Third Kind Elliptic Integrals and 1-Motives

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with an appendix by M. Waldschmidt

Abstract. In [5] we have showed that the Generalized Grothendieck's Conjecture of Periods applied to 1-motives, whose underlying semi-abelian variety is a product of elliptic curves and of tori, is equivalent to a transcendental conjecture involving elliptic integrals of the first and second kind, and logarithms of complex numbers.

In this paper we investigate the Generalized Grothendieck's Conjecture of Periods in the case of 1-motives whose underlying semi-abelian variety is a *non trivial extension* of a product of elliptic curves by a torus. This will imply the introduction of *elliptic integrals of the third kind* for the computation of the period matrix of M and therefore the Generalized Grothendieck's Conjecture of Periods applied to M will be equivalent to a transcendental conjecture involving elliptic integrals of the first, second and third kind.

Contents

Introduction

Let $\mathcal{E}$ be an elliptic curve defined over $\mathbb{C}$ with Weierstrass coordinate functions x and y. On $\mathcal{E}$ we have the differential of the first kind $\omega = \frac{dx}{y}$, which is holomorphic, the differential of the second kind $\eta = -\frac{xdx}{y}$, which has a double pole with residue zero at each point of the lattice $\mathrm{H}_1(\mathcal{E}(\mathbb{C}),\mathbb{Z})$ and no other pole, and the differential of the third kind

$$\xi_Q = \frac{1}{2}\frac{y - y(Q)}{x - x(Q)}\frac{dx}{y},$$

for any point Q of $\mathcal{E}(\mathbb{C}), Q \neq 0$, whose residue divisor is $D = -(0) + (-Q)$. Let γ_1, γ_2 be two closed paths on $\mathcal{E}(\mathbb{C})$ which build a basis for the lattice $\mathrm{H}_1(\mathcal{E}(\mathbb{C}),\mathbb{Z})$. In his Peccot lecture at the Collège de France in 1977, M. Waldschmidt observed that the periods of the Weierstrass $\wp$-function (1.4) are the elliptic integrals of the first kind $\int_{\gamma_i} \omega = \omega_i$ $(i = 1, 2)$, the

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quasi-periods of the Weierstrass ζ-function (1.5) are the elliptic integrals of the second kind $\int_{\gamma_i} \eta = \eta_i$ $(i = 1, 2)$, but *there is no function whose quasi-quasi-periods are elliptic integrals of the third kind*. J.-P. Serre answered this question furnishing the function

$$f_q(z) = \frac{\sigma(z+q)}{\sigma(z)\sigma(q)} e^{-\zeta(q)z} \qquad \text{with } q \in \mathbb{C} \setminus \Lambda$$

whose *quasi-quasi periods* (1.8) are *the exponentials of the elliptic integrals of the third kind* $\int_{\gamma_i} \xi_Q = \eta_i q - \omega_i \zeta(q)$ $(i = 1, 2)$, where q is an elliptic logarithm of the point Q.

Consider now an extension G of $\mathcal{E}$ by $\mathbb{G}_m$ parameterized by the divisor $D = (-Q) - (0)$ of $\mathrm{Pic}^0(\mathcal{E}) \cong \mathcal{E}^* = \underline{\mathrm{Ext}}^1(\mathcal{E}, \mathbb{G}_m)$. Since the three differentials $\{\omega, \eta, \xi_Q\}$ build a basis of the De Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(G)$ of the extension G, elliptic integrals of the third kind play a role in Grothendieck's Conjecture of Periods, more precisely in its generalization (0.4). The aim of this paper is to understand this role applying the Generalized Grothendieck's Conjecture of Periods to 1-motives whose underlying semi-abelian variety is a *non trivial extension* of a product of elliptic curves by a torus.

We start recalling Grothendieck's Conjecture of Periods (0.2) and its generalization (0.4). Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and let K be an algebraically closed sub-field of the field of complex numbers $\mathbb{C}$ which is not necessarily algebraic over $\mathbb{Q}$. Consider a smooth and projective algebraic variety X defined over K. The *periods of* X are the coefficients of the matrix which represents (with respect to K-bases) the canonical isomorphism given by the integration of differentials forms

$$\begin{aligned} \beta_X : \mathrm{H}^*_{\mathrm{dR}}(X) \otimes_K \mathbb{C} &\longrightarrow \mathrm{H}^*_{\mathrm{sing}}(X(\mathbb{C}), K) \otimes_K \mathbb{C} \\ \omega &\longmapsto \Big[\gamma \mapsto \int_\gamma \omega\Big] \end{aligned} \tag{0.1}$$

between the algebraic De Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X)$ and the singular cohomology $\mathrm{H}^*_{\mathrm{sing}}(X(\mathbb{C}), K) = \mathrm{H}^*_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} K$ of X. In Note 10 of [13], Grothendieck conjectures that *any polynomial relation with rational coefficients between the periods of* X *should have a geometrical origin*. More precisely, any algebraic cycle on X and on the products of X with itself, will give rise to a polynomial relation with rational coefficients among the periods of X (see [15, Chp. IV, Historical Note]). We can reformulate this in the following way: *the existence of algebraic cycles on* X *and on the products of* X *with itself, should affect the transcendence degree of the field generated by the periods of* X.

Grothendieck has never written down a precise statement for this conjecture on periods of X. In [1, §7.5], André does it using the notion of *motivic Galois group of* X, whose dimension is strictly related to the existence of algebraic cycles on X and on the products of X with itself. Grothendieck's dream about motives was first to construct the tannakian category of mixed motives, and then, by tannakian duality, to define the motivic Galois group of mixed motives as the group pro-scheme whose category of representations is equivalent to the tannakian category of mixed motives (in other words, as the fundamental group of the tannakian category of mixed motives, see [11, 6.1] or [12, 8.13]). This dream remained unachieved for several years (except for some special cases as abelian varieties, 1-motives, ...). Recently, in two different, independent and equivalent ways, Nori and Ayoub have furnished a definition of the tannakian category of mixed motives with rational coefficients using a weak version of the tannakian duality (see [3] and [19]). More precisely, they construct first a group pro-scheme $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\mathcal{MM})$ over $\mathbb{Q}$, that they call the motivic Galois group of mixed motives, and then they define the tannakian category of mixed motives (with rational coefficients over a sub-field of $\mathbb{C}$) as the category of representations of this group pro-scheme $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\mathcal{MM})$. The inclusion $i : < X >^{\otimes} \rightarrow \mathcal{MM}$ of the tannakian sub-category generated by a smooth and projective algebraic variety X in $\mathcal{MM}$ corresponds to a surjective morphism $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\mathcal{MM}) \rightarrow i\mathcal{G}\mathrm{al}_{\mathrm{mot}}(X)$ of group pro-schemes, that is the motivic Galois group of X is a quotient of the motivic Galois group of the tannakian category of mixed motives (see [7, §2]). With these notation, André states the Conjecture of Periods in the following way

Conjecture 0.1 (Grothendieck's Conjecture of Periods)**.** *Let X be a smooth and projective algebraic variety defined over $\overline{\mathbb{Q}}$, then*

$$\mathrm{tran.deg}_{\mathbb{Q}}\, \overline{\mathbb{Q}}(\mathrm{periods}(X)) = \dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(X), \tag{0.2}$$ {eq:CP}

where $\overline{\mathbb{Q}}(\mathrm{periods}(X))$ is the field generated over $\overline{\mathbb{Q}}$ by the periods of X.

This conjecture is independent of the choice of the K-bases that we do in order to compute the periods of X. André extends Grothendieck's Conjecture of Periods (0.2) to smooth and projective algebraic varieties defined over an algebraically closed sub-field K of $\mathbb{C}$ which is not necessarily algebraic over $\mathbb{Q}$ ([1, §23.4]), and also to mixed motives defined over K (see end of [1, §23.4.1]). In this paper we are involved with this last generalization applied to 1-motives.

A 1-motive $M = [u : X \rightarrow G]$ over K consists of a finitely generated free $\mathbb{Z}$-module X, an extension G of an abelian variety by a torus, and a homomorphism $u : X \rightarrow G(K)$. Denote by $M_{\mathbb{C}}$ the 1-motive defined over $\mathbb{C}$ obtained from M extending the scalars from K to $\mathbb{C}$. In [10] Deligne associates to the 1-motive M

- its De Rham realization $\mathrm{T}_{\mathrm{dR}}(M)$: it is the finite dimensional K-vector space $\mathrm{Lie}(G^\natural)$, with $M^\natural = [u : X \rightarrow G^\natural]$ the universal extension of M by the vector group $\mathrm{Hom}(\mathrm{Ext}^1(M, \mathbb{G}_a), \mathbb{G}_a)$,
- its Hodge realization $\mathrm{T}_{\mathbb{Q}}(M_{\mathbb{C}})$: it is the finite dimensional $\mathbb{Q}$-vector space $\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, with $\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})$ the fibered product of $\mathrm{Lie}(G)$ and X over G via the exponential map $\exp : \mathrm{Lie}(G) \rightarrow G$ and the homomorphism $u : X \rightarrow G$. The $\mathbb{Z}$-module is in fact endowed with a structure of $\mathbb{Z}$-mixed Hodge structure, without torsion, of level ≤ 1 and of type $\{(0,0),(0,-1),(-1,0),(-1,-1)\}$.

Since the Hodge realizations attached to 1-motives are mixed Hodge structures, 1-motives are mixed motives. In particular they are the mixed motives coming geometrically from varieties of dimension ≤ 1. In [10, (10.1.8)], Deligne shows that the De Rham and the Hodge realizations of M are isomorphic

$$\beta_M : \mathrm{T}_{\mathrm{dR}}(M) \otimes_K \mathbb{C} \longrightarrow \mathrm{T}_{\mathbb{Q}}(M_{\mathbb{C}}) \otimes_K \mathbb{C}, \tag{0.3}$$ {eq:betaM}

generalizing the isomorphism (0.1) to 1-motives. We can then define the *periods of M* as the coefficients of the matrix which represents the isomorphism (0.3) with respect to K-bases.

In the particular case of 1-motives, Grothendieck's dream came true: using the category of mixed realizations (see [11, 2.3] and [9, (2.2.5)]), it is possible to endow the category of 1-motives with a tannakian structure with rational coefficients, and therefore to define the motivic Galois group

$$\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$$

of a 1-motive M as the fundamental group of the tannakian category $< M >^{\otimes}$ generated by M (see [11, Def 6.1] or [12, Def 8.13]). By [2, Thm 1.2.1], Nori and Ayoub's motivic Galois groups of a 1-motive coincide with that of Grothendieck. Applying the generalizations of Grothendieck's Conjecture of Periods proposed by André to 1-motives we get

Conjecture 0.2 (Generalized Grothendieck's Conjecture of Periods by Y. André)**.** *Let M be a 1-motive defined over an algebraically closed sub-field K of $\mathbb{C}$ which is not necessarily algebraic over $\mathbb{Q}$, then*

$$\mathrm{tran.deg}_{\mathbb{Q}}\, K(\mathrm{periods}(M)) \geq \dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \tag{0.4}$$ {eq:GCP}

where $K(\text{periods}(M))$ *is the field generated over* K *by the periods of* M*.*

In [5] we showed that the conjecture (0.4) applied to a 1-motive of type

$$M = [u : \mathbb{Z}^r \rightarrow \Pi_{j=1}^n \mathcal{E}_j \times \mathbb{G}_m^s]$$

is equivalent to the elliptico-toric conjecture (see [5, 1.1]) which involves elliptic integrals of the first and second kind and logarithms of complex numbers.

Consider now the 1-motive

$$M = [u : \mathbb{Z}^r \longrightarrow G] \tag{0.5}$$

where G is a *non trivial* extension of a product $\Pi_{j=1}^n \mathcal{E}_j$ of pairwise not isogenous elliptic curves by the torus $\mathbb{G}_m^s$. In this paper we introduce *the 1-motivic elliptic conjecture* (§4) which involves elliptic integrals of the first, second and third kind. Our main Theorem is that this 1-motivic elliptic conjecture is equivalent to the Generalized Grothendieck's Conjecture of Periods applied to the 1-motive (0.5) (Theorem 4.1). The presence of elliptic integrals of the third kind in the 1-motivic elliptic conjecture corresponds to the fact that the extension G underlying M is not trivial. If in the 1-motivic elliptic conjecture we assume that the points defining the extension G are trivial, then this conjecture coincides with the elliptico-toric conjecture stated in[5, 1.1] (see Remarks 4.2). Observe that the 1-motivic elliptic conjecture contains also the Schanuel conjecture (see Remarks 4.3).

In Section 1 we recall basic facts about differential forms on elliptic curves.

In Section 2 we study the short exact sequences which involve the Hodge and De Rham realizations of 1-motives and which are induced by the weight filtration of 1-motives. In Lemma 2.2 we prove that instead of working with the 1-motive (0.5) we can work with a direct sum of 1-motives having $r = n = s = 1$. In [8, §2] D. Bertrand has computed the periods of the 1-motive (0.5) with $r = n = s = 1$ using Deligne's construction of a 1-motive starting from an open singular curve. Putting together Lemma 2.2 and Bertrand's calculation of the periods in the case $r = n = s = 1$, we compute explicitly the periods of the 1-motive (0.5) (see Proposition 2.3).

In section 3 we study the motivic Galois group of 1-motives. We will follow neither Ayoub and Nori's theories nor Grothendieck's theory involving mixed realizations, but using [6] we will work in a completely geometrical setting using *algebraic geometry on tannakian categories.* In Theorem 3.4 we compute explicitly the dimension of the unipotent radical of the motivic Galois group of an arbitrary 1-motive over K. Then, as a corollary, we calculate explicitly the dimension of the motivic Galois group of the 1-motive (0.5) (see Corollary 3.7). For this last result, we restrict to work with a 1-motive whose underlying extension G involves a product of elliptic curves, because only in this case we know explicitly the dimension of the reductive part of its motivic Galois group (in general, the dimension of the motivic Galois group of an abelian variety is not known).

In section 4 we state the 1-motivic elliptic conjecture and we prove our main Theorem 4.1.

In section 5 we compute explicitly the Generalized Grothendieck's Conjecture of Periods in the low dimensional case, that is assuming $r = n = s = 1$ in (0.5). In particular we investigate the cases where $\mathrm{End}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$-linear dependence and torsion properties affect the dimension of the unipotent radical of $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$.

We finish with a remark about the Generalized Grothendieck's Conjecture of Periods: as pointed out by André in in [1, §7.5], the transcendent degree of the field generated over K by the periods of a mixed motive is *always upper-bounded* by the dimension of its motivic Galois group. In fact, if we denote respectively by ω_{H} and ω_{dR} the fibre functors Hodge realization and de Rham realizations of the tannakian category of mixed motives, the affine

K-group scheme $\underline{\mathrm{Isom}}_K^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{H}})$ of isomorphisms of fibre functors is an $\omega_{\mathrm{H}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\mathcal{MM}))$-torsor, called the *torsor of periods*, which is endowed with a $\mathbb{C}$-valued point $\beta : \mathrm{Spec}\,(\mathbb{C}) \to \underline{\mathrm{Isom}}_K^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{H}})$ that defines for each object of $\mathcal{MM}$ the isomorphism between its de Rham realization and its Hodge realization (for the smooth and projective algebraic variety X we get (0.1), for the 1-motive M, we get (0.3), ...). If N is any object of $\mathcal{MM}$ defined over K, the isomorphism β_N is a $K(\mathrm{periods}(N))$-rational point of the torsor of periods $\underline{\mathrm{Isom}}_K^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{H}})$. Therefore for any mixed motive N of $\mathcal{MM}$, we have

$$\mathrm{tran.deg}_K\, K(\mathrm{periods}(N)) \leq \dim \omega_{\mathrm{H}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(N))$$

that is

$$\mathrm{tran.deg}_{\mathbb{Q}}\, K(\mathrm{periods}(N)) \leq \dim \omega_{\mathrm{H}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(N)) + \mathrm{tran.deg}_{\mathbb{Q}}\, K.$$

By [2, Thm 1.2.1], the motivic galois group $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ of a 1-motive M coincides with its Hodge realization $\omega_{\mathrm{H}}\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$, which is the Mumford-Tate group of M, and so in the above inequality we can replace $\omega_{\mathrm{H}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ with $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$. In particular, if $K = \overline{\mathbb{Q}}$, the conjecture (0.4) becomes

$$\mathrm{tran.deg}_{\mathbb{Q}}\, \overline{\mathbb{Q}}(\mathrm{periods}(M)) = \dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M). \qquad (0.6)$$

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Notation

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and let K be an algebraically closed sub-field of the field of complex numbers $\mathbb{C}$ which is not necessarily algebraic over $\mathbb{Q}$.

