Hyperbolic endomorphisms and overlap numbers

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Novembre 2018

IHES/M/18/10

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Abstract

Hyperbolic endomorphisms and overlap numbers for equilibrium measures μ_ψ are studied on lifts of invariant sets. We look into the structure of Rokhlin conditional measures of μ_ψ, with respect to various fiber partitions associated to the lift endomorphism Φ, and find relations between them. We prove an estimate on the box dimension of an invariant measure ν_ψ on the limit set, by using the overlap number of μ_ψ. Then we compute topological overlap numbers in several concrete cases. In particular, we obtain a large class of endomorphisms which asymptotically are irrational-to-1. Topological overlap numbers are then used in dimension estimates.

Mathematics Subject Classification 2000: 37D20, 37D35, 37A35, 37C70.

Keywords: Hyperbolic dynamics of endomorphisms; invariant sets; equilibrium measures; overlap numbers; stable foliations; fibers; entropies.

1 Introduction.

In this paper we give several formulas and applications of overlap numbers of equilibrium measures for lift endomorphisms over invariant sets. Overlap numbers were introduced in [7], and represent asymptotic averages of the numbers of generic consecutive preimages for these endomorphisms. In particular, topological overlap numbers give information about all the preimages.

Consider a finite system $\mathcal{S} = \{\phi_i, i \in I\}$, with $\phi_i : \bar{U} \to \mathbb{R}^d$ conformal and injective maps on the closure $\bar{U}$ of a bounded open set $U \subset \mathbb{R}^d$, and uniformly contracting on $\bar{U}$, i.e $\exists \gamma \in (0,1)$ with $|\phi_i'| < \gamma, i \in I$. Let Σ_I^+ be the one-sided symbolic space $\{\omega = (\omega_1, \omega_2, \ldots), \omega_i \in I, i \geq 1\}$, with canonical metric and topology, and with the shift map $\sigma : \Sigma_I^+ \to \Sigma_I^+$. Let $[\omega_1 \ldots \omega_n]$ be the cylinder $\{\eta \in \Sigma_I^+, \eta_1 = \omega_1, \ldots, \eta_n = \omega_n\}$. In general denote by $\phi_{i_1 \ldots i_p} := \phi_{i_1} \circ \phi_{i_2} \circ \ldots \circ \phi_{i_p}$ for $p \geq 1, i_1, \ldots, i_p \in I$, and by $\phi_{i_1 i_2 \ldots}$ the point given as intersection of the descending sets $\phi_{i_1 \ldots i_p}(U)$, when $p \to \infty$. We denote by Λ the set of all points of type $\phi_{i_1 i_2 \ldots}$, and call it the *limit set* of $\mathcal{S}$; so,

$$\Lambda = \pi(\Sigma_I^+),$$

where $\pi : \Sigma_I^+ \to \Lambda$, $\pi(\omega) = \phi_{\omega_1 \omega_2 \ldots}$, $\omega \in \Sigma_I^+$, is the canonical projection to the limit set. Then consider the following skew product map (see [7]), which we call *the lift endomorphism* of $\mathcal{S}$,

$$\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda, \ \Phi(\omega, x) = (\sigma\omega, \phi_{\omega_1}(x)) \text{ for } (\omega, x) \in \Sigma_I^+ \times \Lambda$$

In general, for any $n \geq 1$, the n-th iterate of Φ looks like:

$$\Phi^n(\omega, x) = (\sigma^n(\omega), \phi_{\omega_n \ldots \omega_1}(x)), \ (\omega, x) \in \Sigma_I^+ \times \Lambda$$

Due to the expansion of σ and the contraction of $\phi_i, i \in I$, the skew product endomorphism Φ has a hyperbolic behaviour and a stable foliation with leaves $\{\omega\} \times \Lambda$, $\omega \in \Sigma_I^+$. Consider now a Hölder continuous potential $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$. Then there exists a unique *equilibrium measure* μ_ψ on $\Sigma_I^+ \times \Lambda$, (for eg [1], [4], [15]), i.e if $P_\Phi : \mathcal{C}(\Sigma_I^+ \times \Lambda, \mathbb{R}) \longrightarrow \mathbb{R}$ is the pressure functional for Φ on $\Sigma_I^+ \times \Lambda$, then μ_ψ is the unique Φ-invariant probability for which

$$P_\Phi(\psi) = h_\Phi(\mu_\psi) + \int_{\Sigma_I^+ \times \Lambda} \psi \, d\mu_\psi = \sup\{h_\Phi(\mu) + \int_{\Sigma_I^+ \times \Lambda} \psi \, d\mu, \ \mu \ \Phi-\text{invariant probability on } \Sigma_I^+ \times \Lambda\},$$

where $h_\Phi(\mu)$ is the measure-theoretic entropy of μ with respect to Φ. Denote also by

$$\nu_\psi := \pi_{2*}\mu_\psi,$$

the projection of μ_ψ on the second coordinate. Then ν_ψ is a probability measure on the limit set Λ, and we want to study the metric properties of this measure. Notice that, in general, ν_ψ is not equal to the classical projection $\pi_*(\pi_{1*}(\mu_\psi))$ of the measure μ_ψ from $\Sigma_I^+ \times \Lambda$ to the limit set Λ.

For a Φ-invariant probability measure μ on $\Sigma_I^+ \times \Lambda$, we define its *Lyapunov exponent* by,

$$\chi(\mu) = \int_{\Sigma_I^+ \times \Lambda} -\log |\phi'_{\omega_1}(x)| \, d\mu(\omega, x) \ > 0$$

Notice that since the skew product Φ is contracting in the second coordinate, the entropy of μ is actually equal to the entropy of its projection on the first coordinate, $h_\Phi(\mu) = h_\sigma(\pi_{1*}\mu)$.

In [14], [13] Ruelle introduced a notion of folding entropy of a measure μ, denoted in our case by $F_\Phi(\mu)$, defined as the conditional entropy $H_\mu(\epsilon|\Phi^{-1}\epsilon)$. If $\Phi^{-1}(\epsilon)$ is the measurable partition of $\Sigma_I^+ \times \Lambda$ with the fibers of Φ, and if μ is an Φ-invariant probability measure on $\Sigma_I^+ \times \Lambda$, then we obtain a system of conditional measures of μ denoted by $(\mu_{(\omega,x)})_{(\omega,x)\in\Sigma_I^+\times\Lambda}$, where $\mu_{(\omega,x)}$ is a probability supported on the finite fiber $\Phi^{-1}(\omega, x)$.

Also recall that the *Jacobian* of an invariant measure introduced in [8], as the local Radon-Nikodym derivative of the push-forward with respect to the measure. If μ is a Φ-invariant measure on $\Sigma_I^+ \times \Lambda$, then we denote by $J_\Phi(\mu)$ its Jacobian; from definition, $J_\Phi(\mu) \geq 1$. In our case,

$$F_\Phi(\mu_\psi) = \int_{\Sigma_I^+ \times \Lambda} \log J_\Phi(\mu)(\omega, x) \, d\mu(\omega, x) \tag{1}$$

In [7] we introduced a notion of *overlap number* $o(\mathcal{S}, \mu_\psi)$ for an equilibrium measure μ_ψ of a Hölder continuous potential on the lift $\Sigma_I^+ \times \Lambda$. This overlap number is an average asymptotic number of the μ_ψ-generic preimages in Λ (since the points in Λ can be covered multiple times by the images $\phi_{i_1 \ldots i_m}(\Lambda)$ if the system $\mathcal{S}$ has overlaps). More precisely, for an arbitrary number $\tau > 0$ denote the set of μ_ψ-generic preimages having the same iterates as (ω, x) by

$$\Delta_n((\omega, x), \tau, \mu_\psi) = \{(\eta_1, \ldots, \eta_n) \in I^n, \exists y \in \Lambda, \phi_{\omega_n \ldots \omega_1}(x) = \phi_{\eta_n \ldots \eta_1}(y), \ |\frac{S_n\psi(\eta, y)}{n} - \int_{\Sigma_I^+ \times \Lambda} \psi \, d\mu_\psi| < \tau\},$$

where $(\omega, x) \in \Sigma_I^+ \times \Lambda$ and $S_n\psi(\eta, y)$ is the consecutive sum of ψ with respect to Φ. Denote by

$$b_n((\omega, x), \tau, \mu_\psi) := Card \Delta_n((\omega, x), \tau, \mu_\psi)$$

Then, in [7] we proved that the following limit exists and defines the *overlap number* of μ_ψ,

$$o(\mathcal{S}, \mu_\psi) = \lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_I^+ \times \Lambda} \log b_n((\omega, x), \tau, \mu_\psi) \, d\mu_\psi(\omega, x)$$

A particular case is the *topological overlap number* $o(\mathcal{S}) := o(\mathcal{S}, \mu_{max})$, which gives information about *all the preimages*, since all preimages are generic with respect to the measure of maximal entropy μ_{max} for Φ on $\Sigma_I^+ \times \Lambda$.

In [7] we proved a connection between the overlap number and the folding entropy of μ_ψ, namely

$$o(\mathcal{S}, \mu_\psi) = \exp(F_\Phi(\mu_\psi)) \tag{2}$$

We found the following estimate for the Hausdorff dimension of the projection $\nu_\psi = \pi_{2*}\mu_\psi$, where $\pi_2 : \Sigma_I^+ \times \Lambda \to \Lambda$, $\pi_2(\omega, x) = x$. The measure ν_ψ is not usually equal to the other projection $\pi_*(\pi_{1*}(\mu_\psi))$ of μ_ψ from $\Sigma_I^+ \times \Lambda$ onto the limit set Λ.

