A characterization of complex quasi-projective manifolds uniformized by unit balls

Ya DENG

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Juillet 2020

IHES/M/20/08

A CHARACTERIZATION OF COMPLEX QUASI-PROJECTIVE MANIFOLDS UNIFORMIZED BY UNIT BALLS

YA DENG

With an appendix written jointly with Benoît Cadorel

Abstract. In 1988 Simpson extended the Donaldson-Uhlenbeck-Yau theorem to the context of Higgs bundles, and as an application he proved a uniformization theorem which characterizes complex projective manifolds and quasi-projective curves whose universal coverings are complex unit balls. In this paper we give a necessary and sufficient condition for quasi-projective manifolds to be uniformized by complex unit balls. This generalizes the uniformization theorem by Simpson. Several byproducts are also obtained in this paper.

Contents

0. Introduction 1
Notations and conventions 6
1. Log Higgs bundles and system of log Hodge bundles 6
2. Principal system of log Hodge bundles 11
3. Tannakian consideration 16
4. Uniformization of quasi-projective manifolds by unit balls 18
5. Higgs bundles associated to non-compact ball quotients 28
6. Conjugate non-compact ball quotient 38
Appendix A. Metric rigidity for toroidal compactification of non-compact ball quotients 39
References 47

0. Introduction

0.1. **Main result.** The main goal of this paper is to characterize complex quasi-projective manifolds whose universal coverings are complex unit balls.

Theorem A (=Theorem 4.8.(i))**.** *Let X be an n-dimensional complex projective manifold and let D be a* smooth *divisor on X (which might contain several disjoint components). Let L be an ample polarization on X. For the log Higgs bundle $(\Omega_X^1(\log D)\oplus \mathcal{O}_X,\theta)$ on (X,D) with the Higgs field θ defined by*

$$\theta:\Omega_X^1(\log D)\oplus \mathcal{O}_X\longrightarrow (\Omega_X^1(\log D)\oplus \mathcal{O}_X)\otimes \Omega_X^1(\log D)\qquad (0.1.1)$$
$$(a,b)\mapsto (0,a),$$

Date: Monday 13th July, 2020.

2010 *Mathematics Subject Classification.* 14D07, 14C30, 32M15.

Key words and phrases. Uniformization, non-compact ball quotient, Simpson-Mochizuki correspondence, log Higgs bundles, principal variation of Hodge structures, toroidal compactification.

if it is μ_L-polystable (see § 1.4 for the definition), then one has the following inequality

$$\big(2c_2(\Omega_X^1(\log D)) - \frac{n}{n+1}c_1(\Omega_X^1(\log D))^2\big)\cdot c_1(L)^{n-2} \geq 0. \tag{0.1.2}$$

When the equality holds, then $X - D \simeq \mathbb{B}^n/\Gamma$ for some torsion free lattice $\Gamma \subset PU(n,1)$ acting on $\mathbb{B}^n$. Moreover, X is the (unique) toroidal compactification of $\mathbb{B}^n/\Gamma$, and each connected component of D is the smooth *quotient of an Abelian variety A by a finite group acting freely on A.*

Let us stress here that the *smoothness* of D in Theorem A is indeed necessary if one would like to characterize non-compact ball quotients: in Theorem 4.8.(ii) we prove that the universal cover of $X - D$ is not the complex unit ball $\mathbb{B}^n$ if D is assumed to be simple normal crossing but not smooth, leaving other conditions in Theorem A unchanged. Thus, it might be more appropriate to say that in this paper we give a characterization of *smooth toroidal compactification* of non-compact ball quotients.

Note that when D is empty or when $\dim X = 1$, Theorem A has already been proved by Simpson [Sim88, Proposition 9.8]. As we will see later, we follow his strategy closely to prove the above theorem. Let us also mention that the inequality (0.1.2) is a direct consequence of Mochizuki's deep work on the Bogomolov-Gieseker inequality for parabolic Higgs bundles [Moc06, Theorem 6.5]. Our main contribution is the uniformization result when the equality in (0.1.2) is achieved. The proof builds on Simpson's ingenious ideas [Sim88] on characterizations of complete varieties uniformized by Hermitian symmetric spaces, as well as Mochizuki's celebrated work on Simpson correspondence for tame harmonic bundles [Moc06]. Since the Kobayashi-Hitchin correspondence for general slope polystable parabolic Higgs bundles is still unproven, we need some additional methods to prove the above uniformization result (see § 0.3 for rough ideas).