A 1-motive $M = [u : X \to G]$ over K consists of a group scheme X which is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, an extension G of an abelian variety A by a torus T, and a homomorphism $u : X \to G(K)$. In this paper we will consider above all 1-motives in which $X = \mathbb{Z}^r$, and G is an extension of a finite product $\Pi_{j=1}^n \mathcal{E}_j$ of elliptic curves by the torus $\mathbb{G}_m^s$ (here r, n and s are integers bigger or equal to 0).

There is a more symmetrical definition of 1-motives. In fact to have the 1-motive $M = [u : \mathbb{Z}^r \to G]$ is equivalent to have the 7-tuple $(\mathbb{Z}^r, \mathbb{Z}^s, \Pi_{j=1}^n \mathcal{E}_j, \Pi_{j=1}^n \mathcal{E}_j^*, v, v^*, \psi)$ where

- $\mathbb{Z}^s$ is the character group of the torus $\mathbb{G}_m^s$ underlying the 1-motive M.
- $v : \mathbb{Z}^r \to \Pi_{j=1}^n \mathcal{E}_j$ and $v^* : \mathbb{Z}^s \to \Pi_{j=1}^n \mathcal{E}_j^*$ are two morphisms of K-group varieties (here $\mathcal{E}_j^* := \underline{\mathrm{Ext}}^1(\mathcal{E}_j, \mathbb{G}_m)$ is the Cartier dual of the elliptic curve $\mathcal{E}_j$). To have the morphism v is equivalent to have r points $P_k = (P_{1k}, \ldots, P_{nk})$ of $\Pi_{j=1}^n \mathcal{E}_j(K)$ with $k = 1, \ldots, r$, whereas to have the morphism v^* is equivalent to have s points $Q_i = (Q_{1i}, \ldots, Q_{ni})$ of $\Pi_{j=1}^n \mathcal{E}_j^*(K)$ with $i = 1, \ldots, s$. Via the isomorphism $\underline{\mathrm{Ext}}^1(\Pi_{j=1}^n \mathcal{E}_j, \mathbb{G}_m^s) \cong (\Pi_{j=1}^n \mathcal{E}_j^*)^s$, to have the s points $Q_i = (Q_{1i}, \ldots, Q_{ni})$ is equivalent to have the extension G of $\Pi_{j=1}^n \mathcal{E}_j$ by $\mathbb{G}_m^s$.
- ψ is a trivialization of the pull-back $(v, v^*)^*\mathcal{P}$ via (v, v^*) of the Poincaré biextension $\mathcal{P}$ of $(\Pi_{j=1}^n \mathcal{E}_j, \Pi_{j=1}^n \mathcal{E}_j^*)$ by $\mathbb{G}_m$. To have this trivialization ψ is equivalent to have points $R_k \in G(K)$ with $k = 1, \ldots, r$ such that the image of R_k via the projection $G \to \Pi_{j=1}^n \mathcal{E}_j$ is $P_k = (P_{1k}, \ldots, P_{nk})$, and so to have the morphism $u : \mathbb{Z}^r \to G$.

The index k, $0 \leq k \leq r$, is related to the lattice $\mathbb{Z}^r$, the index j, $0 \leq j \leq n$, is related to the elliptic curves, and the index i, $0 \leq i \leq s$, is related to the torus $\mathbb{G}_m^s$. For $j = 1, \ldots, n$, we index with a j all the data related to the elliptic curve $\mathcal{E}_j$: for example we denote by $\wp_j(z)$ its Weierstrass $\wp$-function of $\mathcal{E}_j$, by ω_{j1}, ω_{j2} its periods, ...

On a 1-motive $M = [u : X \to G]$ is defined an increasing filtration $\mathrm{W}_\bullet$, called the *weight filtration* of M: $\mathrm{W}_0(M) = M, \mathrm{W}_{-1}(M) = [0 \to G], \mathrm{W}_{-2}(M) = [0 \to T]$. If we set $\mathrm{Gr}_n^{\mathrm{W}} := \mathrm{W}_n/\mathrm{W}_{n-1}$, we have $\mathrm{Gr}_0^{\mathrm{W}}(M) = [X \to 0], \mathrm{Gr}_{-1}^{\mathrm{W}}(M) = [0 \to A]$ and $\mathrm{Gr}_{-2}^{\mathrm{W}}(M) = [0 \to T]$.

Two 1-motives $M_i = [u_i : X_i \to G_i]$ over K (for $i = 1, 2$) are isogeneous is there exists a morphism of complexes $(f_X, f_G) : M_1 \to M_2$ such that $f_X : X_1 \to X_2$ is injective with finite cokernel, and $f_G : G_1 \to G_2$ is surjective with finite kernel. Now, since [10, Thm (10.1.3)] is true modulo isogenies, two isogeneous 1-motives have the same periods. Moreover, two isogeneous 1-motives build the same tannakian category and so they have the same motivic Galois group. Hence in this paper *we can work modulo isogenies.* In particular the elliptic curves $\mathcal{E}_1, \ldots, \mathcal{E}_n$ will be pairwise not isogenous.

{EllipticIntegral}

1. Elliptic integrals of third kind

Let $\mathcal{E}$ be an elliptic curve defined over $\mathbb{C}$ with Weierstrass coordinate functions x and y. Set $\Lambda := \mathrm{H}_1(\mathcal{E}(\mathbb{C}), \mathbb{Z})$. Let $\wp(z)$ be the Weierstrass $\wp$-function relative to the lattice Λ: it is a meromorphic function on $\mathbb{C}$ having a double pole with residue zero at each point of Λ and no other poles. Consider the elliptic exponential

$$\begin{aligned} \exp_{\mathcal{E}} : \mathbb{C} &\longrightarrow \mathcal{E}(\mathbb{C}) \subseteq \mathbb{P}^2(\mathbb{C}) \\ z &\longmapsto \exp_{\mathcal{E}}(z) = [\wp(z), \wp(z)', 1] \end{aligned}$$

whose kernel is the lattice Λ. In particular the map $\exp_{\mathcal{E}}$ induces a complex analytic isomorphism between the quotient $\mathbb{C}/\Lambda$ and the $\mathbb{C}$-valuated points of the elliptic curve $\mathcal{E}$. In this paper, we will use small letters for elliptic logarithms of points on elliptic curves which are written with capital letters, that is $\exp_{\mathcal{E}}(p) = P \in \mathcal{E}(\mathbb{C})$ for any $p \in \mathbb{C}$.

Let $\sigma(z)$ be the Weierstrass σ-function relative to the lattice Λ: it is a holomorphic function on all of $\mathbb{C}$ and it has simple zeros at each point of Λ and no other zeros. Finally let $\zeta(z)$ be the Weierstrass ζ-function relative to the lattice Λ: it is a meromorphic function on $\mathbb{C}$ with simple poles at each point of Λ and no other poles. We have the well-known equalities

$$\frac{d}{dz} \log \sigma(z) = \zeta(z) \quad \text{and} \quad \frac{d}{dz}\zeta(z) = -\wp(z).$$

Recall that a meromorphic differential 1-form is of the *first kind* if it is holomorphic everywhere, of the *second kind* if the residue at any pole vanishes, and of the *third kind* in general. On the elliptic curve $\mathcal{E}$ we have the following differential 1-forms:

(1) the differential of the first kind

$$\omega = \frac{dx}{y}, \tag{1.1}$$

which has neither zeros nor poles and which is invariant under translation. We have that $\exp_{\mathcal{E}}^*(\omega) = dz$.

(2) the differential of the second kind

$$\eta = -\frac{xdx}{y}. \tag{1.2}$$

In particular $\exp_{\mathcal{E}}^*(\eta) = -\wp(z)dz$ which has a double pole with residue zero at each point of Λ and no other poles.

(3) the differential of the third kind

$$\xi_Q = \frac{1}{2}\frac{y-y(Q)}{x-x(Q)}\frac{dx}{y} \tag{1.3}$$

for any point Q of $\mathcal{E}(\mathbb{C}), Q \neq 0$. The residue divisor of ξ_Q is $-(0)+(-Q)$. If we denote $q \in \mathbb{C}$ an elliptic logarithm of the point Q, that is $\exp_{\mathcal{E}}(q) = Q$, we have that

$$\exp_{\mathcal{E}}^*(\xi_Q) = \frac{1}{2}\frac{\wp'(z)-\wp'(q)}{\wp(z)-\wp(q)}dz$$

which has residue -1 at each point of Λ.

The 1-dimensional $\mathbb{C}$-vector space of differentials of the first kind is $\mathrm{H}^0(\mathcal{E},\Omega^1_{\mathcal{E}})$, the 1-dimensional $\mathbb{C}$-vector space of differentials of the second kind modulo holomorphic differentials and exact differentials is $\mathrm{H}^1(\mathcal{E},\mathcal{O}_{\mathcal{E}})$. In particular the first De Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E})$ of the elliptic curve $\mathcal{E}$ is the direct sum $\mathrm{H}^0(\mathcal{E},\Omega^1_{\mathcal{E}}) \oplus \mathrm{H}^1(\mathcal{E},\mathcal{O}_{\mathcal{E}})$ of these two spaces and it has dimension 2. The $\mathbb{C}$-vector space of differentials of the third kind is infinite dimensional.

The inverse map of the complex analytic isomorphism $\mathbb{C}/\Lambda \to \mathcal{E}(\mathbb{C})$ induced by the elliptic exponential is given by the integration $\mathcal{E}(\mathbb{C}) \to \mathbb{C}/\Lambda, P \to \int_O^P \omega \mod \Lambda$, where O is the neutral element for the group law of the elliptic curve.

Let γ_1, γ_2 be two closed paths on $\mathcal{E}(\mathbb{C})$ which build a basis of $\mathrm{H}_1(\mathcal{E}_{\mathbb{C}},\mathbb{Q})$. Then *the elliptic integrals of the first kind* $\int_{\gamma_i}\omega = \omega_i$ $(i=1,2)$ are *the periods of the Weierstrass $\wp$-function*:

(1.4) {eq:periods-wp} $\wp(z+\omega_i) = \wp(z)$ for $i=1,2$.

Moreover *the elliptic integrals of the second kind* $\int_{\gamma_i}\eta = \eta_i$ $(i=1,2)$ are *the quasi-periods of the Weierstrass ζ-function*:

(1.5) {eq:periods-zeta} $\zeta(z+\omega_i) = \zeta(z)+\eta_i$ for $i=1,2$.

Consider Serre's function

(1.6) {eq:def-fq} $f_q(z) = \frac{\sigma(z+q)}{\sigma(z)\sigma(q)}e^{-\zeta(q)z}$ with $q \in \mathbb{C}\setminus\Lambda$

whose logarithmic differential is

(1.7) {eq:expEXiq} $\frac{f_q'(z)}{f_q(z)}dz = \frac{1}{2}\frac{\wp'(z)-\wp'(q)}{\wp(z)-\wp(q)}dz = \exp_{\mathcal{E}}^*(\xi_Q)$

(see [22] and [8, §2]). *The exponentials of the elliptic integrals of the third kind* $\int_{\gamma_i}\xi_Q = \eta_i q - \omega_i\zeta(q)$ $(i=1,2)$ are *the quasi-quasi periods* of the function $f_q(z)$:

(1.8) {eq:periods-fq} $f_q(z+\omega_i) = f_q(z)e^{\eta_i q-\omega_i\zeta(q)}$ for $i=1,2$.

As observed in [22], we have that

(1.9) {eq:fq-sigma} $\frac{f_q(z_1+z_2)}{f_q(z_1)f_q(z_2)} = \frac{\sigma(q+z_1+z_2)\sigma(q)\sigma(z_1)\sigma(z_2)}{\sigma(q+z_1)\sigma(z_1+z_2)\sigma(q+z_2)}.$

Consider now an extension G of our ellitic curve $\mathcal{E}$ by $\mathbb{G}_m$, which is defined over $\mathbb{C}$. Via the isomorphism $\mathrm{Pic}^0(\mathcal{E}) \cong \mathcal{E}^* = \underline{\mathrm{Ext}}^1(\mathcal{E},\mathbb{G}_m)$, to have the extension G is equivalent to have a divisor $D=(-Q)-(0)$ of $\mathrm{Pic}^0(\mathcal{E})$ or a point $-Q$ of $\mathcal{E}^*(\mathbb{C})$. In this paper we identify $\mathcal{E}$ with $\mathcal{E}^*$. A basis of the first De Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(G)$ of the extension G is given by $\{\omega,\eta,\xi_Q\}$. Consider the semi-abelian exponential

$$\exp_G : \mathbb{C}^2 \longrightarrow G(\mathbb{C}) \subseteq \mathbb{P}^5(\mathbb{C})$$

$$(w,z) \longmapsto \exp_G(w,z) = \sigma(z)^3\Big[\wp(z),\wp(z)',1,e^w f_q(z),e^w f_q(z)\Big(\wp(z)+\frac{\wp'(z)-\wp'(q)}{\wp(z)-\wp(q)}\Big)\Big]$$

whose kernel is $\mathrm{H}_1(G(\mathbb{C}),\mathbb{Z})$. A basis of the Hodge realization $\mathrm{H}_1(G(\mathbb{C}),\mathbb{Q})$ of the extension G is given by a closed path δ_Q around Q on $G(\mathbb{C})$ and two closed paths $\tilde{\gamma}_1,\tilde{\gamma}_2$ on $G(\mathbb{C})$ which lift a basis $\{\gamma_1,\gamma_2\}$ of $\mathrm{H}_1(\mathcal{E}_{\mathbb{C}},\mathbb{Q})$ via the surjection $\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Q})\to\mathrm{H}_1(\mathcal{E}_{\mathbb{C}},\mathbb{Q})$. We have that

$$\exp_G^*(\xi_Q)=dw+\frac{f_q'(z)}{f_q(z)}dz. \tag{1.10}$$

{periods}

2. Periods of 1-motives involving elleptic curves

Let $M=[u:X\to G]$ be a 1-motive over K with G an extension of an abelian variety A by a torus T. As recalled in the introduction, to the 1-motive $M_{\mathbb{C}}$ obtained from M extending the scalars from K to $\mathbb{C}$, we can associate its Hodge realization $\mathrm{T}_{\mathbb{Q}}(M_{\mathbb{C}})=(\mathrm{Lie}(G_{\mathbb{C}})\times_G X)\otimes\mathbb{Q}$ which is endowed with the weight filtration (defined over the integers) $\mathrm{W}_0\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})=\mathrm{Lie}(G_{\mathbb{C}})\times_G X,\mathrm{W}_{-1}\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})=\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Z}),\mathrm{W}_{-2}\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})=\mathrm{H}_1(T_{\mathbb{C}},\mathbb{Z})$. In particular we have that $\mathrm{Gr}_0^{\mathrm{W}}\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})\cong X,\mathrm{Gr}_{-1}^{\mathrm{W}}\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})\cong\mathrm{H}_1(A_{\mathbb{C}},\mathbb{Z})$ and $\mathrm{Gr}_{-2}^{\mathrm{W}}\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})\cong\mathrm{H}_1(T_{\mathbb{C}},\mathbb{Z})$.

