# Big Picard theorem and algebraic hyperbolicity for varieties admitting a variation of Hodge structures

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# BIG PICARD THEOREM AND ALGEBRAIC HYPERBOLICITY FOR VARIETIES ADMITTING A VARIATION OF HODGE STRUCTURES

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Abstract. For a complex smooth log pair $(Y, D)$, if the quasi-projective manifold $U = Y - D$ admits a complex polarized variation of Hodge structures with local unipotent monodromies around $D$ or admits an integral polarized variation of Hodge structures, whose period map is quasi-finite, then we prove that $(Y, D)$ is algebraically hyperbolic in the sense of Demailly, and that the generalized big Picard theorem holds for $U$: any holomorphic map $f : \Delta - \{0\} \to U$ from the punctured unit disk to $U$ extends to a holomorphic map of the unit disk $\Delta$ into $Y$. This result generalizes a recent work by Bakker-Brunebarbe-Tsimerman, in which they proved that if the monodromy group of the above variation of Hodge structures is arithmetic, then $U$ is Borel hyperbolic: any holomorphic map from a quasi-projective variety to $U$ is algebraic.

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## 0. Introduction

0.1. **Main results.** The classical big Picard theorem says that any holomorphic map from the punctured disk $\Delta^*$ into $\mathbb{P}^1$ which omits three points can be extended to a

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holomorphic map $\Delta \to \mathbb{P}^1$, where $\Delta$ denotes the unit disk. Therefore, we introduce a new notation of hyperbolicity which generalizes the big Picard theorem.

**Definition 0.1** (Picard hyperbolicity)**.** A quasi-projective variety $U$ is *Picard hyperbolic* if for some (thus any) projective compactification $Y$ of $U$, any holomorphic map $f : \Delta^* \to U$ extends to a holomorphic map $\bar{f} : \Delta \to X$.

Picard hyperbolic varieties fascinate the author a lot because of the recent interesting work [JK18b] by Javanpeykar-Kucharczyk on the *algebraicity of analytic maps*. In [JK18b, Definition 1.1], they introduce a new notion of hyperbolicity: a quasi-projective variety $U$ is *Borel hyperbolic* if any holomorphic map from a quasi-projective variety to $U$ is necessarily *algebraic*. In [JK18b, Corollary 3.11] they prove that a Picard hyperbolic variety is *Borel hyperbolic*. We refer the readers to [JK18b, §1] for their motivation on the Borel hyperbolicity. By A. Borel [Bor72] and Kobayashi-Ochiai [KO71], it has long been known to us that the quotients of bounded symmetric domains by torsion free arithmetic groups are hyperbolically embedded into their Baily-Borel-Satake compactification, and thus they are Picard hyperbolic (see [Kob98, Theorem 6.1.3]). An analogue of bounded symmetric domains is the rich theory of period domain, which was first introduced by Griffiths [Gri68a] and was later systematically studied by him in the seminal work [Gri68b, Gri70a, Gri70b]. Griffiths further conjectured that the image of a 'period map' is algebraic and that the period map is algebraic. In [JK18b, §1.1] Javanpeykar-Kucharczyk formulated an inspiring variant of Griffiths' conjecture as follows.

**Conjecture 0.2** (Griffiths, Javanpeykar-Kucharczyk)**.** An algebraic variety $U$ which admits a quasi-finite period map $U \to \mathcal{D}/\Gamma$ is Borel hyperbolic.

Unlike Hermitian symmetric spaces, except the classical cases (abelian varieties, and K3 type), the quotient of period domain $\mathcal{D}/\Gamma$ in Conjecture 0.2 is never an algebraic variety, and the global monodromy groups $\Gamma$ is not arithmetic in general. However, it is still expected and conjectured by Griffiths that there is a 'partial compactification' for $\mathcal{D}/\Gamma$ analogous to the Baily-Borel-Satake compactification in the sense of [Gri70b, Conjecture 9.2] or [GGLR17, Conjecture 1.2.2]. For a period map $p : U \to \mathcal{D}/\Gamma$, in [GGLR17] Green-Griffiths-Lazza-Robles constructed Hodge theoretic completion for the image $p(U)$ when $\dim p(U) = 1, 2$.

In a recent remarkable work [BBT18], Bakker-Brunebarbe-Tsimerman proved (among others) that a variety (or more generally Deligne-Mumford stacks) admitting a quasi-finite $\mathbb{R}_{\rm an,exp}$-period map is Borel hyperbolic. Since they applied the tools from o-minimal structures, they have to assume that the monodromy group of variation of Hodge structures they studied are *arithmetic*. In this paper, we extend their theorem to the Picard hyperbolicity, and we also remove their arithmeticity condition for monodromy groups. The first result is the following.

**Theorem A.** *Let $Y$ be a complex projective manifold and let $D$ be a simple normal crossing divisor on $Y$. Assume that there is a complex polarized variation of Hodge structures over $U := Y - D$ with local unipotent monodromies around $D$ whose period map is quasi-finite (*i.e.* every fiber is a finite set). Then $U$ is both algebraically hyperbolic, and Picard hyperbolic. In particular, $U$ is Borel hyperbolic.*

We refer the reader to § 1.1 for complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short), and to Definition 3.1 for the definition of algebraic hyperbolicity. As a consequence of Theorem A, we obtain the following result for varieties admitting an integral variation of Hodge structures, which in particular confirms Conjecture 0.2.

**Theorem B.** *Let $U$ be a quasi-projective manifold and let $(V,\nabla,F^\bullet,Q)$ be an integral polarized variation of Hodge structures over $U$, whose period map is quasi-finite. Then $U$ is both algebraically hyperbolic and Picard hyperbolic. In particular, $U$ is Borel hyperbolic.*

Let us mention that when the monodromy group of polarized variation of Hodge structures $(V,\nabla,F^\bullet,Q)$ in Theorem B is assumed to be *arithmetic*, Borel hyperbolicity of the quasi-projective manifold $U$ in Theorem B has been proven in [BBT18, Corollary 7.1]. Our proofs of Theorems A and B are based on complex analytic and Hodge theoretic methods, and it does not use the delicate o-minimal geometry in [PS08, PS09, BKT18, BBT18]. Let us also mention that using Mochizuki's norm estimate for tame harmonic bundles in [Moc07] instead of the estimate for Hodge norms in [CKS86], we can even remove the assumption of 'unipotent monodromies around $D$' in Theorem A. However, it will make the paper more involved and we shall work on it in another paper.

## 0.2. Main strategy.

0.2.1. *Why not Hodge metric?* Let $Y$ be a projective manifold and let $D$ be a simple normal crossing divisor on $Y$. Assume that there is a complex polarized variation of Hodge structures $(V,\nabla,F^\bullet,Q)$ on $U=Y-D$. Then there is a natural holomorphic map, so-called *period map*, $p:U\to \mathcal{D}/\Gamma$ where $\mathcal{D}$ is the *period domain* associated to $(V,\nabla,F^\bullet,Q)$ (see [CMSP17] or [KKM11, §4.3] for the definition) and $\Gamma$ is the monodromy group. The period domain $\mathcal{D}$ admits a canonical ($\Gamma$-invariant) hermitian metric $h_{\mathcal{D}}$, and by Griffiths-Schmid [GS69] its holomorphic sectional curvatures along horizontal directions are bounded from above by a negative constant. One can thus easily show the Kobayashi hyperbolicity of $U$ if $p$ is immersive everywhere. Indeed, since $p$ is tangent to the horizontal subbundle of $T_{\mathcal{D}}$ by the Griffiths transversality, one can pull back the metric $h_{\mathcal{D}}$ to $U$ by $p$ and by the curvature decreasing property, the holomorphic sectional curvature of the hermitian (moreover Kähler) metric $h_U:=p^*h_{\mathcal{D}}$ on $U$ is also bounded from above by a negative constant. This Kähler metric $h_U$ is quite useful in proving that the log cotangent bundle $\Omega_Y(\log D)$ is big and that $(Y,D)$ is of log general type in the work [Zuo00,Bru18,BC17]. However, such metric $h_U$ is not sufficient to prove the Picard hyperbolicity of $U$ since $h_U$ might *degenerate* in a bad way near the boundary $D$ and thus its curvature behavior near $D$ is unclear to us. To the best of our knowledge, it should be quite difficult to prove that $U$ is Picard hyperbolic or algebraically hyperbolic without knowing the precise information of $h_U$ near $D$.

0.2.2. *A Finsler metric on the compactification.* The recent works [LSZ19,Den19] on the Borel and Picard hyperbolicity of moduli of polarized manifolds by Lu, Sun, Zuo and the author motivated us to prove Theorem A. An important tool (amongs others) in these works, is a particular Higgs bundle constructed by Viehweg-Zuo [VZ02, VZ03] (later developed by Popa el al. [PS17, PTW18] using mixed Hodge modules), which contains a *globally* positive line bundle over the compactification $Y$ rather than $U$. This positive line bundle originates from Kawamata's deep work [Kaw85] on the Iitaka conjecture: for an algebraic fiber space $f:X\to Y$ between projective manifolds whose geometric generic fiber admits a good minimal model, $\det f_*(mK_{X/Y})$ is big for $m\gg 0$ if $f$ has maximal variation. In an ingenious way, Viehweg-Zuo [VZ02, VZ03] applied Viehweg's fiber product and cyclic cover tricks to transfer Kawamata's positivity $\det f_*(mK_{X/Y})$ to their Higgs bundles.

