g_1$ is in H modulo D, or equivalently modulo t^{m_x} for each x in D, which is precisely the definition of J_x.

The map Δ_W is defined so that the trivialization $t_2 = h \circ t_1 \circ \varphi_D^{-1}$ so that $h = t_2 \circ \varphi|_D \circ t_1^{-1}$. The projection onto H is given by extracting h, which is which is $t_2 \circ \varphi|_D \circ t_1^{-1} = g_2^{-1}\gamma g_1$.

The final claim should follow from the comparison between the geometric and classical Satake isomorphisms [44, p. 449]. The normalizing factor $q^{d(W)/2}$ arises from the fact that the Satake isomorphism works with perverse sheaves pure of weight 0 on a cell of the affine Grassmannian while we work with middle extensions of the constant sheaf on a cell, which are pure of weight the dimension of the cell, so we must Tate twist by half the dimension of the cell to compare them. □

Definition 9.6. For x a closed point of X, let f_x^W on $G(F_x)$ equal:

- If x is not contained in D or the support of W, the characteristic function of $G(\mathfrak{o}_x)$.
- If x is contained in the support of W, the function associated by the Satake isomorphism to the character of the representation of $\widehat{G}$ whose highest weight corresponds to W_x, times $q^{\deg x \langle W_x, \rho\rangle}$.
- If x is contained in D, the function that vanishes outside of J_x and is equal to χ_x on J_x.

Lemma 9.7. *Let g_1, g_2 be two elements of $G(\mathbb{A}_F)$. Let (α, t_1) and (α, t_2) be the points of $\mathrm{Bun}_{G(D)}(k)$ corresponding to the double cosets of g_1 and g_2 respectively. Then the trace of* Frob_k *on the stalk of K_W at $((\alpha_1, t_1), (\alpha_2, t_2))$ is*

$$\sum_{\gamma \in G(F)} \prod_{x \in |X|} f_x^W (g_2^{-1} \gamma g_1).$$

Proof. By the Lefschetz formula, the trace is the sum of the trace function of $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}^*$ over the inverse image under Δ^W of $((\alpha_1, t_1), (\alpha_2, t_2))$. (This fiber is an affine scheme of finite type, so we do not need to apply the Lefschetz formula for stacks here.)

By Definition 6.12, this fiber consists of isomorphisms $\varphi : \alpha_1 \to \alpha_2$ away from the support of $f_x^{W_1}$, that expressed as elements of $G((t))$ by local coordinates near each point x in the support of W are in the closed cell of the affine Grassmannian associated to W_x, such that $t_2 \circ \varphi|_D \circ t_1^{-1}$ is contained in H.

By Lemma 9.5, such maps φ are in bijection with γ in $G(F)$ such that $g_2^{-1} \gamma g_1$ is in $G(\mathcal{O}_{F_v})$ at all places outside the support of W and the support of D, is in the closure of the cell of the Bruhat decomposition of $G(F_x)$ for each place x associated to W_x for each point x in the support of W, and lies in J_x for each point x of D.

Furthermore, the trace function of $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}$ is is equal to the product of the trace function of $IC_{\mathcal{H}k_{G(D),W}}$ and the trace function of $\mathcal{L}$. The trace function of $IC_{\mathcal{H}k_{G(D),W}}$ is the product over the places lying in the support of W of the function associated to the corresponding representation of $\widehat{G}$ in the Satake isomorphism times $q^{\langle W_x, \rho\rangle}$ by Lemma 9.5. The trace function of $\mathcal{L}$ is a character of $H(k)$, which by definition is $\prod_{x \in D} \chi_x$.

Examining, we see that the trace of the point associated to an element γ is precisely $\prod_{x \in |X|} f_x^W (g_2^{-1} \gamma g_1)$. Summing over γ, we obtain the stated sum. □

Definition 9.8. For $g_1, g_2 \in G(\mathbb{A}_F)$, let

$$\mathsf{K}_W(g_1, g_2) = \sum_{\gamma \in G(F)} \prod_{x \in |X|} f_x^W (g_2^{-1} \gamma g_1)$$

be the trace function of K_W.

9.2. Cohomological interpretation of the trace.

Lemma 9.9. *Assume that $p > 2$ and some $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal. Then we have*

$$\sum_{g_1,g_2\in G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)} \frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$$
$$= q^{(\dim G)(g+|D|-1)+d(W_1)+\dim H}\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\operatorname{Frob}_q, H^i_c(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2}))$$

where the sum on the left is finitely supported and the sum on the right is absolutely convergent.

Proof. By an application due to Katz of a result of Gabber [29, Lemma 1.8.1(1)], because K_{W_1} is pure and perverse of weight $(\dim G)(g+|D|-1)+d(W_1)+\dim H$, the trace function of its dual is the complex conjugate of its trace function divided by $q^{(\dim G)(g+|D|-1)+d(W_1)+\dim H}$. Furthermore, by Lemma 9.3, the order of the automorphism group of a k-points of $\operatorname{Bun}_{G(D)}\times\operatorname{Bun}_{G(D)}$ given by the double coset of (g_1,g_2) is $|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|$. Hence the sum on the left is the sum over k-points of $\operatorname{Bun}_{G(D)}\times\operatorname{Bun}_{G(D)}$ of $q^{((\dim G)(g+|D|-1)+d(W_1)+\dim H)}$ times the trace function of $DK_{W_1}\otimes K_{W_2}$, weighted by the inverse of the order of the automorphism group.

By Lemma 8.9, this trace function vanishes outside $U(k)\times U(k)$. In particular, it has finite support, and so the sum on the left is absolutely convergent. By the Lefschetz formula for algebraic stacks [49, Theorem 4.2(ii)] the sum of this trace function over $U(k)\times U(k)$ is equal to the alternating sum of the trace of Frobenius on $H^i_c(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2})$, as desired. The absolute convergence of the sum follows from [49, Theorem 4.2(i)]. □

Let n be a natural number. Define $F_n=\mathbb{F}_{q^n}(X)$ and $X_n=X_{\mathbb{F}_{q^n}}$. We can base change the data $(G,D,H,\mathcal{L},W_1,W_2)$ from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$ in the following way: We pull back G from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$, we pull back D from X to X_n, we compose W_1 and W_2 with the projection $|X_n|\to|X|$, and we base change H and $\mathcal{L}$ from $G\langle\mathcal{O}_D\rangle$ to $(G\langle\mathcal{O}_D\rangle)_{\mathbb{F}_{q^n}}$. Let $f_x^{W_i,n}$ be the local factors defined by this new data and $\mathsf{K}_{W_i,n}(g_1,g_2)=\sum_{\gamma\in G(F_n)}\prod_{x\in|X|}f_x^{W_i,n}(g_2^{-1}\gamma g_1)$ for $g_1,g_2\in G(\mathbb{A}_{F_n})$. Let $\mathbf{K}(D)_n$ be defined also in terms of this base-changed data.