Moreover to M we can associate its De Rham realization $\mathrm{T}_{\mathrm{dR}}(M)=\mathrm{Lie}(G^\natural)$, where $M^\natural=[X\to G^\natural]$ is the universal vectorial extension of M, which is endowed with the Hodge filtration $\mathrm{F}^0\mathrm{T}_{\mathrm{dR}}(M)=\ker\big(\mathrm{Lie}(G^\natural)\to\mathrm{Lie}(G)\big)$.

The weight filtration induces for the Hodge realizations the short exact sequence

$$0\longrightarrow\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Z})\longrightarrow\mathrm{T}_{\mathbb{Z}}(M_{\mathbb{C}})\longrightarrow\mathrm{T}_{\mathbb{Z}}(X)\longrightarrow 0 \tag{2.1}$$

which is not split in general. On the other hand, for the De Rham realizations we have that

Lemma 2.1. *The short exact sequence, induced by the weight filtration,*

$$0\longrightarrow\mathrm{T}_{\mathrm{dR}}(G)\longrightarrow\mathrm{T}_{\mathrm{dR}}(M)\longrightarrow\mathrm{T}_{\mathrm{dR}}(X)\longrightarrow 0 \tag{2.2}$$

is canonically split.

Proof. Consider the short exact sequence $0\to G\to M\to X[1]\to 0$. Applying $\mathrm{Hom}(-,\mathbb{G}_a)$ we get the short exact sequence of finitely dimensional K-vector spaces

$$0\longrightarrow\mathrm{Hom}(X,\mathbb{G}_a)\longrightarrow\mathrm{Ext}^1(M,\mathbb{G}_a)\to\mathrm{Ext}^1(G,\mathbb{G}_a)\longrightarrow 0$$

Taking the dual we obtain the short exact sequence

$$0\longrightarrow\mathrm{Hom}(\mathrm{Ext}^1(G,\mathbb{G}_a),\mathbb{G}_a)\longrightarrow\mathrm{Hom}(\mathrm{Ext}^1(M,\mathbb{G}_a),\mathbb{G}_a)\longrightarrow X\to 0$$

which is split since $\mathrm{Ext}^1(X,\mathbb{G}_a)=0$. Now consider the composite of the section $X\to\mathrm{Hom}(\mathrm{Ext}^1(M,\mathbb{G}_a),\mathbb{G}_a)$ with the inclusion $\mathrm{Hom}(\mathrm{Ext}^1(M,\mathbb{G}_a),\mathbb{G}_a)\to G^\natural$. Recalling that $\mathrm{F}^0\mathrm{T}_{\mathrm{dR}}(M)\cong\mathrm{Hom}(\mathrm{Ext}^1(M,\mathbb{G}_a),\mathbb{G}_a)$, taking Lie algebras we get the arrow $\mathrm{T}_{\mathrm{dR}}(X)=X\otimes K\to\mathrm{F}^0\mathrm{T}_{\mathrm{dR}}(M)\to\mathrm{T}_{\mathrm{dR}}(M)=\mathrm{Lie}(G^\natural)$ which is a section of the exact sequence (2.2). □

By the above Lemma, if we denote by $\mathrm{H}_{\mathrm{dR}}(M)$ the dual K-vector space of $\mathrm{T}_{\mathrm{dR}}(M)$ we have that

$$\mathrm{H}_{\mathrm{dR}}(M)=\mathrm{H}^1_{\mathrm{dR}}(G)\oplus\mathrm{H}^1_{\mathrm{dR}}(X). \tag{2.3}$$

Consider now a 1-motive $M=[u:\mathbb{Z}^r\to G]$ defined over K, where G is an extension of a finite product $\Pi_{j=1}^n\mathcal{E}_j$ of elliptic curves by the torus $\mathbb{G}_m^s$. Let $\{z_k\}_{k=1,\dots,r}$ a basis of $\mathbb{Z}^r$ and let $\{t_i\}_{i=1,\dots,s}$ a basis of the character group $\mathbb{Z}^s$ of $\mathbb{G}_m^s$. For the moment, in order to simplify notation, denote by A the product of elliptic curves $\Pi_{j=1}^n\mathcal{E}_j$. Denote by G_i the push-out of G by $t_i:\mathbb{G}_m^s\to\mathbb{G}_m$, which is the extension of A by $\mathbb{G}_m$ parameterized by the point $v^*(t_i)=Q_i=(Q_{1i},\dots,Q_{ni})$, and by R_{ik} the K-rational point of G_i above $v(z_k)=P_k=(P_{1k},\dots,P_{nk})$. Consider the 1-motive defined over K,

$$M_{ik}=[u_{ik}:z_k\mathbb{Z}\to G_i]$$

with $u_{ik}(z_k) = R_{ik}$ for $i = 1, \ldots, s$ and $k = 1, \ldots, r$. In [4, Thm 1.7] we have proved geometrically that the 1-motives $M = [u : \mathbb{Z}^r \to G]$ and $\oplus_{i=1}^s \oplus_{k=1}^r M_{ik}$ generate the same tannakian category. Via the isomorphism $\underline{\mathrm{Ext}}^1(\Pi_{j=1}^n \mathcal{E}_j, \mathbb{G}_m) \cong \Pi_{j=1}^n \underline{\mathrm{Ext}}^1(\mathcal{E}_j, \mathbb{G}_m)$, for $i = 1, \ldots, s$, the extension G_i of A by $\mathbb{G}_m$ parameterized by the point $v^*(y_i) = Q_i = (Q_{1i}, \ldots, Q_{ni})$ corresponds to the product of extensions $G_{1i} \times G_{2i} \times \cdots \times G_{ni}$ where G_{ji} is an extension of $\mathcal{E}_j$ by $\mathbb{G}_m$ parameterized by the point Q_{ji}. Via the above isomorphism the point R_{ik} the K-rational point of G_i living above $P_k = (P_{1k}, \ldots, P_{nk})$ corresponds to the K-rational points $(R_{1ik}, \ldots, R_{nik})$ with $R_{jik} \in G_{ji}(K)$ living above $P_{jk} \in \mathcal{E}_j(K)$ for $j = 1, \ldots, n$. Consider the 1-motive defined over K,

$$M_{jik} = [u_{jik} : z_k\mathbb{Z} \to G_{ji}] \qquad (2.4) \quad \{\texttt{eq:jik}\}$$

with $u_{jik}(z_k) = R_{jik}$ for $i = 1, \ldots, s$, $k = 1, \ldots, r$ and $j = 1, \ldots, n$. Let $(l_{jik}, p_{jk}) \in \mathbb{C}^2$ be a semi-abelian logarithm of R_{jik}, that is

$$\exp_{G_{ji}}(l_{jik}, p_{jk}) = R_{jik}. \qquad (2.5) \quad \{\texttt{eq:l}\}$$

{lem:decomposition

Lemma 2.2. *The 1-motives M and $\oplus_{i=1}^s \oplus_{k=1}^r \oplus_{j=1}^n M_{jik}$ generate the same tannakian category.*

Proof. As in [4, Thm 1.7] we will work geometrically and because of loc. cit. it is enough to show that the 1-motives $\oplus_{i=1}^s \oplus_{k=1}^r M_{ik}$ and $\oplus_{i=1}^s \oplus_{k=1}^r \oplus_{j=1}^n M_{jik}$ generate the same tannakian category. Clearly

$$\oplus_{j=1}^n \Big(\oplus_{i=1}^s \oplus_{k=1}^r M_{ik} / [0 \to \Pi_{\substack{1 \leq l \leq n \\ l \neq j}} G_{li}] \Big) = \oplus_{i=1}^s \oplus_{k=1}^r \oplus_{j=1}^n M_{jik}$$

and so $< \oplus_{i=1}^s \oplus_{k=1}^r \oplus_{j=1}^n M_{jik} >^{\otimes} \; \subset \; < \oplus_{i=1}^s \oplus_{k=1}^r M_{ik} >^{\otimes}$. On the other hand, if $\mathrm{d}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}^n$ is the diagonal morphism, for fixed i and k we have that

$$\oplus_{j=1}^n M_{jik} / [\mathbb{Z}^n / \mathrm{d}_{\mathbb{Z}}(\mathbb{Z}) \to 0] = [\Pi_j u_{jik} : \mathrm{d}_{\mathbb{Z}}(\mathbb{Z}) \longrightarrow G_{1i} \times G_{2i} \times \cdots \times G_{ni}] = [u_{ik} : \mathbb{Z} \longrightarrow G_i] = M_{ik}$$

and so

$$\oplus_{i=1}^s \oplus_{k=1}^r \Big(\oplus_{j=1}^n M_{jik} / [\mathbb{Z}^n / \mathrm{d}_{\mathbb{Z}}(\mathbb{Z}) \to 0] \Big) = \oplus_{i=1}^s \oplus_{k=1}^r M_{ik}$$

that is $< \oplus_{i=1}^s \oplus_{k=1}^r M_{ik} >^{\otimes} \; \subset \; < \oplus_{i=1}^s \oplus_{k=1}^r \oplus_{j=1}^n M_{jik} >^{\otimes}$. □

The matrix, which represents the isomorphism (0.3) for the 1-motive $M = [u : \mathbb{Z}^r \to G]$, where G is an extension of $\Pi_{j=1}^n \mathcal{E}_j$ by $\mathbb{G}_m^s$, is a huge matrix difficult to write down, but the above Lemma implies that, instead of studying this huge matrix, it is enough to study the rsn matrices which represent the isomorphism (0.3) for the rsn 1-motives $M_{jik} = [u_{jik} : z_k\mathbb{Z} \to G_{ji}]$.

Following [8, §2], now we compute explicitly the periods of the 1-motive $M = [u : \mathbb{Z} \to G]$, where G is an extension of one elliptic curve $\mathcal{E}$ by the torus $\mathbb{G}_m$. We need Deligne's construction of M starting from an open singular curve (see [10, (10.3.1)-(10.3.2)-(10.3.3)]) that we recall briefly.

Via the isomorphism $\mathrm{Pic}^0(\mathcal{E}) \cong \mathcal{E}^* = \underline{\mathrm{Ext}}^1(\mathcal{E}, \mathbb{G}_m)$, to have the extension G of $\mathcal{E}$ by $\mathbb{G}_m$ underlying the 1-motive M is equivalent to have the divisor $D = (-Q) - (0)$ of $\mathrm{Pic}^0(\mathcal{E})$ or the point $-Q$ of $\cong \mathcal{E}^*$. We assume Q to be a non torsion point. According to [18, page 227], to have the point $u(1) = R \in G(K)$ is equivalent to have a couple

$$(P, g_R) \in \mathcal{E}(K) \times K(\mathcal{E})^*$$

where $\pi(R) = P \in \mathcal{E}(K)$ (here $\pi : G \to \mathcal{E}$ the surjective morphism of group varieties underlying the extension G), and where $g_R : \mathcal{E} \to \mathbb{G}_m, x \mapsto R + \rho(x) - \rho(x + P)$ (here $\rho : \mathcal{E} \to G$ a section of π), is a rational function on $\mathcal{E}$ whose divisor is $T_P^* D - D = (-Q + P) -$

$(P)-(-Q)+(0)$ (here $T_P:\mathcal{E}\to\mathcal{E}$ is the translation by the point P). We assume also R to be a non torsion point. Let $p,q\in\mathbb{C}$ be elliptic logarithms of the points P,Q respectively.

Now pinch the elliptic curve $\mathcal{E}$ at the two points $-Q$ and O and puncture it at two K-rational points P_2 and P_1 whose difference (according to the group law of $\mathcal{E}$) is P, that is $P=P_2-P_1$. The motivic H^1 of the open singular curve obtained in this way from $\mathcal{E}$ is the 1-motive $M=[u:\mathbb{Z}\to G]$, with $u(1)=R$. We will apply Deligne's construction to each 1-motive $M_{jik}=[u_{jik}:z_k\mathbb{Z}\to G_{ji}]$ with $u_{jik}(z_k)=R_{jik}$.

{proof-periods}

Proposition 2.3. *Choose the following basis of the $\mathbb{Q}$-vector space $\mathrm{T}_{\mathbb{Q}}(M_{jik\,\mathbb{C}})$:*

- *a closed path $\delta_{Q_{ji}}$ around $-Q_{ji}$ on $G_{ji}(\mathbb{C})$;*
- *two closed paths $\tilde\gamma_{j1},\tilde\gamma_{j2}$ on $G_{ji}(\mathbb{C})$ which lift the basis $\{\gamma_{j1},\gamma_{j2}\}$ of $\mathrm{H}_1(\mathcal{E}_{j\,\mathbb{C}},\mathbb{Q})$ via the surjection $\mathrm{H}_1(G_{ji\,\mathbb{C}},\mathbb{Q})\to\mathrm{H}_1(\mathcal{E}_{j\,\mathbb{C}},\mathbb{Q})$; and*
- *a closed path $\beta_{R_{jik}}$, which lifts the basis $\{z_k\}$ of $\mathrm{T}_{\mathbb{Q}}(z_k\mathbb{Z})$ via the surjection $\mathrm{T}_{\mathbb{Q}}(M_{jik\,\mathbb{C}})\to\mathrm{T}_{\mathbb{Q}}(z_k\mathbb{Z})$, and whose restriction to $\mathrm{H}_1(G_{ji\,\mathbb{C}},\mathbb{Q})$ is a closed path $\beta_{R_{jik}|G_{ji}}$ on $G_{ji}(\mathbb{C})$ having the following properties: $\beta_{R_{jik}|G_{ji}}$ lifts a path $\beta_{P^1_{jk}P^2_{jk}}$ on $\mathcal{E}_j(\mathbb{C})$ from P^1_{jk} to P^2_{jk} (with $P^2_{jk}-P^1_{jk}=P_{jk}$) via the surjection $\mathrm{H}_1(G_{ji\,\mathbb{C}},\mathbb{Q})\to\mathrm{H}_1(\mathcal{E}_{j\,\mathbb{C}},\mathbb{Q})$, and its restriction to $\mathrm{H}_1(\mathbb{G}_m,\mathbb{Q})$ is a path β_{jik} on $\mathbb{G}_m(\mathbb{C})=\mathbb{C}^*$ from 1 to l_{jik}(2.5);*

and the following basis of the K-vector space $\mathrm{H}_{\mathrm{dR}}(M_{jik})$:

- *the differentials of the first kind $\omega_j=\frac{dx_j}{y_j}$ (1.1) and of the second kind $\eta_j=-\frac{x_jdx_j}{y_j}$ (1.2) of $\mathcal{E}_j$;*
- *the differential of the third kind $\xi_{Q_{ji}}=\frac{1}{2}\frac{y_j-y_j(Q_{ji})}{x_j-x_j(Q_{ji})}\frac{dx_i}{y_j}$ (1.3), whose residue divisor is $D=(-Q_{ji})-(0)$ and which lifts the basis $\{\frac{dt_i}{t_i}\}$ of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m)$ via the surjection $\mathrm{H}^1_{\mathrm{dR}}(G_{ji})\to\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m)$;*
- *the differential df_j of a rational function f_j on $\mathcal{E}_j$ such that $f_j(P^2_{jk})$ differs from $f_j(P^1_{jk})$ by 1.*

These periods of the 1-motive $M=[u:\mathbb{Z}^r\to G]$, where G is an extension of $\Pi^n_{j=1}\mathcal{E}_j$ by $\mathbb{G}^s_m$, are then

$$1,\omega_{j1},\omega_{j2},\eta_{j1},\eta_{j2},p_{jk},\zeta_j(p_{jk}),\eta_{j1}q_{ji}-\omega_{j1}\zeta_j(q_{ji}),\eta_{j2}q_{ji}-\omega_{j2}\zeta_j(q_{ji}),\log f_{q_{ji}}(p_{jk})+l_{jik},2i\pi$$

with $e^{l_{jik}}\in K^*$, for $j=1,\dots,n,k=1,\dots,r$ and $i=1,\dots,s$.