We first note that in the case that there is a $\mathbb{C}$-PVHS $(V,\nabla,F^\bullet,Q)$ over $Y-D$ where $(Y,D)$ is a log pair, one also has a strictly positive line bundle on $U$ if the period map is generically immersive, which was constructed by Griffiths in [Gri70a] half century ago! Based on the work [CKS86, Kas85] on the asymptotic estimate for Hodge metrics at infinity, Bakker-Brunebarbe-Tsimerman [BBT18] showed that this Griffiths line bundle extends to a big line bundle $L_{\rm Gri}$ over $Y$ if the monodromies of $(V, \nabla, F^\bullet, Q)$ around $D$ are unipotent (see Lemma 1.4). As we will see later, the Griffiths line bundle plays a similar role as the Kawamata positivity described above. Indeed, based on the above $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ we construct a Higgs bundle $(E, \theta) = (\oplus_{p+q=m} E^{p,q}, \oplus_{p+q=m}\theta_{p,q})$ on the log pair $(Y, D)$ so that the Griffiths line bundle $L_{\rm Gri}$ is contained in some higher stage $E^{p_0,q_0}$ of $E$. This Higgs bundle shares some similarities with the Viehweg-Zuo Higgs bundle in [VZ02, VZ03] (see Remark 1.6). Inspired by our previous work [Den18b] on the proof of Viehweg-Zuo's conjecture on Brody hyperbolicity of moduli of polarized manifolds, in Theorem 1.8 we show that $(E, \theta)$ still enjoys a 'partially' infinitesimal Torelli property. This enables us construct a negatively curved, and generically positively definite Finsler metric on $U$, in a similar vein as [Den18a, Den19].

**Theorem C** (=Theorem 1.5+Theorem 2.6)**.** *Let $Y$ be a projective manifold and let $D$ be a simple normal crossing divisor on $Y$. Assume that there is a complex polarized variation of Hodge structures over $Y-D$ with local unipotent monodromies around $D$, whose period map is generically immersive. Then there are a* Finsler metric *$h$ (see Definition 2.1) on $T_Y(-\log D)$ which is positively definite on a dense Zariski open set $U^\circ$ of $Y-D$, and a smooth Kähler form $\omega$ on $Y$ such that for any holomorphic map $\gamma : C \to U$ from an open set $C \subset \mathbb{C}$ to $U$, one has*

$$\sqrt{-1}\partial\bar\partial \log |\gamma'(t)|_h^2 \geq \gamma^*\omega. \tag{0.2.1}$$

Let us mention that, though we only construct (possibly degenerate) Finsler metric over $T_Y(-\log D)$, it follows from (0.2.1) that we know exactly the behavior of its curvature near the boundary $D$ since $\omega$ is a smooth Kähler form over $Y$. The proof of Theorem A is then based on Theorem C and the following criteria for big Picard theorem established in the appendix whose proof is Nevanlinna theoretic.

**Theorem D.** *Let $Y$ be a projective manifold and let $D$ be a simple normal crossing divisor on $Y$. Let $f : \Delta^* \to Y - D$ be a holomorphic map. Assume that there is a (possibly degenerate) Finsler metric $h$ of $T_Y(-\log D)$ such that $|f'(t)|_h^2 \not\equiv 0$, and*

$$\frac{1}{\pi}\sqrt{-1}\partial\bar\partial \log |f'(t)|_h^2 \geq f^*\omega \tag{0.2.2}$$

*for some smooth Kähler metric $\omega$ on $Y$. Then $f$ extends to a holomorphic map $\bar f : \Delta \to Y$.*

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## 1. Construction of special Higgs bundles

1.1. **Preliminary on complex variation of Hodge structures.** A *log pair* $(Y, D)$ consists of a smooth projective manifold and a simple normal crossing divisor $D$, and such log pair $(Y, D)$ is called a *log-compactification* of the quasi-projective manifold $Y - D$.

**Definition 1.1.** A Higgs bundle on a log pair $(Y, D)$ is a pair $(E, \theta)$ consisting of a holomorphic vector bundle $E$ on $Y$ and an $\mathscr{O}_Y$-linear map

$$\theta : E \to E \otimes \Omega_Y(\log D)$$

so that $\theta \wedge \theta = 0$. Such $\theta$ is called *Higgs field.*

Following Simpson [Sim88], a complex polarized variation of Hodge structures of weight $m$ over $U = Y - D$ is a $C^\infty$-vector bundle $V = \oplus_{p+q=m} V^{p,q}$ and a flat connection $\nabla$ satisfying Griffiths' transversality condition.

$$\nabla : V^{p,q} \to A^{0,1}(V^{p+1,q-1}) \oplus A^1(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1}) \tag{1.1.1}$$

and such that a polarization exists; this is a sesquilinear form $Q(\bullet,\bullet)$ over $V$, hermitian symmetric or antisymmetric as $m$ is even or odd, invariant under $\nabla$, such that the Hodge decomposition $V = \oplus_{p+q=m} V^{p,q}$ is orthogonal and such that

$$h := (\sqrt{-1})^{p-q} Q(\bullet,\bullet) > 0$$

on $V^{p,q}$.

Let us decompose $\nabla$ into operators of $(1,0)$ and $(0,1)$

$$\nabla = \nabla' + \nabla''$$

and thus $\nabla''$ induces a complex structure on $V$. We define a filtration

$$F^p V := V^{p,q} \oplus V^{p+1,q-1} \oplus \cdots \oplus V^{m,0}$$

and by (1.1.1) $F^p V$ is invariant under $\nabla''$. Hence $F^p V$ can be equipped with the complex structure inherited from $(V, \nabla'')$, and the filtration

$$F^\bullet : V = F^0 V \supset F^1 V \supset \cdots \supset F^m V \supset F^{m+1} V = \{0\}$$

is called the *Hodge filtration*. Such data $(V, \nabla, F^\bullet, Q)$ is called a complex polarized variation of Hodge structures ($\mathbb{C}$-PVHS for short) on $U$.

Note that the flat connection $\nabla$ in (1.1.1) induces an $\mathcal{O}_U$-linear map

$$\eta_{p,q} : F^p V / F^{p+1} V \to (F^{p-1} V / F^p V) \otimes \Omega_U.$$

Let us denote by $F := \oplus_p (F^p V / F^{p+1} V)$ and $\eta = \oplus_p \eta_{p,q}$. Then $(F, \eta)$ is a Higgs bundle on $U$.

We say the $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ on $U$ has *unipotent monodromies* around $D$ if local monodromies around $D$ of the local system on $U$ induced by the flat bundle $(V, \nabla)$ are all *unipotent*.

For two $\mathbb{C}$-PVHS $(V_1, \nabla_1, F^\bullet V_1, Q_1)$ and $(V_2, \nabla_2, F^\bullet V_2, Q_2)$ of weight $m_1$ and $m_2$ over $Y - D$, one can define their tensor product, which is still $\mathbb{C}$-PVHS with weight $m_1 + m_2$. Moreover, if they both have unipotent monodromies around $D$, so is their tensor product.

*Remark* 1.2. It is well-known that $\mathbb{C}$-PVHS are quite close to real variation of Hodge structures ($\mathbb{R}$-PVHS for short, see [CKS86] for a previse definition). Indeed, one can obtain a $\mathbb{R}$-PVHS by adding the $\mathbb{C}$-PVHS with its conjugate. In particular, the estimate of Hodge metric at infinity of a $\mathbb{R}$-PVHS in [CKS86] also holds true for $\mathbb{C}$-PVHS.

For a $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ defined over $U = Y - D$ with unipotent monodromies around $D$, there is a *canonical* way to extend it to a Higgs bundle over the log pair $(Y, D)$. By Deligne, $V$ has a locally free extension $\overline{V}$ to $Y$ such that $\nabla$ extends to a logarithmic connection

$$\overline{\nabla} : \overline{V} \to \overline{V} \otimes \Omega_Y(\log D)$$

with nilpotent residues. For each $p$ we set

$$\overline{F}^p V := \iota_* F^p V \cap \overline{V}$$

where $\iota : U \hookrightarrow Y$ is the inclusive map. By Schmid's nilpotent orbit theorem [Sch73], both $\overline{F}^p V$ and the graded term $\overline{F}^{p,q} = \overline{F}^p V/\overline{F}^{p+1} V$ are locally free, and $\overline{\nabla}$ induces an $\mathcal{O}_Y$-linear map

$$\overline{\eta}_{p,q} : \overline{F}^{p,q} \to \overline{F}^{p-1,q+1} \otimes \Omega_Y(\log D).$$

Hence the pair

$$(\overline{F}, \overline{\eta}) := (\oplus_{p+q=m} \overline{F}^{p,q}, \oplus_{p+q=m} \overline{\eta}_{p,q}) \tag{1.1.2}$$

is a Higgs bundle on the log pair $(Y, D)$, which extends $(F, \eta)$ defined over $U$.

**Definition 1.3.** We say that Higgs bundle $(\overline{F}, \overline{\eta})$ over $(Y, D)$ in (1.1.2) is *canonically* induced by the $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$.

1.2. **Griffiths line bundle.** For the $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ defined over $U$ as above, in [Gri70a], Griffiths constructed a line bundle $L_{\rm Gri}$ on $U$, which he called the *canonical bundle* of $(V, \nabla, F^\bullet, Q)$. In [BBT18, Lemma 6.4] Bakker-Brunebarbe-Tsimerman proved that the Griffiths line bundle indeed extends to a big line bundle on $Y$.

**Lemma 1.4** ( [BBT18, Lemma 6.4]). *Let $(Y, D)$ be a log pair. Let $(V, \nabla, F^\bullet, Q)$ be a $\mathbb{C}$-PVHS of weight $m$ over $Y - D$ with unipotent monodromies around $D$, whose period map is generically immersive. Then the* Griffiths line bundle

$$L_{\rm Gri} := (\det \overline{F}^{m,0})^{\otimes m} \otimes (\det \overline{F}^{m-1,1})^{\otimes (m-1)} \otimes \cdots \otimes \det \overline{F}^{1,m-1}$$

*is a big and nef line bundle on $Y$. Here $(\oplus_{p+q=m} \overline{F}^{p,q}, \oplus_{p+q=m} \overline{\eta}_{p,q})$ is the Higgs bundle on $(Y, D)$ canonically induced by $(V, \nabla, F^\bullet, Q)$ defined in Definition 1.3.*

1.3. **Special Higgs bundles induced by $\mathbb{C}$-PVHS.** Let $(Y, D)$ be a log pair. Let $(V, \nabla, F^\bullet, Q)$ be a $\mathbb{C}$-PVHS of weight $m$ over $Y - D$ with unipotent monodromies around $D$, whose period map is generically immersive. Let $(\overline{F}, \overline{\eta})$ be the Higgs bundle over the log pair $(Y, D)$ canonically induced by $(V, \nabla, F^\bullet, Q)$ defined in Definition 1.3. Let us denote by $r_p := \operatorname{rank} F^{p,q}$, and $r := m r_m + (m-1) r_{m-1} + \cdots + r_1$.