Theorem ([7]). *If $\mathcal{S}$ is a finite conformal iterated function system as above, and if $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$ is Hölder continuous with its equilibrium measure μ_ψ, and if $\nu_\psi := \pi_{2*}(\mu_\psi)$, then*

$$HD(\nu_\psi) \leq t(\mathcal{S}, \psi),$$

where $t(\mathcal{S}, \psi)$ is the unique zero of the pressure $t \to P_\sigma(t \log |\phi_{\omega_1}(\pi(\sigma\omega))| - \log o(\mathcal{S}, \mu_\psi))$.

In the sequel, we first compare in Theorem 1 the conditional measures obtained from μ_ψ by taking certain special measurable partitions of the skew product Φ into fibers. We apply this to a formula for overlap numbers, by using families of conditional measures on fibers (which may be easier to study).

Then, in Theorem 2 we find an upper bound for the lower box dimension of ν_ψ, with the help of the overlap number of μ_ψ, and using the Bounded Distortion Property for conformal systems of contractions. We give a *constructive* method to find a set of large ν_ψ-measure in Λ whose lower box dimension is bounded with the help of overlap numbers, namely is less than

$$\frac{h_\sigma(\pi_{1*}(\mu_\psi)) - \log o(\mathcal{S}, \mu_\psi))}{\chi(\mu_\psi)}$$

This is done by careful estimates of the proportion of the measure of generic points within the measure of balls, and by studying Jacobians of iterates and consecutive preimages of sets.

Then, in Theorems 3 and 4, we compute/estimate topological overlap numbers in several concrete cases, namely for Bernoulli convolution systems associated to reciprocals of Garsia and Pisot numbers ([3], [10]). If $\frac{1}{\lambda}$ is a Garsia number and $\lambda \in (\frac{1}{2}, 1)$, we prove that the topological overlap number $o(\mathcal{S}_\lambda)$ is equal to 2λ. In particular, we obtain endomorphisms which, surprisingly, are *asymptotically irrational-to-1* on their invariant sets. More precisely, for any $n \geq 1$, we obtain systems which asymptotically are $\sqrt[n]{2^{n-1}}$ - to - 1.
Also, in Proposition 1, and Corollaries 2, 3 and 4, we compute topological overlap numbers for general systems with eventual exact (or partial) overlaps. The above results are applied in Corollaries 5 and 6 to give dimension estimates for projection measures.

2 Various conditional measures associated to the lift.

We now study several families of conditional measures associated to the lift Φ and to the equilibrium state $\mu := \mu_\psi$ and various fiber partitions. We look at the relations between them, and find in particular a formula for the folding entropy.

Thus let the following measurable partitions:

i) Consider the skew product map $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$ and its fibers $\Phi^{-1}(\omega, x)$ for $(\omega, x) \in \Sigma_I^+ \times \Lambda$. They form a partition which is clearly measurable, and according to Rokhlin ([12]) there exists a canonical family of conditional measures of $\mu := \mu_\psi$, so for μ-a.e $(\omega, x) \in \Sigma_I^+ \times \Lambda$, the conditional measure $\mu_{(\omega,x)}$ is supported on the finite set $\Phi^{-1}(\omega, x)$. Notice that

$$\Phi^{-1}(\omega, x) = \{(i\omega, \phi_i^{-1}x),\ i \in I,\ \text{if } x \in \phi_i(\Lambda)\},$$

where we denote $\mu_{(\omega,x)}(i) := \mu_{(\omega,x)}(i\omega, \phi_i^{-1}x)$ if $x \in \phi_i(\Lambda)$, and $\mu_{(\omega,x)}(i\omega, \phi_i^{-1}(x)) = 0$ if $x \notin \phi_i(\Lambda)$.

ii) Denote by $\mu^+ := \pi_{1*}\mu$ on Σ_I^+, where $\pi_1 : \Sigma_I^+ \times \Lambda \to \Sigma_I^+$ is the projection on the first coordinate, $\pi_1(\omega, x) = \omega$. Consider the partition of Σ_I^+ with the fibers of σ, and the associated family of conditional measures μ_ω^+ on the finite set $\sigma^{-1}\omega$, for μ^+-a.e $\omega \in \Sigma_I^+$. We also denote $\mu_\omega^+(i\omega)$ by $\mu_\omega^+(i)$.

iii) Consider the partition of $\Sigma_I^+ \times \Lambda$ with the fibers of $\pi_1 : \Sigma_I^+ \times \Lambda \to \Sigma_I^+$, i.e the partition with the leaves $\{\omega\} \times \Lambda$, $\omega \in \Sigma_I^+$, of the stable foliation of Φ. Let then the associated family of conditional measures of μ with respect to this partition, μ_ω on $\pi_1^{-1}(\omega) = \{\omega\} \times \Lambda$, for μ-a.e $\omega \in \Sigma_I^+$. So μ_ω can be considered a probability measure on Λ.

From [7] we know that, for an equilibrium measure μ_ψ on $\Sigma_I^+ \times \Lambda$, the overlap number is

$$o(\mathcal{S}, \mu_\psi) = \exp(F_\Phi(\mu_\psi))$$

We prove now a formula, which gives $F_\Phi(\mu)$ (and thus $o(\mathcal{S}, \mu_\psi)$) in terms of the conditional measures μ_ω and μ_ω^+:

Theorem 1. *Let $\Phi : \Sigma_I^+ \times \Lambda \to \Sigma_I^+ \times \Lambda$ be the lift endomorphism as above, and let a Hölder continuous potential ψ on $\Sigma_I^+ \times \Lambda$ with equilibrium measure $\mu := \mu_\psi$. Then the overlap number $o(\mathcal{S}, \mu)$ of μ_ψ is determined by the corresponding conditional families $(\mu_\omega)_\omega, (\mu_\omega^+)_\omega$ by:*

$$\log o(\mathcal{S}, \mu) = -\sum_{i \in I} \int_{\Sigma_I^+ \times \Lambda} \frac{\mu_\omega^+(i)}{\sum\limits_{j \in I} \mu_\omega^+(j) \cdot \lim\limits_{A_2 \to x} \frac{\mu_{j\omega}(\phi_j^{-1}\phi_i A_2)}{\mu_{i\omega}(A_2)}} \cdot \log \Big(\frac{\mu_\omega^+(i)}{\sum\limits_{j \in I} \mu_\omega^+(j) \cdot \lim\limits_{A_2 \to x} \frac{\mu_{j\omega}(\phi_j^{-1}\phi_i A_2)}{\mu_{i\omega}(A_2)}} \Big)\ d\mu(\omega, x)$$

Proof. From the properties of conditional measures, if $\tilde{g} : \Sigma_I^+ \times \Lambda \to \mathbb{R}$ is μ-integrable, then

$$\begin{aligned} \int_{\Sigma_I^+ \times \Lambda} \tilde{g}(\omega, x)\ d\mu(\omega, x) &= \int_{\Sigma_I^+ \times \Lambda} \int_{\Phi^{-1}(\omega,x)} \tilde{g}(\omega', x') d\mu_{(\omega,x)}(\omega', x')\ d\mu(\omega, x) \\ &= \sum_{i \in I} \int_{\Sigma_I^+ \times \Lambda} \tilde{g}(i\omega, \phi_i^{-1}x) \cdot \mu_{(\omega,x)}(i)\ d\mu(\omega, x) \end{aligned} \tag{3}$$

Notice that since our system has overlaps, a point $x \in \Lambda$ may belong to several sets of type $\phi_i(\Lambda)$. But μ also decomposes after the fibers of π_1, so for any real-valued function $\tilde{g}$ μ-integrable on $\Sigma_I^+ \times \Lambda$,

$$\begin{aligned}\int_{\Sigma_I^+ \times \Lambda} \tilde{g}(\omega, x)\, d\mu(\omega, x) &= \int_{\Sigma_I^+} \int_{\{\omega\}\times\Lambda} \tilde{g}(\omega, x) d\mu_\omega(x)\, d\mu^+(\omega) = \int_{\Sigma_I^+} \Gamma(\omega) d\mu^+(\omega) \\ &= \int_{\Sigma_I^+} \int_{\sigma^{-1}\omega} \Gamma(\omega') d\mu_\omega^+(\omega')\, d\mu^+(\omega) = \sum_{i\in I} \int_{\Sigma_I^+} \Gamma(i\omega) \mu_\omega^+(i)\, d\mu^+(\omega) \\ &= \sum_{i \in I} \int_{\Sigma_I^+ \times \Lambda} \mu_\omega^+(i) \cdot \int_{\{i\omega\}\times\Lambda} \tilde{g}(i\omega, x)\, d\mu_{i\omega}(x)\, d\mu(\omega, x),\end{aligned} \tag{4}$$

where $\Gamma(\omega) := \int_{\{\omega\}\times\Lambda} \tilde{g}(\omega, x) d\mu_\omega(x)$. By taking $\tilde{g}$ such that $\tilde{g}|_{[j]} = 0$ for $j \neq i$, we obtain from (3) and (4) that:

$$\int_{\Sigma_I^+ \times \Lambda} \tilde{g}(i\omega, \phi_i^{-1} x) \mu_{(\omega,x)}(i)\, d\mu(\omega, x) = \int_{\Sigma_I^+ \times \Lambda} \mu_\omega^+(i) \cdot \int_{\{i\omega\}\times\Lambda} \tilde{g}(i\omega, x) d\mu_{i\omega}(x)\, d\mu(\omega, x) \tag{5}$$