Theorem 9.10. *Assume that $p>2$ and some $(G,m_u,H_u,\mathcal{L}_u)$ is geometrically supercuspidal. Then*

$$\sum_{g_1,g_2\in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)_n} \frac{\overline{\mathsf{K}_{W_1,n}(g_1,g_2)}\mathsf{K}_{W_2,n}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|} = O((q^n)^{(\dim G)(g+|D|-1)+\frac{d(W_1)}{2}+\frac{d(W_2)}{2}+\dim H}).$$

Proof. By Lemma 8.3, K_{W_2} is pure of weight $w_2=(\dim G)(g+|D|-1)+d(W_2)+\dim H$ and K_{W_1} is pure of weight $w_2=(\dim G)(g+|D|-1)+d(W_1)+\dim H$, so $w_2-w_1=d(W_2)-d(W_1)$.

By Lemma 2.20, taking $j=0$, it follows that

$$\sum_{i=-\infty}^{0}(-1)^i \operatorname{tr}(\operatorname{Frob}_{q^n}, H^i_c(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2})) = O\left((q^n)^{\frac{d(W_2)-d(W_1)}{2}}\right).$$

Applying Lemma 2.19(2), this cohomology group vanishes for $i>0$, so

$$\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\operatorname{Frob}_{q^n}, H^i_c(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2})) = O\left((q^n)^{\frac{d(W_2)-d(W_1)}{2}}\right).$$

Then we apply Lemma 9.9 over $\mathbb{F}_{q^n}$. It is clear that base changing all the data in this way is equivalent to base-changing $\mathcal{H}k_{G(D),W}\times H$ and thus to base-changing K_{W_1}, K_{W_2}, so we obtain

$$\sum_{g_1,g_2\in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)_n} \overline{\mathsf{K}_{W_1,n}(g_1,g_n)}\mathsf{K}_{W_2,n}(g_1,g_2)$$

$$= (q^n)^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n}, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2}))$$

$$= O\left((q^n)^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} (q^n)^{\frac{d(W_2)-d(W_1)}{2}} \right) = O\left((q^n)^{(\dim G)(g+|D|-1)+\frac{d(W_1)}{2}+\frac{d(W_2)}{2}+\dim H} \right).$$

□

9.3. **Integrality and Weil numbers.** Let m be the order of the arithmetic monodromy group of $\mathcal{L}$, which is equal to the order of the character χ by Lemma 2.12. It is also stable under finite field extension by Lemma 2.12, as the arithmetic and geometric monodromy groups are equal.

Lemma 9.11. *For all $x \in |X|$ and all $W : |X| \to \Lambda^+$, the function f_x^W takes values in the $\mathbb{Z}[\mu_m]$.*

Proof. If x lies in D, this follows from the fact that χ is an eiegnvalue of Frobenius on $\mathcal{L}$ and hence is a root of unity in the monodromy group. If x does not lie in D or the support of W, then f_x takes the values zero and one, both integers. If x lies in the support of W, then the value is a polynomial in q by the Kazhdan-Lusztig purity theorem. □

Lemma 9.12. *For all $g_1, g_2 \in G(\mathbb{A}_F)$, $\mathsf{K}(g_1, g_2)$ is divisible in $\mathbb{Z}[\mu_m]$ by $|\operatorname{Aut}_{D,H}(g_1)|$ and by $|\operatorname{Aut}_{D,H}(g_2)|$.*

Proof. Let γ' be an element of $\operatorname{Aut}_{D,H}(g_1)$. Then for all $x \in |X - D|$, $g_1^{-1}\gamma g_1 \in G(\kappa_x[[t]])$ and so $f_x^W(g_2^{-1}\gamma\gamma' g_1) = f_x^W(g_2^{-1}\gamma g_1)$. For $x \in D$, $g_1^{-1}\gamma g_1 \in J_x$ and so $f_x^W(g_2^{-1}\gamma\gamma' g_1) = f_x^W(g_2^{-1}\gamma g_1)\chi_x(g_1^{-1}\gamma' g_1)$. Hence right multiplication by γ' multiplies $\prod_{x\in|X|} f_x^W(g_2^{-1}\gamma g_1)$ by $\prod_{x\in D}\chi_x(g_1^{-1}\gamma' g_1)$. It follows that $\mathsf{K}(g_1, g_2) = \mathsf{K}(g_1, g_2)\chi_x(g_1^{-1}\gamma' g_1)$ and hence $\mathsf{K}(g_1, g_2) = 0$, and we are done, unless $\prod_{x\in D}\chi_x(g_1^{-1}\gamma' g_1) = 1$. So we may assume that $\prod_{x\in D}\chi_x(g_1^{-1}\gamma' g_1) = 1$ for all $\gamma' \in \operatorname{Aut}_{D,H}(g_1)$

This implies that $\prod_{x\in|X|} f_x^W(g_2^{-1}\gamma g_1)$ is invariant under right multiplication of γ by elements of $\operatorname{Aut}_{D,H}(g_1)$. We can write $\sum_{\gamma\in G(F)}\prod_{x\in|X|} f_x^W(g_2^{-1}\gamma g_1)$ as a sum over orbits of this right multiplication action. Because the action is by multiplication in a group, its orbits are cosets of $\operatorname{Aut}_{D,H}(g_1$, , and so the size of each orbit is $|\operatorname{Aut}_{D,H}(g_1)|$, and by Lemma 9.11, the sum over each orbit is an element of $\mathbb{Z}[\mu_m]$ times $|\operatorname{Aut}_{D,H}(g_1)|$, so the final (finite) sum is divisible by $|\operatorname{Aut}_{D,H}(g_1)|$.

A symmetrical argument works for $\operatorname{Aut}_{D,H}(g_2)$, using left multiplication instead. □

Lemma 9.13. *The sum*

$$\frac{1}{|H(k)|^2} \sum_{g_1,g_2\in G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)} \frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$$

is an element of $\mathbb{Z}[\mu_m]$.

Proof. Break the sum into a sum over pairs of orbits under the action of $H(k)$ on $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ by right multiplication. It suffices to show that the sum over each orbit, divided by $|H(k)|^2$, lies in $\mathbb{Z}[\mu_m]$.