Proof. By Lemma 2.2, the 1-motives $M=[u:\mathbb{Z}^r\to G]$ and $\oplus^s_{i=1}\oplus^r_{k=1}\oplus^n_{j=1}[u_{jik}:z_k\mathbb{Z}\to G_{ji}]$ have the same periods and therefore, we are reduced to prove the case $r=n=s=1$.

Consider the 1-motive $M=[u:z\mathbb{Z}\to G]$, where G is an extension of an elliptic curve $\mathcal{E}$ by $\mathbb{G}_m$ parameterized by $v^*(t)=-Q\in\mathcal{E}(K)$, and $u(z)=R$ is a point of $G(K)$ living over $v(z)=P\in\mathcal{E}(K)$. Let $(l,p)\in\mathbb{C}^2$ be a semi-abelian logarithm of R, that is

$$\exp_G(l,p)=R.$$

In particular $\exp_{\mathcal{E}}(p)=P$. Let P_2 and P_1 K-rational points whose difference is P.
Because of the weight filtration of M we have the non-split short exact sequence

$$0\longrightarrow\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E})\longrightarrow\mathrm{H}^1_{\mathrm{dR}}(G)\longrightarrow\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m)\longrightarrow 0$$

As K-basis of $\mathrm{H}^1_{\mathrm{dR}}(G)$ we choose the differentials of the first kind ω and of the second kind η of $\mathcal{E}$, and the differential of the third kind ξ_Q, which lifts the only element $\frac{dt}{t}$ of the basis of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_m)$. Then, because of the decomposition (2.3), we complete the basis of $\mathrm{H}_{\mathrm{dR}}(M)$ with the differential df of a rational function f on $\mathcal{E}$ such that $f(P_2)$ differs from $f(P_1)$ by 1.
Always because of the weight filtration of M we have the non-split short exact sequence

$$0\longrightarrow\mathrm{H}_1(\mathbb{G}_m,\mathbb{Z})\longrightarrow\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Z})\longrightarrow\mathrm{H}_1(\mathcal{E}_{\mathbb{C}},\mathbb{Z})\longrightarrow 0$$

As $\mathbb{Q}$-basis of $\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Q})$ we choose a closed path δ_Q around $-Q$, and two closed paths $\tilde{\gamma}_1,\tilde{\gamma}_2$ which lift the basis γ_1,γ_2 of $\mathrm{H}_1(\mathcal{E}_{\mathbb{C}},\mathbb{Q})$. Because of the non-split exact sequence (2.1), we complete the basis of $\mathrm{T}_{\mathbb{Q}}(M)$ with a closed path β_R, which lifts the only element z of the basis of $\mathrm{T}_{\mathbb{Q}}(z\mathbb{Z})=\mathbb{Z}\otimes\mathbb{Q}$ via the surjection $\mathrm{T}_{\mathbb{Q}}(M_{\mathbb{C}})\to\mathrm{T}_{\mathbb{Q}}(z\mathbb{Z})$, and whose restriction to $\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Q})$ is a closed path $\beta_{R|G}$ on $G(\mathbb{C})$ having the following properties: $\beta_{R|G}$ lifts a path $\beta_{P_1P_2}$ on $\mathcal{E}(\mathbb{C})$ from P_1 to P_2, and its restriction to $\mathrm{H}_1(\mathbb{G}_m,\mathbb{Q})$ is a path β_l on $\mathbb{G}_m(\mathbb{C})=\mathbb{C}^*$ from 1 to l. With respect to these bases of $\mathrm{T}_{\mathbb{Q}}(M)$ and $\mathrm{H}_{\mathrm{dR}}(M)$, the matrix which represents the isomorphism (0.3) for the 1-motive $M=[u:z\mathbb{Z}\to G]$ is

$$\begin{pmatrix} \int_{\beta_R} df & \int_{\beta_{P_1P_2}}\omega & \int_{\beta_{P_1P_2}}\eta & \int_{\beta_{R|G}}\xi_Q \\ \int_{\tilde{\gamma}_1} df & \int_{\gamma_1}\omega & \int_{\gamma_1}\eta & \int_{\tilde{\gamma}_1}\xi_Q \\ \int_{\tilde{\gamma}_2} df & \int_{\gamma_2}\omega & \int_{\gamma_2}\eta & \int_{\tilde{\gamma}_2}\xi_Q \\ \int_{\delta_Q} df & \int_{\delta_Q}\omega & \int_{\delta_Q}\eta & \int_{\delta_Q}\xi_Q \end{pmatrix} \tag{2.6}$$

{eq:matrix-integrales}

Recalling that $\exp_{\mathcal{E}}^*(\omega)=dz, \exp_{\mathcal{E}}^*(\eta)=d\zeta(z)$, (1.7) and (1.10) we can now compute explicitly all these integrals:

- $\int_{\beta_R} df = f(P_2)-f(P_1)=1$,
- $\int_{\tilde{\gamma}_1} df=\int_{\tilde{\gamma}_2} df=\int_{\delta_Q} df=0$ because of the decomposition (2.3),
- $\int_{\beta_{P_1P_2}}\omega=\int_{p_1}^{p_2} dz=p_2-p_1=p$,
- $\int_{\gamma_i}\omega=\int_0^{\omega_i} dz=\omega_i$ for $i=1,2$,
- $\int_{\delta_Q}\omega=\int_{\delta_Q}\eta=0$ since the image of δ_Q via $\mathrm{H}_1(G_{\mathbb{C}},\mathbb{Q})\to\mathrm{H}_1(\mathcal{E}_{\mathbb{C}},\mathbb{Q})$ is zero,
- $\int_{\gamma_i}\eta=\int_0^{\omega_i} d\zeta=\zeta(\omega_i)-\zeta(0)=\eta_i$ for $i=1,2$,
- $\int_{\beta_{P_1P_2}}\eta=\int_{p_1}^{p_2} d\zeta(z)=\zeta(p_2)-\zeta(p_1)$.

By the pseudo addition formula for the Weierstrass ζ-function (see [24, Example 2, p 451]), $\zeta(z+y)-\zeta(z)-\zeta(y)=\frac{1}{2}\frac{\wp'(z)-\wp'(y)}{\wp(z)-\wp(y)}\in K(\mathcal{E})$, and so it exists a rational function g on $\mathcal{E}$ such that $g(p_2)-g(p_1)=-\zeta(p+p_1)+\zeta(p)+\zeta(p_1)$. Since the differential of the second kind η lives in the quotient space $\mathrm{H}^1(\mathcal{E},\mathcal{O}_{\mathcal{E}})$, we can add to the class of η the exact differential dg, getting

- $\int_{\beta_{P_1P_2}}(\eta+dg)=\int_{p_1}^{p_2}(d\zeta(z)+dg)=\zeta(p_2)-\zeta(p_1)+g(p_2)-g(p_1)=\zeta(p)$,
- $\int_{\beta_{R|G}}\xi_Q=\int_0^l dw+\int_{p_1}^{p_2}\frac{f_q'(z)}{f_q(z)}dz=l+\int_{p_1}^{p_2} d\log f_q(z)=l+\log\frac{f_q(p_2)}{f_q(p_1)}$.

By [24, 20-53], the quotient of σ-functions is a rational function on $\mathcal{E}$, and so from the equality (1.9) it exists a rational function $g_q(z)$ on $\mathcal{E}$ such that $\frac{g_q(p_2)}{g_q(p_1)}=(\frac{f_q(p+p_1)}{f_q(p)f_q(p_1)})^{-1}$, getting

- $\int_{\beta_{R|G}}(\xi_Q+d\log g_q(z))=\int_0^l dw+\int_{p_1}^{p_2}(d\log f_q(z)+d\log g_q(z))=l+\log\big(\frac{f_q(p_2)}{f_q(p_1)}\frac{g_q(p_2)}{g_q(p_1)}\big)=l+\log\big(\frac{f_q(p_2)}{f_q(p_1)}\frac{f_q(p)f_q(p_1)}{f_q(p_2)}\big)=l+\log(f_q(p))$, with $e^l\in K^*$,
- $\int_{\tilde{\gamma}_i}\xi_Q=\int_0^{\omega_i}\frac{f_q'(z)}{f_q(z)}dz=\int_0^{\omega_i} d\log f_q(z)=\log\frac{f_q(\omega_i)}{f_q(0)}=\eta_i q-\omega_i\zeta(q)$ by (1.8) for $i=1,2$,
- $\int_{\delta_{-Q}}\xi_Q=2i\pi\mathrm{Res}_{-Q}\xi_Q=2i\pi$.

The addition of the differential $d\log g_q(z)$ to the differential of the third kind ξ_Q will modify the last two integrals by an integral multiple of $2i\pi$ (see [21, Thm 10-7]) and this is irrelevant for the computation of the field generated by the periods of M.

Explicitly the matrix, which represents the isomorphism (0.3) for $M = [u : z\mathbb{Z} \to G], u(z) = R$, and whose coefficients are the periods of M, is therefore

$$
(2.7) \quad \texttt{\{eq:matrix-periods\}} \quad \begin{pmatrix} 1 & p & \zeta(p) & \log f_q(p) + l \\ 0 & \omega_1 & \eta_1 & \eta_1 q - \omega_1 \zeta(q) \\ 0 & \omega_2 & \eta_2 & \eta_2 q - \omega_2 \zeta(q) \\ 0 & 0 & 0 & 2i\pi \end{pmatrix}.
$$

with $e^l \in K^*$. □

Remark 2.4. The determination of the complex and elliptic logarithms, which appear in the first line of the matrix (2.7), are not well-defined since they depend on the lifting $\beta_{P_1 P_2}$ of the basis of $\mathrm{T}_{\mathbb{Q}}(z\mathbb{Z})$ (recall that the short exact sequence (2.1) is not split). Nevertheless, the field $K(\text{periodes}(M))$, which is involved in the Generalized Grothendieck's Conjecture of Periods, is totally independent of this choice since it contains $2i\pi$, the periods of the Weierstrass $\wp$-function, the quasi-periods of the Weierstrass ζ-function, and finally the quasi-quasi-periods of Serre's function $f_q(z)$ (1.6).

We finish this section with an example: Consider the 1-motive $M = [u : \mathbb{Z}^2 \to G]$, where G is an extension of $\mathcal{E}_1 \times \mathcal{E}_2$ by $\mathbb{G}_m^3$ parameterized by the K-rational points $Q_1 = (Q_{11}, Q_{21}), Q_2 = (Q_{12}, Q_{22}), Q_3 = (Q_{13}, Q_{23})$ of $\mathcal{E}_1^* \times \mathcal{E}_2^*$, and the morphism u corresponds to two K-rational points R_1, R_2 of G leaving over two points $P_1 = (P_{11}, P_{21}), P_2 = (P_{12}, P_{22})$ of $\mathcal{E}_1 \times \mathcal{E}_2$. The more compact way to write the matrix which represents the isomorphism (0.3) for our 1-motive $M = [u : \mathbb{Z}^2 \to G]$ is to consider the 1-motive

$$
M' = M/[0 \longrightarrow \mathcal{E}_1] \oplus M/[0 \longrightarrow \mathcal{E}_2],
$$

that is, with the above notation $M' = [u_1 = \mathbb{Z}^2 \to \Pi_{i=1}^3 G_{1i}] \oplus [u_2 = \mathbb{Z}^2 \to \Pi_{i=1}^3 G_{2i}]$ with u_1 corresponding to two K-rational points $(R_{111}, R_{121}, R_{131})$ and $(R_{112}, R_{122}, R_{132})$ of $\Pi_{i=1}^3 G_{1i}$ living over P_{11} and P_{12}, and u_2 corresponding to two K-rational points $(R_{211}, R_{221}, R_{231})$ and $(R_{212}, R_{222}, R_{232})$ of $\Pi_{i=1}^3 G_{2i}$ living over P_{21} and P_{22}. The 1-motives M and M' generate the same tannakian category: in fact, it is clear that $< M' >^{\otimes} \subset < M >^{\otimes}$ and in the other hand $M = M'/[\mathbb{Z}^2/\mathrm{d}_{\mathbb{Z}}(\mathbb{Z}) \to 0]$. The matrix representing the isomorphism (0.3) for the 1-motive M' with respect to the bases chosen in the above Corollary is

$$
\begin{pmatrix}
 & p_{11} & \zeta_1(p_{11}) & 0 & 0 & \log f_{q_{11}}(p_{11}) + l_{111} & \log f_{q_{12}}(p_{11}) + l_{121} & \log f_{q_{13}}(p_{11}) + l_{131} \\
\mathrm{Id}_{4\times 4} & p_{12} & \zeta_1(p_{12}) & 0 & 0 & \log f_{q_{11}}(p_{12}) + l_{112} & \log f_{q_{12}}(p_{12}) + l_{121} & \log f_{q_{13}}(p_{12}) + l_{131} \\
 & 0 & 0 & p_{21} & \zeta_2(p_{21}) & \log f_{q_{21}}(p_{21}) + l_{211} & \log f_{q_{22}}(p_{21}) + l_{221} & \log f_{q_{23}}(p_{21}) + l_{231} \\
 & 0 & 0 & p_{22} & \zeta_2(p_{22}) & \log f_{q_{21}}(p_{22}) + l_{212} & \log f_{q_{22}}(p_{22}) + l_{222} & \log f_{q_{23}}(p_{22}) + l_{232} \\
 & \omega_{11} & \eta_{11} & & & \eta_{11} q_{11} - \omega_{11} \zeta_1(q_{11}) & \eta_{11} q_{12} - \omega_{11} \zeta_1(q_{12}) & \eta_{11} q_{13} - \omega_{11} \zeta_1(q_{13}) \\
 & \omega_{12} & \eta_{12} & & & \eta_{12} q_{11} - \omega_{12} \zeta_1(q_{11}) & \eta_{12} q_{12} - \omega_{12} \zeta_1(q_{12}) & \eta_{12} q_{13} - \omega_{12} \zeta_1(q_{13}) \\
 & & & \omega_{21} & \eta_{21} & \eta_{21} q_{21} - \omega_{21} \zeta_2(q_{21}) & \eta_{21} q_{22} - \omega_{21} \zeta_2(q_{22}) & \eta_{21} q_{23} - \omega_{21} \zeta_2(q_{23}) \\
 & & & \omega_{22} & \eta_{22} & \eta_{22} q_{21} - \omega_{22} \zeta_2(q_{21}) & \eta_{22} q_{22} - \omega_{22} \zeta_2(q_{22}) & \eta_{22} q_{23} - \omega_{22} \zeta_2(q_{23}) \\
 & & & & & & 2i\pi \mathrm{Id}_{3\times 3} &
\end{pmatrix}
$$

In general, for a 1-motive of the kind $M = [u : \mathbb{Z}^r \to G]$ where G is an extension of a finite product $\Pi_{j=1}^n \mathcal{E}_j$ of elliptic curves by the torus $\mathbb{G}_m^s$, we will consider the 1-motive

$$
M' = \oplus_{j=1}^n \big(M/[0 \longrightarrow \Pi_{\substack{1 \leq l \leq n \\ l \neq j}} \mathcal{E}_j] \big)
$$

whose matrix representing the isomorphism (0.3) with respect to the bases chosen in the above Corollary is

$$\begin{pmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{pmatrix}$$

with $A = \mathrm{Id}_{rn \times rn}, B$ the $rn \times 2n$ matrix involving the periods coming from the morphism $v : \mathbb{Z}^r \to \Pi_{j=1}^n \mathcal{E}_j$, C the $rn \times s$ matrix involving the periods coming from the trivialization Ψ of the pull-back via (v, v^*) of the Poincaré biextension $\mathcal{P}_j$ of $(\mathcal{E}_j, \mathcal{E}_j^*)$ by $\mathbb{G}_m$, D the $2n \times 2n$ matrix having in the diagonal the period matrix of each elliptic curves $\mathcal{E}_j$, E the $2n \times s$ matrix involving the periods coming from the morphism $v^* : \mathbb{Z}^s \to \Pi_{j=1}^n \mathcal{E}_j^*$, and finally $F = 2i\pi \mathrm{Id}_{s \times s}$ the period matrix of $\mathbb{G}_m^s$.

