Graphon Models in Quantum Physics

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Février 2020

IHES/M/20/03

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Abstract. In this work we explain some new applications of Infinite Combinatorics to Quantum Physics. We investigate the use of the theory of graphons in non-perturbative Quantum Field Theory and Deformation Quantization which lead us to discover some new interrelationships between these fundamental topics. In one direction, we study Dyson–Schwinger equations in the context of the graph function theory of sparse graphs which enables us to analyze non-perturbative parameters of strongly coupled Quantum Field Theories via cut-distance compact topological regions of Feynman diagrams, Kontsevich's $\star$-product and other new mathematical settings. In another direction, we initiate a theory of graph function representations for Kontsevich admissible graphs to formulate a new topological Hopf algebraic formalism for the study of these graphs which brings some new useful mathematical tools to relate Deformation Quantization program with non-perturbative renormalization program in Quantum Field Theory models.

Contents

1. Introduction

The research achievements of this work focus on some new applications of the theory of analytic graphs known as graphons, which has been initiated and studied in Infinite Combinatorics, to some fundamental topics in Quantum Physics namely, Quantum Field Theory and Deformation Quantization. Recent research achievements in Mathematical Physics clarified the appearance of a deep relation between the Connes–Kreimer Hopf algebraic approach to the BPHZ perturbative renormalization process of Feynman diagrams and the Kontsevich's $\star$-product deformation machinery in a graphical calculus setting. In this work we aim

Date: February 7, 2020.
2010 *Mathematics Subject Classification.* 03G12; 05C63; 16T05; 46L65; 81T16; 81T75.
Key words and phrases. Combinatorial Dyson–Schwinger Equations; Connes–Kreimer–Marcolli Theory; Kontsevich's Deformation Quantization; Graphons; Non-Perturbative Topological Hopf Algebraic Renormalization; Quantum Logics.

to build some new mathematical tools which are useful to relate non-perturbative topological Hopf algebraic renormalization of Dyson–Schwinger equations with the foundations of Kontsevich's Deformation Quantization formalism.

In Quantum Field Theory, the integral equations such as $\mathbf{G} = 1 + \int I_\gamma \mathbf{G}$ (with respect to some integral kernel I_γ) are obtained by the fixed point equations for Green's functions with the general form

$$\mathbf{G} = 1 + \int I_\gamma + \int\int I_\gamma I_\gamma + \int\int\int I_\gamma I_\gamma I_\gamma + ... \tag{1.1}$$

in a given (strongly coupled) physical theory Φ. These fixed point equations are known as Dyson–Schwinger equations and their solutions can be presented as formal power series in running coupling constants. Therefore they could provide fundamental information for the characterization of non-perturbative situations of gauge field theories on the basis of the strength of running coupling constants. Thanks to the Connes–Kreimer approach to perturbative renormalization, these recursive equations have been reformulated in the language of Hochschild cohomology of (commutative) bialgebras where we need to work on the chain complexes such as $(\bigoplus_{n\geq 0} C_n, \mathbf{b})$ derived from the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams $H_{\rm FG}(\Phi)$. For each n, C_n is the vector space generated by linear maps from $H_{\rm FG}(\Phi)$ to $H_{\rm FG}(\Phi)^{\otimes n}$ while the coboundary operator $\mathbf{b}$ is defined in terms of the renormalization coproduct $\Delta_{\rm FG}$ such that the degree one homogeneous linear endomorphism B^+_γ (known as the grafting operator) is the corresponding Hochschild one-cocycle with respect to a given (1PI) primitive Feynman diagram γ. [1, 9, 40, 41, 42]

Dyson–Schwinger equations in physical theories with the vanishing β-function (such as Conformal Field Theories) can be studied under a linear setting where Hopf subalgebras generated by solutions of these equations are cocommutative. In physical theories with non-zero β-functions such as (low energy) QCD, QFT-models with multi-flavors or theories beyond Standard Model, we need to deal with much more complicated version versions of these equations namely, (non-linear) Dyson–Schwinger equations. Asymptotic freedom property in high energy QCD is useful to study these equations under higher order perturbation theory while in low energy QCD the behavior of the physical system can only be understood under a non-perturbative regime. The Hopf algebraic version of these equations has been applied to study them in the context of some geometric objects encoded by some objects of the Connes–Marcolli universal category of flat equi-singular vector bundles where at the end of the day, we can analyze non-perturbative counterterms and other non-perturbative parameters derived from the BPHZ renormalization of Dyson–Schwinger equations via systems of differential equations together with (ir-)regular singularities. Thanks to this geometric treatment and the Manin renormalization Hopf algebra of Halting problem, the complexity of non-perturbative computations in (systems) of Dyson–Schwinger equations has also been considered. In addition, under an algebraic combinatorial setting, a new class of noncommutative differential calculi has been formulated which characterize quantum integrable systems in non-perturbative parts of gauge field theories. This noncommutative geometric approach has been developed recently to build a new class of spectral triples which encode fundamental geometric information of non-perturbative quantum motions. [10, 22, 24, 27, 28, 32, 37, 38, 39]

Having no complete control on solutions of Dyson–Schwinger equations in strong running coupling constants (which include infinite formal expansions of Feynman diagrams together with increasing powers of running coupling constants) is the most difficult challenge in Theoretical and Mathematical Physics. The renormalization of these expansions could generate infinite number of counterterms which should be added to the original Lagrangian of the physical theory and therefore non-perturbative quantum field theories are non-renormalizable in this context. Lattice models, numerical methods, higher order perturbation techniques, theory of instantons and AdS/CFT correspondence are incapable for a complete analysis of real time dynamical processes in strongly coupled systems. In a separate setting, recently, a new application of Infinite Combinatorics to Quantum Field Theory has been discovered which enables us to use graph functions for the analysis of the behavior of sequences of Feynman diagrams and infinite formal expansions of Feynman diagrams which contribute to Green's functions. Real time dynamical process can be encoded by components or terms of these sequences such that the application of the topology of graphons to these models enable us to provide an accurate mathematical formalism for the real time study of Dyson–Schwinger equations. Graphons, which give us a new view for the study of limits of sequences of Feynman diagrams, can be useful to compute non-perturbative parameters in terms of the renormalization of a particular class of graph functions (for sparse graphs) and some new other mathematical structures. As we know edge densities in increasing sequences of sparse graphs converge to zero where by rescaling the density matrix in terms of a non-zero function of the number of vertices we can determine asymptotic behavior of those graphs. This technique is useful for us when we want to describe the unique solution of a given Dyson–Schwinger equation as the limit of the sequence of its partial sums (as linear combinations of decorated sparse graphs) which converges to a non-zero graph function. The main achievement is to interpret solutions of Dyson–Schwinger equations as boundary objects of a compact topological space of finite graphs where thanks to the Connes–Kreimer Hopf algebraic renormalization, an algebraic non-perturbative renormalization program for Dyson–Schwinger equations have already been formulated. These new tools could lead us to bring some alternative advanced mathematical modelings for the study of the phenomenology of non-perturbative situations of Quantum Field Theories with strong running coupling constants. [32, 33, 34, 35]

In this work we explain the graphon model approach to Dyson–Schwinger equations to show the impact of these analytic graphs in the mathematical formulation of a non-perturbative renormalization program. We show that the non-perturbative parameters derived from this renormalization program can be also interpreted via Baker–Campbell–Hausdorff quantization formula and Kontsevich's $\star$-product.

In Quantum Physics, Deformation Quantization gives the required mathematical model for the description of quantum systems under Dirac's correspondence principle on the basis of quantizing the space of observables on a fixed Poisson manifold via defining a new associative multiplication as a deformation of pointwise multiplication in the direction of the Poisson structure. The Kontsevich approach has provided a universal local deformation quantization for any open domain $\mathcal{U} \subset \mathbb{R}^d$ in the context of a graphical representation for bidifferential operators where we have the star product on $C^\infty(\mathcal{U})$ with the following Taylor series presentation

$$f \star_\alpha g := \sum_{n=0}^{\infty} \frac{\hbar}{n!} \sum_{K \in \mathbf{G}^1_{n,2}} \omega_K B_{K,\alpha}(f,g) \tag{1.2}$$

such that α is the Poisson structure of the configuration space, the interior sum is over all Kontsevich admissible graphs of degree one with n internal and two boundary vertices, $B_{K,\alpha}$ are bidifferential operators and coefficients ω_K are weights which satisfy the cocycle equation. Kontsevich admissible graphs, which are in an one to one correspondence with bidifferential operators, are useful tools in working on graphical calculus for derivations. We can rewrite the star product (1.2) as the following way

$$f \star_\alpha g = \sum_{K \in \mathfrak{g}^\bullet(\mathbb{R}^d)} \hbar^{|K|} \omega_K B_{K,\alpha}(f,g), \quad B_{n,\alpha}(f,g) := \sum_{K \in \mathfrak{g}^n(\mathbb{R}^d)} B_{K,\alpha}(f,g) \tag{1.3}$$

such that $\mathfrak{g}^\bullet(\mathbb{R}^d)$ contains all Kontsevich admissible graphs with finite orders while $\mathfrak{g}^n(\mathbb{R}^d)$ contains those graphs of order n. The main challenge is to determine the weights ω_K to obtain an associative product where analytic and combinatorial techniques have been considered to deal with it. Kontsevich admissible graphs can be visualized by nodes and geodesics in a closed disk under some conditions. We can characterize these graphs via two integers namely, the number of internal vertices decorated by polyvector fields in terms of the action of a bidifferential operator (which maps a graph and a set of compatible polyvector fields to a multidifferential operator) and the number of boundary vertices decorated by smooth functions. Thanks to this class of graphs, the existence of a morphism between the differential graded Lie algebra of the deformation complex of the associative algebra of smooth functions on $\mathbb{R}^d$ and the Chevalley–Eilenberg differential graded Lie algebra of linear homomorphisms between polyvector fields and polydifferential operators has been approved. [17, 18, 19, 20, 21, 30]

In this setting, some interesting interconnections between (universal) Kontsevich's Deformation Quantization ([18, 21]) and the Hopf–Birkhoff factorization in the Connes–Kreimer approach to perturbative Quantum Field Theory ([1, 9, 42]) have been discovered where the Connes–Kreimer BPHZ Hopf algebraic perturbative renormalization has been reformulated on the basis of the Baker–Campbell–Hausdorff formula and the Kontsevich's bidifferential symplectic operator for quantum deformations. It is shown that the Hopf–Birkhoff factorization of each Feynman rules character can be interpreted as a deformation of the pointwise multiplication of some exponential functions via the Kontsevich's $\star$-product in the direction of the linear Poisson bracket. [19, 31]

In this work we explain a new theory of graph functions for Kontsevich admissible graphs to build an infinite version of these graphs which can be encoded by boundary objects of a compact topological Hopf algebra structure on the space of finite Kontsevich admissible graphs. This new Hopf algebraic formalism enables us to study Deformation Quantization in the context of the Connes–Marcolli universal category of flat equi-singular vector bunldes. In addition, it leads us to suggest a non-local generalization for Kontsevich's Deformation Quantization which can work at the level of infinite dimensional manifolds. Furthermore, we formulate the Kontsevich's $\star$-product for a class of noncommutative associative unital algebras derived from solutions of Dyson–Schwinger equations and their renormalization procedure.