Because this action corresponds to right multiplication by $\prod_{x\in D} J_x$, it multiplies $\prod_{x\in|X|} f_x^W(g_1^{-1}\gamma g_2)$ by $\prod_{x\in D}\chi_x(h)$, so it multiplies $\mathsf{K}_{W_i}(g_1, g_2)$ by $\prod_{x\in D}\chi_x(h)$, which is a root of unity, so it fixes $\overline{\mathsf{K}_{W_1}(g_1, g_2)}\mathsf{K}_{W_2}(g_1, g_2)$. Hence the sum over each orbit is the size of that orbit times $\frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$ for some g_1, g_2 in that orbit. By the orbit-stabilizer theorem and Lemma 9.3, the size of the orbit is $\frac{|H(k)|^2|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}{|\operatorname{Aut}_{D,H}(g_1)||\operatorname{Aut}_{D,H}(g_2)|}$. Hence the sum over the orbit, divided by $|H(k)|^2$, is $\frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_{D,H}(g_1)||\operatorname{Aut}_{D,H}(g_2)|}$, which is an algebraic integer by Lemma 9.12. □

We use the convention (following [49, Definition 10.1]) that Weil q-numbers are algebraic numbers whose absolute values are a power of $\sqrt{q}$ independent of the choice of complex embedding, while Weil q-integers are algebraic integers with the same property.

Lemma 9.14. *All the eigenvalue of* Frob_q *on* $H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2})$ *are* q*-Weil numbers.*

Proof. By Lemma 8.4 have

$$DK_{W_1} = \Delta^{W_1}_! \left(IC_{\mathcal{H}k_{G(D),W_1}} \boxtimes \mathcal{L}^{-1} \right) [\dim H]((\dim G)(g + |D| - 1) + d(W_1)) + \dim H).$$

Then if we form a Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{p_1} & \mathcal{H}k_{G(D),W_1} \times H \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \Delta^{W_1}} \\ \mathcal{H}k_{G(D),W_2} \times H & \xrightarrow{\Delta^{W_2}} & U \times U \end{array}$$

by the Künneth formula,

$$H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2})$$
$$= H^{i+\dim H}_c(Y_{\overline{k}}, p_1^*(IC_{\mathcal{H}k_{G(D),W_1}} \boxtimes \mathcal{L}^{-1}) \otimes p_2^*(IC_{\mathcal{H}k_{G(D),W_2}} \boxtimes \mathcal{L})).$$

We can stratify $\mathcal{H}k^{W_i}_{G(D)}$ into strata, the inverse images of Bruhat-Tits cells, on which the Kazhdan-Lusztig purity theorem implies that $IC_{\mathcal{H}k_{G(D),W_1}}$ is a shift of a Tate twist of a constant sheaf. It suffices to prove this on the inverse image of these strata. We can remove the $\mathcal{L}$ and $\mathcal{L}^{-1}$ terms by noting that these are summands of the pushforward of the constant sheaf along the Lang isogeny (Lemma 2.11), so the whole cohomology group is a summand of the cohomology of the inverse image of one of these strata under the Lang isogeny of $H \times H$. Because this is an algebraic stack, it follows from [49, Lemma 10.2] that all eigenvalues of Frobenius on its cohomology are q-Weil numbers. $\square$

Theorem 9.15. *There exists a natural number* N*,* q*-Weil integers* $\alpha_1, \dots, \alpha_N$ *of weight* $\leq (\dim G)(g + |D| - 1) + \frac{d(W_1)}{2} + \frac{d(W_2)}{2} - \dim H$*, and signs* $\epsilon_1, \dots, \epsilon_N \in \{\pm 1\}$*, such that for all* n*,*

$$\frac{1}{|H(\mathbb{F}_{q^n})|^2} \sum_{g_1, g_2 \in G(F_n) \backslash G(\mathbb{A}_{F_n}) / \mathbf{K}(D)_n} \frac{\overline{\mathsf{K}_{W_1,n}(g_1, g_2)} \mathsf{K}_{W_2,n}(g_1, g_2)}{|\operatorname{Aut}_D(g_1)| |\operatorname{Aut}_D(g_2)|} = \sum_{i=1}^N \epsilon_i \alpha_i^n.$$

Furthermore, we may arrange such that

- $\alpha_1, \dots, \alpha_{\dim \mathrm{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})}$ *are* $q^{(\dim G)(g+|D|-1)+d(W_1)-\dim H}$ *times the eigenvalues of* Frob_q *on* $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})$*, which are of weight* $d(W_2) - d(W_1)$*,*
- $\epsilon_1, \dots, \epsilon_{\dim \mathrm{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})}$ *are all equal to* 1*,*
- α_i *has weight* $< 2(\dim G)(g+|D|-1)+d(W_1)+d(W_2)-\dim H$ *for* $i > \dim \mathrm{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})$*.*

Proof. Let S_n be the left-hand side of the formula. We apply Lemma 9.9 over $\mathbb{F}_{q^n}$. It is clear that base-changing all the data in this way is equivalent to base-changing $\mathcal{H}k_{G(D),W} \times H$ and thus to base-changing K_{W_1}, K_{W_2}, so we obtain

$$|H(\mathbb{F}_q^n)|^2 \cdot S_n = (q^n)^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n}, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2}))$$

By Lemma 8.3, K_{W_2} is pure of weight $w_1 = (\dim G)(g+|D|-1)+d(W_2)+\dim H$ and K_{W_1} is pure of weight $w_2 = (\dim G)(g+|D|-1)+d(W_1)+\dim H$, so $w_2-w_1 = d(W_2)-d(W_1)$. Hence the eigenvalue of Frob_q on $H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2})$ are Weil numbers of weight $\leq d(W_2)-d(W_1)+i$.

By Lemma 2.19(2), this cohomology group vanishes for $i > 0$. Hence we can write S_n as a convergent signed sum of nth powers of Weil numbers, with the largest possible weight being

$$\begin{aligned} &2(\dim G)(g+|D|-1) + 2d(W_1) + 2\dim H + d(W_2) - d(W_1) \\ &= 2(\dim G)(g+|D|-1) + 2\dim H + d(W_1) + d(W_2), \end{aligned}$$

and appearing in H^0.

Now $|H(\mathbb{F}_{q^n})|$ is a finite signed sum of nth powers of Weil numbers, with the largest weight $2\dim H$ appearing with multiplicity 1 and sign 1, because H is smooth and connected. Hence $\frac{1}{|H(\mathbb{F}_{q^n})|^2}$ is a convergent signed sum of nth powers of Weil numbers, with the largest weight $-4\dim H$ appearing with multiplicity 1 and sign 1.

Hence their product S_n is also a convergent signed sum of nth powers of Weil numbers. By Lemma 2.20 this convergence is uniform in n. Thus the generating function $\sum_{n=1}^{\infty} u^n S_n$ is a signed sum of terms of the form $\frac{\alpha_i u}{1-\alpha_i u}$, with the α_i Weil numbers. In particular, it is a meromorphic function with poles of order 1 at inverses of Weil numbers α_i and with residues integer multiples of $1/\alpha_i$.