We will show that the conditions in Theorem A is indeed necessary, by proving the following slope stability (with respect to a more general polarization) result for the natural log Higgs bundles associated to toroidal compactification of non-compact ball quotient by torsion free lattice.

Theorem B (=§ 5.4)**.** *Let $\Gamma \subset PU(n,1)$ be a torsion free lattice with only unipotent parabolic elements. Let X be the (smooth) toroidal compactification of the ball quotient $\mathbb{B}^n/\Gamma$. Write $D := X - \mathbb{B}^n/\Gamma$ for the boundary divisor, which is a disjoint union of Abelian varieties. Let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big and nef cohomology $(1,1)$-class on X containing a positive closed $(1,1)$-current $T \in \alpha$ so that $T|_{X-D}$ is a smooth Kähler form and has at most* Poincaré growth *near D (for example, $\alpha = c_1(K_X + D)$ or α contains a Kähler form ω). Then one has the following equality for Chern classes*

$$2c_2(\Omega_X^1(\log D)) - \frac{n}{n+1}c_1(\Omega_X^1(\log D))^2 = 0. \tag{0.1.3}$$

The log Higgs bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ defined in (0.1.1) is μ_α-polystable for the above big and nef polarization α. In particular, it is slope polystable with respect to any Kähler polarization and the polarization by the big and nef class $c_1(K_X + D)$.

As a consequence of Theorems A and B, following [Sim88, Corollary 9.5] in the compact setting, we give a new proof for the following rigidity result of ball quotient under the automorphism of complex number field $\mathbb{C}$ to its coefficients of defining equations.

Corollary C (=§ 6)**.** *Let $\Gamma \subset PU(n,1)$ be a torsion free lattice, and let $X := \mathbb{B}^n/\Gamma$ be the ball quotient, which carries a unique algebraic structure, denoted by X_{alg}. For any* *automorphism* $\sigma \in \mathrm{Aut}(\mathbb{C})$*, let* $X^{\sigma}_{\mathrm{alg}} := X_{\mathrm{alg}} \times_{\sigma} \mathrm{Spec}(\mathbb{C})$ *be the conjugate variety of* X_{alg} *under the automorphism* σ*, and denote by* X^{σ} *the analytification of* $X^{\sigma}_{\mathrm{alg}}$*. Then* X^{σ} *is also a ball quotient, namely there is another torsion free lattice* $\Gamma^{\sigma} \subset PU(n,1)$ *so that* $X^{\sigma} = \mathbb{B}^n / \Gamma^{\sigma}$*.*

When Γ is arithmetic, Corollary C has been proved by Kazhdan [Kaz83]. When Γ is non-arithmetic, it was proved by Mok-Yeung [MY93, Theorem 1] and by Baldi-Ullmo [BU20, Theorem 8.4.2].

We also obtain some byproducts, and let us mention a few. We prove the Simpson-Mochizuki correspondence for principal system of log Hodge bundles over projective log pairs (see Theorem 3.1). We give a characterization of slope stability with respect to big and nef classes for log Higgs bundles on Kähler log pairs (see Theorem 5.7). We also give a very simple proof of the negativity of kernels of Higgs fields of tame harmonic bundles by Brunebarbe [Bru17] (originally by Zuo [Zuo00] for system of log Hodge bundles), using some extension theorems of plurisubharmonic functions in complex analysis (see Theorem 4.6). In the appendix written jointly with Benoît Cadorel, we prove a metric rigidity result for toroidal compactification of non-compact ball quotients (see Theorem A.8).

0.2. **A few histories.** Since the main purpose of this paper is to prove the uniformization result rather than the Miyaoka-Yau type inequality (0.1.2), we shall only recall some earlier work related to the characterization of ball quotient, and we refer the readers to [GKT16, GT16] for more references on the Miyaoka-Yau type inequalities.

Based on his proof of the Calabi conjecture [Yau78], Yau established the inequality (0.1.2) when X is a projective manifold and $D = \varnothing$ with K_X ample. He proved that X is uniformized by the complex unit ball in case of equality. Miyaoka-Yau inequality and uniformization result were extended to the context of compact Kähler varieties with quotient singularities by Cheng-Yau [CY86] using orbifold Kahler-Einstein metrics. A partial uniformization result for smooth minimal models of general type have been obtained by Zhang [Zha09]. More recently, uniformization result has been extended to projective varieties with klt singularities in the series of work [GKPT19b, GKPT19a] by Greb-Kebekus-Peternell-Taji.