Generally speaking, passing from Classical Mechanics to Quantum Mechanics can be described mathematically on the basis of changing the geometry and the logic. In Classical Mechanics we have manifolds, groups and points together with the category of sets as fundamental tools for the analysis of classical systems. Points in a topological space together with some additional structures (such as Poisson brackets, symplectic forms, ...) determine a space of states. Then each physical quantity has its value and each proposition of the form $A \in \mathcal{M}$, represented in terms of some Borel subsets of the state space, has a truth-value true or false. The Borel subsets of the state space form a natural Boolean σ-algebra which means that the Logic of classical systems is Boolean. In Quantum Mechanics we have Hilbert spaces, operators, noncommutative (Hopf) algebras and homomorphisms. The Kochen–Specker Theorem tells us that there is no state space of a quantum system analogous to the classical state space where physical quantities are represented as real-valued functions on the hypothetical state space of a quantum system. This Theorem informs us that such a space does not exist and it is impossible to assign values to all physical quantities at once and therefore it is also impossible to assign true or false values to all propositions. However Birkhoff and von Neumann built the foundations of an instrumentalist approach to Quantum Logic ([2]) where upon measurement of the physical quantity A, we can find the result belong to $\mathcal{M}$ with a determined probability. In their approach, pure states are represented by unit vectors in one particular Hilbert space and propositions with the general form $A \in \mathcal{M}$ are represented by projection operators on this Hilbert space. These projections form a non-distributive lattice. Non-distributive property, dependence on measurement tools and the use of real numbers (as continuum) are the most fundamental problems of this instrumentalist approach and its generalizations [8]. Thanks to modern categorical approaches to foundations of logics, Quantum Logic has been rebuilt in terms of a topos of presheaves on the base category of observables where we can reconstruct physical theories on the basis of search for a suitable representation in a topos of a certain formal language. The base category of this topos model is the category of von Neumann subalgebras of $\mathcal{B}(H)$. In this setting, the first order logic (or propositional calculus) enables us to logically evaluate propositions with the general form "the physical quantity such as A in a given physical system has a value in the subset $\mathcal{M}$ of real numbers." The key step is to find what truth-values such propositions have in a given state of the system and how the truth-value changes with the state in time. This topos model for the analysis of quantum systems, which has been developed by Isham, Doring and Butterfield [8, 13, 14, 15, 16], can also provide higher-order logics for these systems. However this topos model does not recognize the intrinsic difference between Quantum Mechanics and (non-perturbative) Quantum Field Theory and we need to pass from this topos model to other models which are capable to recover real time processes in strongly coupled systems with infinite degrees of freedom. Thanks to the graphon representation model for Dyson–Schwinger equations, an alternative topos model for the logically analysis of (strongly coupled) gauge field theories has been offered such that this new topos model (named it as non-perturbative topos) is capable to show the impact of the strength of running couplings in changing the logical evaluation procedure of physical quantities. This new topos model can provide the logical notion for the understanding of non-locality and other intrinsic foundations of non-perturbative regions of physical systems [34].

In this work we address this new topos model and then we provide a new modification of this topos model which works for Kontsevich's Deformation Quantization.

2. From sparse graphs to graphons

In Infinite Combinatorics, graphons, which were introduced and developed as graph limits of sequences of weighted finite (dense or sparse) graphs, posets, etc, have been applied in several topics such as Graph Theory, Statistics, Machine Learning and Computer Science. These analytic objects are useful for the study of extremely large graphs or networks in terms of functional modelings, statistical estimation and random graph models. [4, 5, 6, 7, 25, 26]

In addition, the space of graphons together with additional topological and geometric structures on it have already provided some useful mathematical tools for the study of non-perturbative behavior of strongly coupled quantum systems. [12, 34, 35, 36]

A (bi)graphon can be introduced as a generalization of an edge weighted graph with a continuum of vertices or as the convergent limit of a sequence of weighted finite graphs. We can describe these analytic graphs in terms of real valued (symmetric) measurable functions on $\Omega\times\Omega$ for a given probability space (Ω, μ_Ω) where the graphon space is the quotient space of all these (symmetric) (Lebesgue) measurable functions with respect to an equivalence relation which identifies almost everywhere equal graphons. Bigraphons are graphons without the symmetric property and they can be useful for us whenever we want to rebuild any Feynman diagram in terms of the combinatorial information of its pixel picture presentation or other labeled graphon models.

Suppose Ω be a separable atomless probability space with the probability measure μ_Ω. Set $\mathcal{W}^{[0,1]}$ as the topological space of all bounded (symmetric) measurable functions from $\Omega \times \Omega$ to $[0, 1]$ (up to almost everywhere equal as the equivalence relation) with respect to the semi-norm

$$\|W\|_{\text{cut}} := \sup_{A\times B\subset\Omega\times\Omega}\Big|\int_{A\times B} W(x,y)d\mu_\Omega(x)d\mu_\Omega(y)\Big| \tag{2.1}$$

with the corresponding metric

$$d_{\text{cut}}(V,W) := \inf_{\rho,\sigma}\|V^\rho - W^\sigma\|_{\text{cut}} \tag{2.2}$$

known as cut-distance metric. The infimum in (2.2) is with respect to all measure-preserving bijections of Ω such that $V^\rho(x,y) := V(\rho(x), \rho(y))$, $W^\sigma(x,y) := W(\sigma(x), \sigma(y))$ are versions of the graphons V, W obtained by the adjacency matrix of a graph in which the vertices are reordered. It is called relabling process.

Lemma 2.1. *For any sequence $\{W_n\}_{n\geq1}$ of labeled graphons there exists a subsequence $\{W_{n_i}\}_{i\geq1}$ and a labeled graphon W such that $d_{\text{cut}}(W_{n_i}, W)$ converges to 0 when n tends to infinity.* [25, 26]

Relabeling process generates weakly isomorphic graphons and in fact, it defines an equivalence relation $\approx$ on the space of labeled graphons. Set $[W]$ (called unlabeled graphon class) as the class of all labeled graphons V which are weakly isomorphic to W. In other words, $V \in [W]$, iff there exist measure-preserving maps σ, τ such that $V^\sigma = W^\tau$ almost everywhere. Therefore an unlabeled graphon is a representative of an equivalence class of graphons modulo relabeling.

Theorem 2.2. *The quotient space $\mathcal{W}^{[0,1]}_{\approx}$ with respect to weakly isomorphic relation is a compact Hausdorff metric space.* [25, 26]

Pixel picture presentations generated by finite simple graphs are the most fundamental examples of graphons. The theory of graphons developed for dense graphs very fast and then it has been also developed for the study of sparse graphs under density approximation methods, different rescaling measure spaces and other metric spaces such as L^p. The edge densities of convergent sequences of sparse graphs tend to zero density graphs (which is almost everywhere the same as the 0-graphon) whenever the number of vertices goes to infinity. If we want to obtain non-zero graphons from these sequences, then we can work on rescaled or stretched versions of the canonical graphons which is closely related to changing the scale of the density matrix (by a function of the number of vertices) which in no longer goes to zero. The application of these methods shows that sequences of sparse graphs without dense spots can converge to graphons with respect to the cut-distance metric after rescaling. In this setting, it is possible to characterize sequences of sparse graphs on the basis of their asymptotic densities and their limiting graphons. Therefore two graphs with different densities may still have similar structure. [3, 4, 5, 6, 7]

Graphons are the key tools for the study of infinite graphs via the theory of random graphs. A simple random graph $G(n,p)$ is defined by taking n nodes and then connecting any two of them with the probability p under an independent decision about each pair. Assigning different probabilities enable us to build different random graph models. For example, the uniform random graph is the result of inserting m edges in such a way that all possible $\binom{\binom{n}{2}}{m}$ choices are equally likely. We can use a given graph function as a functional which assigns a probability value to add an edge for building a random graph.

Lemma 2.3. *Each simple (finite) graph generates a random graph.*

Proof. For a given simple graph G, we need to consider its graphon representation $[W_G]$ which is determined by using the adjacency matrix and pixel picture presentation. The simple random graph $R(G)$ can be defined in terms of adding an edge with probability equals to its weight which is given by W_G. Now for a finite subset $S_n := \{s_1, ..., s_n\}$ in $[0,1]$, build a new weighted graph $G(S, W_G)$ with n vertices such that the edge $s_i s_j$ has the weight $W_G(s_i, s_j)$. The graph

$$R(n, W_G) := R(G(S_n, W_G)) \tag{2.3}$$

is a simple random graph model. □

The probability of the graph G with the vertex set $[n]$ and the edge set $E(G)$ is obtained by integrating over all possible choices of $x_1, ..., x_n \in [n]$ and the chosen graphon function model. In other words,

$$\mathbb{P}_W(G) := \int_{[0,1]^n} \prod_{(i,j)\in E(G)} W(x_i, x_j) \prod_{(i,j)\notin E(G)} [1 - W(x_i, x_j)] \prod_{i\in[n]} dx_i. \tag{2.4}$$

In general, a random graph model is a sequence of random variables $R_1, R_2, ...$ such that for each n, R_n could be a graph with vertex set $[n]$ such that its distribution is invariant under relabeling of the vertices. In other words, isomorphic graphs have the same probability. In graphon model, the distribution over graphs is determined by graph functions and therefore a graphon W can be seen as the weight matrix of an infinite graph with the unit interval as the vertex set while $W(x, y)$ informs the weight of the edge between x, y.