However, it is also a power series with coefficients in the ring of integers of a number field $\mathbb{Q}(\mu_m)$. A variant due to Dwork of a result of E. Borel implies that it is a rational function [19, Theorem 3, p. 645], so all but finitely many of the α_i occur with zero multiplicity, and we have the stated claim, except with q-Weil numbers rather than q-Weil integers. To check they are algebraic integers, it is sufficient to check that they are ℓ-adic integers for each prime ℓ. The ℓ-adic radius of convergence of this rational function is at least one, because all its coefficients are algebraic integers, so all its poles have ℓ-adic norm at least one, and the α_i are the inverses of its poles.

The maximum weight of the Weil numbers occurring is

$$\begin{aligned} &-4\dim H + 2(\dim G)(g+|D|-1) + 2\dim H + d(W_1) + d(W_2) \\ &= 2(\dim G)(g+|D|-1) - 2\dim H + d(W_1) + d(W_2). \end{aligned}$$

A Weil number meets that bound only if it a Weil number from H^0 multiplied by the constant $q^{(\dim G)(g+|D|-1)+d(W_1)+\dim H}$ and then multiplied by $q^{-2\dim H}$. By Lemma 2.19(3), H^0 is isomorphic to $\mathrm{Hom}(K_1, K_2)$. Because K_1 and K_2 are pure, all eigenvalues on $\mathrm{Hom}(K_1, K_2)$ actually have size $q^{\frac{d(W_1)-d(W_2)}{2}}$, so a Weil number meets that bound if and only if it comes from H^0 in this way. Bringing these numbers to the front of the line we obtain the stated claim. □

10. q-aspect families

We continue with the same previous set-up: G is a split semisimple group over k, D is an effective divisor on X, $\mathcal{L}$ is a character sheaf on the factorizable subgroup H of $G\langle \mathcal{O}_D \rangle$. Suppose $k = \mathbb{F}_q$. We shall define the q-aspect family $\mathcal{V} = \mathcal{V}(G, X, D, H, \mathcal{L})$.

For every $n \geq 1$, let $F_n := F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$. We define $\mathcal{V}_n$ as consisting of automorphic representations π of $G(\mathbb{A}_{F_n})$, that are $G(\mathfrak{o}_y)$-unramified for every $y \in |X_n - D|$, and at each place y of F_n lying over a place $x \in D$, and with residue field κ_y/κ_x, admit a vector on which the preimage $J_y \subset G(\kappa_y[\![t]\!])$ of $H(\kappa_y)$ acts by the character χ_y associated to the sheaf $\mathcal{L}$. The automorphic representations are counted with multiplicity, more precisely it is the product of the automorphic multiplicity of π with the dimension of the space of (J_y, χ_y)-invariant vectors in π_y for each place $y \in D$.

10.1. **Spectral expansion of the trace.** For any $n \geq 1$, $\pi \in \mathcal{V}_n$, and $x \in |X_n - D|$, the representation π_x is $G(\mathfrak{o}_x)$-spherical. Recall from §2.2 that to every $G(\mathfrak{o}_x)$-unramified irreducible representation π_x is attached a Satake parameter $t_{\pi_x} \in \widehat{T}(\mathbb{C})/W$. For a dominant weight $\lambda \in \Lambda^+$, we have defined

$$\mathrm{tr}_\lambda(\pi_x) := \mathrm{tr}(\pi_x)(a_\lambda) = \mathrm{tr}(t_{\pi_x}|V_\lambda).$$

For a function $W : |X_n| \to \Lambda^+$ of finite support disjoint from D, let

$$\mathrm{tr}_W(\pi) := \prod_{x \in \mathrm{supp}(W)} \mathrm{tr}_{W_x}(\pi_x).$$

Proposition 10.1. *For every $n \geq 1$, and every $W_1, W_2 : |X_n| \to \Lambda^+$ with finite support disjoint from D,*

$$|H(\mathbb{F}_{q^n})|^2 \sum_{\pi \in \mathcal{V}_n} \mathrm{tr}_{W_1}(\pi)\overline{\mathrm{tr}_{W_2}(\pi)} = \frac{1}{(q^n)^{\frac{d(W_1)+d(W_2)}{2}}} \sum_{g_1, g_2 \in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)} \frac{\mathsf{K}_{W_1}(g_1,g_2)\overline{\mathsf{K}_{W_2}(g_1,g_2)}}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$$

Proof. Recall Definition 9.6 of the test functions f_x^W for $x \in |X|$, and let $f := \prod_{x \in |X|} f_x^W$. By Definition 9.8, $\mathsf{K}_W(g_1, g_2)$ is the kernel of the convolution operator $*f$ on the vector space of all forms. Here, and below, we shall work with the counting measure on $G(\mathbb{A}_{F_n})/\mathbf{K}(D)$ when forming convolutions.

Since $f_u^W = f_u$ is a cuspidal function for the place $u \in D$, the operator $*f$ has image inside the space of cusp forms. More precisely, consider an orthonormal Hecke basis $\mathcal{B}_n = \{\varphi\}$ of the space of cuspidal automorphic forms on $G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)$, where the inner product is

$$\frac{1}{|H(\mathbb{F}_{q^n})|} \sum_{g \in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)} \frac{|\varphi(g)|^2}{|\operatorname{Aut}_D(g)|}.$$

Since automorphic representations in $\mathcal{V}_n$ are counted with multiplicity, this implies that we can arrange the basis so that there is an injection $\mathcal{V}_n \hookrightarrow \mathcal{B}_n$, which we shall denote by $\pi \mapsto \varphi_\pi$. In other words, $\{\varphi_\pi\}$ is a basis of the subspace of automorphic functions on $\mathrm{Bun}_{G(D)}(\mathbb{F}_q)$ which are $(\prod_{y\in D} J_y, \prod_{y \in D} \chi_y)$-equivariant. We can also arrange so that $\varphi * f = 0$ if $\varphi \in \mathcal{B}_n - \mathcal{V}_n$ (for this consider the case $W = 0$, in which case the operator $*f$ is idempotent, and its kernel form an orthogonal complementary subspace).