Let us take now $\tilde{g} = \chi_A$, where $A = A_1 \times A_2$ is the product of two Borelian sets, and $A_1 \subset [i] \subset \Sigma_I^+$. Then if $i\omega \in A_1$, we have

$$\int_{\{i\omega\}\times\Lambda} \tilde{g}(i\omega, x) d\mu_{i\omega}(x) = \mu_{i\omega}(A_2)$$

Let us denote $A_1(i) := \{\omega \in \Sigma_I^+,\ i\omega \in A_1\}$. Thus, with the above choice of $\tilde{g}$,

$$\int_{\Sigma_I^+ \times \Lambda} \tilde{g}(i\omega, \phi_i^{-1} x) \mu_{(\omega,x)}(i)\, d\mu(\omega, x) = \int_{A_1(i)\times\phi_i(A_2)} \mu_{(\omega,x)}(i)\, d\mu(\omega, x)$$

So from the last two displayed equalities and (5), it follows that

$$\int_{A_1(i)\times\phi_i(A_2)} \mu_{(\omega,x)}(i)\, d\mu(\omega, x) = \int_{A_1(i)\times\Lambda} \mu_{i\omega}(A_2) \cdot \mu_\omega^+(i)\, d\mu(\omega, x) \tag{6}$$

Since $\mu = \mu_\psi$ is the equilibrium measure of a Hölder continuous potential, and since the Bowen balls in $\Sigma_I^+ \times \Lambda$ are of type $[\omega_1 \dots \omega_n] \times B(x, r_0)$, it follows that μ^+ is a doubling measure on Σ_I^+. Hence from Borel Density Lemma ([9]), if $A_1(i)$ is a ball around some fixed $\bar{\omega}$ in Σ_I^+, we obtain:

$$\begin{aligned}\frac{1}{\mu^+(A_1(i))} \int_{A_1(i)} \int_{\phi_i A_2} \mu_{(\omega,x)}(i) d\mu_\omega(x)\, d\mu(\omega) &= \frac{1}{\mu^+(A_1(i))} \int_{A_1(i)\times\phi_i A_2} \mu_{(\omega,x)}(i)\, d\mu(\omega, x) \\ &\underset{A_1(i)\to\bar{\omega}}{\longrightarrow} \int_{\phi_i(A_2)} \mu_{(\bar{\omega},x)}(i)\, d\mu_{\bar{\omega}}(x)\end{aligned} \tag{7}$$

But $\int_{A_1(i)\times\Lambda} \mu_{i\omega}(A_2) \cdot \mu_\omega^+(i) d\mu(\omega, x) = \int_{A_1(i)} \mu_{i\omega}(A_2) \mu_\omega^+(i)\, d\mu^+(\omega)$. So from Borel Density Lemma,

$$\frac{1}{\mu^+(A_1(i))} \int_{A_1(i)} \mu_{i\omega}(A_2) \cdot \mu_\omega^+(i)\, d\mu^+(\omega) \underset{A_1(i)\to\bar{\omega}}{\longrightarrow} \mu_{i\bar{\omega}}(A_2) \cdot \mu_{\bar{\omega}}^+(i),$$

for μ^+-a.e $\bar{\omega} \in \Sigma_I^+$. Therefore from (6) and (7) it follows that, for μ^+-a.e $\bar{\omega} \in \Sigma_I^+$,

$$\int_{\phi_i(A_2)} \mu_{(\bar{\omega},x)}(i)\, d\mu_{\bar{\omega}}(x) = \mu_{i\bar{\omega}}(A_2) \cdot \mu_{\bar{\omega}}^+(i) \tag{8}$$

On the other hand from the Φ-invariance of μ, it follows that

$$\int_{\Sigma_I^+\times\Lambda} \tilde{g}(\omega,x)\ d\mu(\omega,x) = \int_{\Sigma_I^+\times\Lambda} \tilde{g}\circ\Phi(\omega,x) d\mu(\omega,x) = \int_{\Sigma_I^+\times\Lambda} \tilde{g}(\sigma\omega,\phi_{\omega_1}x) d\mu(\omega,x)$$

Hence using the conditional decomposition of μ along the fibers of π_1,

$$\int_{\Sigma_I^+}\int_{\{\omega\}\times\Lambda} \tilde{g}(\omega,x) d\mu_\omega(x)\ d\mu^+(\omega) = \int_{\Sigma_I^+}\int_{\{\omega\}\times\Lambda} \tilde{g}(\sigma\omega,\phi_{\omega_1}x) d\mu_\omega(x)\ d\mu^+(\omega)$$

Let us take now again $\tilde{g} = \chi_{A_1\times A_2}$, and notice that $\sigma\omega \in A_1$ and $\phi_{\omega_1}x \in A_2$, if and only if $\omega\in\sigma^{-1}A_1$ and $x\in\phi_{\omega_1}^{-1}A_2$. So from above,

$$\int_{A_1} \mu_\omega(A_2)\ d\mu^+(\omega) = \int_{\sigma^{-1}A_1} \mu_\omega(\phi_{\omega_1}^{-1}A_2)\ d\mu^+(\omega)$$

Since μ^+ is σ-invariant on Σ_I^+, it follows then that:

$$\int_{\sigma^{-1}A_1} \mu_{\sigma\omega}(A_2)\ d\mu^+(\omega) = \int_{\sigma^{-1}A_1} \mu_\omega(\phi_{\omega_1}^{-1}A_2)\ d\mu^+(\omega)$$

Taking $A_1\to\omega$, we obtain from above that, for any Borelian set $A_2\subset\Lambda$, $i\in I$ and μ^+-a.e $\omega\in\Sigma_I^+$,

$$\mu_\omega(\phi_i A_2) = \sum_{j\in I} \mu_{j\omega}(\phi_j^{-1}\phi_i(A_2))\cdot\mu_\omega^+(j) \tag{9}$$

But we can apply Borel Density Lemma for the measure $\phi_*\mu_\omega$ on $\phi_i(\Lambda)$ in (8), and we see that for any $x\in\phi_i(\Lambda)$ and any $r>0$ small, $B(x,r)\cap\phi_i\Lambda = \phi_i(B(\phi_i^{-1}x,r')\cap\Lambda)$ for some $r'>0$ since ϕ_i is injective. Thus by taking A_2 to be a neighbourhood of x, we obtain from (7), (8), (9), that $\lim\limits_{A_2\to x}\frac{\mu_{j\omega}(\phi_j^{-1}\phi_i A_2)}{\mu_{i\omega}(A_2)}$ exist, and that for μ-a.e $(\omega,x)\in\Sigma_I^+\times\Lambda$ and any $i\in I$,

$$\mu_{(\omega,x)}(i) = \frac{\mu_\omega^+(i)}{\sum\limits_{j\in I}\mu_\omega^+(j)\cdot\lim\limits_{A_2\to x}\frac{\mu_{j\omega}(\phi_j^{-1}\phi_i A_2)}{\mu_{i\omega}(A_2)}} \tag{10}$$

So from (10) and the fact that $F_\Phi(\mu) = -\int_{\Sigma_I^+\times\Lambda}\mu_{(\omega,x)}\log\mu_{(\omega,x)}d\mu(\omega,x)$, we obtain the formula for the folding entropy $F_\Phi(\mu)$, and thus from (27) the formula for the overlap number $o(\mathcal{S},\mu)$. □

3 Box dimension estimates.

The notions of lower box dimension and Hausdorff dimension for sets are well-known (for eg [2], [9]). For μ a Borel finite measure on $\mathbb{R}^d$, recall ([2], [9], [11]) that the *lower box dimension* of μ is:

$$\underline{dim}_B(\mu) = \lim_{\delta\to 0}\inf\{\underline{dim}_B(Z), \mu(Z)\geq 1-\delta\}$$

Also denote the Hausdorff dimension of μ by $HD(\mu)$. The following inequality holds,

$$HD(\mu)\leq\underline{dim}_B(\mu)$$

Various aspects of dimensions were studied for eg in [2], [5], [6], [9], [10], [11], [16]. We are now ready to prove the estimate for the lower box dimension of the projection $\nu_\psi := \pi_{2*}(\mu_\psi)$; recall that ν_ψ is *not* the usual projection measure $\pi_*\pi_{1*}\mu_\psi$. The next Theorem gives a *constructive method* to obtain sets Z of large ν_ψ-measure whose box dimensions will then be estimated using overlap numbers. In the process, we estimate also the maximal number of disjoint balls of radius r with centers in Z.