We define a new Higgs bundle $(E, \theta)$ on $(Y, D)$ by setting $(E, \theta) := (\overline{F}, \bar{\eta})^{\otimes r}$. Precisely, $E := \overline{F}^{\otimes r}$, and

$$\theta := \bar{\eta} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-1)-\text{tuple}} + \mathbb{1} \otimes \bar{\eta} \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-2)-\text{tuple}} + \cdots + \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{(r-1)-\text{tuple}} \otimes \bar{\eta}.$$

We have the (Hodge) decomposition

$$E = \oplus_{P+Q=rm} E^{P,Q}$$

with

$$E^{P,Q} := \oplus_{p_1+\cdots+p_r=P; q_1+\cdots+q_r=Q} \overline{F}^{p_1,q_1} \otimes \cdots \otimes \overline{F}^{p_r,q_r} \tag{1.3.1}$$

Hence

$$\theta : E^{P,Q} \to E^{P-1,Q+1} \otimes \Omega_Y(\log D).$$

One can easily show that $(E, \theta)$ is *canonically induced* by the $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)^{\otimes r}$ in the sense of Definition 1.3. Note that the tensor product $(V, \nabla, F^\bullet, Q)^{\otimes r}$ has weight $m \cdot r$, and also has unipotent monodromies around $D$.

Note that $\det \overline{F}^{p,q} = \wedge^{r_p} \overline{F}^{p,q} \subset (\overline{F}^{p,q})^{\otimes r_p} \subset \overline{F}^{\otimes r_p}$. Hence

$$L_{\rm Gri} := (\det \overline{F}^{m,0})^{\otimes m} \otimes (\det \overline{F}^{m-1,1})^{\otimes (m-1)} \otimes \cdots \otimes \det \overline{F}^{1,m-1} \subset (\overline{F}^{m,0})^{\otimes m r_m} \otimes \cdots \otimes (\overline{F}^{1,m-1})^{\otimes r_1} \subset E$$

Moreover, by (1.3.1), one has

$$L_{\rm Gri} \subset E^{P_0,Q_0}$$

with $P_0 = r_m m^2 + r_{m-1}(m-1)^2 + \cdots + r_1$, and $P_0 + Q_0 = rm$.

In summary, we construct a special Higgs bundle on the log pair $(Y, D)$ as follows.

**Theorem 1.5.** *Let $(Y, D)$ be a log pair. Let $(V, \nabla, F^\bullet, Q)$ be a $\mathbb{C}$-PVHS over $Y - D$ with unipotent monodromies around $D$, whose period map is generically immersive. Then there is a Higgs bundle $(E, \theta) = (\oplus_{p+q=\ell} E^{p,q}, \theta)$ on the log pair $(Y, D)$ satisfying the following conditions.*

(i) *The Higgs field $\theta$ satisfies*

$$\theta : E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega_Y(\log D)$$

(ii) *$(E, \theta)$ is canonically induced (in the sense of Definition 1.3) by some $\mathbb{C}$-PVHS over $Y - D$ of weight $\ell$ with unipotent monodromies around $D$.*

(iii) *There is a big and nef line bundle $L$ over $Y$ such that $L \subset E^{p_0,q_0}$ for some $p_0 + q_0 = \ell$.* □

*Remark* 1.6. The interested readers can compare the Higgs bundle in Theorem 1.5 with the Viehweg-Zuo Higgs bundle in [VZ02, VZ03] (see also [PTW18]). Loosely speaking, a *Viehweg-Zuo Higgs bundle* for a log pair $(Y, D)$ is a Higgs bundle $(E = \oplus_{p+q=m} E^{p,q}, \theta)$ over $(Y, D+S)$ induced by some (geometric) $\mathbb{Z}$-PVHS defined over a Zariski open set of $Y - (D \cup S)$, where $S$ is another divisor on $Y$ so that $D + S$ is simple normal crossing. The extra data is that there is a sub-Higgs sheaf $(F = \oplus_{p+q=m} F^{p,q}, \eta) \subset (E, \theta)$ such that the *first stage* $F^{n,0}$ is a big line bundle, and

$$\eta : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega_Y(\log D).$$

As we explained in § 0.2.2, the positivity $F^{n,0}$ comes in a sophisticated way from the Kawamata's big line bundle $\det f_*(mK_{X/Y})$ where $f : X \to Y$ is some algebraic fiber space between projective manifolds. For our Higgs bundle $(E = \oplus_{p+q=m} E^{p,q}, \theta)$ over the log pair $(Y, D)$ in Theorem 1.5, the global positivity is the Griffiths line bundle which is contained in some *intermediate stage* $E^{p_0,q_0}$ of $(E = \oplus_{p+q=m} E^{p,q}, \theta)$.

1.4. **Iterating Higgs fields.** Let $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ be a Higgs bundle on a log pair $(Y, D)$ satisfying the three conditions in Theorem 1.5. We apply ideas by Viehweg-Zuo [VZ02, VZ03] to iterate Higgs fields.

Since $\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega_Y(\log D)$, one can iterate $\theta$ by $k$-times to obtain

$$E^{p_0,q_0} \longrightarrow E^{p_0-1,q_0+1} \otimes \Omega_Y(\log D) \longrightarrow \cdots \longrightarrow E^{p_0-k,q_0+k} \otimes \otimes^k \Omega_Y(\log D)$$

Since $\theta \wedge \theta = 0$, the above morphism factors through

$$E^{p_0,q_0} \longrightarrow E^{p_0-k,q_0+k} \otimes \mathrm{Sym}^k \Omega_Y(\log D)$$

Since $L$ is a subsheaf of $E^{p_0,q_0}$, it induces

$$L \longrightarrow E^{p_0-k,q_0+k} \otimes \mathrm{Sym}^k \Omega_Y(\log D)$$

which is equivalent to a morphism

$$\tau_k : \mathrm{Sym}^k T_Y(-\log D) \to L^{-1} \otimes E^{p_0-k,q_0+k} \tag{1.4.1}$$

The readers might be worried that all $\tau_k$ might be trivial so that the above construction will be meaningless. In the next subsection, we will show that this indeed cannot happen.

1.5. **An infinitesimal Torelli-type theorem.** We first follow ideas in [VZ03, §7] to give some "proper" metric on the special Higgs bundle $(E,\theta)$ constructed in Theorem 1.5. A more general result for $\mathbb{Z}$-PVHS with quasi-unipotent monodromies are obtained by Popa-Taji-Wu [PTW18].

Let $(E=\oplus_{p+q=\ell}E^{p,q},\theta)$ be a Higgs bundle on a log pair $(Y,D)$ satisfying the three conditions in Theorem 1.5. Write the simple normal crossing divisor $D=D_1+\cdots+D_k$. Let $f_{D_i}\in H^0\big(Y,\mathcal{O}_Y(D_i)\big)$ be the canonical section defining $D_i$. We fix a smooth hermitian metrics $g_{D_i}$ on $\mathcal{O}_Y(D_i)$. After rescaling $g_{D_i}$, we assume that $|f_{D_i}|_{g_{D_i}}<1$ for $i=1,\ldots,k$. Set

$$r_D:=\prod_{i=1}^{k}(-\log|f_{D_i}|^2_{g_{D_i}}).$$

Let $g$ be a singular hermitian metric with analytic singularities of the big and nef line bundle $L$ such that $g$ is smooth on $Y\setminus\mathbf{B}_+(L)$ where $\mathbf{B}_+(L)$ is the augmented base locus of $L$, and the curvature current $\sqrt{-1}\Theta_g(L)\geqslant\omega$ for some smooth Kähler form $\omega$ on $Y$. For $\alpha\in\mathbb{N}$, define

$$h_L:=g\cdot(r_D)^{\alpha}$$

The following proposition is a variant of [VZ03, §7] (see also [PTW18, §3] for a more general statement).

**Proposition 1.7.** *When $\alpha\gg 0$, after rescaling $f_{D_i}$, there exists a continuous, positively definite hermitian form $\omega_\alpha$ on $T_Y(-\log D)$ such that*

(i) *the curvature form*

$$\sqrt{-1}\Theta_{h_L}(L)_{\restriction U_0}\geqslant r_D^{-2}\cdot\omega_{\alpha\restriction U_0},\quad \sqrt{-1}\Theta_{h_L}(L)\geq\omega$$

*where $\omega$ is a smooth Kähler metric on $Y$, and $U_0:=Y\setminus\big(D\cup\mathbf{B}_+(L)\big)$.*

(ii) *The singular hermitian metric $h:=h_L^{-1}\otimes h_{\rm hod}$ on $L^{-1}\otimes E$ is locally bounded on $Y$, and smooth outside $D\cup\mathbf{B}_+(L)$, where $h_{\rm hod}$ is the Hodge metric for the Higgs bundle $(E,\theta)|_U$. Moreover, $h$ vanishes on $D\cup\mathbf{B}_+(L)$.*

(iii) *The singular hermitian metric $r_D^2h$ on $L^{-1}\otimes E$ is also locally bounded on $Y$ and vanishes on $D$.* □

Let us explain the idea of the proof for Proposition 1.7. Proposition 1.7.(i) follows from an easy computation. Recall that local monodromies around $D$ of the local system induced by $\mathbb{C}$-PVHS $(E,\theta)|_U$ are assumed to be unipotent. By the deep work by Cattani-Kaplan-Schmid [CKS86] (see also [VZ03, Claim 7.8]) on the estimate of Hodge metrics, we know that the Hodge norms for local sections of $E$ have at most logarithmic growth near $D$, which can be controlled by $r_D^{-\alpha}$ if $\alpha\gg 0$.