The convolution operator $*f$ is an integral operator with kernel

$$\sum_{\varphi \in \mathcal{B}_n} (\varphi * f)(g_1)\overline{\varphi(g_2)} = \sum_{\pi \in \mathcal{V}_n} (\varphi_\pi * f)(g_1)\overline{\varphi_\pi(g_2)}$$

We can show that we have $\varphi_\pi * f = |H(\mathbb{F}_{q^n})|\, q^{\frac{d(W)}{2}}\, \mathrm{tr}_W(\pi)\varphi_\pi$. To do this, observe that for every $x \in \mathrm{supp}(W)$, $*f_x^W$ acts on the representation π_x by scalar multiplication by $\mathrm{tr}_{W_x}(\pi_x)\,(q^n)^{\langle W_x, \rho\rangle}$, and that for $x \in D$, $*f_x^W$ acts on φ_π by a volume factor. Precisely,

$$\sum_{g_2 \in G(\mathbb{A}_{F_n})/\mathbf{K}(D)} \prod_{x \in |X_n|} f_x^W(g_2^{-1}g_1)\varphi_\pi(g_2) = \frac{1}{\mathrm{vol}(\mathbf{K}(D))} \int_{g_2 \in G(\mathbb{A}_{F_n})} \prod_{x\in|X_n|} f_x^W(g_2^{-1}g_1)\varphi_\pi(g_2)$$

$$= \frac{1}{\mathrm{vol}(\mathbf{K}(D))} \prod_{x \in |X_n|} \int_{h \in G(F_x)} f_x^W(h)\varphi_\pi(g_1h^{-1})$$

$$= \frac{\prod_{x\in D}\mathrm{vol}(J_x)}{\mathrm{vol}(\mathbf{K}(D))} \cdot \prod_{x \in \mathrm{supp}(W)} \mathrm{tr}_{W_x}(\pi_x)\,(q^n)^{\langle W_x,\rho\rangle} \cdot \varphi_\pi(g_1),$$

and the ratio of volumes is equal to $|H(\mathbb{F}_{q^n})|$ by Definition 6.6. We deduce the identity

$$K_W(g_1, g_2) = |H(\mathbb{F}_{q^n})| \, (q^n)^{d(W)/2} \sum_{\pi \in \mathcal{V}_n} \mathrm{tr}_W(\pi) \varphi_\pi(g_1) \overline{\varphi_\pi(g_2)}.$$

The proposition now follows from orthogonality relations for the orthonormal basis $\mathcal{B}_n$. □

10.2. **Average Ramanujan bound.** We fix a place $v \in |X - D|$.

Theorem 10.2. *Let $n \geq 1$, and let $\lambda \in \Lambda^+$ be a dominant weight.*

$$\sum_{\pi \in \mathcal{V}_n} \prod_{w|v} |\operatorname{tr}_\lambda(\pi_w)|^2 \ll q^{n(\dim G(g+|D|-1) - \dim H)}.$$

The multiplicative constant depends only on $(G, \lambda, X, D, H, \mathcal{L})$, and is independent of n.

Proof. Let $W : |X| \to \Lambda^+$ be defined by

$$W(x) = \begin{cases} \lambda, & \text{if } x = v, \\ 0, & \text{if } x \neq v. \end{cases}$$

Let K_W be the function defined in Definition 9.8. Then by Proposition 10.1 and Theorem 9.10

$$\sum_{\pi \in \mathcal{V}_n} \mathrm{tr}_W(\pi) = \frac{1}{(q^n)^{\frac{d(W)}{2}} |H(\mathbb{F}_{q^n})|^2} \sum_{g_1, g_2 \in G(F_n) \backslash G(\mathbb{A}_{F_n}) / \mathbf{K}(D)} \frac{\mathsf{K}_W(g_1, g_2) \overline{\mathsf{K}_0(g_1, g_2)}}{|\operatorname{Aut}_D(g_1)| |\operatorname{Aut}_D(g_2)|}$$

$$= \frac{O\left((q^n)^{(\dim G)(g+|D|-1)+d(W)+\dim H}\right)}{(q^n)^{\frac{d(W)}{2}} |H(\mathbb{F}_{q^n})|^2} = O\left((q^n)^{(\dim G)(g+|D|-1)-\dim H}\right) \qquad \square$$

Corollary 10.3. *Let $n \geq 1$, $\pi \in \mathcal{V}_n$, and let λ be a dominant weight of G. Then*

$$|\operatorname{tr}_\lambda(\pi_w)|^2 \ll q^{n(\dim G(g+|D|-1) - \dim H)}$$

with the constant independent of n.

Proof. This follows from Theorem 10.2 because the left side is a sum of squares and hence any term is bounded by the whole. □

10.3. **Sums of Weil numbers.** In the course of the proof above we have shown that several spectral quantities are sums of Weil numbers. Such results are of independent interest, and we spell it out in more details in this subsection.

Proposition 10.4. *There exist q-Weil integers α_i of weight $\leq (\dim G)(g + |D| - 1) - \dim H$, such that*

$$|\mathcal{V}_n| = \sum\nolimits_i \alpha_i^n, \quad \textit{for every } n \geq 1.$$

Proof. This follows from Theorem 9.15 and Proposition 10.1, taking $W_1 = W_2 = 0$. In this case $d(W_1) = d(W_2) = 0$ so the factor of $q^{d(W_1)/2 + d(W_2)/2}$ may be ignored. □

Proposition 10.5. *For every $W : |X| \to \Lambda^+$ of finite support disjoint from D, there exist q-Weil integers β_j of weight $\leq (\dim G)(g + |D| - 1) - \dim H + d(W)$, such that*

$$q^{n\frac{d(W)}{2}} \sum_{\pi \in \mathcal{V}_n} \mathrm{tr}_W(\pi) = \sum\nolimits_j \beta_j^n, \quad \textit{for every } n \geq 1.$$

Proof. This follows from Theorem 9.15 and Proposition 10.1, taking $W_1 = W$ and $W_2 = 0$. □

10.4. **The main theorem.** To prove the main theorem, we shall embed the automorphic representation π of $G(\mathbb{A}_F)$ in a suitable automorphic family $\mathcal{V}_n$ in the q-aspect as in Section 10: at the place u, we shall use the mgs datum, and at the other ramified places, we shall choose a datum with trivial character, and with sufficient depth that π and its base changes Π have a nonzero invariant vector.

Lemma 10.6. *Let G be a reductive group over a local field. There is a constant c such that for any two points x, y in the Bruhat-Tits building, for all depths r, the Moy-Prasad subgroup $G_{x,r}$ contains a conjugate of $G_{y,r+c}$. If G is split, we can take c to depend only on the root data of G and not on the base field.*

Proof. After conjugation, we may assume that x and y are contained in the same apartment. Define a metric on this apartment where the distance $d(x, y)$ is the max over all roots of the absolute value of the difference between the evaluations of the linear function associated to this root on x and y. Then by construction, it is clear that $G_{x,r}$ contains $G_{y,d(x,y)}$. Take c to be the supremum over pairs x, y of the minimum distance between x any any conjugate of y under the affine Weyl group action. Because this action is cocompact, a finite supremum in fact exists. Because the metric on the apartment and the affine Weyl group can be defined combinatorially, c depends only on the underlying root data. □

Lemma 10.7. *Let G be a split semisimple algebraic group. Let $F = \mathbb{F}_q(X)$. Let π be an automorphic representation of $G(\mathbb{A}_F)$, mgs at a place u, with Condition BC. Then there exists a divisor D on X, a subgroup $H \subseteq G\langle \mathcal{O}_D \rangle$, and a character sheaf $\mathcal{L}$ on H, that is geometrically supercuspidal on U, and such that for all n, the base change Π_n of π to F_n is contained in the associated family $\mathcal{V}_n$.*

Proof. By the definition of Condition BC, there exists mgs datum $(G_{\kappa_u}, m_u, H_u, \mathcal{L}_u)$ such that for all n, for all places u' of F_n lying over u with local field $E_{u'}$, $\Pi_{n,u'}$ is a quotient of $\text{c-ind}_{J_{u,E}}^{G(E_{u'})} \chi_{u,E}$.