Theorem 2. *Consider the conformal system $\mathcal{S} = \{\phi_i, i \in I\}$ wih limit set Λ, and the Hölder continuous potential $\psi : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, with its equilibrium measure μ_ψ, and let $\nu_\psi := \pi_{2*}\mu_\psi$. Then,*

$$\underline{dim}_B(\nu_\psi) \leq \frac{h_\sigma(\pi_{1*}(\mu_\psi)) - \log o(\mathcal{S}, \mu_\psi))}{\chi(\mu_\psi)}$$

Proof. Recall that by definition, $\chi(\mu_\psi) > 0$. For $n \geq 1$, let $S_n\psi(\omega, x) := \psi(\omega, x) + \psi(\Phi(\omega, x)) + \ldots + \psi(\Phi^{n-1}(\omega, x))$. For all $(\omega, x) \in \Sigma_I^+ \times \Lambda$, $\Phi^n(\omega, x) = (\sigma^n(\omega), \phi_{\omega_n \ldots \omega_1}(x))$. From Chain Rule, $J_{\Phi^n}(\mu_\psi)(\omega, x) = J_\Phi(\mu_\psi)(\omega, x) \ldots J_\Phi(\mu_\psi)(\Phi^{n-1}(\omega, x))$. We know from the Birkhoff Ergodic Theorem, from the formula for folding entropy (1) and the fact that μ_ψ is ergodic that,

$$\frac{1}{n} \log |\phi'_{\omega_n \ldots \omega_1}(x)| \underset{n\to\infty}{\to} \int_{\Sigma_I^+ \times \Lambda} \log |\phi'_{\omega_1}(x)| d\mu_\psi(\omega, x), \text{ and } \frac{1}{n} \log J_{\Phi^n}(\mu_\psi)(\omega, x) \underset{n\to\infty}{\to} F_\Phi(\mu_\psi), \text{ and}$$

$$\frac{1}{n} S_n\psi(\omega, x) \underset{n\to\infty}{\to} \int_{\Sigma_I^+ \times \Lambda} \psi(\omega, x) d\mu_\psi(\omega, x)$$

For an integer $n \geq 1$ and an arbitrary number $\tau > 0$, consider therefore the Borelian set

$$D_n(\tau) := \{(\omega, x) \in \Sigma_I^+ \times \Lambda, \text{with } |\frac{1}{p} \log J_{\Phi^p}(\mu_\psi)(\omega, x) - F_\Phi(\mu_\psi)| < \tau, \text{ and}$$
$$|\frac{1}{p} \log |\phi'_{\omega_p \ldots \omega_1}(x)| - \int \log |\phi'_{\omega_1}|(x) d\mu_\psi(\omega, x)| < \tau, \ \ |\frac{1}{p} S_p\psi(\omega, x) - \int \psi d\mu_\psi| < \tau, \ \forall p \geq n\}$$

From above, $\mu_\psi(D_n(\tau)) \underset{n\to\infty}{\to} 1$ for all $\tau > 0$, and moreover,

$$D_1(\tau) \subset \ldots \subset D_n(\tau) \subset D_{n+1}(\tau) \subset \ldots \tag{11}$$

On the other hand, notice that a Bowen ball in $\Sigma_I^+ \times \Lambda$ has the form $[\omega_1 \ldots \omega_n] \times B(x, r_0)$, and from the estimates of equilibrium measures on Bowen balls (for eg [4]), we have:

$$\mu_\psi([\omega_1 \ldots \omega_n] \times B(x, r_0)) \approx \exp(S_n\psi(\omega, x) - nP_\Phi(\psi)), n \geq 1,$$

where $\approx$ means that the two quantities are comparable with a comparability constant which depends only on ψ and is independent of n, x, ω. Now, if $\omega' \in [\omega_1 \ldots \omega_n]$ and if $(\eta, y) \in \Phi^{-n}\Phi^n(\omega, x)$, then $(\eta, y) \in \Phi^{-n}\Phi^n(\omega', x)$, and viceversa. But we proved in [7] that for μ_ψ-a.e $(\omega, x) \in \Sigma_I^+ \times \Lambda$,

$$J_{\Phi^n}(\mu_\psi)(\omega, x) \approx \frac{\sum_{(\eta,y) \in \Phi^{-n}\Phi^n(\omega,x)} e^{S_n\psi(\eta,y)}}{e^{S_n\psi(\omega,x)}}, \tag{12}$$

with comparability constant independent of ω, x, n. Therefore, if $\omega' \in [\omega_1 \ldots \omega_n]$, it follows from (12) that there exists a constant $C > 0$ so that for μ_ψ-a.e (ω, x) and all $n \geq 1$,

$$\frac{1}{C} J_{\Phi^n}(\mu_\psi)(\omega', x) \le J_{\Phi^n}(\mu_\psi)(\omega, x) \le C J_{\Phi^n}(\mu_\psi)(\omega', x) \tag{13}$$

This means that $D_n(\tau)$ is basically a product set, or more precisely that there exists a set $E_n(\tau) \subset \Lambda$ such that $D_n(\tau/2) \subset [\omega_1 \ldots \omega_n] \times E_n(\tau) \subset D_n(\tau)$. Notice now that the map Φ^n is injective on the set $[\omega_1 \ldots \omega_n] \times B(x, r_0)$, for some fixed r_0, since the composition map $\phi_{\omega_n \ldots \omega_1}$ is injective on U. Thus from the properties of Jacobians of measures on sets of injectivity, we get

$$\begin{aligned} \mu_\psi(\Phi^n([\omega_1 \ldots \omega_n] \times B(x, r_0) \cap D_n(\tau))) &= \int_{[\omega_1 \ldots \omega_n] \times B(x, r_0) \cap D_n(\tau)} J_{\Phi^n}(\mu_\psi)(\eta, y) \, d\mu_\psi(\eta, y) \\ &\ge C e^{n(F_\Phi(\mu_\psi) - \tau)} \cdot \mu_\psi\big([\omega_1 \ldots \omega_n] \times B(x, r_0) \cap D_n(\tau)\big) \end{aligned} \tag{14}$$

We now want to estimate $\mu_\psi\big([\omega_1 \ldots \omega_n] \times B(x, r_0) \cap D_n(\tau)\big)$. First notice that, since ψ is Hölder continuous, the consecutive sum $S_n\psi(\omega, x)$ with respect to Φ, does not really depend on x, but only on ω. So there exists a constant $C > 0$ such that for any $x, y \in \Lambda$, $\omega \in \Sigma_I^+$,

$$|S_n\psi(\omega, x) - S_n\psi(\omega, y)| \le C$$

Thus one can fix $y = x_0$ above in Λ. We want to show that for any Borel set $A \subset \Lambda$ and any n,

$$\mu_\psi([\omega_1 \ldots \omega_n] \times A) \approx e^{S_n\psi(\omega, x_0) - nP_\Phi(\psi)} \cdot \nu_\psi(A), \tag{15}$$

with comparability constants independent of ω, n, A. Since μ_ψ is a Borel measure, it is enough to show (15) for open balls $A = B(y, r)$. Let us also recall that all the contractions ϕ_i are conformal, thus we have a Bounded Distortion property, namely there exist constants $C > 0, 0 < r_0 < 1$, such that for any $x, y \in \Lambda$ with $d(x, y) < r_0$, any integer n and any sequence $\underline{i} \in \Sigma_I^+$,

$$C^{-1}\phi'_{i_1 \ldots i_n}(x) \le \phi'_{i_1 \ldots i_n}(y) \le C\phi'_{i_1 \ldots i_n}(x) \tag{16}$$

By using this Bounded Distortion property, we take the backward iterates of $B(y, r)$ along various prehistories until reaching diameter r_0 for the respective preimage. We consider such maximal prehistories with $\phi^{-1}_{\eta_1 \ldots \eta_p} B(y, r) = B(z(y, \eta_1, \ldots, \eta_p), r_0)$, for some preimage in Λ denoted by $z(y, \eta_1, \ldots, \eta_p)$. We do this for all prehistories of order p, then if one of the preimages $\phi^{-1}_{\eta_1 \ldots \eta_p} B(y, r)$ has diameter smaller than r_0, we continue the process to order $p + q$, where q is the largest integer such that $\phi^{-1}_{\eta_1 \ldots \eta_p \ldots \eta_{p+q}} B(y, r)$ has diameter smaller than r_0. Assume without loss of generality that $\eta_1, \ldots, \eta_p$ is such a maximal trajectory (this can happen for various p's). Since μ_ψ is Φ-invariant, we can add succesively the measures of the maximal preimages $[\eta_p \ldots \eta_1] \times \phi^{-1}_{\eta_1 \ldots \eta_p} B(y, r)$, first those with $p = 1$, then with $p = 2$, etc. These measures sum up to the measure of $\Sigma_I^+ \times B(y, r)$, so

$$\sum_{p \ge 1,\ \eta_1 \ldots \eta_p\ maximal} \mu_\psi\big([\eta_p \ldots \eta_1] \times B(z(\eta_1, \ldots, \eta_p), r_0)\big) = \mu_\psi(\Sigma_I^+ \times B(y, r)) = \nu_\psi(B(y, r)) \tag{17}$$

We get a similar formula for $\mu_\psi([\eta_p \ldots \eta_1\omega_1 \ldots \omega_n] \times B(y, r))$, since the pulling back along prehistories of $B(y, r)$ does not depend on $\omega_1, \ldots, \omega_n$. Hence by using maximal prehistories as above,

$$\sum_{p \ge 1,\ \eta_1 \ldots \eta_p\ maximal} \mu_\psi\big([\eta_p \ldots \eta_1\omega_1 \ldots \omega_n] \times B(z(\eta_1, \ldots, \eta_p), r_0)\big) = \mu_\psi([\omega_1 \ldots \omega_n] \times B(y, r)) \tag{18}$$