All the above works dealt with *compact* varieties. A strong uniformization result was established by Kobayashi [Kob84, Kob85] in the case of *open orbifold surfaces* (see also [CY86]). In [CY86] Cheng-Yau also gave a differential geometric characterization of quasi-projective ball quotients of any dimensions using the method of bounded geometry in [CY80]. At almost the same time, based on [CY86], Tian-Yau [TY87] and Tsuji [Tsu88] independently established similar *algebraic geometric* characterizations of non-compact ball quotient of any dimension. To the best of author's knowledge, [TY87, Tsu88] are the only known works of algebraic geometric characterization of high dimensional quasi-projective manifolds whose universal covers are unit balls. See also [Yau93] for more details.

All these aforementioned uniformization results are built on the positivity of the (log) canonical sheaf of the varieties together with existence of Kähler-Einstein metrics. In [Sim88], Simpson established a remarkable uniformization result in terms of stability of Higgs bundles. We essentially follow his approaches in this paper. In next subsection, we shall recall his ideas and discuss main difficulties in generalizing his methods to the context of *non-compact varieties*.

0.3. **Main strategy.** Let us briefly recall Simpson's strategy for the proof of Theorem A when $D = \varnothing$. In [Sim88, Theorem 1], Simpson proved that Higgs bundles over compact Kähler manifolds are *polystable* if and only if they admit Hermitian-Yang-Mills metrics. He then introduced the important notion of *principal system of Hodge bundles*, which is closed related to *principal variation of Hodge structures*. Based on the *Donaldson heat flow methods* in his proof of [Sim88, Theorem 1], in [Sim88, Proposition 8.2] he proved that a principal system of Hodge bundles with vanishing second Chern classes gives rise to a principal variation of Hodge structures, and vice versa. Assume now the boundary divisor D of X in Theorem A is empty. In [Sim88, p. 901] Simpson defined a principal system of Hodge bundles associated to $(\Omega_X^1 \oplus \mathcal{O}_X, \theta)$ whose second Chern class vanishes by [Sim88, Proposition 9.8]. By [Sim88, Proposition 8.2], this gives rise to a principal variation of Hodge structures on the universal covering of X, whose *period map* is biholomorphic to the complex unit ball of dim X since X is *compact*. This is the rough idea of Simpson's proof for Theorem A when $D = \varnothing$.

Let us explain our rough ideas in the proof of Theorem A when the equality in (0.1.2) holds.

Step 1: Following Simpson in the compact setting, we first define systems of log Hodge bundles over log pairs. We prove that, a system of log Hodge bundles on a projective with vanishing first and second Chern classes admits a Hodge metric, which is adapted to the trivial parabolic structure (see Proposition 1.16). The proof is different from Simpson's method since its is not clear for us that Donaldson's heat flow can give the desired Hermitian-Yang-Mills metric in the log setting. Instead, we first apply Mochizuki's celebrated theorem [Moc06, Theorem 9.4] to show the existence of harmonic metric, and we then use some C^*-action invariant property of log Hodge bundles to show that this harmonic metric is moreover a *Hodge metric*.

Step 2: We generalize the result in Step 1 to the context of *principal bundles*. Fix a Hodge group G_0. Following Simpson again, we define a principal system of log Hodge bundles (P, τ) on log pairs (X, D) with the structure group $K \subset G$, where G is the complexification of G_0. Based on the result in Step 1 together with some similar Tannakian arguments in [Sim90], in Theorem 3.1 we prove that if there is a faithful Hodge representation $\rho : G \to GL(V)$ for some polarized Hodge structure $(V = \oplus_{i+j=w} V^{i,j}, h_V)$ so that the system of log Hodge bundles $(P \times_K V, d\rho(\tau))$ is μ_L-polystable with $\int_X ch_2(P \times_K V) \cdot c_1(L)^{\dim X - 2} = 0$, then there is a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ gives rise to a *principal variation of Hodge structures* on $X - D$.