3. Dimension of the unipotent radical of the motivic Galois group of a 1-motive

{motivicGaloisgrou

Denote by $\mathcal{MM}_{\leq 1}(K)$ the category of 1-motives defined over K. Using mixed realizations (see [11, 2.3] and [9, (2.2.5)]) or Nori ans Ayoub's works (see [3] and [19]), it is possible to endow the category of 1-motives with a *tannakian structure with rational coefficients* (roughly speaking a tannakian category $\mathcal{T}$ with rational coefficients is an abelian category endowed with a functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ defining the tensor product of two objects of $\mathcal{T}$, and with a fibre functor over $\mathrm{Spec}(\mathbb{Q})$ - see [12, 2.1, 1.9, 2.8] for details). We use neither Nori and Ayoub's theories nor mixed realizations: we work in a completely geometrical setting using algebraic geometry on tannakian category and defining as one goes along the objects, the morphisms and the tensor products that we will need (essentially we tensorize motives with pure motives of weight 0, and as morphisms we use projections and biextensions).

The unit object of the tannakian category $\mathcal{MM}_{\leq 1}(K)$ is the 1-motive $\mathbb{Z}(0) = [\mathbb{Z} \to 0]$. In this section we use the notation $Y(1)$ for the torus whose cocharacter group is Y. In particular $\mathbb{Z}(1) = [0 \to \mathbb{G}_m]$. If M is a 1-motive, we denote by $M^\vee \cong \underline{\mathrm{Hom}}(M, \mathbb{Z}(0))$ its dual and by $ev_M : M \otimes M^\vee \to \mathbb{Z}(0), \delta_M : \mathbb{Z}(0) \to M^\vee \otimes M$ the arrows of $\mathcal{MM}_{\leq 1}(K)$ characterizing this dual. The Cartier dual of M is $M^* = M^\vee \otimes \mathbb{Z}(1)$. If M_1, M_2 are two 1-motives, we set

(3.1) {eq:BiextHom} $$\mathrm{Hom}_{\mathcal{MM}_{\leq 1}(K)}(M_1 \otimes M_2, M_3) := \mathrm{Biext}^1(M_1, M_2; M_3)$$

where $\mathrm{Biext}^1((M_1, M_2; M_3)$ is the abelian group of isomorphism classes of biextensions of (M_1, M_2) by M_3. In particular the isomorphism class of the Poincaré biextension $\mathcal{P}$ of (A, A^*) by $\mathbb{G}_m$ is the Weil pairing $P_{\mathcal{P}} : A \otimes A^* \to \mathbb{Z}(1)$ of A.

The tannakian sub-category $< M >^\otimes$ generated by the 1-motive M is the full sub-category of $\mathcal{MM}_{\leq 1}(K)$ whose objects are sub-quotients of direct sums of $M^{\otimes\, n} \otimes M^{\vee \,\otimes\, m}$, and whose fibre functor is the restriction of the fibre functor of $\mathcal{MM}_{\leq 1}(K)$ to $< M >^\otimes$. Because of the tensor product of $< M >^\otimes$, we have the notion of commutative Hopf algebra in the category Ind $< M >^\otimes$ of Ind-objects of $< M >^\otimes$ and so we can define the category of affine $< M >^\otimes$-group schemes, just called *motivic affine group schemes*, as the opposite of the category of commutative Hopf algebra in Ind $< M >^\otimes$. The Lie algebra of a motivic affine group scheme is a pro-object L of $\langle M \rangle^\otimes$ endowed with a Lie algebra structure, i.e. L is endowed with an anti-symmetric application $[\,,\,] : \mathrm{L} \otimes \mathrm{L} \to \mathrm{L}$ satisfying the Jacobi identity.

The *motivic Galois group* $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ of M is the fundamental group of the tannakian category $< M >^\otimes$ generated by M, i.e. the motivic affine group scheme $\mathrm{Sp}(\Lambda)$ where Λ is the object of $< M >^\otimes$ universal for the following property: for any object X of $< M >^\otimes$, it exists a morphism

(3.2) {eq:lambdaX} $$\lambda_X : X^\vee \otimes X \longrightarrow \Lambda$$

functorial on X, i.e. such that for any morphism $f : X \to Y$ in $< M >^{\otimes}$ the diagram

$$\begin{array}{ccc} Y^{\vee} \otimes X & \overset{f^t \otimes 1}{\longrightarrow} & X^{\vee} \otimes X \\ {\scriptstyle 1\otimes f} \downarrow & & \downarrow \lambda_X \\ Y^{\vee} \otimes Y & \overset{\lambda_Y}{\longrightarrow} & \Lambda \end{array}$$

is commutative. The universal property of Λ is that for any object U of $< M >^{\otimes}$, the map

$$\begin{aligned} \mathrm{Hom}(\Lambda, U) &\longrightarrow \big\{ u_X : X^{\vee} \otimes X \to U, \text{ functorial on } X \big\} \\ f &\longmapsto f \circ \lambda_X \end{aligned}$$

is bijective. The morphisms (3.2), which can be rewritten as $X \to X \otimes \Lambda$, define the action of the motivic Galois group $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ on each object X of $< M >^{\otimes}$.

If $\omega_{\mathbb{Q}}$ is the fibre functor Hodge realization realization of the tannakian category $< M >^{\otimes}$, $\omega_{\mathbb{Q}}(\Lambda)$ is the Hopf algebra whose spectrum $\mathrm{Spec}(\omega(\Lambda))$ is the $\mathbb{Q}$-group scheme $\underline{\mathrm{Aut}}^{\otimes}_{\mathbb{Q}}(\omega_{\mathbb{Q}})$, i.e. the Mumford-Tate group $\mathrm{MT}(M)$ of M. In other words, the motivic Galois group of M is *the geometric interpretation* of the Mumford-Tate group of M. By [2, Thm 1.2.1] these two group schemes coincides, and in particular they have the same dimension

$$\dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) = \dim \mathrm{MT}(M). \tag{3.3}$$

Let $M = [u : X \to G]$ be a 1-motive defined over K, with G an extension of an abelian variety A by a torus T. The weight filtration $\mathrm{W}_{\bullet}$ of M induces a filtration on its motivic Galois group $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ ([20, Chp IV §2]):

$\mathrm{W}_0(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$

$\mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \big\{ g \in \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \mid (g - id)M \subseteq \mathrm{W}_{-1}(M), (g - id)\mathrm{W}_{-1}(M) \subseteq \mathrm{W}_{-2}(M), (g - id)\mathrm{W}_{-2}(M) = 0 \big\}$,

$\mathrm{W}_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \big\{ g \in \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \mid (g - id)M \subseteq \mathrm{W}_{-2}(M), (g - id)\mathrm{W}_{-1}(M) = 0 \big\}$,

$\mathrm{W}_{-3}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = 0.$

Clearly $\mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ is unipotent. Denote by $\mathrm{UR}(M)$ the unipotent radical of $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$.

Consider the graduated 1-motive

$$\widetilde{M} = \mathrm{Gr}^{\mathrm{W}}_{*}(M) = X + A + T$$

associated to M and let $< \widetilde{M} >^{\otimes}$ be the tannakian sub-category of $< M >^{\otimes}$ generated by $\widetilde{M}$. The functor "take the graduated" $\mathrm{Gr}^{\mathrm{W}}_{*} :< M >^{\otimes} \twoheadrightarrow < \widetilde{M} >^{\otimes}$, which is a projection, induced the inclusion of motivic affine group schemes

$$\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M}) \hookrightarrow \mathrm{Gr}^{\mathrm{W}}_{*}\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M). \tag{3.4}$$

{eq:dimGr0}

Lemma 3.1. *Let $M = [u : X \to G]$ be a 1-motive defined over K, with G an extension of an abelian variety A by a torus T. The quotient $\mathrm{Gr}^{\mathrm{W}}_0(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ is reductive and the inclusion of motivic group schemes (3.4) identifies $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M})$ with this quotient.*

Moreover, if $X = \mathbb{Z}^r$ and $T = \mathbb{G}_m^s$

$$\dim \mathrm{Gr}^{\mathrm{W}}_0\big(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)\big) = \dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M}) = \begin{cases} \dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(A) & \text{if } A \neq 0, \\ 1 & \text{if } A = 0, T \neq 0, \\ 0 & \text{if } A = T = 0. \end{cases}$$

Proof. By a motivic analogue of [9, §2.2], $\mathrm{Gr}_0^{\mathrm{W}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ acts via $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Gr}_0^{W}(M)$, by homotheties on $\mathrm{Gr}_{-2}^{\mathrm{W}}(M)$, and its image in the group of authomorphisms of $\mathrm{Gr}_{-1}^{\mathrm{W}}(M)$ is the motivic Galois group $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(A)$ of the abelian variety A underlying M. Therefore $\mathrm{Gr}_0^{\mathrm{W}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ is reductive, and via the inclusion (3.4) it coincides with $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M})$. To conclude, observe that $\mathrm{Lie}\,\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\mathbb{G}_m) = \mathbb{G}_m$, which has dimension 1, and $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(\mathbb{Z}) = \mathrm{Sp}(\mathbb{Z}(0))$ which has dimension 0. □

The inclusion $< \widetilde{M} >^{\otimes} \hookrightarrow < M >^{\otimes}$ of tannakian categories induces the following surjection of motivic affine group schemes, which is the restriction $g \mapsto g_{|\widetilde{M}}$,

(3.5) {eq:RestrictionGr_0} $$\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \twoheadrightarrow \mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M}).$$

As an immediate consequence of the above Lemma we have

{eq:DecomDim}

Corollary 3.2. *Let $M = [u : X \to G]$ be a 1-motive defined over K. Then*

$$\mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \ker\left[\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \twoheadrightarrow \mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M})\right].$$

In particular, $\mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ *is the unipotent radical* $\mathrm{UR}(M)$ *of* $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ *and*

$$\dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) = \dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M}) + \dim \mathrm{UR}(M).$$

Observe that we can prove the equality $\mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \ker\left[\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \twoheadrightarrow \mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M})\right]$ directly using the definition of the weight filtration:

$$\begin{aligned} g \in \mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) &\iff (g-id)\mathrm{Gr}_0^{\mathrm{W}}(M) = 0, (g-id)\mathrm{Gr}_{-1}^{\mathrm{W}}(M) = 0, (g-id)\mathrm{Gr}_{-2}^{\mathrm{W}}(M) = 0 \\ &\iff g_{|\mathrm{Gr}_*^{\mathrm{W}}(M)} = \mathrm{id}, \text{ i.e. } g = \mathrm{id} \text{ in } \mathcal{G}\mathrm{al}_{\mathrm{mot}}(\widetilde{M}). \end{aligned}$$

The inclusion $< M + M^{\vee}/\mathrm{W}_{-2}(M + M^{\vee}) >^{\otimes} \hookrightarrow < M >^{\otimes}$ of tannakian categories induces the following surjection of motivic affine group schemes, which is the restriction $g \mapsto g_{|M+M^{\vee}/\mathrm{W}_{-2}(M+M^{\vee})}$,

(3.6) {eq:Gr_1} $$\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \twoheadrightarrow \mathcal{G}\mathrm{al}_{\mathrm{mot}}\big(M + M^{\vee}/\mathrm{W}_{-2}(M + M^{\vee})\big).$$

{eq:DecomRU}

Lemma 3.3. *Let $M = [u : X \to G]$ be a 1-motive defined over K. Then*

$$\mathrm{W}_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \ker\left[\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) \twoheadrightarrow \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M + M^{\vee}/\mathrm{W}_{-2}(M + M^{\vee}))\right].$$

In particular, the quotient $\mathrm{Gr}_{-1}^{\mathrm{W}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ *of the unipotente radical* $\mathrm{UR}(M)$ *is the unipotent radical* $\mathrm{W}_{-1}\big(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M + M^{\vee}/\mathrm{W}_{-2}(M + M^{\vee}))\big)$ *of* $\mathcal{G}\mathrm{al}_{\mathrm{mot}}\big(M + M^{\vee}/\mathrm{W}_{-2}(M + M^{\vee})\big)$.

Proof. Using the definition of the weight filtration, we have:

$$\begin{aligned} g \in \mathrm{W}_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) &\iff (g-id)M/\mathrm{W}_{-2}(M) = 0,\ (g-id)\mathrm{W}_{-1}(M) = 0 \\ &\iff g_{|M/\mathrm{W}_{-2}(M)} = \mathrm{id},\ g_{|M^{\vee}/\mathrm{W}_{-2}(M^{\vee})} = \mathrm{id} \\ &\iff g = \mathrm{id} \text{ in } \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M + M^{\vee}/\mathrm{W}_{-2}(M + M^{\vee})). \end{aligned}$$

Since the surjection of motivic affine group schemes (3.6) respects the weight filtration, $\mathrm{W}_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ is in fact the kernel of $\mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) \twoheadrightarrow \mathrm{W}_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M+M^{\vee}/\mathrm{W}_{-2}(M+M^{\vee})))$. Hence we get the second statement. □

From the definition of weight filtration, we observe that

$$\mathrm{W}_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) \subseteq \underline{\mathrm{Hom}}(X, Y(1)) \cong X^{\vee} \otimes Y(1).$$

By the above Lemma, we have that

$$\mathrm{Gr}_{-1}^{\mathrm{W}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) \subseteq \underline{\mathrm{Hom}}(X + Y^{\vee}, A + A^*) \cong X^{\vee} \otimes A + Y \otimes A^*.$$

In order to compute the dimension of the unipotent radical $\mathrm{UR}(M)$ of $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ we use notations of [6, §3], that we recall briefly. Let $(X, Y^\vee, A, A^*, v : X \to A, v^* : Y^\vee \to A^*, \psi : X \otimes Y^\vee \to (v \times v^*)^*\mathcal{P})$ be the 7-tuple defining the 1-motive $M = [u : X \to G]$ over K, where G an extension of A by the torus $Y(1)$. Let

$$E = \mathrm{W}_{-1}(\underline{\mathrm{End}}(\widetilde{M})).$$

It is the direct sum of the pures motives $E_{-1} = X^\vee \otimes A + A^\vee \otimes Y(1)$ and $E_{-2} = X^\vee \otimes Y(1)$ of weight -1 and -2. As observed in [6, §3], the composition of endomorphisms furnishes a ring structure to E given by the arrow $P : E \otimes E \to E$ of $\langle M \rangle^\otimes$ whose only non trivial component is

$$E_{-1} \otimes E_{-1} \longrightarrow (X^\vee \otimes A) \otimes (A^* \otimes Y) \longrightarrow \mathbb{Z}(1) \otimes X^\vee \otimes Y = E_{-2}$$

where the first arrow is the projection from $E_{-1} \otimes E_{-1}$ to $(X^\vee \otimes A) \otimes (A^* \otimes Y)$ and the second arrow is the Weil pairing $P_{\mathcal{P}} : A \otimes A^* \to \mathbb{Z}(1)$ of A.