Now the Bowen balls for the map Φ are sets of type $[\omega_1 \dots \omega_n] \times B(z, r_0)$. Then from the estimates of equilibrium measures on Bowen balls ([4]),

$$\mu_\psi([\eta_p \dots \eta_1 \omega_1 \dots \omega_n] \times B(z, r_0)) \approx e^{S_{n+p}\psi(\eta_p \dots \eta_1 \omega_1 \dots \omega_n, z) - (n+p)P_\Phi(\psi)}, \tag{19}$$

where the comparability constants do not depend on z, ω, η, n, p. We write also $S_n\psi(\omega_1 \dots \omega_n)$ for $S_n\psi(\omega, x_0)$, since by Hölder continuity this sum does not depend on x_0, modulo a constant. But

$$e^{S_{n+p}\psi(\eta_p \dots \eta_1 \omega_1 \dots \omega_n, z) - (n+p)P_\Phi(\psi)} = e^{S_p\psi(\eta_p \dots \eta_1) + S_n\psi(\omega_1 \dots \omega_n) - (n+p)P_\Phi(\psi)} =$$
$$= e^{S_p\psi(\eta_p \dots \eta_1) - pP_\Phi(\psi)} \cdot e^{S_n\psi(\omega_1 \dots \omega_n) - nP_\Phi(\psi)}$$

Hence from the last displayed formula and (19), we see that for any z,

$$\mu_\psi([\eta_p \dots \eta_1 \omega_1 \dots \omega_n] \times B(z, r_0)) \approx \mu_\psi([\eta_p \dots \eta_1] \times B(z, r_0)) \cdot e^{S_n\psi(\omega_1 \dots \omega_n) - nP_\Phi(\psi)}$$

So this last formula together with (17) and (18) imply that there exists a constant $C > 0$ independent of n, x, y, r, ω, such that

$$\frac{1}{C}\nu_\psi(B(y, r))e^{S_n\psi(\omega, x_0) - nP_\Phi(\psi)} \le \mu_\psi([\omega_1 \dots \omega_n] \times B(y, r)) \le C\nu_\psi(B(y, r))e^{S_n\psi(\omega, x_0) - nP_\Phi(\psi)}$$

This proves (15), by replacing any Borel set A with a union of disjoint balls of type $B(y, r)$.

Now recall that $\mu_\psi(D_n(\tau)) \to 1$ when $n \to \infty$; hence for any $\delta > 0$ small, there exists $n(\delta) \ge 1$ such that $\mu_\psi(D_n(\tau)) \ge 1 - \delta$ for all $n \ge n(\delta)$; hence from the Φ-invariance of μ_ψ, $\mu_\psi(\Phi^n(D_n(\tau))) \ge 1 - \delta$. Moreover there exists a strictly increasing sequence of integers $(k_n)_n$, with $k_n \ge n$, such that,

$$\mu_\psi(D_{k_n}) \ge 1 - \alpha_n, \text{ and } \sum_{n \ge 1} \alpha_n < \infty \tag{20}$$

Denote now by $Y_n(\tau) := \pi_2 D_n(\tau) \subset \Lambda$. We want to apply a version of Borel Density Lemma ([9] pg 293), in order to estimate the portion of the ν_ψ-measure of the intersection between a ball and $Y_n(\tau)$. Indeed for any $\delta > 0$ it follows that for any $n \ge n(\delta)$, there exists a borelian subset $\tilde{Y}_n(\tau) \subset Y_n(\tau)$ and $\rho_n > 0$, such that $\nu_\psi(\tilde{Y}_n(\tau)) \ge 1 - 2\delta$, and for any $x \in \tilde{Y}_n(\tau)$ and any $r \le \rho_n$,

$$\nu_\psi(B(x, r) \cap Y_n(\tau)) \ge \frac{1}{2}\nu_\psi(B(x, r)) \tag{21}$$

Let $Z_n(\tau) := \pi_2\Phi^n(D_n(\tau))$ and $\tilde{Z}_n(\tau) := \bigcap\limits_{\ell \ge n} Z_{k_\ell}(\tau)$, for $n \ge 1$. Then, since $\mu_\psi(\Phi^n(D_n(\tau))) \ge \mu_\psi(D_n(\tau))$, it follows from (20) that

$$\nu_\psi(\tilde{Z}_n(\tau)) \ge 1 - \sum_{m \ge n} \alpha_m, \text{ and } \nu_\psi(\tilde{Z}_n(\tau)) \underset{n \to \infty}{\to} 1$$

Given the radius ρ_n above, we can find an integer $s_n \ge n$, such that any ball $B(y, \frac{\rho_n}{2})$ with $y \in \Lambda$, intersects the set $\tilde{Y}_{s_n}(\tau)$. This is true since $\nu_\psi(\tilde{Y}_n(\tau)) \to 1$, and since μ_ψ is the equilibrium measure of a Hölder continuous potential, thus it is positive on balls of radius $\rho_n/2$. Denote now

$$r_n := e^{n(-\chi(\mu_\psi) + \tau)}, \ n \ge 1$$

Consider an arbitrary family $\mathcal{F}_{k_\ell}$ of mutually disjoint balls of radii $\rho_n r_{k_\ell}$ with centers in $\pi_2\Phi^{k_\ell}(D_{k_\ell}(\tau))$, for $\ell \geq s_n$, and assume the balls in $\mathcal{F}_{k_\ell}$ contain images of type $\phi_{i_{k_\ell}\ldots i_1}(B(z,\rho_n))$ for z in a family of centers F_{k_ℓ}. But from above, for all $\ell \geq s_n$ and $z \in F_{k_\ell}$, the ball $B(z,\rho_n/2)$ must contain a point $\xi_z \in \tilde{Y}_{s_n}(\tau)$. Hence $B(\xi_z,\rho_n/2) \subset B(z,\rho_n)$, and thus $\phi_{i_{k_\ell}\ldots i_1}(B(\xi_z,\rho_n/2)) \subset \phi_{i_{k_\ell}\ldots i_1}(B(z,\rho_n))$ for all $z \in F_{k_\ell}$. So we obtain a family $\mathcal{G}_{k_\ell}$ of disjoint sets $\phi_{i_{k_\ell}\ldots i_1}(B(\xi_z,\rho_n/2)),\ z \in F_{k_\ell}$. From our construction,

$$N(\mathcal{G}_{k_\ell}) := Card(\mathcal{G}_{k_\ell}) = N(\mathcal{F}_{k_\ell}) := Card(\mathcal{F}_{k_\ell})$$

However $\tilde{Y}_{s_n}(\tau) \subset Y_{s_n}(\tau) \subset \pi_2 D_{k_\ell}(\tau)$, if $\ell \geq s_n \geq n$, so from the above properties of the set $\tilde{Y}_{s_n}(\tau)$ and (21), it follows that $\nu_\psi(\tilde{Y}_{s_n}(\tau)) \geq 1-2\delta$ and,

$$\nu_\psi(B(\xi_z,\rho_n/2)\cap Y_{s_n}(\tau)) \geq \frac{1}{2}\nu_\psi(B(\xi_z,\rho_n/2))$$

But now from (11), $Y_{s_n}(\tau) \subset Y_k(\tau) = \pi_2 D_k(\tau)$ for all $k \geq s_n$, and recall $\ell \geq s_n \geq n$; hence from the last inequality,

$$\nu_\psi(B(\xi_z,\rho_n/2)\cap Y_{k_\ell}(\tau)) \geq \frac{1}{2}\nu_\psi(B(\xi_z,\rho_n/2)) \tag{22}$$

Let us estimate now the ν_ψ-measure of a set from $\mathcal{G}_{k_\ell}$, for $\ell \geq s_n$. Since Φ^{k_ℓ} is injective on $[i_1\ldots i_{k_\ell}]\ \times \Lambda$, we obtain from (15) and (22),

$$\begin{aligned}
&\nu_\psi(\phi_{i_{k_\ell}\ldots i_1}B(\xi_z,\rho_n/2)\cap Y_{k_\ell}(\tau)) = \mu_\psi(\Phi^{k_\ell}([i_1\ldots i_{k_\ell}]\times B(\xi_z,\rho/2)\cap D_{k_\ell})) = \\
&\quad = \int_{[i_1\ldots i_{k_\ell}]\times(B(\xi_z,\rho_n/2)\cap Y_{k_\ell}(\tau))} J_{\Phi^{k_\ell}}(\mu_\psi)(\omega,x)\ d\mu_\psi(\omega,x) \\
&\quad \geq C\exp(k_\ell(F_\Phi(\mu_\psi)-\tau))\cdot\exp(S_{k_\ell}\psi(\omega,x_0)-k_\ell P_\Phi(\psi))\cdot\nu_\psi(B(\xi_z,\rho_n/2)\cap Y_{k_\ell}) \\
&\quad \geq \tilde{C}_n\exp(k_\ell(F_\Phi(\mu_\psi)-\tau)\cdot\exp(k_\ell(-h_\Phi(\mu_\psi)-\tau)) = \tilde{C}_n\exp(k_\ell(F_\Phi(\mu_\psi)-h_\Phi(\mu_\psi)-2\tau)),
\end{aligned} \tag{23}$$

for some constants $C_n, \tilde{C}_n > 0$, where we used the estimate on the Jacobian of Φ^{k_ℓ} on D_{k_ℓ}, the estimate on the equilibrium measure μ_ψ of a Bowen ball $[i_1\ldots i_{k_\ell}]\times B(\xi_z,\rho_n/2)$, and the behaviour of $S_{k_\ell}\psi$ on the generic points from D_{k_ℓ}. Since the balls in $\mathcal{F}_{k_\ell}$ are disjoint, and each of them contains a set of type $\phi_{i_{k_\ell}\ldots i_1}B(\xi_z,\rho_n/2)\cap Y_{k_\ell}$, it follows that for all integers $\ell \geq s_n$,

$$\sum_{\xi_z\in G_{k_\ell}} \nu_\psi(\phi_{i_{k_\ell}\ldots i_1}B(\xi_z,\rho_n/2)\cap Y_{k_\ell}(\tau)) \leq 1$$