3. A graphon model approach to Dyson–Schwinger equations

A single Feynman diagram presents a finite number of possible interactions among virtual or elementary particles where its on-shell part obeys the mass-energy equation and conservation of momenta while its off-shell part obeys no special rules or measurements. Feynman rules allow us to associate an ill-defined iterated integral to each Feynman diagram. Infinite formal expansions of Feynman diagrams (as polynomials with respect to running coupling constants) are capable to encode all possible interactions among virtual and elementary particles in a physical theory. These expansions, which derived originally from Green's functions, can be studied on the basis of the self-similar nature of Green's functions and their fixed point equations known as Dyson–Schwinger equations. The Connes–Kreimer renormalization Hopf algebra $H_{\rm FG}(\Phi) = \bigoplus_{n\geq 0} H_{\rm FG}^{(n)}(\Phi)$ of Feynman diagrams of a given gauge field theory Φ is a connected graded commutative non-cocommutative Hopf algebra such that for each n, $H_{\rm FG}^{(n)}(\Phi)$ is the vector space of divergent 1PI n-loop Feynman diagrams and products of Feynman diagrams with the overall loop number n. It is also possible to apply another graduation parameter on Feynman diagrams with respect to the number of internal edges to obtain a finite type graded structure on Feynman diagrams. [1, 9, 40, 42]

Lemma 3.1. *For a fixed probability space (Ω, μ_Ω), we can identify a unique unlabeled graphon class with respect to each Feynman diagram Γ in a physical theory Φ.*

Proof. For simplicity we work on the closed interval $[0,1]$ as the Lebesgue measure space. The renormalization Hopf algebra $H_{\rm FG}(\Phi)$ can be embedded in the Connes–Kreimer Hopf algebra H_{CK} of non-planar rooted trees via applying decorations on trees. These decorations encode some fundamental information of the physical theory Φ such as types of particles and interactions [1, 9]. For each Feynman diagram Γ, set t_Γ as its corresponding non-planar decorated rooted tree in $H_{CK}(\Phi)$. We can build the unlabeled graphon class $[W_\Gamma]$ in terms of the pixel picture presentation of the finite simple weighted graph t_Γ. If $V(t_\Gamma)$ as the set of all vertices in the tree has n elements, then divide the measure space $[0,1]$ into subintervals $I_i = [\frac{i-1}{n}, \frac{i}{n})$. The boxes $I_i \times I_j$ are in one to one correspondence with the boxes in the pixel picture presentation of t_Γ. Define the labeled graphon $W_\Gamma(x,y) = 1$ for $(x,y) \in I_i \times I_j$ whenever there exists an edge between vertices v_i, v_j in t_Γ and define $W_\Gamma(x,y) = 0$ whenever there is no edge between v_i and v_j in t_Γ. □

We name W_Γ as the Feynman graphon corresponding to $\Gamma \in H_{\rm FG}(\Phi)$ on the probability space (Ω, μ_Ω). The vector space $\mathcal{S}^{\Phi}_{\rm graphon}$ generated by all this type of graphons can be equipped with the renormalization Hopf algebraic structure. [34, 35]

Thanks to Feynman graphons, a sequence $\Gamma_1, \Gamma_2, ...$ of Feynman diagrams is convergent iff the corresponding sequence $W_{\Gamma_1}, W_{\Gamma_2}, ...$ of Feynman graphons is cut-distance convergent to a graphon W_∞ when n tends to infinity. We use the notation Γ_∞ for the infinite graph with the corresponding graph function model W_∞. In other words, $W_{\Gamma_\infty} \in [W_\infty]$ and $W_\infty \in [W_{\Gamma_\infty}]$. We call this type of graphs "large Feynman diagrams".

Dyson–Schwinger equations are the main tools in dealing with infinite formal expansions of Feynman diagrams. The Connes–Kreimer renormalization Hopf algebra is useful to reformulate Dyson–Schwinger equations in (strongly coupled) physical theories as recursive equations in Hochschild Cohomology Theory such that as the result, recently we have formulated a new non-perturbative renormalization program for these equations in the language of Noncommutative Geometry and graphons. [1, 24, 32, 40, 41, 42]

Primitive (1PI) Feynman diagrams such as γ can determine a particular class of Hochschild one cocycles B^+_γ with respect to the coboundary operator $\mathbf{b}$ defined on the basis of the Kreimer renormalization coproduct such that we have

$$< \Gamma_1 \otimes ... \otimes \Gamma_{n+1}, \mathbf{b}T(\Gamma) >:=< \mathbf{b}\rho_T(\Gamma_1, ..., \Gamma_{n+1}), \Gamma >, \ \rho_T(\Gamma_1, ..., \Gamma_n) := T^t(\Gamma_1 \otimes ... \otimes \Gamma_n)$$

$$\text{i.e.} \quad < \rho_T(\Gamma_1, ..., \Gamma_j\Gamma_{j+1}, ..., \Gamma_{n+1}), \Gamma >=< \Gamma_1 \otimes ... \otimes \Gamma_{n+1}, \Delta_j(T(\Gamma)) > . \tag{3.1}$$

This coboundary operator, which can be rewritten by

$$\mathbf{b}T(\Gamma) := (\mathrm{id} \otimes T)\Delta_{\mathrm{FG}}(\Gamma) + \sum_{j=1}^{n} (-1)^j \Delta_j(T(\Gamma)) + (-1)^{n+1} T(\Gamma) \otimes \mathbb{I}, \tag{3.2}$$

contributes to the reformulation of fixed point equations of Green's functions. For a given family $\{\gamma_n\}_{n\geq 1}$ of primitive (1PI) Feynman diagrams, it is possible to reformulate a class of Dyson–Schwinger equations via the combinatorial equation

$$X(c(g)) = \mathbb{I} + \sum_{n\geq 1} (c(g))^n \omega_n B^+_{\gamma_n}(X^{n+1}) \tag{3.3}$$

as a recursive equation in $H_{\mathrm{FG}}(\Phi)[c(g)]$ with respect to any running coupling constant $c(g)$ as a function of the bare coupling constant g. The unique solution of this equation has a general form

$$X(c(g)) = \sum_{n\geq 0} (c(g))^n X_n \tag{3.4}$$

such that for each $n \geq 1$, X_n is a graph in $H_{\mathrm{FG}}(\Phi)$ and X_0 is the empty graph. Each X_n is a symbol for those Feynman diagrams which contribute to the order n of the (non-)perturbative expansion of the Dyson–Schwinger equation (3.3). It is possible to build each X_n under a recursive Hochschild machinery in terms of graphs X_j with lower orders. In other words, for each $n \geq 1$, we have

$$X_n = \sum_{j=1}^{n} \omega_j B^+_{\gamma_j}\Big(\sum_{k_1+...+k_{j+1}=n-j, \ k_i\geq 0} X_{k_1} ... X_{k_{j+1}} \Big). \tag{3.5}$$

The renormalization Hopf algebra is not enough to encode the infinite object $X(c(g))$ under strong couplings and for this reason a new topological enrichment of $H_{\mathrm{FG}}(\Phi)$ has been defined such that the resulting compact topological Hopf algebra can recover solutions of Dyson–Schwinger equations as objects of the boundary region [35]. The collection $\{X_n\}_{n\geq 0}$ provides generators for a free commutative graded connected Hopf subalgebra such that the behavior of running couplings (controlled by the β-function of the physical theory) could change its (non-)cocommutativity. It can be seen that the amount of $c(g)$ has a direct influence on the behavior of the formal expansion $X(c(g))$ such that for small enough running couplings, this expansion can be studied by higher order perturbation theory. In physical theories with vanishing β function, we need only deal with linear Dyson–Schwinger equations which generate cocommutative Hopf subalgebras however in physical theories with non-zero β-functions, we need to deal with non-linear Dyson–Schwinger equations which generate non-cocommutative Hopf subalgebras [1, 22, 23, 24]. We use the notation $H_{\rm DSE}$ for this type of Hopf subalgebras.

Theorem 3.2. *Thanks to the compactness of the space of graphons, we can interpret the unique solution of each Dyson–Schwinger equation DSE as the cut-distance convergent limit of a sequence of random graphs associated to those Feynman diagrams which contribute to the equation DSE.*

Proof. The representation of Feynman diagrams via the space of graphons is the key tool to study Dyson–Schwinger equations in the context of random graph models. Thanks to the n-adic metric,

$$d_{\rm adic}(\Gamma_1,\Gamma_2) := 2^{-{\rm val}(\Gamma_1-\Gamma_2)}, \quad {\rm val}(\Gamma) := {\rm Max}\{n \in \mathbb{N} : \ \Gamma \in \bigoplus_{k\geq n} H^{(k)}_{\rm FG}(\Phi)\} \tag{3.6}$$

on Feynman diagrams in $H_{\rm FG}(\Phi)$, we can build a sequence of random graphs with respect to the sequence $\{Y_m\}_{m\geq 1}$ of partial sums

$$Y_m := \mathbb{I} + X_1 + ... + X_{k_i} + ... + X_{k_j} + ... + X_m \tag{3.7}$$

of the given equation DSE. A decorated version of the Connes–Kreimer Hopf algebra of non-planar rooted trees provides a universal toy model for the Hopf algebra $H_{\rm DSE}$ which allows us to represent finite formal expansions Y_m of Feynman diagrams via linear combinations of decorated rooted trees [1, 9]. We build our model based on the rooted tree representation of these partial sums. For each m, the random graph R_m is given by using vertices of the rooted tree t_{Y_m} which is embedded into the closed interval (via a poset embedding ρ_m) such that with the probability $d_{\rm adic}(\Gamma_{k_i},\Gamma_{k_j})$ there exists an edge between v_i and v_j whenever $\rho_m^{-1}(v_i) \in X_{k_i}$ and $\rho_m^{-1}(v_j) \in X_{k_j}$ in the partial sum Y_m. The cut-distance convergent limit of the sequence $\{R_m\}_{m\geq 1}$ when m tends to infinity is the non-zero Feynman graphon W_X corresponding to the unique solution X of DSE. □

The next step is to explicitly identify graphon classes corresponding to solutions of Dyson–Schwinger equations and for this purpose we need to apply rescaling methods in the theory of graphons.

Theorem 3.3. *For a fixed probability space (Ω,μ_Ω), we can identify a unique unlabeled graphon class with respect to each Dyson–Schwinger equation DSE in a (strongly coupled) gauge field theory Φ.*

Proof. For simplicity we work on the closed interval $[0,1]$ as the Lebesgue measure space. Theorem 3.2 and Proposition 4.6 in [35] tell us that for a given combinatorial Dyson–Schwinger equation DSE with the unique solution X and the corresponding sequence $\{Y_m\}_{m\geq 1}$ of its partial sums, the sequence $\{Y_m\}_{m\geq 1}$ is cut-distance convergent to X. In other words, the sequence $\{W_{Y_m}\}_{m\geq 1}$ is convergent to W_X with respect to the cut-distance topology. The density of the sparse graph X is almost zero while by rescaling of the probability space or renormalizing the graphon models, we can remove this problem and obtain a non-zero unlabeled graphon class $[W_X]$. In the rest of the proof we build this non-zero Feynman graphon.