Now let us prove the following result which is a variant of [Den18b, Theorem C]. It in particular answers the question in last subsection, and this result is crucial in constructing negatively curved Finsler metric over $T_Y(-\log D)$ in Theorem C.

**Theorem 1.8** (Infinitesimal Torelli-type property)**.** *The morphism $\tau_1:T_Y(-\log D)\to L^{-1}\otimes E^{p_0-1,q_0+1}$ defined in (1.4.1) is always generically injective.*

The proof is almost the same at that of [Den18b, Theorem C]. We provide it here for completeness sake.

*Proof of Theorem 1.8.* By Theorem 1.5.(iii), the inclusion $L\subset E^{p_0,q_0}$ induces a global section $s\in H^0(Y,L^{-1}\otimes E^{p_0,q_0})$, which is *generically* non-vanishing over $U=Y-D$. Set

$$U_1:=\{y\in Y-(D\cup\mathbf{B}_+(L))\mid s(y)\neq 0\}\tag{1.5.1}$$

which is a non-empty Zariski open set of $U$. Since the Hodge metric $h_{\rm hod}$ is a direct sum of metrics $h_p$ on $E^{p,q}$, the metric $h$ for $L^{-1}\otimes E$ is a direct sum of metrics $h_L^{-1}\cdot h_p$ on $L^{-1}\otimes E^{p,q}$, which is smooth over $U_0:=Y-(D\cup \mathbf{B}_+(L))$. Let us denote $D'$ to be the $(1,0)$-part of its Chern connection over $U_1$, and $\Theta$ to be its curvature form. Then by the Griffiths curvature formula of Hodge bundles (see [CMSP17, p. 363]), over $U_0$ we have

$$\begin{aligned}\Theta &= -\Theta_{L,h_L}\otimes \mathbb{1}+\mathbb{1}\otimes \Theta_{h_{p_0}}(E^{p_0,q_0})\\ &= -\Theta_{L,h_L}\otimes \mathbb{1}-\mathbb{1}\otimes(\theta^*_{p_0,q_0}\wedge\theta_{p_0,q_0})-\mathbb{1}\otimes(\theta_{p_0+1,q_0-1}\wedge\theta^*_{p_0+1,q_0-1})\\ &= -\Theta_{L,h_L}\otimes \mathbb{1}-\tilde{\theta}^*_{p_0,q_0}\wedge\tilde{\theta}_{p_0,q_0}-\tilde{\theta}_{p_0+1,q_0-1}\wedge\tilde{\theta}^*_{p_0+1,q_0-1} \qquad (1.5.2)\end{aligned}$$

where we set

$$\theta_{p,q}=\theta|_{E^{p,q}}:E^{p,q}\longrightarrow E^{p-1,q+1}\otimes\Omega_Y(\log D)$$

and

$$\tilde{\theta}_{p,q}=\mathbb{1}\otimes\theta_{p,q}:L^{-1}\otimes E^{p,q}\to L^{-1}\otimes E^{p-1,q+1}\otimes\Omega_Y(\log D)$$

and define $\tilde{\theta}^*_{p,q}$ to be the adjoint of $\tilde{\theta}_{p,q}$ with respect to the metric $h_L^{-1}\cdot h$. Hence over $U_1$ one has

$$\begin{aligned}-\sqrt{-1}\partial\bar{\partial}\log|s|_h^2 &= \frac{\{\sqrt{-1}\Theta(s),s\}_h}{|s|_h^2}+\frac{\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h}{|s|_h^4}-\frac{\sqrt{-1}\{D's,D's\}_h}{|s|_h^2}\\ &\leqslant \frac{\{\sqrt{-1}\Theta(s),s\}_h}{|s|_h^2} \qquad (1.5.3)\end{aligned}$$

thanks to Cauchy-Schwarz inequality

$$\sqrt{-1}|s|_h^2\cdot\{D's,D's\}_h\geqslant\sqrt{-1}\{D's,s\}_h\wedge\{s,D's\}_h.$$

Putting (1.5.2) to (1.5.3), over $U_1$ one has

$$\begin{aligned}\sqrt{-1}\Theta_{L,h_L}-\sqrt{-1}\partial\bar{\partial}\log|s|_h^2 &\leqslant -\frac{\{\sqrt{-1}\tilde{\theta}^*_{p_0,q_0}\wedge\tilde{\theta}_{p_0,q_0}(s),s\}_h}{|s|_h^2}\\ &\quad-\frac{\{\sqrt{-1}\tilde{\theta}_{p_0+1,q_0-1}\wedge\tilde{\theta}^*_{p_0+1,q_0-1}(s),s\}_h}{|s|_h^2}\\ &= \frac{\sqrt{-1}\{\tilde{\theta}_{p_0,q_0}(s),\tilde{\theta}_{p_0,q_0}(s)\}_h}{|s|_h^2}+\frac{\{\tilde{\theta}^*_{p_0+1,q_0-1}(s),\tilde{\theta}^*_{p_0+1,q_0-1}(s)\}_h}{|s|_h^2}\\ &\leq \frac{\sqrt{-1}\{\tilde{\theta}_{p_0,q_0}(s),\tilde{\theta}_{p_0,q_0}(s)\}_h}{|s|_h^2} \qquad (1.5.4)\end{aligned}$$

where $\tilde{\theta}_{p_0,q_0}(s)\in H^0\big(Y,L^{-1}\otimes E^{p_0-1,q_0+1}\otimes\Omega_Y(\log D)\big)$. By Proposition 1.7.(ii), one has $|s|_h^2(y)=0$ for any $y\in D\cup\mathbf{B}_+(L)$. Therefore, there exists $y_0\in U_0$ so that $|s|_h^2(y_0)\geqslant |s|_h^2(y)$ for any $y\in U_0$. Hence $|s|_h^2(y_0)>0$, and by (1.5.1), $y_0\in U_1$. Since $|s|_h^2$ is smooth over $U_0$, $\sqrt{-1}\partial\bar{\partial}\log|s|_h^2$ is semi-negative at $y_0$ by the maximal principle. By Proposition 1.7.(i), $\sqrt{-1}\Theta_{L,h_L}$ is strictly positive at $y_0$. By (1.5.4) and $|s|_h^2(y_0)>0$, we conclude that $\sqrt{-1}\{\tilde{\theta}_{p_0,q_0}(s),\tilde{\theta}_{p_0,q_0}(s)\}_h$ is strictly positive at $y_0$. In particular, for any non-zero $\xi\in T_{Y,y_0}$, $\tilde{\theta}_{p_0,q_0}(s)(\xi)\neq 0$. For

$$\tau_1: T_Y(-\log D)\to L^{-1}\otimes E^{p_0-1,q_0+1}$$

in (1.4.1), over $U$ it is defined by $\tau_1(\xi) := \tilde{\theta}_{p_0,q_0}(s)(\xi)$, which is thus *injective at* $y_0 \in U_1$. Hence $\tau_1$ is *generically injective*. The theorem is thus proved. □

## 2. Construction of negatively curved Finsler metric

We first introduce the definition of Finsler metric.

**Definition 2.1** (Finsler metric)**.** Let $E$ be a holomorphic vector bundle on a complex manifold $X$. A *Finsler metric* on $E$ is a real non-negative *continuous* function $h : E \to [0, +\infty[$ such that

$$h(av) = |a|h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. The metric $h$ is *positively definite* at a subset $U \subset X$ if $h(v) > 0$ for any nonzero $v \in E_x$ and any $x \in U$.

We shall mention that our definition is a bit different from that in [Kob98, Chapter 2, §3], which requires *convexity*, and the Finsler metric therein can be upper-semi continuous.

Let $(E = \oplus_{p+q=\ell} E^{p,q}, \theta)$ be a Higgs bundle on a log pair $(Y, D)$ satisfying the three conditions in Theorem 1.5. We adopt the same notations as those in Theorem 1.5 and § 1.5 throughout this section. Let us denote by $n$ the largest non-negative number for $k$ so that $\tau_k$ in (1.4.1) is not trivial. By Theorem 1.8, $n > 0$. Following [Den18a, §3.4] we construct Finsler metrics $F_1, \ldots, F_n$ on $T_Y(-\log D)$ as follows. By (1.4.1), for each $k = 1, \ldots, n$, there exists

$$\tau_k : \mathrm{Sym}^k T_Y(-\log D) \to L^{-1} \otimes E^{p_0-k, q_0+k}.$$

Then it follows from Proposition 1.7.(ii) that the (Finsler) metric $h$ on $L^{-1} \otimes E^{p_0-k,q_0+k}$ induces a Finsler metric $F_k$ on $T_Y(-\log D)$ defined as follows: for any $e \in T_Y(-\log D)_y$,

$$F_k(e) := h\big(\tau_k(e^{\otimes k})\big)^{\frac{1}{k}} \tag{2.0.1}$$

Let $C \subset \mathbb{C}$ be any open set of $\mathbb{C}$. For any holomorphic map $\gamma : C \to U := Y - D$, one has

$$d\gamma : T_C \to \gamma^* T_U \hookrightarrow \gamma^* T_Y(-\log D). \tag{2.0.2}$$

We denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector fields in $C \subset \mathbb{C}$, $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. The Finsler metric $F_k$ induces a continuous Hermitian pseudo-metric on $C$, defined by

$$\gamma^* F_k^2 = \sqrt{-1} G_k(t) dt \wedge d\bar{t}. \tag{2.0.3}$$

Hence $G_k(t) = |\tau_k\big(d\gamma(\partial_t)^{\otimes k}\big)|_h^{\frac{2}{k}}$, where $\tau_k$ is defined in (1.4.1).