Thus, using (23) and the fact that $N(\mathcal{G}_{k_\ell}) = N(\mathcal{F}_{k_\ell})$, we obtain for any family $\mathcal{F}_{k_\ell}$ as above,

$$N(\mathcal{F}_{k_\ell}) \leq C_n^{-1}\exp(-k_\ell(F_\Phi(\mu_\psi)-h_\Phi(\mu_\psi)-2\tau)) \tag{24}$$

So if for some $\ell \geq s_n$ we take a disjointed family $\mathcal{W}$ of balls of radii $\rho_n\cdot r_{k_\ell}$ with centers in $\tilde{Z}_{s_n}(\tau) = \bigcap_{\ell\geq s_n}\pi_2\Phi^{k_\ell}D_{k_\ell}(\tau)$, then its cardinality $N(\mathcal{W})$ is less than the cardinality of some family $\mathcal{F}_{k_\ell}$ from above, hence from (24) we obtain an estimate for the lower box dimension,

$$\underline{dim}_B(\tilde{Z}_{s_n}(\tau)) \leq \frac{h_\Phi(\mu_\psi)-F_\Phi(\mu_\psi)+2\tau}{\chi(\mu_\psi)-\tau}$$

But on the other hand, we know from construction that $\nu_\psi(\tilde{Z}_{s_n}(\tau)) \geq 1 - \sum\limits_{j \geq s_n} \alpha_j \to 1$, when $n \to \infty$. So from the above, using the definition of lower box dimension of a measure, it follows

$$\underline{dim}_B(\nu_\psi) \leq \frac{h_\Phi(\mu_\psi) - F_\Phi(\mu_\psi) + 2\tau}{\chi(\mu_\psi) - \tau},$$

for any small number $\tau > 0$, and thus the conclusion follows, namely $\underline{dim}_B(\nu_\psi) \leq \frac{h_\Phi(\mu_\psi) - F_\Phi(\mu_\psi)}{\chi(\mu_\psi)}$. □

4 Formulas and computation of topological overlap numbers.

In this section we compute the topological overlap number in several concrete significant cases. Then, these formulas are applied to box dimension estimates. The *topological overlap number* of a conformal system $\mathcal{S} = \{\phi_i, i \in I\}$ is defined (see [7]) in relation to the lift endomorphism Φ, as the overlap number of the measure of maximal entropy μ_{max} for Φ on $\Sigma_I^+ \times \Lambda$, and is denoted by $o(\mathcal{S})$,

$$o(\mathcal{S}) = o(\mathcal{S}, \mu_{max})$$

Consider now a probabilistic vector $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_{|I|}})$ and its associated Bernoulli measure $\mu^+_{\mathbf{p}}$ on Σ_I^+. Then the classical projection of $\mu^+_{\mathbf{p}}$ on the limit set Λ of $\mathcal{S}$ is $\pi_*\mu^+_{\mathbf{p}}$. The Bernoulli measure $\mu^+_{\mathbf{p}}$ is the equilibrium measure with respect to σ of the potential $g : \Sigma_I^+ \to \mathbb{R},\ g(\omega) = \log p_{\omega_1},\ \omega \in \Sigma_I^+$. Let $\psi := g \circ \pi_1 : \Sigma_I^+ \times \Lambda \to \mathbb{R}$, and μ_ψ be its equilibrium measures with respect to the endomorphism Φ. Then we proved in [7] that for this choice of ψ, $\pi_{2*}\mu_\psi = \pi_*\pi_{1*}\mu_\psi$. But from estimates of equilibrium measures on Bowen balls, it follows that for a constant r_0, $\mu_\psi([\omega_1 \ldots \omega_n] \times B(x, r_0)) \approx e^{S_n\psi(\omega,x) - nP_\Phi(\psi)}$, where the comparability constant is independent of n, x, ω. Thus by summing up,

$$\mu_\psi([\omega_1 \ldots \omega_n] \times \Lambda) \approx e^{S_n g(\omega) - nP_\sigma(g)},$$

since Φ is contracting in the second coordinate and since ψ depends only on ω. Denote $\mu_{g\circ\pi_1}$ by $\mu_{\mathbf{p}}$, which can be considered a lift of $\mu^+_{\mathbf{p}}$ to $\Sigma_I^+ \times \Lambda$. So $\pi_{1*}\mu_{\mathbf{p}}$ satisfies the same estimates on cylinders as the Bernoulli measure $\mu^+_{\mathbf{p}}$, and thus from above, we obtain $\pi_{1*}\mu_{\mathbf{p}} = \mu^+_{\mathbf{p}}$. Therefore,

$$\pi_{2*}\mu_{\mathbf{p}} = \pi_*\mu^+_{\mathbf{p}} \tag{25}$$

In particular, if μ^+_{max} denotes the measure of maximal entropy for the shift on Σ_I^+, i.e the Bernoulli measure associated to the probability vector $(\frac{1}{|I|}, \ldots, \frac{1}{|I|}))$, we obtain

$$\pi_{2*}\mu_{max} = \pi_*\mu^+_{max} \tag{26}$$

We showed in [7] that, if $\pi : \Sigma_I^+ \to \Lambda$ is the canonical projection to the limit set of $\mathcal{S}$ and if

$$\beta_n(x) := Card\{(\eta_1, \ldots, \eta_n) \in I^n, x \in \phi_{\eta_1 \ldots \eta_n}(\Lambda)\},\ n \geq 1,$$

then the topological overlap number of $\mathcal{S}$ is given by the formula:

$$o(\mathcal{S}) = \exp\big(\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_I^+} \log \beta_n(\pi\omega)\ d\mu^+_{max}(\omega)\big) \tag{27}$$

3.1. Consider the system $\mathcal{S}_\lambda = \{\phi_{-1}, \phi_1\}$, where $\phi_{-1}(x) = \lambda x - 1$, $\phi_1(x) = \lambda x + 1$. When $\lambda \in (\frac{1}{2}, 1)$ this system has overlaps, and its limit set is the interval $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$. When there is no confusion about λ, this limit set will also be denoted by Λ. We consider then the measure of maximal entropy μ_{max} for Φ on $\Sigma_2^+ \times \Lambda$.

3.1a. Let us look first at reciprocals of *Garsia numbers*. A number γ is called a *Garsia number* if it is an algebraic integer in $(1, 2)$ whose minimal polynomial has constant coefficient ± 2 and so that γ and all of its conjugates have absolute value strictly greater than 1 (see [3]). Examples of such minimal polynomials are $x^{n+p} - x^n - 2$ for $n, p \geq 1$, with $\max\{p, n\} \geq 2$. For instance $2^{\frac{1}{n}}, n \geq 2$, are Garsia numbers.

Theorem 3. *The topological overlap number $o(\mathcal{S}_\lambda)$ of the system $\mathcal{S}_\lambda$ for $\lambda \in (\frac{1}{2}, 1)$ with $\frac{1}{\lambda}$ a Garsia number, is equal to 2λ.*

Proof. Recall that the limit set of $\mathcal{S}_\lambda$ is the interval $I_\lambda = [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$. From [3] it follows that, if λ is the reciprocal of a Garsia number, then all 2^n sums of type $\sum\limits_0^{n-1} \pm\lambda^k$ are distinct and at least $\frac{C}{2^n}$ apart, for some constant $C > 0$. Let us order increasingly these 2^n numbers $\sum\limits_0^{n-1} \pm\lambda^k$, and denote them by $\zeta_1, \ldots, \zeta_{2^n}$. Hence from [3] these points ζ_i are distinct, and

$$|\zeta_i - \zeta_j| \geq \frac{C}{2^n}, i \neq j \tag{28}$$

Since there are 2^n points ζ_j in the interval I_λ, there is a constant $C' > 0$ so that, for any $i \neq j$,

$$\frac{C'}{2^n} \geq |\zeta_i - \zeta_j| \geq \frac{C}{2^n}$$

Now the numbers of type $\zeta_j + \sum\limits_{k \geq n} r_k \lambda^k$, where $\zeta_j = \sum\limits_{0 \leq k \leq n-1} \omega_k \lambda^k$ and $\omega_k \in \{-1, 1\}$, form the interval $I_j := \pi([\omega_0, \ldots \omega_{n-1}]$. The length of I_j is $C_1 \lambda^n$, for some fixed constant $C_1 > 0$. Since $\lambda > \frac{1}{2}$, it follows from (28) that any interval I_j contains at least $C_2(2\lambda)^n$ points ζ_j and at most $C_3(2\lambda)^n$ points ζ_j, for some constants $C_3 > C_2 > 0$. With the possible exception of an interval J_1 of length $C_4\lambda^n$ with left endpoint $-\frac{1}{1-\lambda}$ (i.e the left endpoint of I_λ), and an interval J_2 of same length with right endpoint $\frac{1}{1-\lambda}$ (i.e the right endpoint of I_λ), we see that any point x belongs to at least $C_5(2\lambda)^n$ intervals I_j and to at most $C_6(2\lambda)^n$ intervals I_j, where the constants $C_1, \ldots, C_6$ do not depend on n. Recall that $I_j = \pi([\omega_0, \ldots, \omega_{n-1}]$ for some $\omega_k \in \{-1, 1\}, 0 \leq k \leq n-1$, and that $\mu^+_{max}([\omega_0, \ldots, \omega_{n-1}]) = \frac{1}{2^n}$, where μ^+_{max} is the measure of maximal entropy on Σ_2^+. From above (27) we know that,