The n-adic metric (3.6) defines a new function $F_{\text{adic},X}$ on the set $V(X)$ of all vertices of X as follows

$$F_{\text{adic},X} : V(X) \times V(X) \to \mathbb{R}, \quad (v_i, v_j) \longmapsto d_{\text{adic}}(Y_{i_0}, Y_{j_0}) \tag{3.8}$$

such that

$$i_0 := \text{Min}\{s : \ \ v_i \in Y_s\}, \quad j_0 := \text{Min}\{t : \ \ v_j \in Y_t\}. \tag{3.9}$$

The value $F_{\text{adic},X}(v_i, v_j)$ can be seen as the weight of the edge $v_i v_j$ in the large Feynman diagram X. In addition, for every vertex $v_i \in V(X)$ define

$$w_i := d_{\text{adic}}(Y_{i_0}, \mathbb{I}) \in [0, 1] \tag{3.10}$$

as the weight of v_i. Finite expansions $a_n := \sum_{1 \leq k \leq n} w_k$ (for each $n \geq 1$) determine subintervals $I_n := [a_{n-1}, a_n)$ for each n which define a partition for the closed interval $[0, 1]$.

The non-zero Feynman graphon class $[W_X]$ corresponding to the large Feynman diagram X can be defined by graph functions with the general form

$$W_X : [0, 1] \times [0, 1] \to [0, 1], \quad W_X(x, y) := d_{\text{adic}}(Y_{i_0}, Y_{j_0}) \tag{3.11}$$

whenever $(x, y) \in I_{i_0} \times I_{j_0}$. □

Remark 3.4. It is also possible to build other non-zero Feynman graphons with respect to solutions of Dyson–Schiwnger equations in terms of changing the measure, the probability space or rescaling methods.

Corollary 3.5. *We can describe the unique solution X of a given Dyson–Schwinger equation DSE as a sequence of random graphs or non-zero graphons which lead us to characterize the original equation DSE in terms of its asymptotic densities or limiting graphons.*

Corollary 3.6. *The Connes–Kreimer renormalization Hopf algebra $H_{\text{FG}}(\Phi)$ can be topologically completed with respect to the cut-distance topology. The distance between Feynman diagrams can be determined by their corresponding Feynman graphons. In other words,*

$$d(\Gamma_1, \Gamma_2) := d_{\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}). \tag{3.12}$$

The resulting compact topological Hopf algebra (denoted by $H^{\text{cut}}_{\text{FG}}(\Phi)$) involves all Feynman diagrams and solutions of Dyson–Schwinger equations under different running couplings $c(g)$ in the physical theory Φ.

Thanks to Theorems 3.2, 3.3 and Corollary 3.6, it is now possible to introduce a distance between Dyson–Schwinger equations. For given equations DSE_1 and DSE_2, define

$$d(\text{DSE}_1, \text{DSE}_2) := d_{\text{cut}}(W_{X_{\text{DSE}_1}}, W_{X_{\text{DSE}_2}}) = \lim_{m \to \infty} d_{\text{cut}}(W_{Y_m(\text{DSE}_1)}, W_{Y_m(\text{DSE}_2)}). \tag{3.13}$$

Theorem 3.7. *There exists a non-perturbative generalization of the Connes–Kreimer BPHZ Hopf algebraic renormalization which works on the unique solution of a given Dyson–Schwinger equation in a (strongly coupled) gauge field theory Φ.*

Proof. The dual of the free commutative connected graded Hopf algebra $\mathcal{S}^{\Phi}_{\text{graphon}}$ of Feynman graphons is the complex infinite dimensional pro-unipotent Lie group $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$. Set $\text{Loop}(\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C}), \mu)$ as the space of loops γ_μ on the infinitesimal punctured disk $\mathbf{\Delta}^*$ around the origin in the complex plane with values in $\mathbb{G}^{\Phi}_{\text{graphon}}(\mathbb{C})$. The disk $\mathbf{\Delta}^*$ is determined by the dimensional regularization parameter. These loops can encode unrenormalized regularized characters which act on Feynman graphons. The Hopf–Birkhoff factorization (γ_-, γ_+) gives us the factorization $(\tilde{\phi}_-, \tilde{\phi}_+)$ for the character $\tilde{\phi}$ such that $\tilde{\phi}^z$ is defined via the corresponding Feynman rules character. For each finite Feynman diagram Γ, we have

$$\tilde{\phi}^z(W_\Gamma) := \phi^z(\Gamma). \tag{3.14}$$

For each large Feynman diagram X as the solution of a given Dyson–Schwinger equation DSE, $\tilde{\phi}^z(W_X)$ is defined as the cut-distance convergent limit of the sequence $\{\tilde{\phi}^z(W_{Y_m})\}_{m\geq 0}$ such that for each $m \geq 0$, W_{Y_m} is the Feynman graphon corresponding to the partial sum Y_m of X.

Applying these new regularized Feynman rules characters allows us to make the new sequence $\{S^{\tilde{\phi}}_{R_{\rm ms}}(W_{Y_m})\}_{m\geq 1}$ of Feynman graphons which is convergent with respect to the cut-distance topology and we have

$$S^{\tilde{\phi}}_{R_{\rm ms}}(W_X) = \lim_{m\to\infty} S^{\tilde{\phi}}_{R_{\rm ms}}(W_{Y_m}) = \lim_{m\to\infty} \sum_{i=1}^{m} S^{\tilde{\phi}}_{R_{\rm ms}}(W_{X_i})$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m} (-R_{\rm ms}(\tilde{\phi}(W_{X_i}) - R_{\rm ms}(\sum S^{\tilde{\phi}}_{R_{\rm ms}}(W_\gamma)\tilde{\phi}(W_{X_i/\gamma}))). \tag{3.15}$$

The Feynman graphon $S^{\tilde{\phi}}_{R_{\rm ms}}(W_X)$ determines the non-perturbative counterterms generated by the non-perturbative BPHZ renormalization of X.

In addition, we can also make the sequence $\{S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}(W_{Y_m})\}_{m\geq 1}$ of Feynman graphons which is cut-distance convergent and we have

$$S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}(W_X) = \lim_{m\to\infty} S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}(W_{Y_m}) = \lim_{m\to\infty} \sum_{i=1}^{m} S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}(W_{X_i}). \tag{3.16}$$

$S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}(W_X)$ determines the corresponding renormalized values generated by the non-perturbative BPHZ renormalization of X. □

Corollary 3.8. *The universal Connes–Marcolli category $\mathcal{E}^{\rm CM}$ of flat equi-singular vector bundles ([10]) encodes the geometric information of the non-perturbative renormalization group corresponding to the non-perturbative BPHZ renormalization of large Feynman diagrams.*

Proof. We consider the category $\mathcal{E}^{\Phi}_{\rm graphon}$ of flat equi-singular $\mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C})$-connections on the regularization bundle. This category is equivalent to the category ${\rm Rep}_{\mathbb{G}^{\Phi,*}_{\rm graphon}}$ of finite dimensional representations of the affine group scheme $\mathbb{G}^{\Phi,*}_{\rm graphon}$. The universality of the category $\mathcal{E}^{\rm CM}$, which is a neutral Tannakian category and equivalent to the category ${\rm Rep}_{\mathbb{U}^*}$, is applied to recover $\mathcal{E}^{\Phi}_{\rm graphon}$ as a subcategory. Therefore we can obtain graded representations

$$\eta : \mathbb{U}(\mathbb{C}) \longrightarrow \mathbb{G}^{\Phi}_{\rm graphon}(\mathbb{C}) \tag{3.17}$$

such that the composition map $\eta \circ {\rm rg}$ (for ${\rm rg} : \mathbb{G}_a \to \mathbb{U}$) encapsulates our non-perturbative renormalization group in terms of objects of the category $\mathcal{E}^{\rm CM}$. □

The Isham–Doring topos approach does not recognize the impact of the strength of running couplings in strongly coupled physical systems in the logical foundations of physical theories. Thanks to our new graphon model approach, cut-distance topological Hopf subalgebras generated by solutions of Dyson–Schwinger equations can be applied for the construction of a new topos model (named as non-perturbative topos) for the logical analysis of non-perturbative parts of strongly coupled gauge field theories. Objects of the base category of the non-perturbative topos allow us to logically evaluate propositions about topological regions of Feynman diagrams which contribute to Dyson–Schwinger equations in a given physical theory under different running couplings in real time process. This new topos model enables us to logically recognize any change in physical systems during time process in terms of changing the amount of running couplings. Representations in the non-perturbative topos model can be applied to modify it for the description of higher-order logics in different gauge field theories. This new topos model can lead us to understand better the non-locality of non-perturbative regions. [34]

Theorem 3.9. *There exists a topos which encodes the logical information about cut-distance topological regions of Feynman diagrams which contribute to fixed point equations of Green's functions in a given (strongly coupled) gauge field theory* Φ.

Proof. The non-perturbative topos $\mathbf{T}_{\Phi}^{\mathrm{non},g}$ is the topos of presheaves on a new base category $\mathcal{C}_{\Phi}^{\mathrm{non},g}$. Objects of this base category are cut-distance compact topological Hopf subalgebras $H^{\mathrm{cut}}_{\mathrm{DSE}(c(g))}$ generated by solutions of Dyson–Schwinger equations under different running couplings $c(g)$ (with respect to the bare coupling constant g). The subobject classifier of this topos, which has a natural Heyting algebraic structure, enables us to evaluate truth-values propositions about topological regions of Feynman diagrams which contribute to Dyson–Schwinger equations [34]. The structures of the spectral presheaf, the outer presheaf and other logical properties of this topos have been discussed in another submitted work by the author where this topos model has also been generalized for physical theories with multiple bare coupling constants. □

4. A graphon model approach to Kontsevich's Deformation Quantization

In this section, we explain a new graph function representation model for Kontsevich admissible graphs to build a new Hopf algebraic formalism which can be topologically enriched to recover an infinite version of these graphs. We then use these Kontsevich graphons to formulate a new generalization of the Kontsevich's $\star$-product. We apply the Hopf algebra of Kontsevich admissible graphs to address the foundations of a differential Galois theory and a topos model for Kontsevich's Deformation Quantization. We also show that non-perturbative parameters generated by Theorem 3.7 can be computed via the Kontsevich's $\star$-product. Finally, we give a modified version of the Kontsevich's $\star$-product on a class of noncommutative differential calculi originated from renormalization of Dyson–Schwinger equations.

Definition 4.1. A Kontsevich admissible graph is a simple oriented graph (with no multiple edges or self-loops) which contains two classes of totally ordered disjoint sets of vertices called internal vertices and boundary vertices (or leaves). There is also a total order on the set of all edges. It is possible to present each Kontsevich graph via nodes and geodesics in a closed disk such that internal vertices are points inside the disk and boundary vertices are points on the boundary region of the disk.