Step 3: For the system of log Hodge bundles $(E := \Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ in Theorem A, we first associate it a principal system of log Hodge bundles (P, τ) in Proposition 2.10, whose Hodge group $G_0 = PU(n, 1)$ is of Hermitian type (see Definition 2.4). One can easily show that $c_2(P \times_K \mathfrak{g}) = c_2(\mathrm{End}(E)^\perp) = 0$ when the equality in (0.1.2) holds, where $\mathrm{End}(E)^\perp$ denotes the trace free part of $\mathrm{End}(E)$. By a theorem of Mochizuki in Theorem 1.11, the system of log Hodge bundles $(P \times_K \mathfrak{g}, d(Ad)(\tau)) = (\mathrm{End}(E)^\perp, \theta_{End(E)^\perp})$ is also slope polystable if (E, θ) is slope polystable. Since the adjoint representation $G \to GL(\mathfrak{g})$ is a faithful Hodge representation, by the result in Step 2, there is a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ gives rise to a *principal variation of Hodge structures* on $X - D$. Since $\tau : T_X(-\log D) \to P \times_K \mathfrak{g}^{-1,1}$ is an isomorphism, this implies that the period map $p : \widetilde{X - D} \to {}^{PU(n,1)}\!/_{U(n)}$ associated to $(P|_{X-D}, \tau|_{X-D}, P_H)$ from the universal cover $\widetilde{X - D}$ of $X - D$ to the period domain ${}^{G_0}\!/_{K_0} = {}^{PU(n,1)}\!/_{U(n)}$ is *locally biholomorphic*. For more details, see Step one of the proof of Theorem 4.8

Step 4: We have to prove that the period map p in Step 3 is moreover a biholomorphism. Note that when $D = \varnothing$, this step is quite easy. In Remark 2.6 we show that it suffices to prove that the hermitian metric $\tau^* h_H$ on $X - D$ is *complete*, where h_H is the hermitian metric on $P \times_K \mathfrak{g}^{-1,1}|_{X-D}$ induced by the metric reduction P_H together with the Killing form of $\mathfrak{g}$. This step is slightly involved and the readers can find it in Step two of the proof of Theorem 4.8. To be brief, we establish a precise model metric (ansatz) for $(E, \theta) \otimes (E^*, \theta^*)$ locally around D with at most log growth, and we prove that this local metric is indeed *mutually bounded* with h_H using similar ideas in [Sim90, §4]. Based on this model metric, we obtain a precise norm estimates for h_H near D, so that we can prove that $\tau^* h_H$ is a complete metric on $X - D$. This concludes that the universal cover of $X - D$ is the unit ball ${}^{PU(n,1)}\!/_{U(n)}$.

0.4. **Further perspectives.** In this paper we only consider log Higgs bundles, namely parabolic Higgs bundles with trivial parabolic structures. If one allows non-trivial parabolic structures in Theorem A, we expect that there is a ramified covering of X by the complex unit ball which is only ramified over D.

Theorem A gives a characterizations for ball quotients admitting a smooth toroidal compactification. It is certainly an interesting question to extend this characterization for ball quotients whose toroidal compactification is only an orbifold or even for singular ball quotients. The first step towards this question is to extend Theorem 3.1 to the *stacky* setting as [Sim11].

In Theorem A, we consider the ample polarization for log Higgs bundles. In the last decades, after the sequel work by Campana-Peternell [CP11], Greb-Kebekus-Peternell [GKP16] and Campana-Păun [CP19], for applications in birational geometry it is quite important to consider more general polarization by *big and nef line bundles* or even *movable curves*. In Theorem B we establish such generalization for log Higgs bundles associated to toroidal compactifications of ball quotients. In a future project we would like to extend Theorem A to this context.

In [Sim88, Theorem 2], Simpson established a characterization of hermitian symmetric spaces of non-compact type. In Corollary 3.2 we only partially generalize his result to the log setting. The missing point is the precise norm estimate of the Hodge metric as Step 4 in § 0.3. We will consider this problem in a future work.

0.5. **Acknowledgments.** This work owes a lot to the deep work [Sim88, Sim90, Sim92, Moc06], to which I express my deepest gratitude. I sincerely thank Professor Carlos Simpson for answering my questions, as well as his suggestions and encouragements. I thank Professor Takuro Mochizuki for sending me his personal notes on the proof of Theorem 1.11. I also thank Professors Jean-Pierre Demailly, Henri Guenancia, Emmanuel Ullmo, Shing-Tung Yau, and Gregorio Baldi, Jiaming Chen, Chen Jiang, Jie Liu, Mingchen Xia for very helpful discussions and their remarks on this paper. My special thanks go to Benoît Cadorel for his very fruitful discussions on the toroidal compactification, which lead to a joint appendix with him in this paper. This work is supported by "le fond Chern" à l'IHES.