Because of the definition (3.1), the product $P : E_{-1} \otimes E_{-1} \to E_{-2}$ defines a biextension $\mathcal{B}$ of (E_{-1}, E_{-1}) by E_{-2}, whose pull-back $d^*\mathcal{B}$ via the diagonal morphism $d : E_{-1} \to E_{-1} \times E_{-1}$ is a $\Sigma - X^\vee \otimes Y(1)$-torsor over E_{-1}. By [6, Lem 3.3] this $\Sigma - X^\vee \otimes Y(1)$-torsor $d^*\mathcal{B}$ induces a Lie bracket $[\,,\,] : E \otimes E \to E$ on E which becomes therefore a Lie algebra.

The action of $E = W_{-1}(\underline{\mathrm{End}}(\widetilde{M}))$ on $\widetilde{M}$ is given by the arrow $E \otimes \widetilde{M} \to \widetilde{M}$ of $\langle M \rangle^\otimes$ whose only non trivial components are

$$\begin{aligned} \alpha_1 &: (X^\vee \otimes A) \otimes X \longrightarrow A \\ \alpha_2 &: (A^* \otimes Y) \otimes A \longrightarrow Y(1) \\ \gamma &: (X^\vee \otimes Y(1)) \otimes X \longrightarrow Y(1) \end{aligned} \tag{3.7}$$

where the first and the last arrows are induced by $ev_{X^\vee} : X^\vee \otimes X \to \mathbb{Z}(0)$, while the second one is $\mathrm{rk}(Y)$-copies of the Weil pairing $P_{\mathcal{P}} : A \otimes A^* \to \mathbb{Z}(1)$. By [6, Lem 3.3], via the arrow $(\alpha_1, \alpha_2, \gamma) : E \otimes \widetilde{M} \to \widetilde{M}$, the 1-motive $\widetilde{M}$ is in fact a $(E, [,])$-Lie module.

As observed in [6, Rem 3.4 (3)] E acts also on the Cartier dual $\widetilde{M}^* = Y^\vee + A^* + X^\vee(1)$ of $\widetilde{M}$ and this action is given by the arrows

$$\begin{aligned} \alpha_2^* &: (A^* \otimes Y) \otimes Y^\vee \longrightarrow A^* \\ \alpha_1^* &: (X^\vee \otimes A) \otimes A^* \longrightarrow X^\vee(1) \\ \gamma^* &: (X^\vee \otimes Y(1)) \otimes Y^\vee \longrightarrow X^\vee(1) \end{aligned} \tag{3.8}$$

where α_2^* et γ^* are projections, while α_1^* is $\mathrm{rk}(X^\vee)$-copies of the Weil pairing $P_{\mathcal{P}} : A \otimes A^* \to \mathbb{Z}(1)$ of A.

Via the arrows $\delta_{X^\vee} : \mathbb{Z}(0) \to X \otimes X^\vee$ et $\delta_Y : \mathbb{Z}(0) \to Y^\vee \otimes Y$, to have the morphisms $v : X \to A$ and $v^* : Y^\vee \to A^*$ underlying the 1-motive M is the same thing as to have the morphisms $V : \mathbb{Z}(0) \to A \otimes X^\vee$ and $V^* : \mathbb{Z}(0) \to A^* \otimes Y$. Therefore to have v and v^* is equivalent to have a point

$$b = (b_1, b_2) \in E_{-1}(K) = A \otimes X^\vee(K) + A^* \otimes Y(K).$$

Fix now an element $(x, y^\vee)$ in the character group $X \otimes Y^\vee$ of the torus $X^\vee \otimes Y(1)$. By construction of the point b, it exists an element $(s, t) \in X \otimes Y^\vee(K)$ such that

$$\begin{aligned} v(x) &= \alpha_1(b_1, s) \in A(K) \\ v^*(y^\vee) &= \alpha_2^*(b_2, t) \in A^*(K). \end{aligned}$$

Let $i^*_{x,y^\vee} d^*\mathcal{B}$ be the pull-back of $d^*\mathcal{B}$ via the inclusion $i_{x,y^\vee} : \{(v(x), v^*(y^\vee))\} \hookrightarrow E_{-1}$ in E_{-1} of the abelian sub-variety generated by the point $(v(x), v^*(y^\vee))$. The push-down

$(x,y^\vee)_* i^*_{x,y^\vee} d^*\mathcal{B}$ of $i^*_{x,y^\vee} d^*\mathcal{B}$ via the character $(x,y^\vee) : X^\vee \otimes Y(1) \to \mathbb{Z}(1)$ is a $\Sigma - \mathbb{Z}(1)$-torsor over $\{(v(x), v^*(y^\vee))\}$:

$$\begin{array}{ccccc} (x,y^\vee)_* i^*_{x,y^\vee} d^*\mathcal{B} & \longleftarrow & i^*_{x,y^\vee} d^*\mathcal{B} & \longrightarrow & d^*\mathcal{B} \\ \downarrow & & \downarrow & & \downarrow \\ \{(v(x), v^*(y^\vee))\} & = & \{(v(x), v^*(y^\vee))\} & \stackrel{i_{x,y^\vee}}{\longrightarrow} & E_{-1} \end{array}$$

To have the point $\psi(x,y^\vee)$ is equivalent to have a point $(\widetilde{b})_{x,y^\vee}$ of $(x,y^\vee)_* i^*_{x,y^\vee} d^*\mathcal{B}$ over $(v(x), v^*(y^\vee))$, and so to have the trivialization ψ is equivalent to have a point

$$\widetilde{b} \in (d^*\mathcal{B})_b$$

in the fibre of $d^*\mathcal{B}$ over $b = (b_1, b_2)$.

Consider now the following pure motives:

(1) Let B be the *smallest* abelian sub-variety (modulo isogenies) of $X^\vee \otimes A + A^* \otimes Y$ which contains the point $b = (b_1, b_2) \in X^\vee \otimes A(k) \times A^* \otimes Y(K)$. The pull-back $i^* d^*\mathcal{B}$ of $d^*\mathcal{B}$ via the inclusion $i : B \hookrightarrow E_{-1}$ of B on E_{-1}, is a $\Sigma - X^\vee \otimes Y(1)$-torsor over B.

(2) Let Z_1 be the *smallest* $\mathrm{Gal}(\overline{K}/K)$-sub-module of $X^\vee \otimes Y$ such that the torus $Z_1(1)$ contains the image of the Lie bracket $[\,,\,] : B \otimes B \to X^\vee \otimes Y(1)$. The push-down $p_* i^* d^*\mathcal{B}$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $i^* d^*\mathcal{B}$ via the projection $p : X^\vee \otimes Y(1) \twoheadrightarrow (X^\vee \otimes Y/Z_1)(1)$ is a trivial $\Sigma - (X^\vee \otimes Y/Z_1)(1)$-torsor over B, i.e.

$$p_* i^* d^*\mathcal{B} = B \times (X^\vee \otimes Y/Z_1)(1).$$

Note by $\pi : p_* i^* d^*\mathcal{B} \twoheadrightarrow (X^\vee \otimes Y/Z_1)(1)$ the canonical projection and by $s : B \hookrightarrow p_* i^* d^*\mathcal{B}$ the canonical section. We still note $\widetilde{b}$ the points of $i^* d^*\mathcal{B}$ and of $p_* i^* d^*\mathcal{B}$ living over $b \in B$.

(3) Let Z be the *smallest* $\mathrm{Gal}(\overline{K}/K)$-sub-module of $X^\vee \otimes Y$ containing Z_1 and such that the sub-torus $(Z/Z_1)(1)$ of $(X^\vee \otimes Y/Z_1)(1)$ contains $\pi(\widetilde{b})$.

Let $A_\mathbb{C}$ be the abelian variety defined over $\mathbb{C}$ obtained from A extending the scalars from K to the complexes. Denote by g the dimension of A. Consider the abelian exponential

$$\exp_A : \mathrm{Lie} A_\mathbb{C} \longrightarrow A_\mathbb{C}$$

whose kernel is the lattice $\mathrm{H}_1(A_\mathbb{C}(\mathbb{C}), \mathbb{Z})$, and denote by $\log_A$ an abelian logarithm of A, that is a choice of an inverse map of $\exp_A$. Consider the composite

$$P_\mathcal{P} \circ (v \times v^*) : X \otimes Y^\vee \longrightarrow \mathbb{Z}(1)$$

where $P_\mathcal{P} : A \otimes A^* \to \mathbb{Z}(1)$ is the Weil pairing of A. Since we work modulo isogenies, we identify the abelian variety A with its Cartier dual A^*. Let $\omega_1, \dots, \omega_g$ be differentials of the first kind which build a basis of the K-vector space $\mathrm{H}^0(A, \Omega^1_A)$ of holomorphic differentials, and let $\eta_1, \dots, \eta_g$ be differentials of the second kind which build a basis of the K-vector space $\mathrm{H}^1(A, \mathcal{O}_A)$ of differentials of the second kind modulo holomorphic differentials and exact differentials. As in the case of elliptic curves, the first De Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(A)$ of the abelian variety A is the direct sum $\mathrm{H}^0(A, \Omega^1_A) \oplus \mathrm{H}^1(A, \mathcal{O}_A)$ of these two vector spaces and it has dimension $2g$. Let $\gamma_1, \dots, \gamma_{2g}$ be closed paths which build a basis of the $\mathbb{Q}$-vector space $\mathrm{H}_1(A_\mathbb{C}, \mathbb{Q})$. For $n = 1, \dots, g$ and $m = 1, \dots, 2g$, the abelian integrals of the first kind $\int_{\gamma_m} \omega_n = \omega_{nm}$ are the *periods* of the abelian variety A, and the abelian integrals of the second kind $\int_{\gamma_m} \eta_n = \eta_{nm}$ are the *quasi-periods* of A.

{eq:dimUR}

Theorem 3.4. *Let $M = [u : X \to G]$ be a 1-motive defined over K, with G an extension of an abelain variety A by a torus $Y(1)$. Denote by $F = \mathrm{End}(A) \otimes_\mathbb{Z} \mathbb{Q}$ the field of endomorphisms* *of the abelian variety A. Let $x_1, \dots, x_{\mathrm{rk}(X)}$ be generators of the character group X and let $y_1^\vee, \dots, y_{\mathrm{rk}(Y^\vee)}^\vee$ be generators of the character group $Y^\vee$. Then*

$$\dim_{\mathbb{Q}} \mathrm{UR}(M) =$$

$$2 \dim_F \mathcal{A}b\mathcal{L}og\, \mathrm{Im}(v, v^*) + \dim_{\mathbb{Q}} \mathcal{L}og\, \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*)) + \dim_{\mathbb{Q}} \mathcal{L}og\, \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))})$$

where

- $\mathcal{A}b\mathcal{L}og\, \mathrm{Im}(v, v^*)$ *is the F-sub-vector space of $\mathbb{C}/(\sum_{\substack{n=1,\dots,g \\ m=1,\dots,2g}} F\, \omega_{nm})$ generated by the abelian logarithms $\{\log_A v(x_k), \log_A v^*(y_i^\vee)\}_{\substack{k=1,\dots,\mathrm{rk}(X) \\ i=1,\dots,\mathrm{rk}(Y^\vee)}}$;*
- $\mathcal{L}og\, \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*))$ *is the $\mathbb{Q}$-sub-vector space of $\mathbb{C}/2i\pi\mathbb{Q}$ generated by the logarithms $\{\log P_{\mathcal{P}}(v(x_k), v^*(y_i^\vee))\}_{\substack{k=1,\dots,\mathrm{rk}(X) \\ i=1,\dots,\mathrm{rk}(Y^\vee)}}$;*
- $\mathcal{L}og\, \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))})$ *is the $\mathbb{Q}$-sub-vector space of $\mathbb{C}/2i\pi\mathbb{Q}$ generated by the logarithms $\{\log \psi(x_{k'}, y_{i'}^\vee)\}_{\substack{(x_{k'}, y_{i'}^\vee) \in \ker(P_{\mathcal{P}} \circ (v \times v^*)) \\ 1 \leq k' \leq \mathrm{rk}(X),\ 1 \leq i' \leq \mathrm{rk}(Y^\vee)}}$.*

Proof. By the main theorem of [6, Thm 0.1], the unipotent radical $W_{-1}(\mathrm{Lie}\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ is the semi-abelian variety extension of B by $Z(1)$ defined by the adjoint action of the Lie algebra $(B, Z(1), [\,,\,])$ over $B + Z(1)$. Since the tannakian category $< M >^*$ has rational coefficients, we have that $\dim_{\mathbb{Q}} W_{-1}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = 2 \dim B + \dim Z(1)$. Concerning the abelian part we have that

$$\dim B = \dim_F \mathcal{A}b\mathcal{L}og\, \mathrm{Im}(v, v^*).$$

On the other hand, for the toric part we have by construction $\dim Z(1) = \dim(Z/Z_1)(1) + Z_1(1)$. Because of the explicit description the Lie bracket $[\,,\,] : B \otimes B \to X^\vee \otimes Y(1)$ given in [6, (2.8.4)], we have that

$$\dim Z_1(1) = \dim_{\mathbb{Q}} \mathcal{L}og\, \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*)).$$

Finally by construction we have that

$$\dim(Z/Z_1)(1) = \dim_{\mathbb{Q}} \mathcal{L}og\, \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))}).$$

□

Remark 3.5. The dimension of the quotient $\mathrm{Gr}_{-1}^{\mathrm{W}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ of the unipotent radical $\mathrm{UR}(M)$ is equal to the dimension of the abelian sub-variety B of $X^\vee \otimes A + A^* \otimes Y$, that is

$$\dim_{\mathbb{Q}} \mathrm{Gr}_{-1}^{\mathrm{W}}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \dim_F \mathcal{A}b\mathcal{L}og\, \mathrm{Im}(v, v^*).$$

The dimension of $W_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M))$ is the dimension of the sub-torus $Z(1)$ of $X^\vee \otimes Y(1)$, that is

$$\dim_{\mathbb{Q}} W_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = \dim_{\mathbb{Q}} \mathcal{L}og\, \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*)) + \dim_{\mathbb{Q}} \mathcal{L}og\, \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))})$$

Remark 3.6. A 1-motive $M = [u : X \to G]$ defined over K is said to be *deficient* if $\mathrm{W}_{-2}(\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)) = 0$. In [14] Jacquinot and Ribet construct such a 1-motive in the case $\mathrm{rk}(X) = \mathrm{rk}(Y^\vee) = 1$. By the above Theorem we have that M is deficient if and only if for any $(x, y^\vee) \in X \otimes Y^\vee$,

$$P_{\mathcal{P}}(v(x), v^*(y^\vee)) = 1 \quad \text{and} \quad \psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))}(x, y^\vee) = 1,$$

that is if and only if the two arrows $P_{\mathcal{P}} \circ (v \times v^*) : X \otimes Y^\vee \to \mathbb{Z}(1)$ and $\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))} : X \otimes Y^\vee \to \mathbb{Z}(1)$ are the trivial arrow.