The graphical calculus works on Kontsevich admissible graphs to encode actions of poly-differential operators on smooth functions defined on $\mathbb{R}^d$. The number of internal vertices presents the number of polyvector fields and the number of boundary vertices presents the number of smooth functions. Set $\mathfrak{g}^{p,q}(\mathbb{R}^d)$ as the collection of all Kontsevich admissible graphs such as K which has q internal vertices while $|V(K)| - |E(K)| - 1 = p$ and suppose $\mathfrak{g}^{\bullet,\bullet}(\mathbb{R}^d)$ be the bigraded vector space generated by $\bigcup_{p,q\geq 0} \mathfrak{g}^{p,q}(\mathbb{R}^d)$. A normal subgraph G of K is a full subgraph such that the quotient graph $H = K/G$ is a graph in $\mathfrak{g}^{\bullet,\bullet}(\mathbb{R}^d)$. It is possible to rebuild the admissible graph K as an extension of H by G via inserting the subgraph G inside H with respect to the types of vertices. The notation $G \hookrightarrow K \twoheadrightarrow H$ is used to present the normal subgraph and the extension process. The original graph K can be rebuilt by the insertion of the graph G into a vertex of the quotient graph H. The type of the vertex of G which is inserted into determines the type of the extension. [17, 18, 20]

Lemma 4.2. *For a fixed probability space (Ω, μ_Ω), we can identify a unique unlabeled graphon class with respect to the combinatorial information of each Kontsevich admissible graph in $\mathfrak{g}^{\bullet,\bullet}(\mathbb{R}^d)$.*

Proof. For simplicity we work on the closed interval $[0, 1]$ as the Lebesgue measure space. For each graph K with n number of internal vertices $v_1, ..., v_n$ and m number of boundary vertices $v_{n+1}, ..., v_{n+m}$, we can build its corresponding pixel picture presentation by dividing the unit square into $n + m$ small squares or boxes $I_i \times I_j$, $1 \leq i, j \leq n + m$. The box $I_i \times I_j$ is black or white whether there is an edge between corresponding vertices or not. Define the labeled graphon $V_K(x, y) = 1$ for $(x, y) \in I_i \times I_j$ whenever there is an edge (or geodesic) between the vertices v_i and v_j and otherwise define $V_K(x, y) = 0$. □

We name V_K as the labeled Kontsevich graphon corresponding to the graph $K \in \mathfrak{g}^{\bullet,\bullet}(\mathbb{R}^d)$ on the probability space (Ω, μ_Ω). The class $[V_K]$ collects all weakly isomorphic graphons with respect to different relabeling. Set $\mathcal{S}^{\rm Kont}_{\rm graphon}(\mathbb{R}^d)$ as the vector space generated by this type of graphons.

A sequence $K_1, K_2, ...$ of finite Kontsevich admissible graphs is convergent iff the corresponding sequence $V_{K_1}, V_{K_2}, ...$ of Kontsevich graphons is cut-distance convergent to a graphon V_∞ when n tends to infinity. We use the notation K_∞ for the Kontsevich admissible graph with the corresponding graph function model V_∞ and call it "large Kontsevich graph". In other words, $V_{K_\infty} \in [V_\infty]$ and $V_\infty \in [V_{K_\infty}]$. It is important to apply rescaling methods to ignore 0-graphon as the convergent limit of these sparse type graphs.

Remark 4.3. The large Kontsevich graph K_∞ generated by the information of the graphon V_∞ might contain infinite number of internal or boundary vertices or infinite number of edges.

The renormalization Hopf algebra $H_{\rm FG}(\Phi)$ has a Lie algebraic source in terms of the insertion operator on Feynman diagrams. The insertion operator can be described by Hochschild–Kontsevich products $\bullet, \circ$. These products, which are defined on the space $\mathfrak{g}^{\bullet,\bullet}(\mathbb{R}^d)$ in terms of internal and external extensions of Kontsevich admissible graphs, are given by $H \bullet G :=$

$\sum_{G\hookrightarrow K\twoheadrightarrow H,\ \text{internal}} \pm K$ and $H \circ G := \sum_{G\hookrightarrow K\twoheadrightarrow H,\ \text{boundary}} \pm K$ which are $(0,-1)$ degree and bigraded products, respectively. We have the quotient graph $K/G = H$ which is the result of shrinking the normal subgraph G. [9, 18, 19]

We can equip the bigraded vector space $\mathfrak{g}^{\bullet,\bullet}(\mathbb{R}^d)$ with the cut-distance topology to obtain a new topological vector space presented by $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}(\mathbb{R}^d)$.

It is now possible to extend products $\bullet, \circ$ to the level of large Kontsevich graphs which will be useful to extend the insertion operator to the level of large Feynman diagrams and find the Lie algebraic origin of the topological Hopf algebra $H^{\text{cut}}_{\text{FG}}(\Phi)$.

Lemma 4.4. *The products $\bullet, \circ$ are well-defined on large Kontsevich graphs.*

Proof. Suppose $\{K_n\}_{n\geq 0}$ be a sequence of Kontsevich admissible graphs which is convergent to the large Kontsevich graph K_∞ and $\{G_n\}_{n\geq 0}$ be another sequence of Kontsevich admissible graphs such that for each n, G_n is a normal subgraph of K_n. Suppose the sequence $\{G_n\}_{n\geq 0}$ is convergent to the large Kontsevich graph G_∞. We can consider a new sequence $\{H_n\}_{n\geq 0} := \{K_n/G_n\}_{n\geq 0}$ of quotient graphs which is cut-distance convergent to the large Kontsevich graph H_∞. Thanks to Kontsevich graphon representations V_{K_∞}, V_{G_∞} and V_{K_∞/G_∞}, we can show that $V_{H_\infty} \in [V_{K_\infty/G_\infty}]$ and therefore $H_\infty = K_\infty/G_\infty$. Now for each n, we can define

$$H_n \bullet G_n = \sum_{G_n\hookrightarrow K_n\twoheadrightarrow H_n,\ \text{internal}} \pm K_n, \quad H_n \circ G_n = \sum_{G_n\hookrightarrow K_n\twoheadrightarrow H_n,\ \text{boundary}} \pm K_n, \tag{4.1}$$

which lead us to define $H_\infty \bullet G_\infty$ as the cut-distance convergent limit of the sequence $\{H_n \bullet G_n\}_{n\geq 0}$ and define $H_\infty \circ G_\infty$ as the cut-distance convergent limit of the sequence $\{H_n \circ G_n\}_{n\geq 0}$. □

One important note is that $H_\infty \bullet G_\infty$ or $H_\infty \circ G_\infty$ might have infinite terms in their series where thanks to the compactness of the topology of graphons, these infinite series can be interpreted as objects in the boundary of the space $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}(\mathbb{R}^d)$.

Theorem 4.5. *There exists a graded Hopf algebra structure on $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}(\mathbb{R}^d)$ which is compatible with normal subgraphs and cut-distance topology.*

Proof. For a given Kontsevich admissible graph K, define

$$\Delta(K) = \mathbb{I} \otimes K + K \otimes \mathbb{I} + \sum_G G \otimes K/G \tag{4.2}$$

such that the sum is over all normal subgraphs of K and $\mathbb{I}$ is the empty graph. Terms in the above formal expansion are in the one to one correspondence with all possible internal and boundary extensions of normal subgraphs of the original graph K. The bigraded property of $\mathfrak{g}^{\bullet;\bullet}_{\text{cut}}(\mathbb{R}^d)$ allows us to define a new grading structure

$$\mathfrak{g}^{n}_{\text{cut}}(\mathbb{R}^d) := \bigoplus_{p+q=n} \mathfrak{g}^{p,q}_{\text{cut}}(\mathbb{R}^d) \tag{4.3}$$

and then formulate $\mathfrak{g}^{\bullet}_{\text{cut}}(\mathbb{R}^d)$ as the graded vector space generated by $\bigcup_{n\geq 0} \mathfrak{g}^{n}_{\text{cut}}(\mathbb{R}^d)$ and equipped with the cut-distance topology. Thanks to this graduation parameter, we can obtain the required antipode recursively and achieve the promising Hopf algebra structure.

Kontsevich graphons (i.e. Lemma 4.2) are useful to topologically complete this Hopf algebra. For any Kontsevich graphon $V_K \in \mathcal{S}^{\rm Kont}_{\rm graphon}(\mathbb{R}^d)$ corresponding to a finite graph K, define

$$\Delta(V_K) = V_{\mathbb{I}} \otimes V_K + V_K \otimes V_{\mathbb{I}} + \sum V_G \otimes V_{K/G} \tag{4.4}$$

such that $V_{\mathbb{I}}$ is the 0-graphon and the sum is controlled by Kontsevich graphons associated to normal subgraphs of the original graph K under internal and external extensions. For any large Kontsevich graph K_∞ with the corresponding Kontsevich graphon V_{K_∞} as the cut-distance convergent limit of the sequence $\{K_n\}_{n\geq 0}$ of finite Kontsevich admissible graphs, apply this coproduct to define $\Delta(V_{K_\infty})$ as the cut-distance convergent limit of the sequence $\{\Delta(V_{K_n})\}_{n\geq 0}$. Thanks to the graduation parameter and using the same idea, we can also define the antipode for large Kontsevich graphs. In addition, the compactness of the cut-distance topology shows us that the defined coproduct and antipode (as linear operators) are bounded which means that they are continuous operators.

We use the notation $H^{\rm cut}_{\rm Kont}(\mathbb{R}^d)$ for the resulting topological Hopf algebra of Kontsevich admissible graphs corresponding to Deformation Quantization program in $C^\infty(\mathbb{R}^d)$ which is generated by $\mathfrak{g}^{\bullet}_{\rm cut}(\mathbb{R}^d)$ at the vector space level. It is a graded connected free commutative non-cocommutative Hopf algebra. The correspondences $K \mapsto V_K$ and $\{K_n\}_{n\geq 0} \mapsto V_{K_\infty}$ make Hopf algebraic homomorphism between Hopf algebras $H^{\rm cut}_{\rm Kont}(\mathbb{R}^d)$ and $\mathcal{S}^{\rm Kont}_{\rm graphon}(\mathbb{R}^d)$. □

Remark 4.6. The distance between Kontsevich admissible graphs K_1, K_2 can be defined by their corresponding Kontsevich graphons. In other words,

$$d(K_1, K_2) := d_{\rm cut}(V_{K_1}, V_{K_2}). \tag{4.5}$$

The cocycle equation in Deformation Quantization informs us the existence of a fundamental relation between the Kreimer renormalization coproduct on Feynman diagrams (which decomposes each Feynman diagram based on disjoint unions of (1PI) divergent Feynman subdiagrams) and normal subgraphs of Kontsevich admissible graphs [18, 19, 30]. Therefore the Connes–Kreimer Hopf algebra of non-planar rooted trees (equipped with a particular class of decorations) can provide a universal model for Hopf algebras $H^{\rm cut}_{\rm Kont}(\mathbb{R}^d)$ and $\mathcal{S}^{\rm Kont}_{\rm graphon}(\mathbb{R}^d)$.