Let S be the set of ramified places of π other than u. Again by the definition of Condition BC, Π_n is unramified outside $S \cup \{u\}$, with a bound on the depth inside S. Let m be some integer greater than this bound on the depth plus the constant of Lemma 10.6. It follows that for all places x lying over a place in S, Π_n contains a vector invariant under the depth m subgroup of the standard hyperspecial maximal compact, which is the subgroup of elements of $G(\kappa_x[[t]])$ congruent to 1 mod t^m.

It follows that if we let D be the divisor of multiplicity m at each point of S and multiplicity m_u at u, $H = H_u$, and $\mathcal{L} = \mathcal{L}_u$, then $\Pi_n \in \mathcal{V}_n$ for all n.

Finally, $(G, D, H, \mathcal{L})$ is geometrically supercuspidal at u because $(G_{\kappa_u}, m, u_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal. □

To improve the bound of Corollary 10.3 for this family, and obtain the main theorem, we use a variant of the tensor power trick, where bounds for large n will imply stronger bounds for small n.

Theorem 10.8. *Let G be a split semisimple algebraic group. Assume the characteristic of F is not 2. Let π be an automorphic representation of $G(\mathbb{A}_F)$, mgs at a place u, and satisfying Condition BC. Let v be a place at which G is unramified for the standard hyperspecial maximal compact subgroup. Then π is tempered at v.*

Proof. Let $\lambda \in \Lambda^+$ be a dominant weight. We apply Corollary 10.3 to the family produced by Lemma 10.7 to obtain that

$$\prod_{w|v} |\operatorname{tr}_\lambda(\Pi_{n,w})|^2 \ll (q^n)^{(\dim G)(g+|D|-1)-\dim H}.$$

Let $n_0 := \gcd(n, [\kappa_v : k])$, and $n_1 := n/n_0$. All the places $w|v$ have isomorphic residue field κ_w, with $[\kappa_w : \kappa_v] = n_1$, and by the definition of base change, they have the same Hecke eigenvalue. So all of the n_0 terms in the above product are equal to each other, and we deduce

$$\operatorname{tr}_\lambda(\pi_{n,w}) \ll (q^{n_1})^{((\dim G)2(g+|D|-1)-\dim H)/2}.$$

Let $t(\pi_v)$ be the Satake parameter of π_v. Then the Satake parameter of $\pi_{n,w}$ is equal to $t(\pi_v)^{n_1}$, hence $\operatorname{tr}_\lambda(\pi_{n,w}) = \operatorname{tr}(t(\pi_v)^{n_1}|V_\lambda)$. Because all $n_1 \geq 1$ arise for some n (specifically for $n = [\kappa_v : k]n_1$), Lemma 2.21 implies that we have the inequality

$$|\operatorname{tr}(t(\pi_v)^{n_1}|V_\lambda)| \leq \dim V_\lambda \cdot (q^{n_1})^{((\dim G)2(g+|D|-1)-\dim H)/2},$$

in particular

$$|\operatorname{tr}_\lambda(\pi_v)| \leq \dim V_\lambda \cdot q^{((\dim G)2(g+|D|-1)-\dim H)/2}.$$

Since the inequality holds for every $\lambda \in \Lambda^+$, we deduce that in fact $|\operatorname{tr}_\lambda(\pi_v)| \leq \dim V_\lambda$, and π_v is tempered. □

Remark 10.9. A close analogue of the argument may be found in the Bombieri-Stepanov proof of the Riemann hypothesis for curves over finite fields. Weil's proof for a curve C of genus g over $\mathbb{F}_q$ immediately proves in one stroke the Riemann bound $|\#C(\mathbb{F}_q) - q - 1| \leq 2g\sqrt{q}$. The proof of Bombieri-Stepanov, say in the special case of a Galois cover of $\mathbb{P}^1$, involves more steps. One first deduces an estimate $\#C(\mathbb{F}_q) \leq 1 + q + (2g+1)\sqrt{q}$, then by applying this bound to twists of C curve, obtains $\#C(\mathbb{F}_q) - q - 1 \geq 1 + q - O((2g+1))\sqrt{q})$, with a constant depending on the order of the Galois group. To improve the constant from $O(2g+1)$ to the correct value $2g$, it is necessary to use the rationality of the L-function. From the estimate for $\#C(\mathbb{F}_{q^n})$ for n large, one deduces the sharp bound for the zeroes of the zeta function and thus a sharp bound for the number of points.

Our method closely follows the strategy of the last deduction. The role of the zeroes of the zeta function is played by the eigenvalues of the Satake parameter, and the role of the rationality of the zeta function in demonstrating that the counts of points over different finite fields are controlled by a single function can be compared with cyclic base change. The main difference is that, while the bound $(2g+1)\sqrt{q}$ is sufficient for most practical purposes, the constant factor which we amplify away is ineffective, and would render the estimate useless in the λ aspect if not dealt with.

Remark 10.10. We compare our use of the tensor power trick to Rankin's trick, and the closely related observation of Langlands that the Ramanujan bound is consequence of symmetric power functoriality. In both cases, some special case of functoriality is used to amplify a weaker bound into a stronger one. The needed functoriality is rather weak in our case, where it is cyclic base change. However, our argument and Rankin's trick are different in one crucial respect, other than the different versions of functoriality applied. Rankin's trick produces an improvement in the dependence on q in the bound. Speaking geometrically, we may refer to it as an improvement of the weight. In our method, however, the weight is fixed as q varies (unsurprising as it arises geometrically as the weight of a cohomology group), and is not improved directly. Instead, we pass to the large q^n limit to handle a constant term independent of q.

10.5. **Hecke eigenvalues are Weil numbers.** We establish the following strenghtening of the previous Theorem 10.8. Assumptions are as before.