Now let $M = [u : \mathbb{Z}^r \to G]$ be a 1-motive defined over K, with G an extension of a product $\Pi_{j=1}^n \mathcal{E}_j$ of pairwise not isogenous elliptic curves by the torus $\mathbb{G}_m^s$. We go back to the notation used in Section 2. Denote by $\mathrm{pr}_h : \Pi_{j=1}^n \mathcal{E}_j \to \mathcal{E}_h$ and $\mathrm{pr}_h^* : \Pi_{j=1}^n \mathcal{E}_j^* \to \mathcal{E}_h^*$ the projections into the h-th elliptic curve and consider the composites $v_h = \mathrm{pr}_h \circ v : \mathbb{Z}^r \to \mathcal{E}_h$ and $v_h^* = \mathrm{pr}_h^* \circ v^* : \mathbb{Z}^s \to \mathcal{E}_h^*$. Let $\mathcal{P}$ be the Poincaré biextension of $(\Pi_{j=1}^n \mathcal{E}_j, \Pi_{j=1}^n \mathcal{E}_j^*)$ by $\mathbb{G}_m$ and let $\mathcal{P}_j$ be the Poincaré biextension of $(\mathcal{E}_j, \mathcal{E}_j^*)$ by $\mathbb{G}_m$. The category of biextensions is additive in each variable, and so we have that $P_{\mathcal{P}} = \Pi_{j=1}^n P_{\mathcal{P}_j}$, where $P_{\mathcal{P}_j} : \mathcal{E}_j \otimes \mathcal{E}_j^* \to \mathbb{Z}(1)$ is the Weil pairing of the elliptic curve $\mathcal{E}_j$.

{eq:dimGalMot}

Corollary 3.7. *Let $M = [u : \mathbb{Z}^r \to G]$ be a 1-motive defined over K, with G an extension of a product $\Pi_{j=1}^n \mathcal{E}_j$ of pairwise not isogenous elliptic curves by the torus $\mathbb{G}_m^s$. Denote by $k_j = \mathrm{End}(\mathcal{E}_j) \otimes_{\mathbb{Z}} \mathbb{Q}$ the field of endomorphisms of the elliptic curve $\mathcal{E}_j$ for $j = 1, \ldots, n$. Let $x_1, \ldots, x_r$ be generators of the character group $\mathbb{Z}^r$ and let $y_1^\vee, \ldots, y_s^\vee$ be generators of the character group $\mathbb{Z}^s$. Then*

$$\dim_{\mathbb{Q}} \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) = 4 \sum_{j=1}^n (\dim_{\mathbb{Q}} k_j)^{-1} - n + 1 + \sum_{j=1}^n 2 \dim_{k_j} \mathcal{A}b\mathcal{L}og\ \mathrm{Im}(v_j, v_j^*) +$$

$$\dim_{\mathbb{Q}} \mathcal{L}og\ \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*)) + \dim_{\mathbb{Q}} \mathcal{L}og\ \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))})$$

- *$\mathcal{A}b\mathcal{L}og\ \mathrm{Im}(v_j, v_j^*)$ is the k_j-sub-vector space of $\mathbb{C}/k_j\,\omega_{j1} + k_j\,\omega_{j2}$ generated by the elliptic logarithms $\{p_{jk}, q_{ji}\}_{\substack{k=1,\ldots,r \\ i=1,\ldots,s}}$ of the points $\{P_{jk}, Q_{ji}\}_{\substack{k=1,\ldots,r \\ i=1,\ldots,s}}$ for $j = 1, \ldots, n$;*
- *$\mathcal{L}og\ \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*))$ is the $\mathbb{Q}$-sub-vector space of $\mathbb{C}/2i\pi\mathbb{Q}$ generated by the logarithms $\{\log P_{\mathcal{P}_j}(P_{jk}, Q_{ji})\}_{\substack{k=1,\ldots,r,\ i=1,\ldots,s \\ j=1,\ldots,n}}$;*
- *$\mathcal{L}og\ \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}} \circ (v \times v^*))})$ is the $\mathbb{Q}$-sub-vector space of $\mathbb{C}/2i\pi\mathbb{Q}$ generated by the logarithms $\{\log \psi(x_{k'}, y_{i'}^\vee)\}_{\substack{(x_{k'}, y_{i'}^\vee) \in \ker(P_{\mathcal{P}_j} \circ (v_j \times v_j^*)) \\ 1 \leq k' \leq r,\ 1 \leq i' \leq s,\ j=1,\ldots,n}}$.*

Proof. Since the elliptic curves are pairwise not isogenous, by [17, §2] and (3.3) we have that

$$\dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}\big(\Pi_{j=1}^n \mathcal{E}_j\big) = 4 \sum_{j=1}^n (\dim_{\mathbb{Q}} k_j)^{-1} - n + 1.$$

Therefore putting together Corollary 3.2, Lemma 3.1 and Theorem 3.4 we can conclude. □

Remark 3.8. We can express the dimension of the motivic Galois group of a product of elliptic curves also as $3n_1 + n_2 + 1$, where n_1 is the number of elliptic curves without complex multiplication and n_2 is the number of elliptic curves with complex multiplication. Therefore

$$\dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M) = \dim \mathrm{UR}(M) + 3n_1 + n_2 + 1$$

4. Generalized Grothendieck's Conjecture of Periods for 1-motives involving elliptic curves

{conjecture}

The 1-motivic elliptic conjecture

Consider

- $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be elliptic curves pairwise not isogenous. Denote by $k_j = \mathrm{End}(\mathcal{E}_j) \otimes_{\mathbb{Z}} \mathbb{Q}$ the field of endomorphisms of $\mathcal{E}_j$ for $j = 1, \ldots, n$;
- $Q_i = (Q_{1i}, \ldots, Q_{ni})$ be s points of $\Pi_{j=1}^n \mathcal{E}_j^*(\mathbb{C})$ for $i = 1, \ldots, s$. These points determine an extension G of $\Pi_{j=1}^n \mathcal{E}_j$ by $\mathbb{G}_m^s$;
- $R_1, \ldots, R_r$ be r points of $G(\mathbb{C})$. Denote by $(P_{1k}, \ldots, P_{nk}) \in \Pi_{j=1}^n \mathcal{E}_j(\mathbb{C})$ the projection of the point R_k on $\Pi_{j=1}^n \mathcal{E}_j$ for $k = \ldots, r$.

Then

$$\text{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(2i\pi, g_{2j}, g_{3j}, Q_{ji}, R_k, \omega_{j1}, \omega_{j2}, \eta_{j1}, \eta_{j2}, p_{jk}, \zeta_j(p_{jk}),$$
$$\eta_{j1}q_{ji} - \omega_{j1}\zeta_j(q_{ji}), \eta_{j2}q_{ji} - \omega_{j2}\zeta_j(q_{ji}), \log f_{q_{ji}}(p_{jk}) + l_{jik}\Big)_{\substack{j=1,\dots,n \ \ i=1,\dots,s \\ k=1,\dots,r}} \geq$$
$$4\sum_{j=1}^{n}(\dim_{\mathbb{Q}} k_j)^{-1} - n + 1 + \sum_{j=1}^{n} 2\dim_{k_j} \mathcal{A}b\mathcal{L}og\ \mathrm{Im}(v_j, v_j^*) +$$
$$\dim_{\mathbb{Q}} \mathcal{L}og\ \mathrm{Im}(P_{\mathcal{P}} \circ (v \times v^*)) + \dim_{\mathbb{Q}} \mathcal{L}og\ \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}}\circ(v\times v^*))})$$

where

- $\mathcal{A}b\mathcal{L}og\ \mathrm{Im}(v_j, v_j^*)$ is the k_j-sub-vector space of $\mathbb{C}/k_j\,\omega_{j1} + k_j\,\omega_{j2}$ generated by the elliptic logarithms $\{p_{jk}, q_{ji}\}_{\substack{k=1,\dots,r \\ i=1,\dots,s}}$ of the points $\{P_{jk}, Q_{ji}\}_{\substack{k=1,\dots,r \\ i=1,\dots,s}}$ for $j = 1,\dots,n$;
- $\mathcal{L}og\ \mathrm{Im}(P_{\mathcal{P}}\circ(v\times v^*))$ is the $\mathbb{Q}$-sub-vector space of $\mathbb{C}/2i\pi\mathbb{Q}$ generated by the logarithms $\{\log P_{\mathcal{P}_j}(P_{jk}, Q_{ji})\}_{\substack{k=1,\dots,r,\ \ i=1,\dots,s \\ j=1,\dots,n}}$;
- $\mathcal{L}og\ \mathrm{Im}(\psi_{|\ker(P_{\mathcal{P}}\circ(v\times v^*))})$ is the $\mathbb{Q}$-sub-vector space of $\mathbb{C}/2i\pi\mathbb{Q}$ generated by the logarithms $\{\log\psi(x, y^\vee)\}_{\substack{(x,y^\vee)\in\ker(P_{\mathcal{P}_j}\circ(v_j\times v_j^*)) \\ (x,y^\vee)\in\mathbb{Z}^r\otimes\mathbb{Z}^s}}$.

Because of Proposition 2.3 and Corollary 3.7, we can conclude that

{thmMain}

Theorem 4.1. *Let $M = [u : \mathbb{Z}^r \to G]$ be a 1-motive defined over K, with G an extension of a product $\Pi_{j=1}^n \mathcal{E}_j$ of pairwise not isogenous elliptic curves by the torus $\mathbb{G}_m^s$. Then the Generalized Grothendieck's Conjecture of Periods applied to M is equivalent to the 1-motivic elliptic conjecture.*

{Rk1}

Remark 4.2. If $Q_{ji} = 0$ for $j = 1,\dots,n$ and $i = 1,\dots,s$, the above conjecture is the elliptic-toric conjecture stated in [5, 1.1], which is equivalent to the Generalized Grothendieck's Conjecture of Periods applied to the 1-motive $M = [u : \Pi_{k=1}^r z_k\mathbb{Z} \to \mathbb{G}_m^s \times \Pi_{j=1}^n \mathcal{E}_j]$ with $u(z_k) = (R_{1k},\dots,R_{sk}, P_{1k},\dots,P_{nk}) \in \mathbb{G}_m^s(K) \times \Pi_{j=1}^n \mathcal{E}_j(K)$.

{Rk2}

Remark 4.3. If $Q_{ji} = P_{ij} = \mathcal{E}_j = 0$ for $j = 1,\dots,n$ and $i = 1,\dots,s$, the above conjecture is equivalent to the Generalized Grothendieck's Conjecture of Periods applied to the 1-motive $M = [u : \Pi_{k=1}^r z_k\mathbb{Z} \to \mathbb{G}_m^s]$ with $u(z_k) = (R_{1k},\dots,R_{sk}) \in \mathbb{G}_m^s(K)$, which in turn is equivalent to the Schanuel conjecture (see [5, Cor 1.3 and §3]).

{lowDim}

5. Low dimensional case: $r = n = s = 1$

In this section we work with a 1-motive $M = [u : \mathbb{Z} \to G]$ defined over K whose underlying extension G is an extension of just one elliptic curve $\mathcal{E}$ by the torus $\mathbb{G}_m$, i.e. $r = s = n = 1$.

Let $g_2 = 60\ \mathrm{G}_4$ and $g_3 = 140\ \mathrm{G}_6$ with G_4 and G_6 the Eisenstein series relative to the lattice $\Lambda := \mathrm{H}_1(\mathcal{E}(\mathbb{C}), \mathbb{Z})$ of weight 4 and 6 respectively. The field of definition K of the 1-motive $M = [u : \mathbb{Z} \to G], u(1) = R$ is

$$\mathbb{Q}\big(g_2, g_3, Q, R\big).$$

By Proposition 2.3, the field $K(\text{periods}(M))$ generated over K by the periods of M, which are the coefficients of the matrix (2.7), is

$$\mathbb{Q}\Big(g_2, g_3, Q, R, 2i\pi, \omega_1, \omega_2, \eta_1, \eta_2, p, \zeta(p), \eta_1 q - \omega_1\zeta(q), \eta_2 q - \omega_2\zeta(q), \log f_q(p) + l\Big).$$

$\mathrm{End}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$-linear dependence between the points P and Q and torsion properties of the points P, Q, R affect the dimension of the unipotent radical of $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$. By Corollary 3.7 we have the following table concerning the dimension of the motivic Galois group $\mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ of M:

	$\dim \mathrm{UR}(M)$	$\dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ $\mathcal{E}$ CM	$\dim \mathcal{G}\mathrm{al}_{\mathrm{mot}}(M)$ $\mathcal{E}$ not CM	M
Q, R torsion ($\Rightarrow$ P torsion)	0	2	4	$M=[u:\mathbb{Z}\to\mathcal{E}\times\mathbb{G}_m]$ $u(1)=(0,1)$
P,Q torsion (R not torsion)	1	3	5	$M=[u:\mathbb{Z}\to\mathcal{E}\times\mathbb{G}_m]$ $u(1)=(0,R)$
R torsion ($\Rightarrow$ P torsion)	2	4	6	$M=[u:\mathbb{Z}\to G]$ $u(1)=0$
Q torsion (P and R not torsion)	3	5	7	$M=[u:\mathbb{Z}\to\mathcal{E}\times\mathbb{G}_m]$ $u(1)=(P,R)$
P torsion (R and Q not torsion)	3	5	7	$M=[u:\mathbb{Z}\to\mathcal{E}^*\times\mathbb{G}_m]$ $u(1)=(Q,R)$
P,Q $\mathrm{End}(\mathcal{E})\otimes_{\mathbb{Z}}\mathbb{Q}$-lin indep	5	7	9	$M=[u:\mathbb{Z}\to G]$ $u(1)=R$

We can now state explicitly the Generalized Grothendieck's Conjecture of Periods (0.4) for the 1-motives involved on the above table:

- R and Q are torsion: We work with the 1-motive $M=[u:\mathbb{Z}\to\mathcal{E}\times\mathbb{G}_m], u(1)=(0,1)$ or $M=[0\to\mathcal{E}]$. If $\mathcal{E}$ is not CM,

$$\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}\Big(g_2,g_3,\omega_1,\omega_2,\eta_1,\eta_2\Big)\geq 4$$

  that is 4 at least of the 6 numbers $g_2,g_3,\omega_1,\omega_2,\eta_1,\eta_2$ are algebraically independent over $\mathbb{Q}$. If $\mathcal{E}$ is CM,

$$\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}\Big(g_2,g_3,\omega_1,\eta_1\Big)\geq 2$$

  that is 2 at least of the 4 numbers g_2,g_3,ω_1,η_1 are algebraically independent over $\mathbb{Q}$. If $g_2,g_3\in\overline{\mathbb{Q}}$ this is Chudnovsky Theorem: $\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}(\omega_1,\eta_1)=2$
- P and Q are torsion: We work with the 1-motive $M=[u:\mathbb{Z}\to\mathcal{E}\times\mathbb{G}_m], u(1)=(0,R)$ (we deal with this case in author's Ph.D, see [5]). If $\mathcal{E}$ is not CM,

$$\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}\Big(g_2,g_3,\omega_1,\omega_2,\eta_1,\eta_2,R,\log(R)\Big)\geq 5$$

  that is 5 at least of the 8 numbers $g_2,g_3,\omega_1,\omega_2,\eta_1,\eta_2,R,\log(R)$ are algebraically independent over $\mathbb{Q}$. If $\mathcal{E}$ is CM,

$$\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}\Big(g_2,g_3,\omega_1,\eta_1,R,\log(R)\Big)\geq 3$$

  that is 3 at least of the 6 numbers $g_2,g_3,\omega_1,\eta_1,R,\log(R)$ are algebraically independent over $\mathbb{Q}$.
- R is torsion: We work with the 1-motive $M=[u:\mathbb{Z}\to G], u(1)=0$ or $M=[v^*:\mathbb{Z}\to\mathcal{E}^*], v^*(1)=Q$. If $\mathcal{E}$ is not CM,

$$\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}\Big(g_2,g_3,\omega_1,\omega_2,\eta_1,\eta_2,Q,q,\zeta(q)\Big)\geq 6$$

  that is 6 at least of the 9 numbers $g_2,g_3,\omega_1,\omega_2,\eta_1,\eta_2,Q,q,\zeta(q)$ are algebraically independent over $\mathbb{Q}$. If $\mathcal{E}$ is CM,

$$\mathrm{tran.deg}_{\mathbb{Q}}\,\mathbb{Q}\Big(g_2,g_3,\omega_1,\eta_1,Q,q,\zeta(q)\Big)\geq 4$$

  that is 4 at least of the 7 numbers $g_2,g_3,\omega_1,\eta_1,Q,q,\zeta(q)$ are algebraically independent over $\mathbb{Q}$.