Lemma 4.7. *(i) A sequence of Kontsevich admissible graphs is convergent if it is a cut-distance Cauchy sequence.*

(ii) Each large Kontsevich graph K_∞ determines a cut-distance convergent sequence of finite random graphs.

Proof. (i) It is a direct result of the definition.

(ii) Let V_{K_∞} be the corresponding Kontsevich graphon. For each n, we can define a finite random graph $G(S_n, V_{K_\infty})$ which contains n nodes $S_n := \{s_1, ..., s_n\}$ in $[0,1]$ such that the existence of an edge between s_i and s_j is determined by the probability $V_{K_\infty}(s_i, s_j)$. Thanks to [25, 35], the sequence $\{R(n, V_{K_\infty})\}_{n\geq 1}$ (such that $R(n, V_{K_\infty}) := R(G(S_n, V_{K_\infty}))$ is cut-distance convergent to V_{K_∞}. □

Corollary 4.8. *The non-perturbative parameters generated by Theorem 3.7 can be reformulated in terms of the Kontsevich's $\star$-product.*

Proof. Thanks to Lemma 3.1, Theorem 3.7 and Section 6 in [31], Kontsevich graphons in the sequences $\{S^{\tilde{\phi}}_{R_{\rm ms}}(W_{Y_m})\}_{m\geq 1}$ and $\{S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi}(W_{Y_m})\}_{m\geq 1}$ can be rewritten in terms of the Baker–Campbell–Hausdorff formula for the Hausdorff series. This enables us to view the Hopf–Birkhoff factorization $S^{\tilde{\phi}}_{R_{\rm ms}} * \tilde{\phi} = \tilde{\phi}_+$ via the deformation star product $S^{\tilde{\phi}}_{R_{\rm ms}} \star \tilde{\phi}$ of the pointwise multiplication of the exponential functions $S^{\tilde{\phi}}_{R_{\rm ms}}$ and $\tilde{\phi}$. Now thanks to Theorem 4.5, $S^{\tilde{\phi}}_{R_{\rm ms}} \star \tilde{\phi}(W_X)$ can be determined as the convergent limit of the sequence $\{S^{\tilde{\phi}}_{R_{\rm ms}} \star \tilde{\phi}(W_{Y_m})\}_{m\geq 1}$. □

It is possible to equip $\mathfrak{g}^{\bullet,\bullet}_{\rm cut}(\mathbb{R}^d)$ with the Lie algebra structure defined by the commutator $[.,.]_\circ$ which gives us the degree $(1,0)$ differential operator d_1 on large Kontsevich graphs. We can also extend the Kontsevich's vertical differential operator on $\mathfrak{g}^{\bullet,\bullet}_{\rm cut}(\mathbb{R}^d)$ to define the differential operator d_2 on large Kontsevich graphs. For a given large Kontsevich graph K_∞ corresponding to the Kontsevich graphon $[V_{K_\infty}]$, the $(0,1)$ degree differential operator $d_2(K_\infty)$ is defined as the cut-distance convergent limit of the sequence $\{d_2(K_n)\}_{n\geq 0}$ such that for each n, we have $d_2(K_n) := \sum_{e\hookrightarrow L_n \twoheadrightarrow K_n,\ \text{internal}} \pm L_n = K_n \bullet e$ which is expanding the internal vertices of K_n by the insertion of an additional edge.

Corollary 4.9. *The cut-distance topological space of Kontsevich graphons can be equipped with the Hochschild–Kontsevich differential graded Lie algebra structure.*

Proof. We work on the graded topological vector space $\mathfrak{g}^{\bullet}_{\rm cut}(\mathbb{R}^d)$ such that the differential operators d_1, d_2 commute on the total complex $\mathfrak{g}^{\bullet,\bullet}_{\rm cut}(\mathbb{R}^d)$. Therefore $d := d_1 \pm d_2$ is a total differential operator which is compatible with the graded Lie bracket $[.,.]_\circ$. Thanks to Lemma 4.4, we can lift the differential operators d_1, d_2 onto the level of Kontsevich graphons. We have

$$V_K \bullet V_L := V_{K\bullet L}, \quad V_K \circ V_L := V_{K\circ L}, \quad d_1(V_K) := V_{d_1(K)}, \quad d_2(V_K) := V_{d_2(K)}. \tag{4.6}$$

□

Thanks to the built Hopf algebraic formalism, it is now possible to give a new geometric description for Deformation Quantization in the context of differential systems and Riemann-Hilbert correspondence. This alternative geometric setting improves our knowledge about the relation between Connes–Kreimer–Marcolli approach to perturbative Quantum Field Theory and Kontsevich's Deformation Quantization.

Theorem 4.10. *The collection of all Kontsevich admissible graphs which contribute to Deformation Quantization in $C^\infty(\mathbb{R}^d)$ determines a subcategory of the Connes–Marcolli universal category $\mathcal{E}^{\rm CM}$.*

Proof. We work on the graded connected free commutative Hopf algebra $H_{\rm Kont}(\mathbb{R}^d)$ and consider the category of flat equi-singular $\mathbb{G}_{{\rm Kont},\mathbb{R}^d}(\mathbb{C})$-connections on the regularization bundle such that the complex Lie group $\mathbb{G}_{{\rm Kont},\mathbb{R}^d}(\mathbb{C})$ is the space of characters of $H_{\rm Kont}(\mathbb{R}^d)$. This category is equivalent to the category ${\rm Rep}_{\mathbb{G}^*_{{\rm Kont},\mathbb{R}^d}}$ of finite dimensional representations of the affine group scheme $\mathbb{G}^*_{{\rm Kont},\mathbb{R}^d}$. Now thanks to the universal property of the neutral Tannakian category $\mathcal{E}^{\rm CM}$ of flat equi-singular vector bundles with respect to commutative Hopf algebras ([10]), we can embed ${\rm Rep}_{\mathbb{G}^*_{{\rm Kont},\mathbb{R}^d}}$ inside $\mathcal{E}^{\rm CM}$ to identify the subcategory $\mathcal{E}_{{\rm Kont},\mathbb{R}^d}$ on the basis of a new class of graded representations such as $\nu : \mathbb{U}(\mathbb{C}) \longrightarrow \mathbb{G}_{{\rm Kont},\mathbb{R}^d}$ which encode flat equi-singular $\mathbb{G}_{{\rm Kont},\mathbb{R}^d}(\mathbb{C})$-connections. □

Thanks to Kontsevich graphons, it is possible to build a new topos model (originated from non-perturbative topos) which encodes the logical foundations of (non-)local Kontsevich's Deformation Quantization.

Theorem 4.11. *The non-perturbative topos provides a topos model which can evaluate logical propositions about Kontsevich admissible graphs and large Kontsevich graphs which contribute to Deformation Quantization in* $C^{\infty}(\mathbb{R}^d)$.

Proof. We need to change the base category of the non-perturbative topos in the way that it could recover Kontsevich admissible graphs. For this purpose we define the base category $\mathcal{C}^{\rm non}_{{\rm Kont},\mathbb{R}^d}$ as the category which contains $H^{\rm cut}_{\rm Kont}(\mathbb{R}^d)$ and its all topological Hopf subalgebras as objects. Then we can build $\mathbf{T}^{\rm non}_{{\rm Kont},\mathbb{R}^d}$ as the topos of presheaves on $\mathcal{C}^{\rm non}_{{\rm Kont},\mathbb{R}^d}$. The subobject classifier of $\mathbf{T}^{\rm non}_{{\rm Kont},\mathbb{R}^d}$, which has the Heyting algebraic structure, can evaluate logical propositions about compact cut-distance topological regions of Kontsevich admissible graphs and also large Kontsevich graphs which live in the boundary regions. □

In the final step of this section, we are going to apply this graphon model to formulate the Kontsevich's deformation star product in a non-perturbative setting.

For a given Poisson structure α, the Kontsevich star product (1.2) with the general form $f \star_{\alpha} g := P(\alpha)(f \otimes g)$ is an associative product on $C^{\infty}(\mathbb{R}^d)[[\hbar]]$ as a deformation quantization of the commutative pointwise product with respect to α [21]. For finite dimensional configuration spaces, the Poisson tensor is a section of the vector bundle $\Lambda^2(T(M))$ which defines a skew-symmetric form on each contangent space $T^*_m(M)$. For infinite dimensional configuration spaces, it is possible to define the Poisson structure on a unital subalgebra of $C^{\infty}(M)$ (or the class of admissible differentials) with respect to the given locally convex manifold M. [17, 29]

Thanks to Kontsevich graphons, it is now possible to define the star product deformation for the level of infinite dimensional configuration spaces (such as $\mathbb{R}^{\infty}$) which are equipped with weak Poisson structures.

Lemma 4.12. *The Kontsevich's $\star$-product is well-defined for* $\mathbb{R}^{\infty}$.