Theorem 10.11. *For every $\lambda \in \Lambda^+$, the trace $q^{\langle\lambda,\rho\rangle}\operatorname{tr}_\lambda(\pi_v)$ of the λ-Hecke operator is a sum of length $\dim(V_\lambda)$ of q-Weil integers of weight $\langle\lambda, 2\rho\rangle$.*

Proof. Hecke eigenvalues are algebraic numbers because of the finiteness of the support of cuspidal automorphic functions with prescribed local conditions. Next we will prove that the Hecke eigenvalues have size $q^{\langle\lambda,\rho\rangle}$ for every embedding of the coefficient field into $\mathbb{C}$. Every embedding comes from another automorphic form satisfying the same assumptions, possibly with a different mgs datum. Indeed the local mgs condition at u is preserved under $\operatorname{Aut}(\mathbb{C})$, and also the global Condition BC. Thus the previous Theorem 10.8 applies. Finally the integrality follows either from [34, Prop. 2.1], or from Lemma 9.13 by varying $\lambda \in \Lambda^+$. ☐

Example 10.12. Consider the rigid automorphic sheaves constructed in [26, 54]. The Condition BC is satisfied because the trace function over each finite extension $\mathbb{F}_{q^n}$ defines an automorphic function that generates a corresponding automorphic representation (see Remark 6.11). We have seen in Section 3.5 that epipelagic representations are mgs. Thus Theorem 10.8 applies, and the temperedness is consistent with the results of *loc. cit.*, indeed the construction of ℓ-adic sheaves on $\mathbb{P}^1_{\backslash\{0,\infty\}}$ that generalizes Kloosterman sums. The conclusion of Theorem 10.11 on integrality is also consistent with *loc. cit.*, precisely, it follows from [26, (5.8)], which explicates Kl^{V_λ} as an exponential sum, and because each of the Kummer, Artin-Schreier, and IC sheaves is integral. This is analogous to Theorem 9.13.

11. Relationship with Lafforgue-Langlands parameters and Arthur parameters

In this section, we will describe a potential approach to provide a different proof of the main theorem of this paper, using V. Lafforgue's Langlands parameterization, the Lafforgue-Genestier semisimplified local Langlands parameterization, and some conjectural explicit calculations with that parameterization. We will then express the same strategy, or a very similar strategy, in the language of Arthur parameters, and again without direct reference to parameters of any kind, using only the notion of two representations being in the same L-packet.

The starting point of all three approaches will be a guess about the Langlands parameters of mgs representations. We can verify this conjecture in the GL_n case, where the local Langlands correspondence is known by results of Laumon–Rapoport–Stuhler, and Henniart–Lemaire [27].

Proposition 11.1. *Let F_u be a non-archimedean local field and let π_u be a mgs representation of $GL_n(F_u)$. Then its local Langlands parameter $\sigma_u : W_{F_u} \to GL_n(\overline{\mathbb{Q}}_\ell)$ is irreducible when restricted to the inertia group of F_u.*

Proof. For each unramified extension F'_u of F_u, let π'_u be the base change representation of π_u. It follows from [27, Prop. II.2.9], [27, Prop. II.5.15.2], and the orbital integral identity in Theorem 4.11 that π'_u is an mgs representation, with datum compatible with that of π_u. In particular π'_u is supercuspidal.

It is established in [27, Thm. IV.1.5] that the Langlands parameter of π'_u is the restriction of the Langlands parameter σ_u to $W_{F'_u}$. Since π'_u is supercuspidal, we have that σ_u restricts to an irreducible $W_{F'_u}$ representation.

Because $\sigma_u(I_{F_u})$ is a finite group, the action of $\sigma_u(\mathrm{Frob}_u)$ on it, by conjugation has finite order m. Let F'_u be an unramified extension of F_u of degree m. Then $\sigma_u(W_{F'_u})$ is generated by $\sigma_u(I_{F_u})$

and the mth power of Frob_u, which commutes with it. Hence $\sigma_u(\mathrm{Frob}_u^m)$ lies in the center of $\sigma_u(W_{F'_u})$, which acts irreducibly, so $\sigma_u(\mathrm{Frob}_u^m)$ is a scalar, and hence $\sigma_u(I_{F_u})$ acts irreducibly, as desired. □

To conjecturally apply this to general groups, and use it to verify Ramanujan, we use the work of V. Lafforgue and Genestier-Lafforgue on the Langlands correspondence over function fields, which we now review: Recall that D is a divisor on X, and $\mathbf{K}(D)$ is the compact subgroup of the adelic points of the split semisimple G consisting at each place of local sections of the group scheme congruent to the identity modulo D.

Lafforgue [35] defines a $\mathcal{C}_c(\mathbf{K}(D)\backslash G(\mathbb{A}_F)/\mathbf{K}(D))$-module decomposition of $\mathcal{C}^{cusp}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q),\overline{\mathbb{Q}}_\ell)$ indexed by continuous, semisimple representations $\sigma : \mathrm{Gal}(\overline{F}|F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, unramified away from D. Since $\pi^{\mathbf{K}(D)}$ is nonzero, it appears in this decomposition.

Letting ι be an embbedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$, we say a representation $\mathrm{Gal}(\overline{F}|F)$ is ι-pure of weight w if for each unramified place v, the image by ι of the eigenvalues of Frob_v on the representation are complex numbers of norm $|\kappa_u|^{\frac{w}{2}}$. We say that a representation is ι-mixed if it has a filtration whose associated graded components are ι-pure of increasing weights. All representations, σ appearing in this decomposition, composed with any representation of $\widehat{G}$, are ι-mixed. (In fact by a result of L. Lafforgue this is known for any representation, but it has a direct proof in this case.)

Genestier-Lafforgue [22] define for each local representation π_u a semisimple representation $\sigma_{\pi_u} : \mathrm{Gal}(\overline{F_u}|F_u) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, which satisfies the following compatibility condition: Whenever $\pi^{\mathbf{K}(D)}$ appears as an irreducible $\mathcal{C}_c(\mathbf{K}(D)\backslash G(\mathbb{A}_F)/\mathbf{K}(D))$-module inside the summand of $\mathcal{C}^{cusp}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q),\overline{\mathbb{Q}}_\ell)$ indexed by a representation σ, the semisimplification of the restriction of σ to $\mathrm{Gal}(\overline{F_u}|F_u)$ is equal to σ_{π_u}.

The key conjecture, which is expected to generalize Proposition 11.1, is as follows. In the case of an epipelagic representation π_u, it is consistent with the conjectures of [43, §7.1], in which the assertion is expressed in the form $\widehat{\mathfrak{g}}^{\sigma_{\pi_u}(I_{F_u})} = 0$.

Conjecture 11.2. *For π_u a mgs representation, the image of the inertia subgroup I_{F_u} of $\mathrm{Gal}(\overline{F_u}|F_u)$ under the parameter σ_{π_u} is not contained in any proper parabolic subgroup of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.*

It follows from this conjecture that, if π is mgs at one place, then π is tempered at all unramified places. This follows from the below chain of reasoning, which depends on the Lemmas 11.3,11.4, and 11.5 immediately afterwards.