- Q is torsion: We work with the 1-motive $M = [u : \mathbb{Z} \to \mathcal{E} \times \mathbb{G}_m], u(1) = (P, R)$ (we deal with this case in author's Ph.D, see [5]). If $\mathcal{E}$ is not CM,
$$\mathrm{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, P, R, p, \zeta(p), \log(R)\Big) \geq 7$$
that is 7 at least of the 11 numbers $g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, P, R, p, \zeta(p), \log(R)$ are algebraically independent over $\mathbb{Q}$. If $\mathcal{E}$ is CM,
$$\mathrm{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(g_2, g_3, \omega_1, \eta_1, P, R, p, \zeta(p), \log(R)\Big) \geq 5$$
that is 5 at least of the 9 numbers $g_2, g_3, \omega_1, \eta_1, P, R, p, \zeta(p), \log(R)$ are algebraically independent over $\mathbb{Q}$.
- P is torsion: We work with the 1-motive $M = [u : \mathbb{Z} \to G], u(1) = R \in \mathbb{G}_m(K)$ or $M = [u : \mathbb{Z} \to \mathcal{E}^* \times \mathbb{G}_m], u(1) = (Q, R)$. If $\mathcal{E}$ is not CM,
$$\mathrm{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, Q, R, q, \zeta(q), \log(R)\Big) \geq 7$$
that is 7 at least of the 11 numbers $g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, Q, R, q, \zeta(q), \log(R)$ are algebraically independent over $\mathbb{Q}$. If $\mathcal{E}$ is CM,
$$\mathrm{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(g_2, g_3, \omega_1, \eta_1, Q, R, q, \zeta(q), \log(R)\Big) \geq 5$$
that is 5 at least of the 9 numbers $g_2, g_3, \omega_1, \eta_1, Q, R, q, \zeta(q), \log(R)$ are algebraically independent over $\mathbb{Q}$.
- P, Q, R are not torsion and P, Q are $\mathrm{End}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$-linearly independent: We work with the 1-motive $M = [u : \mathbb{Z} \to G], u(1) = R \in G(K)$. If $\mathcal{E}$ is not CM,
$$\mathrm{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(g_2, g_3, Q, R, \omega_1, \omega_2, \eta_1, \eta_2, p, \zeta(p), q, \zeta(q), \eta_1 q - \omega_1 \zeta(q), \eta_2 q - \omega_2 \zeta(q), \log f_q(p) + l\Big) \geq 9$$
that is 9 at least of the 15 numbers $g_2, g_3, Q, R, \omega_1, \omega_2, \eta_1, \eta_2, p, \zeta(p), q, \zeta(q), \eta_1 q - \omega_1 \zeta(q), \eta_2 q - \omega_2 \zeta(q), \log f_q(p)$ are algebraically independent over $\mathbb{Q}$. If $\mathcal{E}$ is CM,
$$\mathrm{tran.deg}_{\mathbb{Q}}\, \mathbb{Q}\Big(g_2, g_3, Q, R, \omega_1, \eta_1, p, \zeta(p), q, \zeta(q), \eta_1 q - \omega_1 \zeta(q), \eta_2 q - \omega_2 \zeta(q), \log f_q(p) + l\Big) \geq 7$$
that is 7 at least of the 13 numbers $g_2, g_3, Q, R, \omega_1, \eta_1, p, \zeta(p), q, \zeta(q), \eta_1 q - \omega_1 \zeta(q), \eta_2 q - \omega_2 \zeta(q), \log f_q(p)$ are algebraically independent over $\mathbb{Q}$.

References

[1] Y. André. Une introduction aux motifs (motifs purs, motifs mixtes, périodes). Panoramas et Synthèses [Panoramas and Syntheses], 17. Société Mathématique de France, Paris, 2004.

[2] Y. André. A note on 1-motives. to appear in Int. Math. Res. Not. (2019).

[3] J. Ayoub. L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle. J. Reine Angew. Math. 693 (2014), I: 1–149, II: 151–226.

[4] C. Bertolin. The Mumford-Tate group of 1-motives. Ann. Inst. Fourier (Grenoble) 52 (2002), no. 4, 1041–1059.

[5] C. Bertolin. Périodes de 1-motifs et transcendance. J. Number Theory 97 (2002), no. 2, 204–221.

[6] C. Bertolin. Le radical unipotent du groupe de Galois motivique d'un 1-motif. Math. Ann. 327 (2003), no. 3, 585–607.

[7] C. Bertolin. Motivic Galois theory for 1-motives Ann. Sci. Math. Québec 32 (2008), no. 2, 105–124.

[8] D. Bertrand. Schanuel's conjecture for non-isoconstant elliptic curves over function fields. In *Model theory with applications to algebra and analysis*, Vol. 1, 41–62, London Math. Soc. Lecture Note Ser., 349, Cambridge Univ. Press, Cambridge, 2008.
[9] J.-L. Brylinski. 1-motifs et formes automorphes (théorie arithmétique des domaines des Siegel). Pub. Math. Univ. Paris VII 15 (1983).
[10] P. Deligne. Théorie de Hodge III. Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.
[11] P. Deligne. Le groupe fondamental de la droite projective moins trois points. Galois group over $\mathbb{Q}$, Math. Sci. Res. Inst. Pub. 16. 1989.
[12] P. Deligne. Catégories tannakiennes. The Grothendieck Festschrift II, Birkhäuser 87, 1990.
[13] A. Grothendieck. On the de Rham cohomology of algebraic varieties. Inst. Hautes Études Sci. Publ. Math. No. 29, 1966, 95–103.
[14] O. Jacquinot and K. Ribet. Deficient points on extensions of abelian varieties by $\mathbb{G}_m$. J. Number Th. No. 25, 1987, 133–151.
[15] S. Lang. Introduction to transcendental numbers. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.
[16] B. Mazur and W. Messing. Universal extensions and one dimensional crystalline cohomology. Lecture Notes in Mathematics, Vol. 370. Springer-Verlag, Berlin-New York, 1974.
[17] B. Moonen. Notes on Mumford-Tate groups. Notes de cours du centre Emile Borel, N. 19 (1999).
[18] D. Mumford. Abelian varieties. Oxford UP, 1974.
[19] M. Nori. Course at Tata Int. Fund. Res. Notes by N. Fakhruddin, Mumbai (2000).
[20] N. Saavedra Rivano. Catégories tannakiennes. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972. ii+418 pp.
[21] G. Springer. Introduction to Riemann Surfaces. Addison-Wesley Publishing Company, Inc., Reading, Mass. 1957. viii+307 pp.
[22] M. Waldschmidt. Nombres transcendants et fonctions sigma de Weierstrass. C. R. Math. Rep. Acad. Sci. Canada 1 (1978/79), no. 2, 111–114.
[23] M. Waldschmidt. Nombres transcendants et groupes algébriques. With appendices by Daniel Bertrand and Jean-Pierre Serre. Astérisque No. 69–70 (1987).
[24] E. T. Whittaker and G. N. Watson. A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions. Reprint of the fourth (1927) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. vi+608 pp.

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THIRD KIND ELLIPTIC INTEGRALS AND TRANSCENDENCE

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Abstract. This short appendix aims at giving references on papers related with transcendence results concerning elliptic integrals of the third kind. So far, results on transcendence and linear independence are known, but there are very few results on algebraic independence.

In his book on transcendental numbers [Sc1957], Th. Schneider proposes eight open problems, the third of which is : *Try to find transcendence results on elliptic integrals of the third kind.*

In [La1966, Historical Note of Chapter IV], S. Lang explains the connections between elliptic integrals of the second kind, Weierstrass zeta function and extensions of an elliptic curve by $\mathbb{G}_a$. He applies the so–called Schneider–Lang criterion to the Weierstrass elliptic and zeta functions and deduces the transcendence results due to Th. Schneider on elliptic integrals of the first and second kind. At that time, it was not known how to use this method for proving results on elliptic integrals of the third kind.

The solution came from [Se1979], where J-P. Serre introduces the functions f_q (with the notation of [B2019]) related to elliptic integrals of the third kind, which satisfy the hypotheses of the Schneider-Lang criterion and are attached to extensions of an elliptic curve by $\mathbb{G}_m$. This is how the first transcendence results on these integrals were obtained [Wa1979a, Wa1979b]. In [BeLau1981], D. Bertrand and M. Laurent give further applications of the Schneider-Lang criterion involving elliptic integrals of the third kind. Applications are given in [Be1983a, Be1983b, S1986], dealing with the Neron–Tate canonical height on an elliptic curve (including the p–adic height) and the arithmetic nature of Fourier coefficients of Eisenstein series. A first generalization to abelian integrals of the third kind is quoted in [Be1983b]. Transcendence measures are given in [R1980a].

Properties of the smooth Serre compactification of a commutative algebraic group and of the exponential map, together with the links with integrals, are studied in [FWü1984]. See also [KL1985]. In [M2016, Chapter 20 – Elliptic functions] (see in particular Theorem 20.11 and exercises 20.104 and 20.105) more details are given on the functions associated with elliptic integrals of the third kind, the associated algebraic groups, which are extensions of an elliptic curve by $\mathbb{G}_m$, and the consequences of the Schneider-Lang criterion.

The first results of linear independence of periods of elliptic integrals of the third kind are due to M. Laurent [Lau1980, Lau1982] (he announced his results in [Lau1979a, Lau1979b]). The proof uses Baker's method. More general results on linear independence are due to G. Wüstholz [Wü1984] (see also [BaWü2007, § 6.2]), including the following one, which answers a conjecture that M. Laurent stated in [Lau1982] where he proved special cases of it. Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2, g_3. Let ζ be the corresponding Weierstrass zeta function, ω a nonzero period of $\wp$ and η the corresponding quasi-period of ζ. Let $u_1, \dots, u_n$ be complex numbers which are not poles of $\wp$, which are $\mathbb{Q}$

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linearly independent modulo $\mathbb{Z}\omega$ and such that $\wp(u_1),\dots,\wp(u_n)$ are algebraic. Define

$$\lambda(u_i,\omega)=\omega\zeta(u_i)-\eta u_i.$$

Then the $n+3$ numbers

$$1,\omega,\eta,\lambda(u_1),\dots,\lambda(u_n)$$

are linearly independent over $\overline{\mathbb{Q}}$.

The question of the transcendence of the nonvanishing periods of a meromorphic differential form on an elliptic curve defined over the field of algebraic numbers is now solved [BaWü2007, Theorem 6.6]. See also [HWü2018], as well as [T2017, § 1.5] for abelian integrals of the first and second kind. A reference of historical interest to a letter from Leibniz to Huygens in 1691 is quoted in [BaWü2007, § 6.3] and [Wü20012].

The only results on algebraic independence related with elliptic integrals of the third kind so far are those obtained by É. Reyssat [R1980b, R1982] and by R. Tubbs [T1987, T1990]. We are very far from anything close to the conjectures in [B2019].

For a survey (with an extensive bibliography including 254 entries), see [Wa2008].

The references below are listed by chronological order.

References

[Sc1957] Schneider, Theodor. Einführung in die transzendenten Zahlen. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[La1966] Lang, Serge. Introduction to transcendental numbers. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966.

[Se1979] Serre, Jean–Pierre. Quelques propriétés des groupes algébriques commutatifs. Appendice II de [Wa1979a], 191 – 202.

[Wa1979a] Waldschmidt, Michel. Nombres transcendants et groupes algébriques. With appendices by Daniel Bertrand and Jean-Pierre Serre. Astérisque No. 69-70 (1979), 218 pp.
https://smf.emath.fr/publications/nombres-transcendants-et-groupes-algebriques-2e-edition

[Wa1979b] Waldschmidt, Michel. Nombres transcendants et fonctions sigma de Weierstrass. C. R. Math. Rep. Acad. Sci. Canada 1 (1978/79), no. 2, 111–114.

[Lau1979a] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 15, 699–701.

[Lau1979b] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. Séminaire Delange-Pisot-Poitou, 20e année: 1978/1979. Théorie des nombres, Fasc. 1, Exp. No. 13, 4 pp.,

[Lau1980] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. J. Reine Angew. Math. 316 (1980), 122–139.

[R1980a] Reyssat, Éric. Approximation de nombres liés à la fonction sigma de Weierstrass. Ann. Fac. Sci. Toulouse Math. (5) 2 (1980), no. 1, 79–91.

[R1980b] Reyssat, Éric. Fonctions de Weierstrass et indépendance algébrique. C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 10, A439–A441.

[BeLau1981] Bertrand, Daniel & Laurent, Michel. Propriétés de transcendance de nombres liés aux fonctions thêta. C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 16, 747–749.

[Lau1982] Laurent, Michel. Transcendance de périodes d'intégrales elliptiques. II. J. Reine Angew. Math. 333 (1982), 144–161.

[R1982] Reyssat, Éric. Propriétés d'indépendance algébrique de nombres liés aux fonctions de Weierstrass. Acta Arith. 41 (1982), no. 3, 291–310.

[Be1983a] Bertrand, Daniel. Problèmes de transcendance liés aux hauteurs sur les courbes elliptiques. Mathematics, pp. 55–63, CTHS: Bull. Sec. Sci., III, Bib. Nat., Paris, 1981.

[Be1983b] Bertrand, Daniel. Endomorphismes de groupes algébriques; applications arithmétiques. Diophantine approximations and transcendental numbers (Luminy, 1982), 1–45, Progr. Math., 31, Birkhäuser Boston, Boston, MA, 1983.

[Wü1984] Wüstholz, Gisbert. Transzendenzeigenschaften von Perioden elliptischer Integrale. J. Reine Angew. Math. 354 (1984), 164–174.

[FWü1984] Faltings, Gert & Wüstholz, Gisbert. Einbettungen kommutativer algebraischer Gruppen und einige ihrer Eigenschaften. J. Reine Angew. Math. 354 (1984), 175–205.

[KL1985] Knop, Friedrich & Lange, Herbert. Some remarks on compactifications of commutative algebraic groups. Comment. Math. Helv. 60 (1985), no. 4, 497–507.

[S1986] Scholl, Antony. Fourier coefficients of Eisenstein series on noncongruence subgroups. Math. Proc. Cambridge Philos. Soc. 99 (1986), no. 1, 11–17.

[T1987] Tubbs, Robert. Algebraic groups and small transcendence degree. I. J. Number Theory 25 (1987), no. 3, 279–307.

[T1990] Tubbs, Robert. Algebraic groups and small transcendence degree. II. J. Number Theory 35 (1990), no. 2, 109–127.

[BaWü2007] Baker, Alan & Wüstholz, Gisbert. Logarithmic forms and Diophantine geometry. New Mathematical Monographs, 9. Cambridge University Press, Cambridge, 2007.

[Wa2008] Waldschmidt, Michel. Elliptic functions and transcendence. Surveys in number theory, 143–188, Dev. Math., 17, Springer, New York, 2008. `https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/SurveyTrdceEllipt2006.pdf`

[Wü20012] Wüstholz, Gisbert. Leibniz' conjecture, periods & motives. Colloquium De Giorgi 2009, 33–42, Colloquia, 3, Ed. Norm., Pisa, 2012.

[M2016] Masser, David. Auxiliary polynomials in number theory. Cambridge Tracts in Mathematics, 207. Cambridge University Press, Cambridge, 2016.

[T2017] Tretkoff, Paula. Periods and special functions in transcendence. Advanced Textbooks in Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[HWü2018] Huber, Annette & Wüstholz, Gisbert. Periods of 1-motives. `https://arxiv.org/abs/1805.10104`

[B2019] Bertolin, Cristiana. Third kind elliptic integrals and 1-motives.

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