By Theorem 1.8, there is a Zariski open set $U^\circ$ of $U$ such that $U^\circ \cap \mathbf{B}_+(L) = \varnothing$, and $\tau_1$ is injective at any point of $U^\circ$. We now fix any holomorphic map $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$. By Proposition 1.7.(ii), the metric $h$ for $L^{-1} \otimes E$ is smooth and positively definite over $U - \mathbf{B}_+(L)$. Hence $G_1(t) \not\equiv 0$. Let $C^\circ$ be an (non-empty) open set of $C$ whose complement $C \setminus C^\circ$ is a *discrete set* so that

- The image $\gamma(C^\circ) \subset U^\circ$.
- For every $k = 1, \ldots, n$, either $G_k(t) \equiv 0$ on $C^\circ$ or $G_k(t) > 0$ for any $t \in C^\circ$.
- $\gamma'(t) \neq 0$ for any $t \in C^\circ$, namely $\gamma|_{C^\circ} : C^\circ \to U^0$ is immersive everywhere.

By the definition of $G_k(t)$, if $G_k(t) \equiv 0$ for some $k > 1$, then $\tau_k(\partial_t^{\otimes k}) \equiv 0$ where $\tau_k$ is defined in (1.4.1). Note that one has $\tau_{k+1}(\partial_t^{\otimes (k+1)}) = \tilde{\theta}\big(\tau_k(\partial_t^{\otimes k})\big)(\partial_t)$, where

$$\tilde{\theta} = \mathbb{1}_{L^{-1}} \otimes \theta : L^{-1} \otimes E \to L^{-1} \otimes E \otimes \Omega_Y(\log D)$$

We thus conclude that $G_{k+1}(t) \equiv 0$. Hence it exists $1 \leq m \leq n$ so that the set $\{k \mid G_k(t) > 0 \text{ over } C^\circ\} = \{1, \ldots, m\}$, and $G_\ell(t) \equiv 0$ for all $\ell = m+1, \ldots, n$. From now on, *all the computations* are made over $C^\circ$ if not specified.

Using the same computations in the proof of [Den18a, Proposition 3.12], we have following curvature formula.

**Theorem 2.2.** *For $k = 1, \ldots, m$, over $C^\circ$ one has*

$$\frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} \geq \Theta_{L,h_L}(\partial_t, \bar{\partial}_t) - \frac{G_2^2}{G_1} \quad \text{if } k = 1, \tag{2.0.4}$$

$$\frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} \geq \frac{1}{k}\Big(\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) + \frac{G_k^k}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^k}\Big) \quad \text{if } k > 1. \tag{2.0.5}$$

*Here we make the convention that $G_{m+1} \equiv 0$ and $\frac{0}{0} = 0$. We also write $\partial_t$ (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively, where $d\gamma$ is defined in (2.0.2).* □

Let us mention that in [Den18a, eq. (3.3.58)] we drop the term $\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$ in (2.0.5), though it can be easily seen from the proof of [Den18a, Lemma 3.9].

We will follows ideas in [Den18a, §3.4] (inspired by [TY15, BPW17, Sch17]) to introduce a new Finsler metric $F$ on $T_Y(-\log D)$ by taking convex sum in the following form

$$F := \sqrt{\sum_{k=1}^{n} k\alpha_k F_k^2}. \tag{2.0.6}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ are some constants which will be fixed later.

For the above $\gamma : C \to U$ with $\gamma(C) \cap U^\circ \neq \varnothing$, we write

$$\gamma^* F^2 = \sqrt{-1} H(t) dt \wedge d\bar{t}.$$

Then

$$H(t) = \sum_{k=1}^{n} k\alpha_k G_k(t), \tag{2.0.7}$$

where $G_k$ is defined in (2.0.3). Recall that for $k = 1, \ldots, m$, $G_k(t) > 0$ for any $t \in C^\circ$.

We first recall a computational lemma by Schumacher.

**Lemma 2.3** ( [Sch17, Lemma 17]). *Let $\alpha_j > 0$ and $G_j$ be positive real numbers for $j = 1, \ldots, n$. Then*

$$\begin{aligned} &\sum_{j=2}^{n} \Big(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^j}{G_{j-1}^{j-2}}\Big) \\ &\geqslant \frac{1}{2}\Bigg(-\frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 + \sum_{j=2}^{n-1}\Bigg(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Bigg) G_j^2\Bigg) \end{aligned} \tag{2.0.8}$$

□

Now we are ready to compute the curvature of the Finsler metric $F$ based on Theorem 2.2.

**Theorem 2.4.** *Fix a smooth Kähler metric $\omega$ on $Y$. There exist* universal *constants $0 < \alpha_1 < \ldots < \alpha_n$ and $\delta > 0$, such that for any $\gamma : C \to U = Y - D$ with $C$ an open set of $\mathbb{C}$ and $\gamma(C) \cap U^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\bar{\partial} \log |\gamma'(t)|_F^2 \geq \delta \gamma^* \omega \tag{2.0.9}$$

*Proof.* By Theorem 1.8 and the assumption that $\gamma(C)\cap U^\circ\neq\varnothing$, $G_1(t)\not\equiv 0$. We first recall a result in [Den18a, Lemma 3.11], and we write its proof here for it is crucial in what follows.

**Claim 2.5.** *There is a universal constant* $c_0>0$ (i.e. *it does not depend on* $\gamma$) *so that* $\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\geq c_0 G_1(t)$ *for all* $t\in C$.

*Proof of Claim 2.5.* Indeed, by Proposition 1.7.(i), it suffices to prove that

$$\frac{|\partial_t|^2_{\gamma^*(r_D^{-2}\cdot\omega_\alpha)}}{|\tau_1(d\gamma(\partial_t))|_h^2}\geqslant c_0 \tag{2.0.10}$$

for some $c_0>0$, where $\omega_\alpha$ is a positively definite Hermitian metric on $T_Y(-\log D)$. Note that

$$\frac{|\partial_t|^2_{\gamma^*(r_D^{-2}\cdot\omega_\alpha)}}{|\tau_1(d\gamma(\partial_t))|_h^2}=\frac{|\partial_t|^2_{\gamma^*(r_D^{-2}\cdot\omega_\alpha)}}{|\partial_t|^2_{\gamma^*\tau_1^*h}}=\frac{|\partial_t|^2_{\gamma^*(\omega_\alpha)}}{|\partial_t|^2_{\gamma^*\tau_1^*(r_D^2\cdot h)}},$$

where $\tau_1^*(r_D^2\cdot h)$ is a Finsler metric (indeed continuous pseudo hermitian metric) on $T_Y(-\log D)$ by Proposition 1.7.(iii). Since $Y$ is compact, there exists a constant $c_0>0$ such that

$$\omega_\alpha\geqslant c_0\tau_1^*(r_D^2\cdot h).$$

Hence (2.0.10) holds for any $\gamma: C\to U$ with $\gamma(C)\cap U^\circ\neq\varnothing$. The claim is proved. □

By [Sch12, Lemma 8],

$$\sqrt{-1}\partial\bar{\partial}\log(\sum_{j=1}^n j\alpha_jG_j)\geqslant\frac{\sum_{j=1}^n j\alpha_jG_j\sqrt{-1}\partial\bar{\partial}\log G_j}{\sum_{i=1}^n j\alpha_jG_i} \tag{2.0.11}$$

Putting (2.0.4) and (2.0.5) to (2.0.11), and making the convention that $\frac{0}{0}=0$, we obtain

$$\begin{aligned}
\frac{\partial^2\log H(t)}{\partial t\partial\bar{t}}&\geq\frac{1}{H}\Big(-\alpha_1G_2^2+\sum_{k=2}^n\alpha_k\big(\frac{G_k^{k+1}}{G_{k-1}^{k-1}}-\frac{G_{k+1}^{k+1}}{G_k^{k-1}}\big)\Big)+\frac{\sum_{k=1}^n\alpha_kG_k}{H}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\\
&=\frac{1}{H}\Big(\sum_{j=2}^n\Big(\alpha_j\frac{G_j^{j+1}}{G_{j-1}^{j-1}}-\alpha_{j-1}\frac{G_j^j}{G_{j-1}^{j-2}}\Big)\Big)+\frac{\sum_{k=1}^n\alpha_kG_k}{H}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\\
&\overset{(2.0.8)}{\geq}\frac{1}{H}\Big(-\frac{1}{2}\frac{\alpha_1^3}{\alpha_2^2}G_1^2+\frac{1}{2}\sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}-\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2+\frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\
&\quad+\frac{\sum_{k=1}^n\alpha_kG_k}{H}\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)\\
&\overset{\text{Claim 2.5}}{\geq}\frac{1}{H}\Big(\frac{\alpha_1}{2}(c_0-\frac{\alpha_1^2}{\alpha_2^2})G_1^2+\frac{1}{2}\sum_{j=2}^{n-1}\Big(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}-\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}\Big)G_j^2+\frac{1}{2}\frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}}G_n^2\Big)\\
&\quad+\frac{1}{H}(\frac{1}{2}\alpha_1G_1+\sum_{k=2}^n\alpha_kG_k)\Theta_{L,h_L}(\partial_t,\bar{\partial}_t)
\end{aligned}$$

One can take $\alpha_1=1$, and choose the further $\alpha_j>\alpha_{j-1}$ inductively so that

$$c_0-\frac{\alpha_1^2}{\alpha_2^2}>0,\quad\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}}-\frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}}>0\quad\forall\, j=2,\ldots,n-1. \tag{2.0.12}$$

Hence

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geq \frac{1}{H}\Big(\frac{1}{2}\alpha_1 G_1 + \sum_{k=2}^{n} \alpha_k G_k\Big)\Theta_{L,h_L}(\partial_t, \bar{\partial}_t) \overset{(2.0.7)}{\geq} \frac{1}{n}\Theta_{L,h_L}(\partial_t, \bar{\partial}_t)$$

over $C^\circ$. By Proposition 1.7.(i), this implies that

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|_F^2 = \sqrt{-1}\partial\bar{\partial}\log H(t) \geq \frac{1}{n}\gamma^*\sqrt{-1}\Theta_{L,h_L} \geq \delta\gamma^*\omega \tag{2.0.13}$$

over $C^\circ$ for some positive constant $\delta$, which does not depend on $\gamma$. Since $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over $C$, by the extension theorem of subharmonic function, (2.0.13) holds over the whole $C$. Since $c_0 > 0$ is a constant which does not depend on $\gamma$, so are $\alpha_1, \ldots, \alpha_n$ by (2.0.12). The theorem is thus proved. □

In summary of results in this subsection, we obtain the following theorem.