$$o(\mathcal{S}_\lambda) = \exp(\lim_n \frac{1}{n} \int_{\Sigma_2^+} \log \beta_n(\pi\omega)\ d\mu^+_{max}(\omega)),$$

where $\beta_n(x) := Card\{(\eta_0, \ldots, \eta_{n-1}) \in \{-1, 1\}^n, x \in \phi_{\eta_0 \ldots \eta_{n-1}}(\Lambda_\lambda)\}$ for $x \in \Lambda_\lambda$ and $n \geq 1$. But from above, we see that for x outside the intervals J_1, J_2 of length $C_4\lambda^n$ at the endpoints of I_λ,

$$C_5(2\lambda)^n \leq \beta_n(x) \leq C_6(2\lambda)^n$$

Thus from the last estimate on $\beta_n(x)$ on the complement of $J_1 \cup J_2$, and using that $\mu_{max}([\omega_0, \ldots, \omega_{n-1}]) = \frac{1}{2^n}$, we obtain that for some constant $C_7 > 0$ (independent of n),

$$(2^n - C_7(2\lambda)^n) \cdot n \log(2\lambda) \frac{1}{2^n} \leq \int_{\Sigma_2^+} \log \beta_n(\pi\omega) \; d\mu_{max}^+(\omega) \leq 2^n \cdot n \log(2\lambda) \cdot \frac{1}{2^n} = n \log(2\lambda)$$

Therefore $o(\mathcal{S}_\lambda) = 2\lambda$, since from the last displayed inequalities it follows that,

$$\lim_{n \to \infty} \frac{1}{n} \int_{\Sigma_2^+} \log \beta_n(\pi\omega) \; d\mu_{max}^+(\omega) = \log(2\lambda)$$

□

Since for any $n \geq 1$, $2^{\frac{1}{n}}$ is a Garsia number (see [3]), we then obtain from Theorem 3 a system which asymptotically is $\sqrt[n]{2^{n-1}}$ -to-1. For these examples the projection $\pi_* \mu_{max}^+$ is absolutely continuous ([3]), and $\pi_* \mu_{max}^+ = \pi_{2*} \mu_{max}$ from (26), hence:

Corollary 1. *For the system $\mathcal{S}_\lambda$ with $\lambda = 2^{-\frac{1}{n}}$, the topological overlap number is $o(\mathcal{S}_\lambda) = \sqrt[n]{2^{n-1}}$, and the measure $\pi_{2*}\mu_{\max}$ is absolutely continuous.*

3.1b. The second example is of Bernoulli convolutions with λ being the reciprocal of a *Pisot number*. A Pisot number is by definition an algebraic integer all of whose conjugates are strictly less than 1 in absolute value (for eg [3], [10]). We prove the following.

Theorem 4. *The topological overlap number of $\mathcal{S}_\lambda$ for $\lambda \in (\frac{1}{2}, 1)$ with $\frac{1}{\lambda}$ a Pisot number, satisfies*

$$o(\mathcal{S}_\lambda) \geq 2\lambda > 1$$

Proof. If $\frac{1}{\lambda}$ is a Pisot number, the distance between any two different polynomial sums of type $P(\omega, \lambda, n) = \sum_{i=0}^{n-1} \omega_i \lambda^i$ for $\omega \in \Sigma_2^+ = \{-1, 1\}^\infty$, is at least $C\lambda^n$, for some constant $C > 0$, which follows from the algebraic properties of $\frac{1}{\lambda}$ (see [3], [10]). Then the number $q(n)$ of all possible values of such polynomials $P(\omega, \lambda, n)$, when n, λ are fixed, satisfies

$$q(n) \leq C_1 \lambda^{-n}, \tag{29}$$

for some constant C_1 independent of n. Since there are 2^n tuples $(\omega_0, \ldots, \omega_{n-1}) \in \{-1, 1\}^n$, but only at most $C_1 \lambda^{-n}$ values for polynomials $P(\omega, \lambda, n)$, and since $\lambda > \frac{1}{2}$, there must be many equalities between such values. Denote by $V_n(\lambda)$ the set of values of polynomials $P(\omega, \lambda, n)$,

$$V_n(\lambda) = \{\alpha_1, \ldots, \alpha_{q(n)}\}, \text{ where } \alpha_1 < \ldots < \alpha_{q(n)}, \tag{30}$$

where $q(n)$ satisfies (29). We know that

$$\pi([\omega_0, \ldots, \omega_{n-1}]) = \{P(\omega, \lambda, n) + \sum_{i=n}^{\infty} \omega_i \lambda^i, \; \omega_i \in \{-1, 1\}, i \geq n\},$$

so $\pi([\omega_0, \dots, \omega_{n-1}])$ is an interval in Λ_λ of length between λ^n and $2\lambda^n$ (depending on its location). Denote by $N_i := Card\{(\omega_0, \dots, \omega_{n-1}) \in \{-1, 1\}^n, P(\omega, \lambda, n) = \alpha_i\}, \ 1 \le i \le q(n)$. From (29) recall that $|\alpha_i - \alpha_j| \ge C_1 \lambda^n$ if $i \ne j$. Since each value α_i is taken N_i times by polynomials $P(\omega, \lambda, n)$, $1 \le i, j \le q(n)$, it follows that there exists a constant $C_2 > 0$ so that for all $n \ge 1$,

$$\beta_n(\pi\omega) \ge C_2 N_i, \text{ whenever } P(\omega, \lambda, n) = \alpha_i, 1 \le i \le q(n) \tag{31}$$

For the measure of maximal entropy μ^+_{max} on Σ^+_2 we have $\mu^+_{max}([\omega_0, \dots, \omega_{n-1}]) = \frac{1}{2^n}$, so from (31),

$$\int_{\Sigma^+_2} \log \beta_n(\pi\omega) \ d\mu^+_{max}(\omega) \ge \sum_{j=1}^{q(n)} (\log C_2 N_j) \cdot \frac{N_j}{2^n} = \log 2^n + \sum_{j=1}^{q(n)} \frac{N_j}{2^n} \log \frac{N_j}{2^n} + \log C_2 \tag{32}$$

However in general for any probability vector $(p_1, \dots, p_m)$, one has the upper bound (for eg [15]),

$$-\sum_{i=1}^{m} p_i \log p_i \le \log m$$

From (30), we know $N_1 + \dots N_{q(n)} = 2^n$, so we can take the probability vector $(\frac{N_1}{2^n}, \dots, \frac{N_{q(n)}}{2^n})$, and from (32) it follows that:

$$\frac{1}{n} \log \int_{\Sigma^+_2} \log \beta_n(\pi\omega) \ d\mu^+_{max}(\omega) \ge \log 2 - \frac{\log C_1 \lambda^{-n}}{n} + \frac{\log C_2}{n}$$

This implies then from (27) that $o(\mathcal{S}_\lambda) \ge 2\lambda$, hence $o(\mathcal{S}_\lambda) > 1$ since $\lambda > \frac{1}{2}$.

□

3.2. We now look at examples with eventual exact or at least substantial overlaps, in which case the topological overlap number will be estimated, or even computed exactly. We consider first the case when there are *exact overlaps*, i.e. when:

$$\phi_{i_1 \dots i_p}(\Lambda) = \phi_{j_1 \dots j_p}(\Lambda),$$

for certain maximal tuples $(i_1, \dots, i_p), (j_1, \dots, j_p)$. Exact overlaps may appear after certain number of iterates, but for simplicity we look first at the case $p = 1$; the generalization is straightforward.

So consider the system $\mathcal{S} = \{\phi_i, 1 \le i \le m\}$ of conformal injective contractions, and assume we have the blocks

$$\phi_1 = \dots = \phi_{k_1}, \ \phi_{k_1+1} = \dots = \phi_{k_2}, \dots, \ \phi_{k_p} = \phi_m, \tag{33}$$

where there are no overlaps between the different blocks, i.e the system $\{\phi_{k_i}, 1 \le i \le p\}$ satisfies the Open Set Condition.