Notations and conventions

- A couple (E,h) is a *Hermitian vector bundle* on a complex manifold X if E a holomorphic vector bundle on X equipped with a smooth hermitian metric h. $\bar{\partial}_E$ denotes the complex structure of E, and we sometimes simply write $\bar{\partial}$ if no confusion arises.
- Two hermitian metrics h and $\tilde{h}$ of a holomorphic vector bundle on X are *mutually bounded* if $C^{-1}h \leq \tilde{h} \leq Ch$ for some constant $C>0$, and we shall denote by $h \sim h'$.
- For a hermitian vector bundle (E,h) on a complex manifold, $d_h = \partial_h + \bar{\partial}_E$ denotes its Chern connection and $R_h(E) = d_h^2$ denotes its Chern curvature.
- For a Higgs bundle (E,θ,h) with a smooth metric h on a complex manifold, $F_h(E) := R_h(E) + [\theta, \overline{\theta}_h]$, where $\overline{\theta}_h$ is the adjoint of θ with respect to h. We denote by $F_h(E)^\perp$ the *trace free part* of $F_h(E)$.
- Let (E,θ) be a log Higgs bundle on a log pair (X,D). For $a,b \in \mathbb{Z}_{\geq 0}$, we denote by $T^{a,b}(E,\theta)$ the tensor product of (E,θ) with $T^{a,b}E := \mathrm{Hom}(E^{\otimes a}, E^{\otimes b})$, and $T^{a,b}\theta$ the induced Higgs field.
- Δ denotes the unit disk in $\mathbb{C}$, and Δ^* denotes the punctured unit disk.
- The complex manifold X in this paper is always assumed to be connected and of dimension n.
- A *log pair* (X,D) consists of a (possibly non-compact) complex manifold X and simple normal crossing divisor D on X. Such a log pair is called *projective* (resp. Kähler) if X is a projective (resp. compact Kähler) manifold.
- P denotes the holomorphic principal K-fiber bundle on a complex manifold or log pairs, and $P_H \subset P$ denotes its metric reduction with the structure group $K_0 \subset K$.
- For a cohomology big $(1,1)$-class α on a compact Kähler manifold, $\mathcal{E}(\alpha)$ denotes the set of closed positive $(1,1)$-currents in α with *full Monge-Ampère mass*.
- For a closed positive $(1,1)$-current T on a complex manifold, locally it can be written as $T = \sqrt{-1}\partial\bar{\partial}\varphi$ with φ some plurisubharmonic function. Such φ is called the *local potential* of T.
- Throughout the paper we always work over the complex number field $\mathbb{C}$.

1. Log Higgs bundles and system of log Hodge bundles

1.1. **Higgs bundles and tame harmonic bundles.** In this section we recall the definition of Higgs bundles and tame harmonic bundles. We refer the readers to [Sim88, Sim90, Sim92, Moc02, Moc07] for further details.

Definition 1.1. Let X be a complex manifold. A *Higgs bundle* on X is a pair (E,θ) where E is a holomorphic vector bundle with $\bar{\partial}_E$ its complex structure, and $\theta : E \to E \otimes \Omega_X^1$ is a holomorphic one form with value in $\mathrm{End}(E)$, say *Higgs field*, satisfying $\theta \wedge \theta = 0$.

Let (E,θ) be a Higgs bundle over a complex manifold X. Write $D'' := \bar{\partial}_E + \theta$. Then $D''^2 = 0$. Suppose h is a smooth hermitian metric of E. Denote by $d_h := \partial_h + \bar{\partial}_E$ the Chern connection with respect to h, and by $\overline{\theta}_h$ the adjoint of θ with respect to h. Write $D'_h := \partial_h + \overline{\theta}_h$. The metric h is *harmonic* if the operator $D_h := D'_h + D''$ is integrable, that is, if $D_h^2 = R_h + [\theta, \overline{\theta}_h] = 0$.

Definition 1.2 (Harmonic bundle)**.** A harmonic bundle on a complex manifold X is triple (E,θ,h) where (E,θ) is a Higgs bundle and h is a harmonic metric for (E,θ).

Let X be an n-dimensional complex manifold, and let D be a simple normal crossing divisor.