**Theorem 2.6.** *Let $(E = \oplus_{p+q=\ell}E^{p,q}, \theta)$ be a Higgs bundle on a log pair $(Y, D)$ satisfying the three conditions in Theorem 1.5. Then there are a Finsler metric $h$ on $T_Y(-\log D)$ which is* positively definite *on a dense Zariski open set $U^\circ$ of $U := Y - D$, and a smooth Kähler form $\omega$ on $Y$ such that for any holomorphic map $\gamma : C \to U$ from any open subset $C$ of $\mathbb{C}$ with $\gamma(C) \cap U^\circ \neq \varnothing$, one has*

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|_h^2 \geq \gamma^*\omega. \tag{2.0.14}$$

□

## 3. Big Picard theorem and Algebraic hyperbolicity

3.1. **Definition of algebraic hyperbolicity.** *Algebraic hyperbolicity* for a compact complex manifold $X$ was introduced by Demailly in [Dem97a, Definition 2.2], and he proved in [Dem97a, Theorem 2.1] that $X$ is algebraically hyperbolic if it is Kobayashi hyperbolicity. The notion of algebraic hyperbolicity was generalized to log pairs by Chen [Che04].

**Definition 3.1** (Algebraic hyperbolicity)**.** Let $(X, D)$ be a log pair. For any reduced irreducible curve $C \subset X$ such that $C \not\subset X$, we denote by $i_X(C, D)$ the number of distinct points in the set $\nu^{-1}(D)$, where $\nu : \tilde{C} \to C$ is the normalization of $C$. The log pair $(X, D)$ is *algebraically hyperbolic* if there is a smooth Kähler metric $\omega$ on $X$ such that

$$2g(\tilde{C}) - 2 + i(C, D) \geq \deg_\omega C := \int_C \omega$$

for all curves $C \subset X$ as above.

Note that $2g(\tilde{C}) - 2 + i(C, D)$ depends only on the complement $X - D$. Hence the above notion of hyperbolicity also makes sense for quasi-projective manifolds: we say that a quasi-projective manifold $U$ is algebraically hyperbolic if it has a log compactification $(X, D)$ which is algebraically hyperbolic.

However, unlike Demailly's theorem, it is unclear to us that Kobayashi hyperbolicity or Picard hyperbolicity of $X - D$ will imply the algebraic hyperbolicity of $(X, D)$. In [PR07] Pacienza-Rousseau proved that if $X - D$ is hyperbolically embedded into $X$, the log pair $(X, D)$ (and thus $X - D$) is algebraically hyperbolic.

3.2. **Proofs of main results.** In this subsection, we will combine Theorem D with Theorem C to prove main results in this paper.

*Proof of Theorem A.* By Theorem C, there exist finite log pairs $\{(X_i, D_i)\}_{i=0,\ldots,N}$ so that

(1) There are morphisms $\mu_i : X_i \to Y$ with $\mu_i^{-1}(D) = D_i$, so that each $\mu_i : X_i \to \mu_i(X_i)$ is a birational morphism, and $X_0 = Y$ with $\mu_0 = \mathbb{1}$.
(2) There are smooth Finsler metrics $h_i$ for $T_{X_i}(-\log D_i)$ which is positively definite over a Zariski open set $U_i^\circ$ of $U_i := X_i - D_i$.
(3) $\mu_i|_{U_i^\circ} : U_i^\circ \to \mu_i(U_i^\circ)$ is an isomorphism.
(4) There are smooth Kähler metrics $\omega_i$ on $X_i$ such that for any curve $\gamma : C \to U_i$ with $C$ an open set of $\mathbb{C}$ and $\gamma(C) \cap U_i^\circ \neq 0$, one has

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|^2_{h_i} \geq \gamma^*\omega_i. \tag{3.2.1}$$

(5) For any $i \in \{0,\ldots,N\}$, either $\mu_i(U_i) - \mu_i(U_i^\circ)$ is zero dimensional, or there exists $I \subset \{0,\ldots,N\}$ so that

$$\mu_i(U_i) - \mu_i(U_i^\circ) \subset \cup_{j\in I}\mu_j(X_j)$$

Let us explain how to construct these log pairs. By the assumption, there is a $\mathbb{C}$-PVHS $(V, \nabla, F^\bullet, Q)$ on $Y - D$ with the period map quasi-finite, which is thus generically immersive. We then apply Theorem C to construct a Finsler metric on $T_Y(-\log D)$ which is positively definite over some Zariski open set $U^\circ$ of $U = Y - D$ with the desired curvature property (2.0.14). Set $X_0 = Y$, $\mu_0 = \mathbb{1}$ and $U_0^\circ = U^\circ$. Let $Z_1, \ldots, Z_m$ be all irreducible varieties of $Y - U^\circ$ which are not components of $D$. Then $Z_1 \cup \ldots \cup Z_m \supset U \setminus U^\circ$. For each $i$, we take a desingularization $\mu_i : X_i \to Z_i$ so that $D_i := \mu_i^{-1}(D)$ is a simple normal crossing divisor in $X_i$. For the $\mathbb{C}$-PVHS $\mu_i^*(V, \nabla, F^\bullet, Q)$ on $U_i = X_i - D_i$ by pulling-back $(V, \nabla, F^\bullet, Q)$ via $\mu_i$, its period map is generically immersive, and it also has unipotent monodromies around $D_i$. We then apply Theorem C to construct the desired Finsler metrics in Item 4 for $T_{X_i}(-\log D_i)$. We iterate this construction, and since each step the dimension of $X_i$ is strictly decreased, this algorithm stops after finite steps.

(i) We will first prove that $U$ is Picard hyperbolic. Fix any holomorphic map $f : \Delta^* \to U$. If $f(\Delta^*) \cap U_0^\circ \neq \varnothing$, then by Theorem D and Item 4, we conclude that $f$ extends to a holomorphic map $\overline{f} : \Delta \to X_0 = Y$.

Assume now $f(\Delta^*) \cap \mu_0(U_0^\circ) = \varnothing$. By Item 5, there exists $I_0 \subset \{0,\ldots,N\}$ so that

$$f(\Delta^*) \subset \mu_0(U_0) - \mu_0(U_0^\circ) \subset \cup_{j\in I_0}\mu_j(X_j)$$

Since $\mu_j(X_j)$ are all irreducible, there exists $k \in I_0$ so that $f(\Delta^*) \subset \mu_k(X_k)$. Note that $U_k := \mu_k^{-1}(U)$. Hence $f(\Delta^*) \subset \mu_k(U_k)$. If $f(\Delta^*) \cap \mu_k(U_k^\circ) \neq \varnothing$, by Item 3 $f(\Delta^*)$ is not contained in the exceptional set of $\mu_k$. Hence $f$ can be lift to $f_k : \Delta^* \to U_k$ so that $\mu_k \circ f_k = f$ and $f_k(\Delta^*) \cap U_k^\circ \neq \varnothing$. By Theorem D and Item 4 again we conclude that $f_k$ extends to a holomorphic map $\overline{f}_k : \Delta \to X_k$. Hence $\mu_k \circ \overline{f}_k$ extends $f$. If $f(\Delta^*) \cap \mu_k(U_k^\circ) = \varnothing$, we apply Item 5 to iterate the above arguments and after finite steps there exists $X_i$ so that $f(\Delta^*) \subset \mu_i(U_i)$ and $f(\Delta^*) \cap \mu_i(U_i^\circ) \neq \varnothing$. By Item 3, $f$ can be lifted to $f_i : \Delta^* \to U_i$ so that $\mu_i \circ f_i = f$ and $f_i(\Delta^*) \cap U_i^\circ \neq \varnothing$. By Theorem D and Item 4 again, $f_i$ extends to the origin, and so is $f$. We prove the Picard hyperbolicity of $U = Y - D$.

(ii) Let us prove the algebraic hyperbolicity of $U$. Fix any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$. By the above arguments, there exists $i \in \{0,\ldots,N\}$ so that $C \subset \mu_i(X_i)$ and $C \cap \mu_i(U_i^\circ) \neq \varnothing$. Let $C_i \subset X_i$ be the strict transform of $C$ under $\mu_i$. By Item 3 $h_i|_{C_i}$ is not identically equal to zero.

Denote by $\nu_i : \tilde{C}_i \to C_i \subset X_i$ the normalization of $C_i$, and set $P_i := (\mu_i \circ \nu_i)^{-1}(D) = \nu_i^{-1}(D_i)$. One has

$$d\nu_i : T_{\tilde{C}_i}(-\log P_i) \to \nu_i^* T_{X_i}(-\log D_i)$$

which induces a (non-trivial) pseudo hermitian metric $\tilde{h}_i := \nu_i^* h_i$ over $T_{\tilde{C}_i}(-\log P_i)$. By (3.2.1), the *curvature current*

$$\frac{\sqrt{-1}}{2\pi}\Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \nu_i^* \omega_i$$

Hence

$$2g(\tilde{C}_i) - 2 + i(C, D) = \int_{\tilde{C}_i} \frac{\sqrt{-1}}{2\pi}\Theta_{\tilde{h}_i^{-1}}(K_{\tilde{C}_i}(\log P_i)) \geq \int_{\tilde{C}_i} \nu_i^* \omega_i$$

Fix a Kähler metric $\omega_Y$ on $Y$. Then there is a constant $\varepsilon_i > 0$ so that $\omega_i \geq \varepsilon_i \mu_i^* \omega_Y$. We thus have

$$2g(\tilde{C}_i) - 2 + i(C, D) \geq \varepsilon_i \int_{\tilde{C}_i} (\mu_i \circ \nu_i)^* \omega_Y = \varepsilon_i \deg_{\omega_Y} C,$$

for $\mu_i \circ \nu_i : \tilde{C}_i \to C$ is the normalization of $C$. Set $\varepsilon := \inf_{i=0,\ldots,N} \varepsilon_i$. Then we conclude that for any reduced and irreducible curve $C \subset Y$ with $C \not\subset D$, one has

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\omega_Y} C$$

where $\tilde{C} \to C$ is its normalization. This shows the algebraic hyperbolicity of $U$.