Proof. We present locally the (weak) Poisson structure α in an open subset $\mathcal{U}$ of $\mathbb{R}^{\infty}$ via the infinite formal expansion

$$\alpha = \sum_{i,j=1}^{\infty} \alpha^{ij}(x)\partial_i \wedge \partial_j \tag{4.7}$$

with respect to local coordinates $x^1, x^2, \ldots$. Kontsevich graphons are useful to describe the space of polydifferential operators which act on real valued smooth functions on $\mathbb{R}^{\infty}$. For each n, let $P_{(x_1,\ldots,x_n)}$ be the orthogonal projection map which projects the points $(x_1, x_2, \ldots)$ in $\mathbb{R}^{\infty}$ into the n-dimensional subspace generated by the components $(x_1, \ldots, x_n)$. For each $f \in C^{\infty}(\mathcal{U})$, we can identify $f_{(x_1,\ldots,x_n)}$ as the smooth function on an open subspace of $\mathbb{R}^n$ such that

$$f = f_{(x_1,\ldots,x_n)} \circ P_{(x_1,\ldots,x_n)}. \tag{4.8}$$

Thanks to Remark 4.6, for each $f, g \in C^{\infty}(\mathcal{U})$, define $f \star_{\alpha} g$ as the cut-distance convergent limit of the sequence $\{f_{(x_1,\ldots,x_n)} \star_{\sum_{i,j=1}^{n} \alpha^{ij}(x)\partial_i \wedge \partial_j} g_{(x_1,\ldots,x_n)}\}_{n \geq 0}$. □

Corollary 4.13. *The Kontsevich's $\star$-product is well-defined for the topological Hopf algebra $H^{\rm cut}_{\rm Kont}(\mathbb{R}^d)$.*

Proof. The unital noncommutative associative convolution algebra $(L(H_{\rm Kont}(\mathbb{R}^d),\mathbb{C}),*)$ contains the Lie algebra $\mathrm{Lie}_{\mathrm{Kont},\mathbb{R}^d}(\mathbb{C})$ of the complex Lie group $\mathbb{G}_{\mathrm{Kont},\mathbb{R}^d}(\mathbb{C})$ of characters. This Lie algebra is generated by infinitesimal characters (or complex derivations) Z_K indexed by Kontsevich admissible graphs K and defined by $Z_K(L)=\delta_{KL}$. The exponential map

$$\mathrm{Lie}_{\mathrm{Kont},\mathbb{R}^d}(\mathbb{C})\to\mathbb{G}_{\mathrm{Kont},\mathbb{R}^d}(\mathbb{C}),\quad \exp^*(Z_K)=\sum_{n\geq 0}\frac{Z_K^{*n}}{n!} \tag{4.9}$$

gives the bijection between this Lie algebra and its Lie group. Thanks to the associative property of the convolution product, we can consider the commutator $[.,.]_*$ (with respect to the convolution product $*$) as the Poisson structure on $L(H_{\rm Kont}(\mathbb{R}^d),\mathbb{C})$ such that infinitesimal characters Z_K can be considered as Hamiltonian derivations.

We can modify Kontsevich admissible graphs for the space of (infinitesimal) characters and polyderivations defined with respect to the exponential map $\exp^*$. Internal vertices are symbols for polyvector fields. Boundary vertices are symbols for characters. The multiplication of m characters is represented by the graph with no internal vertices and m boundary vertices. The identification of m-vector fields with polyderivations is represented by the graph with one internal vertex, m boundary vertices and m edges.

Thanks to the graduation parameter of the Hopf algebra of (large) Kontsevich admissible graphs, for each character ϕ, define $\phi_{(n)}$ as its projection into the subalgebra generated by $H^{(n)}_{\rm Kont}(\mathbb{R}^d)$. Now define the star product $\phi\star_{[.,.]_*}\psi$ on the space of characters as the cut-distance convergent limit of the sequence $\{\phi_{(n)}\star_{[.,.]_*}\psi_{(n)}\}_{n\geq 0}$.

In addition, for a given large Kontsevich graph K_∞ as the convergent limit of the sequence $K_1,K_2,...$ of finite Kontsevich admissible graphs such that for each n, $K_n\in H^{(n)}_{\rm Kont}(\mathbb{R}^d)$, we have

$$\phi\star_{[.,.]_*}\psi(K_\infty)=\phi\star_{[.,.]_*}\psi(\lim_{n\to\infty}K_n)=\lim_{n\to\infty}\phi_{(n)}\star_{[.,.]_*}\psi_{(n)}(K_n). \tag{4.10}$$

This extends the star product $\star_{[.,.]_*}$ on the topological Hopf algebra $H^{\rm cut}_{\rm Kont}(\mathbb{R}^d)$. □

Remark 4.14. For any large Kontsevich graph K_∞ as the convergent limit of the sequence $\{K_n\}_{n\geq 0}$, there exists an infinitesimal character Z_{K_∞} which can be added as a new generator to the Lie algebra $\mathrm{Lie}_{\mathrm{Kont},\mathbb{R}^d}(\mathbb{C})$ to provide a topological enrichment of this Lie algebra with respect to the cut-distance topology.

In [38] a new class of noncommutative differential calculi has been built with respect to the BPHZ renormalization of Dyson–Schwinger equations. They are equipped with a family of Poisson structures originated from renormalization or regularization schemes.

Consider a given Dyson–Schwinger equation $\mathrm{DSE}(c(g))$ in a strongly coupled gauge field theory Φ with the corresponding non-cocommutative Hopf subalgebra $H_{\mathrm{DSE}(c(g))}$. The renormalization program given by Theorem 3.7 is on the basis of dimensional regularization (with the regularization algebra $A_{\rm dr}$ of Laurent series with finite pole parts) and minimal subtraction (with the renormalization map $R_{\rm ms}:A_{\rm dr}\to A_{\rm dr}$ which projects each series into its pole parts). We can deform the convolution algebra $L(H_{\mathrm{DSE}(c(g))},A_{\rm dr})$ of linear maps by the Rota–Baxter algebra $(A_{\rm dr},R_{\rm ms})$. For this purpose we first lift the map $R_{\rm ms}$ onto the space

$L(H_{\mathrm{DSE}(c(g))}, A_{\mathrm{dr}})$ (presented by $\mathbf{R}(\phi) := R_{\mathrm{ms}} \circ \phi$) and then for each real number λ we define a class of Nijenhuis maps on $L(H_{\mathrm{DSE}(c(g))}, A_{\mathrm{dr}})$ given by

$$\mathbf{R}_\lambda := \mathbf{R} - \lambda\tilde{\mathbf{R}} \ \text{ s.t. } \ \tilde{\mathbf{R}} := \mathrm{Id} - \mathbf{R}. \tag{4.11}$$

These Nijenhuis maps enable us to define a new class of noncommutative associative products on $L(H_{\mathrm{DSE}(c(g))}, A_{\mathrm{dr}})$ given by

$$\phi_1 \circ_\lambda \phi_2 := \mathbf{R}_\lambda(\phi_1) * \phi_2 + \phi_1 * \mathbf{R}_\lambda(\phi_2) - \mathbf{R}_\lambda(\phi_1 * \phi_2). \tag{4.12}$$

The commutator with respect to these products can be given by the relation

$$[\phi_1, \phi_2]_\lambda := [\mathbf{R}_\lambda(\phi_1), \phi_2] + [\phi_1, \mathbf{R}_\lambda(\phi_2)] - \mathbf{R}_\lambda[\phi_1, \phi_2] \tag{4.13}$$

which leads us to define new Lie algebraic structures. We use the notation $C_\lambda^{\mathrm{DSE}(c(g))}$ for this class of noncommutative associative unital deformed (Lie) algebras. Thanks to the Dubois-Violette approach to noncommutative differential geometry via Hamiltonian derivations ([11]), we can build a class of differential graded Lie algebras $(\Omega^\bullet(C_\lambda^{\mathrm{DSE}(c(g))}), d_\lambda)$ such that their corresponding noncommutative deRham complexes could determine Poisson structures $\{.,.\}_\lambda$ and related noncommutative symplectic geometries [34, 38]. If we use Kontsevich graphons, then we can define the Kontsevich's $\star$-product with respect to deformed Poisson structures $\{.,.\}_\lambda$ which can be defined weakly on the space $L(H_{\mathrm{DSE}(c(g))}, A_{\mathrm{dr}})$.

Corollary 4.15. *For a given Dyson–Schwinger equation* $\mathrm{DSE}(c(g))$ *in a strongly coupled physical theory, the Kontsevich's $\star$-product is well-defined for the topological Hopf algebra* $H^{\mathrm{cut}}_{\mathrm{DSE}(c(g))}$.

Proof. Consider the subspace $\partial\mathrm{char}_\lambda H_{\mathrm{DSE}(c(g))}$ of the noncommutative associative unital deformed algebra $C_\lambda^{\mathrm{DSE}(c(g))}$ which is the Lie algebra of infinitesimal characters or derivations into A_{dr}. Objects of this Lie algebra are linear maps in $L(H_{\mathrm{DSE}(c(g))}, A_{\mathrm{dr}})$ which obeys the Leibniz rule with respect to the deformed product $\circ_\lambda$. This space is generated by infinitesimal characters Z_Γ indexed by graphs $\Gamma \in H_{\mathrm{DSE}(c(g))}$ and defined by $Z_\Gamma(\Gamma') := \delta_{\Gamma,\Gamma'}$. The exponential map

$$\partial\mathrm{char}_\lambda H_{\mathrm{DSE}(c(g))} \longrightarrow \mathrm{char}_\lambda H_{\mathrm{DSE}(c(g))}, \quad \exp^{\circ_\lambda}(Z) := \sum_{n\geq 0} \frac{Z^{\circ_\lambda n}}{n!} \tag{4.14}$$

gives the bijection between this Lie algebra and its corresponding Lie group. The associative property of the algebra $C_\lambda^{\mathrm{DSE}(c(g))}$ shows that the commutator $[.,.]_\lambda$ (with respect to the product $\circ_\lambda$) is the Poisson structure on $C_\lambda^{\mathrm{DSE}(c(g))}$ such that infinitesimal characters Z_Γ can be considered as Hamiltonian derivations.

Thanks to the cut-distance topological completion of the graded Hopf algebra $H_{\mathrm{DSE}(c(g))}$, namely, $H^{\mathrm{cut}}_{\mathrm{DSE}(c(g))}$, we can now formulate the star product $\phi \star_{[.,.]_\lambda} \psi$ on the space of characters of the Hopf algebra $H_{\mathrm{DSE}(c(g))}$. For each character $\phi \in \mathrm{char}_\lambda H_{\mathrm{DSE}(c(g))}$, define $\phi_{(n)}$ as the projection of the character ϕ into the subalgebra generated by the generators $X_1, ..., X_n$ of the unique solution of the equation DSE with the corresponding infinitesimal characters $Z_{X_1}, ..., Z_{X_n}$. For each $\phi, \psi \in \mathrm{char}_\lambda H_{\mathrm{DSE}(c(g))}$, define $\phi \star_{[.,.]_\lambda} \psi$ as the cut-distance convergent limit of the sequence $\{\phi_{(n)} \star_{[.,.]_\lambda} \psi_{(n)}\}_{n\geq 0}$. For the large Feynman diagram X (as the unique solution of DSE) which is cut-distance convergent limit of the sequence $\{Y_m\}_{m\geq 0}$ of its partial sums, we have

$$\phi \star_{[.,.]_\lambda} \psi(X) = \phi \star_{[.,.]_\lambda} \psi(\lim_{m\to\infty} Y_m) = \lim_{m\to\infty} \phi_{(m)} \star_{[.,.]_\lambda} \psi_{(m)}(Y_m). \tag{4.15}$$

□

Corollary 4.16. *For a given Dyson–Schwinger equation* DSE$(c(g))$ *in a strongly coupled physical theory, the Kontsevich's $\star$-product is well-defined in the differential graded Lie algebra* $(\Omega^\bullet(C_\lambda^{\mathrm{DSE}(c(g))}), d_\lambda)$.