(1) Assume that π_u is mgs.

Then, under Conjecture 11.2:

(2) The image of $\mathrm{Gal}(\overline{F_u}|F_u)$ in σ_{π_u} is not contained in any parabolic subgroup of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.

Thus we deduce:

(3) The image of $\mathrm{Gal}(\overline{F}|F)$ in every Lafforgue-Langlands parameter σ of π is not contained in any parabolic subgroup of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.
(4) The composition of every Lafforgue-Langlands parameter σ of π with every representation of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ is pure of weight 0.
(5) π is tempered at every unramified place.

Indeed the implication (2) $\implies$ (3) is Lemma 11.3, then (3) $\implies$ (4) is Lemma 11.4, and Lemma 11.5 gives (4) $\implies$ (5).

Lemma 11.3. *Let $\sigma : \mathrm{Gal}(\overline{F}|F) \to \widehat{G}(\mathbb{C})$ be a representation with image contained in a parabolic subgroup. Let v be a place. Then the image of the semisimplification of the restriction of σ to $\mathrm{Gal}(\overline{F}_u|F_u)$ is contained in a parabolic subgroup.*

Proof. That the property of being contained in a parabolic subgroup is stable under restriction is obvious. That it is preserved under semisimplification is immediate from the definition of semisimplification - we take a minimal parabolic subgroup containing the image of the representation, if any, and then project onto the Levi of that parabolic, and the semisimplification is independent of which minimal parabolic we take, so as long as some parabolic subgroup contains the image, some Levi subgroup contains the image of the semisimplification. ☐

Lemma 11.4. *Let $\sigma : \mathrm{Gal}(\overline{F}|F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ be a ι-mixed representation, and let $\rho : \widehat{G} \to GL_n$ be a representation. Then either the image of σ is contained in a parabolic subgroup or $\rho(\sigma)$ is pure of weight zero.*

Proof. Because $\rho(\sigma)$ is ι-mixed, it has a canonical filtration into pure representations. The image of σ is contained in the stabilizer of this filtration inside $\widehat{G}$. We will show that either this stabilizer is a parabolic subgroup of $\widehat{G}$ or $\rho(\sigma)$ is pure of weight zero.

Let v be a place at which σ is unramified and let T be a torus containing the semisimplication of Frob_u. Then the generalized eigenspaces of Frob_u are sums of eigenspaces of T. For any character of T, we can take the logarithm of the absolute value of ι applied to the eigenvalue of Frob_u on that character, which defines a linear function on the weight lattice of T. Because each associated graded of the weight filtration is pure of increasing weight, the eigenvalues of Frob_u on each associated graded all have the same absolute value, so each associated graded of the weight filtration is a sum of eigenspaces of T where this linear function takes a fixed value, and this value is increasing in the filtration. Thus an element preserves the weight filtration if and only if it sends eigenspaces of T to eigenspaces of T where the linear function takes equal or lower values on their weights.

This is exactly the subgroup of $\widehat{G}$ generated by all roots where this linear function takes a nonnnegative value on their weights. This subgroup is parabolic unless it contains every root, in which case this linear function is zero on all roots, which because $\widehat{G}$ is semisimple implies it is zero on all characters of T, so the representation is pure of weight zero. ☐

Lemma 11.5. *Let π be a representation of $G(\mathbb{A}_F)$ such that $\pi^{\mathbf{K}(D)}$ is nonzero and appears inside the summand of $\mathcal{C}^{cusp}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by a parameter σ such that $\rho \circ \sigma$ is ι-pure of weight zero for every representation ρ of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$. Then π is tempered at all unramified places.*

Proof. This follows from the compatibility between the action of $\mathcal{H}(G(F_v), G(\mathfrak{o}_v)$ on the summand of $\mathcal{C}^{cusp}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by σ and the conjugacy class of $\sigma(\mathrm{Frob}_v)$. ☐

We now sketch two, more conjectural, analogues of this argument. The first is based on Arthur parameters, and explains how we expect our main theorem can be related to Arthur's conjectures.

- (2') The image of the Weil group in the local Langlands parameter of π_u is not contained in any parabolic subgroup of $\widehat{G}$.
- (3') The image of the Weil group in every global Langlands parameter of π is not contained in any parabolic subgroup of $\widehat{G}$.
- (4') The image of SL_2 in every global Arthur parameter of π is trivial.

Here the implications (2') $\implies$ (3') $\implies$ (4') can be proved directly from the characterizing properties of Arthur and Langlands parameters. For the implication (3') $\implies$ (4'), the proof is similar to the proof of Lemma 11.4, but with a diagonal element in SL_2 replacing the Frobenius element. The implication (4') $\implies$ (5) is part of Arthur's conjectures on Arthur parameters, while the implication (1) $\implies$ (2') is a variant of Conjecture 11.2.

It is clear that if the Arthur-Lafforgue conjecture on the relationship of Lafforgue parameters with Arthur parameters could be proved, then this argument would be essentially the same as the previous argument.

The second analogue avoids mentioning parameters of any kind, except through their L-packets, and relies on conjectures only in terms of automorphic representations.

(2") All representations in the L-packet of π_u are supercuspidal representations.
(3") All global representations π' such that, for each place v, π_v and π'_v are in the same L-packet, are cuspidal.
(4") All global representations π' such that, for all but finitely many places, $\pi_u = \pi'_u$, are cuspidal.

The implication (4") $\implies$ (5) is a restatement of the conjecture that non-tempered cuspidal representations are CAP. The implications (2") $\implies$ (3") $\implies$ (4") are trivial, and the implication (1) $\implies$ (2") is again a variant of Conjecture 11.2.

Our method of proof of the main result is also purely automorphic, and in some respects follows this last strategy. Indeed assertion (4") is necessary to construct a spectral set $\mathcal{V}$, prescribed by local behavior containing π, which is obtained by projection from an automorphic kernel $K(x, y)$ of compact support. See the related discussion in §1.1. Assertions (2") and (3") appear implicitely in Condition BC, since the theory of base change and stabilization of trace formulas is related to the notion L-packet.

Remark 11.6. Many of the reverse implications are known or conjectured. In the Arthur parameter setting, (4') implies (3'), since discrete series representations should have elliptic Arthur parameters, meaning that the Weil group and SL_2 are never both contained in the same parabolic subgroup. The same statement is true in the Lafforgue parameter setting, conditional on the Arthur–Lafforgue conjecture that Lafforgue's parameters come from Arthur parameters. In every setting, (5) is known to imply (4) (resp. (4'), (4")). However (3) never implies (2) as global cuspidality-type conditions cannot imply local supercuspidality conditions. Hence it is not possible to prove the conjecture that (1) implies (2) as a corollary of our main result.

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Index of notation

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