Let μ^+_{max} be the measure of maximal entropy on Σ^+_m, and denote the measure of maximal entropy for Φ on $\Sigma^+_m \times \Lambda$ by μ_{max}. Then, the topological overlap number $o(\mathcal{S}) := o(\mathcal{S}, \mu_{max})$ takes in consideration all preimages of Φ, and we proved in [7] that

$$o(\mathcal{S}) = \exp\big(\lim_{n \to \infty} \frac{1}{n} \int_{\Sigma^+_m} \log \beta_n(\pi\omega) \ d\mu^+_{max}(\omega) \big), \tag{34}$$

where $\beta_n(x) := \text{Card}\{(\eta_1, \ldots, \eta_n) \in I^n,\ x \in \phi_{\eta_1 \ldots \eta_n}(\Lambda)\}$. In this case, if $x \in \phi_{j_1 \ldots j_n}(\Lambda)$ and if $k_{i_\ell - 1} + 1 \leq j_\ell \leq k_{i_\ell}$, then for $x = \pi\omega$ and $\omega = (j_1 j_2 \ldots)$, we have:

$$\beta_n(x) = (k_{i_1} - k_{i_1 - 1}) \cdot \ldots \cdot (k_{i_n} - k_{i_n - 1}), \tag{35}$$

where if $i_\ell = 1$, then the factor $(k_{i_\ell} - k_{i_\ell - 1})$ is replaced by k_1. Let us take the function $\Psi : \Sigma_m^+ \to \mathbb{R}$, $\Psi(\omega) := \log k_1$ for $1 \leq \omega_1 \leq k_1$, and $\Psi(\omega) := \log(k_i - k_{i-1})$ for $k_{i-1} + 1 \leq \omega_1 \leq k_i$. If ω, η are close enough in Σ_m^+, then $\omega_1 = \eta_1$, hence Ψ is Hölder continuous on Σ_m^+.
Notice that, if $\omega \in [j_1 \ldots j_n]$ and $k_{i_s - 1} + 1 \leq j_s \leq k_{i_s}$ if $i_s > 1$, or $1 \leq j_1 \leq k_1$ if $i_s = 1$, then

$$\Psi(\omega) = \log(k_{i_1} - k_{i_1 - 1}),\ \Psi(\sigma\omega) = \log(k_{i_2} - k_{i_2 - 1}), \ldots$$

However from above,

$$\int_{\Sigma_m^+} \log \beta_n(\pi\omega)\, d\mu_{max}^+(\omega) = \sum_{s=1,\ldots n} \sum_{k_{i_s - 1} + 1 \leq j_s \leq k_{i_s}} \int_{[j_1 \ldots j_n]} \log(k_{i_1} - k_{i_1 - 1}) + \ldots \log(k_{i_n} - k_{i_n - 1})\, d\mu_{max}^+(\omega)$$

Thus, if $S_n\Psi$ denotes the consecutive sum of Ψ with respect to σ, we obtain

$$\int_{\Sigma_m^+} \log \beta_n(\pi\omega)\ d\mu_{max}^+(\omega) = \int_{\Sigma_m^+} S_n\Psi(\omega)\ d\mu_{max}^+(\omega) \tag{36}$$

Hence from (36), by Birkhoff Egodic Theorem for the measure of maximal entropy μ_{max}^+ on Σ_m^+,

$$\frac{1}{n} \int \log \beta_n(\pi\omega)\ d\mu_{max}^+(\omega) = \frac{1}{n} \int_{\Sigma_m^+} S_n\Psi(\omega)\ d\mu_{max}^+(\omega) \underset{n\to\infty}{\longrightarrow} \int_{\Sigma_m^+} \Psi(\omega)\ d\mu_{max}^+(\omega)$$

We have thus proved the following:

Proposition 1. *In the above setting from (33), the topological overlap number of the system $\mathcal{S}$ is given by,*

$$o(\mathcal{S}) = o(\mathcal{S}, \mu_{max}) = \exp\big(\frac{k_1 \log k_1 + (k_2 - k_1)\log(k_2 - k_1) + \ldots + (k_p - k_{p-1})\log(k_p - k_{p-1})}{m}\big)$$

As in Corollary 5, the above estimates can be extended for the p-iterated system $\mathcal{S}^p = \{\phi_{i_1 \ldots i_p},\ i_j \in I, 1 \leq j \leq p\}$, and thus we obtain:

Corollary 2. *Assume we have the system of conformal injective contractions $\mathcal{S} = \{\phi_i, i \in I\}$ with $|I| = m$, and let Λ be its limit set. Assume also that there exists a family $\mathcal{F} \subset I^p$ of p-tuples such that $\phi_{i_p \ldots i_1}(\Lambda) = \phi_{j_p \ldots j_1}(\Lambda)$ for $(i_1, \ldots, i_p), (j_1, \ldots, j_p) \in \mathcal{F}$, and denote $Card(\mathcal{F}) = N(\mathcal{F})$. Then*

$$o(\mathcal{S}) \geq \exp\big(\frac{N(\mathcal{F}) \log N(\mathcal{F})}{m^p}\big)$$

However, á priori there may exist only *partial overlaps* at the level of p-iterates, which comprise a positive proportion of the measure. In particular the next Corollaries apply well for Bernoulli convolutions systems $\mathcal{S}_\lambda$, since in this case the limit set is an interval $\Lambda = I_\lambda$ and we can numerically estimate the proportion of overlaps at some iterate p. As above we obtain.

Corollary 3. *In the above setting assume that there is a family $\mathcal{F} \subset I^p$ of p-tuples and $k \geq 1$ so that for any $(i_1, \dots, i_p) \in \mathcal{F}$, there exists $(j_1 \dots j_k) \in I^k$ such that*

$$\phi_{i_1 \dots i_p j_1 \dots j_k}(\Lambda) \subset \bigcap_{(\ell_1, \dots \ell_p) \in \mathcal{F}} \phi_{\ell_1 \dots \ell_p}(\Lambda)$$

Then if $N(\mathcal{F})$ denotes the cardinality of $\mathcal{F}$, we obtain:

$$o(\mathcal{S}) \geq \exp\big(\frac{N(\mathcal{F}) \log N(\mathcal{F})}{m^{p+k}}\big)$$

More generally we have the following:

Corollary 4. *In the above setting assume that there are families $\mathcal{F}_1, \dots, \mathcal{F}_s \subset I^p$ of p-tuples and positive integers $k_1, \dots, k_s$ such that, for any $1 \leq j \leq s$ and for any $(i_{j1}, \dots, i_{jp}) \in \mathcal{F}_j$ there exists some k_j-tuple $(j_1, \dots, j_{k_j}) \in I^{k_j}$ with*

$$\phi_{i_{j1} \dots i_{jp} j_1 \dots j_{k_j}}(\Lambda) \subset \bigcap_{(\ell_1, \dots, \ell_p) \in \mathcal{F}_j} \phi_{\ell_1 \dots \ell_p}(\Lambda)$$

Then if $N(\mathcal{F}_j) := Card \mathcal{F}_j$, $1 \leq j \leq s$, we obtain:

$$o(\mathcal{S}) \geq \exp\big(\frac{N(\mathcal{F}_1) \log N(\mathcal{F}_1)}{m^{p+k_1}} + \dots + \frac{N(\mathcal{F}_s) \log N(\mathcal{F}_s)}{m^{p+k_s}}\big)$$

Recall now from (25) that for Bernoulli measures we have the equality of the two projectional measures, i.e $\pi_{2*}\mu_{\mathbf{p}} = \pi_* \mu_{\mathbf{p}}^+$. Also recall that μ_{max} is the measure of maximal entropy for Φ on $\Sigma_I^+ \times \Lambda$, and μ_{max}^+ is the measure of maximal entropy for the shift on Σ_I^+.

Then, from Proposition 1, Theorem 2 and Corollaries 2, 3 and 4, we obtain the following dimension estimates:

Corollary 5. *Assume we have the system of conformal injective contractions $\mathcal{S} = \{\phi_i, i \in I\}$ with $|I| = m$, and let Λ be its limit set, and denote by μ_{max} the measure of maximal entropy on $\Sigma_I^+ \times \Lambda$. Assume also that there exists a family $\mathcal{F}$ of p-tuples such that $\phi_{i_p \dots i_1}(\Lambda) = \phi_{j_p \dots j_1}(\Lambda)$ for $(i_1, \dots, i_p), (j_1, \dots, j_p) \in \mathcal{F}$, and denote $Card(\mathcal{F}) = N(\mathcal{F})$. Then $o(\mathcal{S}) \geq \exp\big(\frac{N(\mathcal{F}) \log N(\mathcal{F})}{m^p}\big)$, and*

$$\underline{dim}_B(\pi_{2*}\mu_{max}) = \underline{dim}_B(\pi_* \mu_{max}^+) \leq \frac{p \cdot h_\sigma(\mu_{max}^+) - \frac{N(\mathcal{F}) \log N(\mathcal{F})}{m^p}}{p \cdot \chi(\mu_{max})}$$

Corollary 6. *In the above setting assume there are families $\mathcal{F}_1, \dots, \mathcal{F}_s \subset I^p$ of p-tuples and $k_1, \dots, k_s \geq 1$ so that, for any $1 \leq j \leq s$ and any $(i_{j1}, \dots, i_{jp}) \in \mathcal{F}_j$ there exists some k_j-tuple $(j_1, \dots, j_{k_j}) \in I^{k_j}$, with*

$$\phi_{i_{j1} \dots i_{jp} j_1 \dots j_{k_j}}(\Lambda) \subset \bigcap_{(\ell_1, \dots, \ell_p) \in \mathcal{F}_j} \phi_{\ell_1 \dots \ell_p}(\Lambda)$$

Then if $N(\mathcal{F}_j) := Card \mathcal{F}_j$, $1 \leq j \leq s$, we obtain:

$$\underline{dim}_B(\pi_{2*}\mu_{max}) = \underline{dim}_B(\pi_* \mu_{max}^+) \leq \frac{p \cdot h_\sigma(\mu_{max}^+) - \frac{N(\mathcal{F}_1) \log N(\mathcal{F}_1)}{m^{p+k_1}} - \dots - \frac{N(\mathcal{F}_s) \log N(\mathcal{F}_s)}{m^{p+k_s}}}{p \cdot \chi(\mu_{max})}$$

Acknowledgements: This work was supported by grant PN-III-P4-ID-PCE-2016-0823 from UEFISCDI.

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