The proof of the theorem is accomplished. □

To prove Theorem B, we need the following fact on Picard and algebraic hyperbolicity.

**Lemma 3.2.** *Let $U$ be a quasi-projective manifold and let $p : \tilde{U} \to U$ be a finite étale cover. Then if $\tilde{U}$ is Picard hyperbolic or algebraically hyperbolic, so is $U$.*

*Proof.* Let us take log-compactifications $(X, D)$ and $(Y, E)$ for $\tilde{U}$ and $U$ respectively, so that $p$ extends to a morphism $\overline{p} : X \to Y$ with $\overline{p}^{-1}(E) = D$.

(i) Assume now $\tilde{U}$ is Picard hyperbolic. For any holomorphic map $f : \Delta^* \to U$, we claim that there is a finite covering

$$\begin{aligned} \pi : \Delta^* &\to \Delta^* \\ z &\mapsto z^n \end{aligned}$$

so that there is a holomorphic map $\tilde{f} : \Delta^* \to \tilde{U}$ with

$$\begin{array}{ccc} \Delta^* & \xrightarrow{\tilde{f}} & \tilde{U} \\ \downarrow{\pi} & & \downarrow{p} \\ \Delta^* & \xrightarrow{f} & U \end{array}$$

Indeed, fix any based point $z_0 \in \Delta^*$ with $x_0 := f(z_0)$. Pick any $y_0 \in p^{-1}(x_0)$. Then either $f_*(\pi_1(\Delta^*, z_0))$ is a finite group or $f_*(\pi_1(\Delta^*, z_0)) \cap p_*(\pi_1(\tilde{U}, y_0)) \supsetneq \{0\}$ since $p_*(\pi_1(\tilde{U}, y_0))$ is a subgroup of $\pi_1(U, x_0)$ with finite index. Let $\gamma \in \pi_1(\Delta^*, z_0) \simeq \mathbb{Z}$ be a generator. Then $f_*(\gamma^n) \subset p_*(\pi_1(\tilde{U}, y_0))$ for some $n \in \mathbb{Z}_{>0}$. Therefore, $(f \circ \pi)_*(\pi_1(\Delta^*, z_0)) \subset p_*(\pi_1(\tilde{U}, y_0))$, which implies that the lift $\tilde{f}$ of $f \circ \pi$ for the covering map $p$ exists.

Since $\tilde{U}$ is Picard hyperbolic, $\tilde{f}$ extends to a holomorphic map $\overline{\tilde{f}} : \Delta \to X$. The composition $\overline{p} \circ \overline{\tilde{f}}$ extends $f \circ \pi$. Since $\pi$ extends to a map $\overline{\pi} : \Delta \to \Delta$, we thus has

$$\lim_{z \to 0} f(z) = \overline{p} \circ \overline{\tilde{f}}(0).$$

By the *Riemann extension theorem*, $f$ extends to the origin holomorphically.

(ii) Assume that $(X, D)$ is algebraically hyperbolic. Fix smooth Kähler metrics $\omega_X$ and $\omega_Y$ on $X$ and $Y$ so that $\overline{p}^*\omega_Y \leq \omega_X$. Then there is a constant $\varepsilon > 0$ such that for any reduced and irreducible curve $C \subset X$ with $C \not\subset D$, one has

$$2g(\tilde{C}) - 2 + i(C, D) \geq \varepsilon \deg_{\omega_X} C$$

where $\tilde{C} \to C$ is its normalization.

Take any reduced and irreducible curve $C \subset Y$ with $C \not\subset E$. Then there is a reduced and irreducible curve $C'$ of $X$ so that $\overline{p}(C') = C$. Let $\nu : \tilde{C} \to C$ and $\nu' : \tilde{C}' \to C'$ be their normalization respectively, which induces a (possibly ramified) covering map $\pi : \tilde{C}' \to \tilde{C}$ so that

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow{\nu'} & C' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{p}|_{C'}} \\ \tilde{C} & \xrightarrow{\nu} & C \end{array}$$

Set $P := \nu^{-1}(E)$ and $Q := (\nu')^{-1}(D)$. Then $\pi^\circ : \tilde{C}' - Q \to \tilde{C} - P$ is an unramified covering map. By Riemann–Hurwitz formula one has

$$\begin{aligned} 2g(\tilde{C}) - 2 + i(C, E) &= \frac{1}{\deg \pi}\big(2g(\tilde{C}') - 2 + i(C', D)\big) \\ &\geq \frac{\varepsilon}{\deg \pi} \deg_{\omega_X} C' \geq \frac{\varepsilon}{\deg \pi} \deg_{\overline{p}^*\omega_Y} C' = \varepsilon \deg_{\omega_Y} C \end{aligned}$$

Hence $(Y, E)$ is also algebraically hyperbolic, and so is $U$.

The lemma is proved. □

Note that in [JK18a, Proposition 5.2.(1)], Javanpeykar-Kamenova proved that if $X \to Y$ is an finite morphism of projective varieties over an algebraically closed field of characteristic zero, then $Y$ is algebraically hyperbolic provided that $X$ is algebraically hyperbolic.

We now show how to reduce Theorem B to Theorem A by applying Lemma 3.2.

*Proof of Theorem B.* Let $(Y, D)$ be a log-compactification of $U$. Since there is a $\mathbb{Z}$-PVHS $(V, \nabla, F^\bullet, Q)$ on $U$, by a theorem of A. Borel, its local monodromies around $D$ is quasi-unipotent. By [Bru18, §3.2], there is a finite étale cover $p : \tilde{U} \to U$ and a log-compactification $(X, E)$ of $\tilde{U}$ so that $p^*(V, \nabla, F^\bullet, Q)$ has unipotent monodromies around $E$. Since the period map of $(V, \nabla, F^\bullet, Q)$ is assumed to be quasi-finite, so is that of $p^*(V, \nabla, F^\bullet, Q)$. By Theorem A, we know that $\tilde{U}$ is both Picard hyperbolic and algebraically hyperbolic, and it follows immediately from Lemma 3.2 that the same holds for $U$. □

We end this section with the following remark.

*Remark* 3.3. Let $(E, \theta)$ be the Higgs bundle on a log pair $(Y, D)$ as that in Theorem 2.6. One can also use the idea by Viehweg-Zuo [VZ02] in constructing their *Viehweg-Zuo* sheaf (based on the negativity of kernels of Higgs fields by Zuo [Zuo00]) to prove a weaker result than Theorem 2.6: for any holomorphic map $\gamma : C \to U$ from any open subset $C$ of $\mathbb{C}$ with $\gamma(C) \cap U^\circ \neq \varnothing$, there exists a Finsler metric $h_C$ of $T_Y(-\log D)$ (depending on $C$) and a Kähler metric $\omega_C$ for $Y$ (also depending on $C$) so that $|\gamma'(t)|_h^2 \not\equiv 0$ and

$$\sqrt{-1}\partial\bar{\partial}\log|\gamma'|^2_{h_C} \geq \gamma^*\omega_C.$$

It follows from our proof of Theorem A that one can also combine Theorem D with this weaker result to prove Theorem A. We prefer to stating and proving the more general result Theorem C since we expect that it should have further applications.

## Appendix A. Criteria for big Picard theorem

Since [Den19] will not be published, in this appendix we provide the proof of Theorem D using Nevanlinna theory for completeness sake. We first begin with some preliminary in Nevannlina theory.

A.1. **Preliminary in Nevannlina theory.** Let $\mathbb{D}^* := \{t \in \mathbb{C} \mid |t| > 1\}$, and $\mathbb{D} := \mathbb{D}^* \cup \infty$. Then via the map $z \mapsto \frac{1}{z}$, $\mathbb{D}^*$ is isomorphic to the punctured unit disk $\Delta^*$ and $\mathbb{D}$ is isomorphic to the unit disk $\Delta$. Therefore, for any holomorphic map $f$ from the punctured disk $\Delta^*$ into a projective variety $Y$, $f$ extends to the origin if and only if $f(\frac{1}{z}) : \mathbb{D}^* \to Y$ extends to the infinity.

Let $(X, \omega)$ be a compact Kähler manifold, and $\gamma : \mathbb{D}^* \to X$ be a holomorphic map. Fix any $r_0 > 1$. Write $\mathbb{D}_r := \{z \in \mathbb{C} \mid r_0 < |z| < r\}$. The *order function* is defined by

$$T_{\gamma,\omega}(r) := \int_{r_0}^{r} \frac{d\tau}{\tau} \int_{\mathbb{D}_\tau} \gamma^*\omega.$$

As is well-known, the asymptotic behavior of $T_{\gamma,\omega}(r)$ as $r \to \infty$ characterizes whether $\gamma$ can be extended over the $\infty$ (see *e.g.* [Dem97b, 2.11. Cas «local »] or [NW14, Remark 4.7.4.(ii)]).

**Lemma A.1.** *$T_{\gamma,\omega}(r) = O(\log r)$ if and only if $\gamma$ is extended holomorphically over $\infty$.* □

We first recall two useful formulas (the second one is the Jensen formula in [Nog81, eq. (1.1)]).