Proof. Thanks to [38], the symplectic structure generated by noncommutative deRham complex on the space of Hamiltonian derivations of $C_\lambda^{\mathrm{DSE}(c(g))}$ could determine the required Poisson structure $\{.,.\}_\lambda$. The graduation parameter enables us to define $\star_{\{.,.\}_\lambda}$ on $\Omega^\bullet(C_\lambda^{\mathrm{DSE}(c(g))})$ as the cut-distance convergent limit of star products on the components $\Omega^n(C_\lambda^{\mathrm{DSE}(c(g))})$. □

5. Conclusion

This research work was trying to show some new applications of graphon models to fundamental topics in Quantum Physics. On the one hand, Feynman graphons have been concerned to describe a non-perturbative topological Hopf algebraic renormalization program for solutions of Dyson–Schwinger equations. On the other hand, Kontsevich graphons have been introduced to obtain a new topological Hopf algebraic formalism for the study of Kontsevich's $\star$-product in a non-local setting. On the third hand, these graphon models have been applied to show some new interconnections between Kontsevich's Deformation Quantization and Hopf algebraic approach to Quantum Field Theory under perturbative and non-perturbative settings.

Generally speaking, we can classify non-perturbative quantum physical systems in terms of the behavior of running coupling constants which can be encoded by β-functions. In one class we have physical theories with negative β-functions such as high energy QCD, in another class we have physical theories with zero β-function such as Conformal Field Theory and in other class we have physical theories with positive β-functions such as low energy QCD, gauge field theories beyond Standard Model with multi-flavors. Dyson–Schwinger equations in physical theories with zero β-function can be reduced to linear versions such that lattice models, Borel resummation, large N limits, numerical methods, theory of instantons and AdS/CFT correspondence are useful tools in dealing with these equations to compute physical parameters. However these methods can not provide a complete understanding of Dyson–Schwinger equations under strong running coupling constants in physical theories with non-zero β-functions. The Hopf algebraic approach to Quantum Field Theory together with graphon models enable us to reformulate Dyson–Schwinger equations in the context of new mathematical settings where now we can describe these non-perturbative equations as objects of the boundary of a compact topological space of finite graphs. This graphon model approach gives us the opportunity to compute non-perturbative parameters generated by renormalization of Dyson–Schwinger equations under new algebraic and geometric settings. In addition, it provides a new topos model for the analysis of the logical differences in non-perturbative physical systems under changing running coupling constants during time process. Furthermore, graph function theory of sparse graphs has been applied to formulate a new Hopf algebraic formalism for Kontsevich's Deformation Quantization program which already led us to find some new interrelationships between this fundamental theory in Quantum Physics and mathematical foundations of non-perturbative Quantum Field Theory. The achievements of this research effort can also be useful to present a non-local generalization of Kontsevich's Deformation Quantization program.

Acknowledgement. The author is grateful to Institut des Hautes Etudes Scientifiques for the support and hospitality.

References

[1] C. Bergbauer, D. Kreimer, New algebraic aspects of perturbative and non-perturbative Quantum Field Theory, in "New Trends in Mathematical Physics; Selected contributions of the XVth International Congress on Mathematical Physics", 45–58, V. Sidoravicius (Ed.), Springer, 2009.
[2] G. Birkhoff, J. von Neumann, The logic of quantum mechanics, Ann. Math. (2) 37 no. 4, 823–843, 1936.
[3] B. Bollobas, O. Riordan, Metrics for sparse graphs, in S. Huczynska, J. D. Mitchell, and C. M. Roney-Dougal, eds., Surveys in combinatorics 2009, pages 211–287, London Math. Soc. Lecture Note Ser. 365, Cambridge University Press, Cambridge, 2009.
[4] C. Borgs, J.T. Chayes, H. Cohn, N. Holden, Sparse exchangeable graphs and their limits via graphon processes. J. Mach. Learn. Res. 18, Paper No. 210, 71 pp, 2017.
[5] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, An L^p theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions, Trans. Amer. Math. Soc. 372, no. 5, 3019–3062, 2019.
[6] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, An L^p theory of sparse graph convergence II: LD convergence, quotients, and right convergence, Annals of Probability 46, 337–396, 2018.
[7] C. Borgs, J.T. Chayes, L. Lovasz, V.T. Sos, K. Vesztergombi, Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, Adv. Math., 219(6):1801–1851, 2008.
[8] B. Coecke, Quantum logic in intuitionistic perspective, Studia Logica 70, 411–440, 2002.
[9] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. 199, 203–242, 1998.
[10] A. Connes, M. Marcolli, Noncommutative geometry, quantum fields and motives, Colloquium Publications, Amer. Math. Soc., Vol. 55, 2008.
[11] M. Dubois-Violette, Lectures on graded differential algebras and noncommutative geometry, Proc. Workshop on noncommutative differential geometry and its application to physics, Shonan-Kokusaimura 1999, Series: Mathematical Physics Studies, Vol. 23, 2001.
[12] P. Diao, D. Guillot, A. Khare, B. Rajaratnam, Differential calculus on graphon space. J. Comb. Theory Ser. A 133, 183–227, 2015.
[13] A. Doring, C.J. Isham, A topos foundation for theories of Physics, I. formal languages for Physics, J. Math. Phys. 49: 053515, 2008.
[14] A. Doring, C.J. Isham, A topos foundation for theories of Physics, II. daseinisation and the liberation of Quantum Theory, J. Math. Phys. 49: 053516, 2008.
[15] A. Doring, C.J. Isham, A topos foundation for theories of Physics, III. the representation of physical quantities with arrows, J. Math. Phys. 49: 053517, 2008.
[16] A . Doring, C.J. Isham, A topos foundation for theories of Physics, IV. categories of systems, J. Math. Phys. 49: 053518, 2008.
[17] G. Dito, D. Sternheimer, Deformation quantization: genesis, developments and metamorphoses, IRMA Lectures in Math. Theoret. Phys., vol. 1, Walter De Gruyter, Berlin, pp. 9–54, 2002.
[18] D. Fiorenza, L.M. Ionescu, Graph complexes in deformation quantization, Lett. Math. Phys. 73, 193, 2005.
[19] L.M. Ionescu, A combinatorial approach to coefficients in deformation quantization, arXiv: math.QA/0404389, 2004.

[20] V. Kathotia, Kontsevich's universal formula for deformation quantization and the Campbell–Baker–Hausdorff formula, Internat. J. Math., 11(4), 523–551, 2000.
[21] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66, 157–216, 2003.
[22] D. Kreimer, Etude for linear Dyson–Schwinger equations, IHES/P/06/23, 2006.
[23] D. Kreimer, Anatomy of a gauge theory, Ann. Phys. 321, 2757–2781, 2006.
[24] D. Kreimer, K. Yeats, An etude in non-linear Dyson–Schwinger equations, Nuclear Physics B - Proc. Supplements 160: 116–121, 2006.
[25] S. Janson, Graphons, cut norm and distance, couplings and rearrangements, New York Journal of Mathematics, NYJM Monographs 4, State University of New York, University at Albany, Albany, NY, 2013.
[26] L. Lovasz, B. Szegedy. Limits of dense graph sequences, J. Combin. Theory Ser. B, 96(6): 933–957, 2006.
[27] Y. Manin, Renormalization and computation II: time cut-off and the Halting problem, In: Math.Struct. in Comp. Science, vol. 22, Special issue, pp. 729–751, Cambridge UP. math.QA/0908.3430, 2012.
[28] Y. Manin, Renormalization and computation I: motivation and background, In: Proceedings OPERADS 2009, eds. J. Loday and B. Vallette, Seminaires et Congres 26, Soc. Math. de France, 181–222. math.QA/0904.4921, 2013.
[29] KH. Neeb, H. Sahlmann, T. Thiemann, Weak Poisson structures on infinite dimensional manifolds and hamiltonian actions, In: Dobrev V. (eds) Lie Theory and Its Applications in Physics. Springer Proceedings in Mathematics and Statistics, vol 111, pp 105–135, 2014.
[30] M. Polyak, Quantization of linear Poisson structures and degrees of maps, Lett. Math. Phys. 66, 15–35, 2003.
[31] M. Rosenbaum, J. D. Vergara, The Hopf algebra of renormalization, normal coordinates and Kontsevich deformation quantization, J.Phys. A37, 7939–7954, 2004.
[32] A. Shojaei-Fard, Formal aspects of non-perturbative Quantum Field Theory via an operator theoretic setting, Intern. J. Geom. Methods Mod. Phys., Vol. 16, No. 12, 1950192 (23 pages), 2019.
[33] A. Shojaei-Fard, Non-perturbative β-functions via Feynman graphons, Modern Phys. Lett. A, Vol. 34, No. 14, 1950109 (10 pages), 2019.
[34] A. Shojaei-Fard, A mathematical perspective on the phenomenology of non-perturbative Quantum Field Theory, Monograph, The MPIM Preprint Series 2018 (65), MPIM 18-65, arXiv:1811.05333, 2018.
[35] A. Shojaei-Fard, Graphons and renormalization of large Feynman diagrams, Opuscula Mathematica, Vol. 38, no. 3, 427–455, 2018.
[36] A. Shojaei-Fard, A measure theoretic perspective on the space of Feynman diagrams, Bol. Soc. Mat. Mex. (3) 24, no. 2, 507–533, 2018.
[37] A. Shojaei-Fard, A new perspective on intermediate algorithms via the Riemann–Hilbert correspondence, Quantum Stud. Math. Found. 4, no. 2, 127–148, 2017.
[38] A. Shojaei-Fard, Application of deformed Lie algebras to non-perturbative Quantum Field Theory, J. Indian Math. Soc. (N.S.) 84, no. 1–2, 109–129, 2017.
[39] A. Shojaei-Fard, A geometric perspective on counterterms related to Dyson–Schwinger equations, Intern. J. Modern Phys. A 28, No. 32, 1350170 (15 pages), 2013.
[40] W.D. van Suijlekom, The structure of renormalization Hopf algebras for gauge theories I: Representing Feynman graphs on BV-algebras, Commun. Math. Phys. 290, 291–319, 2009.
[41] W.D. van Suijlekom, Renormalization of gauge fields using Hopf algebras, In: Fauser, J.T.B., Zeidler, E., eds., Quantum Field Theory. Basel: Birkhuser Verlag, 2008.
[42] W.D. van Suijlekom, Renormalization of gauge fields: A Hopf algebra approach, Commun. Math. Phys. 276, 773–798, 2007.

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