**Lemma A.2.** *Write* $\log^+ x := \max(\log x, 0)$.

(A.1.1) $$\log^+(\sum_{i=1}^{N} x_i) \leq \sum_{i=1}^{N}\log^+ x_i + \log N, \quad \log^+\prod_{i=1}^{N} x_i \leq \sum_{i=1}^{N}\log^+ x_i \quad \text{for } x_i \geq 0.$$

(A.1.2) $$\begin{aligned}\frac{1}{\pi}\int_{r_0}^{r}\frac{d\tau}{\tau}\int_{\mathbb{D}_\tau}\sqrt{-1}\partial\bar{\partial}v &= \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})d\theta - \frac{1}{2\pi}\int_0^{2\pi} v(r_0e^{i\theta})d\theta \\ &\quad - \frac{\log r}{2\pi}\int_0^{2\pi}\sqrt{-1}(\bar{\partial}-\partial)v(r_0e^{i\theta})d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})d\theta + O(\log r)\end{aligned}$$

*for all functions $v$ so that $\sqrt{-1}\partial\bar{\partial}v$ exists as measures* (e.g. *$v$ is the difference of two subharmonic functions).* □

The following lemma is well-known to experts (see *e.g.* [Dem97b, Lemme 1.6]).

**Lemma A.3.** *Let $X$ be a projective manifold equipped with a hermitian metric $\omega$ and let $u : X \to \mathbb{P}^1$ be a rational function. Then for any holomorphic map $\gamma : \mathbb{D}^* \to X$, one has*

(A.1.3) $$T_{u\circ\gamma,\omega_{FS}}(r) \leq CT_{\gamma,\omega}(r) + O(1)$$

*where $\omega_{FS}$ is the Fubini-Study metric for $\mathbb{P}^1$.* □

The following logarithmic derivative lemma is crucial in the proof of Theorem D.

**Lemma A.4** ( [NW14, Lemma 4.2.9.(i)], [Dem97b, 3.4. Cas local]). *Let $u : \mathbb{D}^* \to \mathbb{P}^1$ be any meromorphic function. Then for any $k \geq 1$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\frac{u^{(1)}(re^{i\theta})}{u(re^{i\theta})}| d\theta \leq C(\log^+ T_{u,\omega_{FS}}(r) + \log r) \quad \|, \tag{A.1.4}$$

*for some constant $C > 0$ which does not depend on $r$. Here the symbol $\|$ means that the inequality holds outside a Borel subset of $(r_0, +\infty)$ of finite Lebesgue measure.* □

We need the lemma by E. Borel.

**Lemma A.5** ( [NW14, Lemma 1.2.1]). *Let $\phi(r) \geq 0 (r \geq r_0 \geq 0)$ be a monotone increasing function. For every $\delta > 0$,*

$$\frac{d}{dr}\phi(r) \leq \phi(r)^{1+\delta} \quad \|. \tag{A.1.5}$$

□

A.2. **Proof of Theorem D.** The ideas we used here mainly follow from that by Siu-Yeung [SY96] and Ru-Wong [RW95] on the vanishing of pullback of jet differential on *entire curves.*

*Proof of Theorem D.* We take a finite affine covering $\{U_\alpha\}_{\alpha \in I}$ of $X$ and rational functions $(x_{\alpha 1}, \ldots, x_{\alpha n})$ on $X$ which are holomorphic on $U_\alpha$ so that

$$dx_{\alpha 1} \wedge \cdots \wedge dx_{\alpha n} \neq 0 \text{ on } U_\alpha$$
$$D \cap U_\alpha = (x_{\alpha, s(\alpha)+1} \cdots x_{\alpha n} = 0)$$

Hence

$$(e_{\alpha 1}, \ldots, e_{\alpha n}) := (\frac{\partial}{\partial x_{\alpha 1}}, \ldots, \frac{\partial}{\partial x_{\alpha s(\alpha)}}, x_{\alpha, s(\alpha)+1}\frac{\partial}{\partial x_{\alpha, s(\alpha)+1}}, \ldots, x_{\alpha n}\frac{\partial}{\partial x_{\alpha n}}) \tag{A.2.1}$$

is a basis for $T_X(-\log D)|_{U_\alpha}$. Write

$$(f_{\alpha 1}(t), \ldots, f_{\alpha n}(t)) := (x_{\alpha 1} \circ f, \ldots, x_{\alpha n} \circ f)$$

so that $f_{\alpha j} : \mathbb{D}^* \to \mathbb{P}^1$ is a meromorphic function over $\mathbb{D}^*$ for any $\alpha$ and $j$. With respect to the trivialization of $T_X(-\log D)$ induced by the basis (A.2.1), $f'(t)$ can be written as

$$f'(t) = f'_{\alpha 1}(t)e_{\alpha 1} + \cdots + f'_{\alpha s(\alpha)}(t)e_{\alpha s(\alpha)} + (\log f_{\alpha, s(\alpha)+1})'(t)e_{\alpha, s(\alpha)+1} + \cdots + (\log f_{\alpha n})'(t)e_{\alpha n}$$

over $U_\alpha$. Let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha \in I}$.

Since $h$ is Finsler metric for $T_X(-\log D)$ which is continuous and locally bounded from above by Definition 2.1, and $I$ is a finite set, there is a constant $C > 0$ so that

$$\rho_\alpha \circ f \cdot |f'(t)|_h^2 \leq C\Big(\sum_{j=1}^{s(\alpha)} \rho_\alpha \circ f \cdot |f'_{\alpha j}(t)|^2 + \sum_{i=s(\alpha)+1}^{n} |(\log f_{\alpha i})'(t)|^2\Big) \quad \forall t \in \mathbb{D}^* \tag{A.2.2}$$

for any $\alpha$. Hence

$$
\begin{aligned}
T_{f,\omega}(r) := \int_{r_0}^{r} \frac{d\tau}{\tau}\int_{\mathbb{D}_\tau} f^*\omega &\overset{(0.2.2)}{\leq} \int_{r_0}^{r} \frac{d\tau}{\tau}\int_{\mathbb{D}_\tau} \frac{1}{\pi}\sqrt{-1}\partial\bar{\partial}\log|f'|_h^2 \\
&\overset{(A.1.2)}{\leq} \frac{1}{2\pi}\int_0^{2\pi} \log|f'(re^{i\theta})|_h d\theta + O(\log r) \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \sum_\alpha |\rho_\alpha \circ f \cdot f'(re^{i\theta})|_h d\theta + O(\log r) \\
&\overset{(A.1.1)}{\leq} \sum_\alpha \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'(re^{i\theta})|_h d\theta + O(\log r) \\
&\overset{(A.2.2)+(A.1.1)}{\leq} \sum_\alpha \sum_{i=s(\alpha)+1}^{n} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |(\log f_{\alpha i})'(re^{i\theta})| d\theta \\
&\quad + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta + O(\log r) \\
&\overset{(A.1.4)}{\leq} C_1 \sum_\alpha \sum_{i=s(\alpha)+1}^{n} \big(\log^+ T_{f_{\alpha i},\omega_{FS}}(r) + \log r\big) \\
&\quad + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta + O(\log r) \quad \| \\
&\overset{(A.1.3)}{\leq} C_2(\log^+ T_{f,\omega}(r) + \log r) + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta \quad \|. \qquad \text{(A.2.3)}
\end{aligned}
$$

Here $C_1$ and $C_2$ are two positive constants which do not depend on $r$.

**Claim A.6.** *For any $\alpha \in I$ and any $j \in \{1, \ldots, s(\alpha)\}$, one has*

$$
\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta \leq C_3(\log^+ T_{f,\omega}(r) + \log r) + O(1) \quad \| \qquad \text{(A.2.4)}
$$

*for a positive constant $C_3$ which does not depend on $r$.*

*Proof of Claim A.6.* The proof of the claim is borrowed from [NW14, eq.(4.7.2)]. Pick $C > 0$ so that $\rho_\alpha^2\sqrt{-1}dx_{\alpha j} \wedge d\bar{x}_{\alpha j} \leq C\omega$. Write $f^*\omega := \sqrt{-1}B(t)dt \wedge d\bar{t}$. Then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})|d\theta = \frac{1}{4\pi}\int_0^{2\pi} \log^+(|\rho_\alpha^2 \circ f| \cdot |f'_{\alpha j}(re^{i\theta})|^2)d\theta \\
&\leq \frac{1}{4\pi}\int_0^{2\pi} \log^+ B(re^{i\theta})d\theta + O(1) \leq \frac{1}{4\pi}\int_0^{2\pi} \log(1 + B(re^{i\theta}))d\theta + O(1) \\
&\leq \frac{1}{2}\log(1 + \frac{1}{2\pi}\int_0^{2\pi} B(re^{i\theta})d\theta) + O(1) = \frac{1}{2}\log(1 + \frac{1}{2\pi r}\frac{d}{dr}\int_{\mathbb{D}_r} rBdrd\theta) + O(1) \\
&= \frac{1}{2}\log(1 + \frac{1}{2\pi r}\frac{d}{dr}\int_{\mathbb{D}_r} f^*\omega) + O(1) \\
&\overset{(A.1.5)}{\leq} \frac{1}{2}\log(1 + \frac{1}{2\pi r}(\int_{\mathbb{D}_r} f^*\omega)^{1+\delta}) + O(1) \quad \| \\
&= \frac{1}{2}\log(1 + \frac{r^\delta}{2\pi}(\frac{d}{dr}T_{f,\omega}(r))^{1+\delta}) + O(1) \quad \| \\
&\overset{(A.1.5)}{\leq} \frac{1}{2}\log(1 + \frac{r^\delta}{2\pi}(T_{f,\omega}(r))^{(1+\delta)^2}) + O(1) \quad \| \\
&\leq 4\log^+ T_{f,\omega}(r) + \delta \log r + O(1) \quad \|.
\end{aligned}$$

Here we pick $0 < \delta < 1$ and the last inequality follows. The claim is proved. □

Putting (A.2.4) to (A.2.3), one obtains

$$T_{f,\omega}(r) \leq C(\log^+ T_{f,\omega}(r) + \log r) + O(1) \quad \|$$

for some positive constant $C$. Hence $T_{f,\omega}(r) = O(\log r)$. We apply Lemma A.1 to conclude that $f$ extends to the $\infty$. □

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