Definition 1.3. (Admissible coordinate) Let p be a point of X, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ be components of D containing p. An *admissible coordinate* around p is the tuple $(U; z_1, \ldots, z_n; \varphi)$ (or simply $(U; z_1, \ldots, z_n)$ if no confusion arises) where

- U is an open subset of X containing p.
- there is a holomorphic isomorphism $\varphi : U \to \Delta^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \ldots, \ell$.

We shall write $U^* := U - D$, $U(r) := \{z \in U \mid |z_i| < r, \forall i = 1, \ldots, n\}$ and $U^*(r) := U(r) \cap U^*$.

Recall that the Poincaré metric ω_P on $(\Delta^*)^\ell \times \Delta^{n-\ell}$ is described as

$$\omega_P = \sum_{j=1}^{\ell} \frac{\sqrt{-1}dz_j \wedge d\bar{z}_j}{|z_j|^2(\log|z_j|^2)^2} + \sum_{k=\ell+1}^{n} \sqrt{-1}dz_k \wedge d\bar{z}_k$$

Note that

$$\omega_P = -\sqrt{-1}\partial\bar{\partial}\log\Big(\prod_{j=1}^{\ell}(-\log|z_j|^2)\cdot\exp\big(\sum_{k=\ell+1}^{n}|z_k|^2\big)\Big).$$

Definition 1.4 (Poincaré growth)**.** For a hermitian metric ω on $(\Delta^*)^\ell \times \Delta^{n-\ell}$, we say it has at most (resp. the same) *Poincaré growth* if there is $C > 0$ so that $\omega \le C\omega_P$ (resp. $\omega \sim \omega_P$). Let (X, D) be a log pair. A hermitian metric ω on $X - D$ has at most (resp. the same) *Poincaré growth near* D if for any point $x \in D$, there is an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x and a constant $C_U > 0$ so that $\omega \le C_U\omega_P$ (resp. $\omega \sim \omega_P$) for the Poincaré metric ω_P on U^*.

Remark 1.5 (Global Kähler metric with Poincaré growth). Let (X, ω) be a compact Kähler manifold and $D = \sum_{i=1}^{\ell} D_i$ is a simple normal crossing divisor on X. By Cornalba-Griffiths [CG75], one can construct a *Kähler current* T over X, whose restriction on $X - D$ is a complete Kähler form, which has the same Poincaré growth near D as follows.

Let σ_i be the section $H^0(X, \mathcal{O}_X(D_i))$ defining D_i, and we pick any smooth metric h_i for the line bundle $\mathcal{O}_X(D_i)$. One can prove that the closed $(1,,1)$-current

$$T := \omega - \sqrt{-1}\partial\bar{\partial}\log(-\prod_{i=1}^{\ell}\log|\varepsilon\cdot\sigma_i|^2_{\cdot h_i}), \tag{1.1.1}$$

the desired Kähler current when $0 < \varepsilon \ll 1$.

For any harmonic bundle (E, θ, h), let p be any point of X, and $(U; z_1, \ldots, z_n)$ be an admissible coordinate around p. On U, we have the description:

$$\theta = \sum_{j=1}^{\ell} f_j d\log z_j + \sum_{k=\ell+1}^{n} g_k dz_k \tag{1.1.2}$$

Definition 1.6 (Tameness)**.** Let t be a formal variable. We have the polynomials $\det(f_j - t)$, and $\det(g_k - t)$, whose coefficients are holomorphic functions defined over U^*. When the functions can be extended to the holomorphic functions over U, the harmonic bundle is called *tame* at p. A harmonic bundle is *tame* if it is tame at each point.

1.2. **Parabolic Higgs bundle.** In this section, we recall the notions of parabolic Higgs bundles. For more details refer to [Moc07]. Let X be a complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a reduced simple normal crossing divisor and $U = X - D$ be the complement of D.

Definition 1.7. A parabolic sheaf $(E, {}_{\boldsymbol{a}}E, \theta)$ on (X, D) is a torsion free $\mathcal{O}_U$-module E, together with an $\mathbb{R}^l$-indexed filtration ${}_{\boldsymbol{a}}E$ (*parabolic structure*) by coherent subsheaves such that

(1) $\boldsymbol{a} \in \mathbb{R}^l$ and ${}_{\boldsymbol{a}}E|_U = E$.
(2) For $1 \leq i \leq l$, ${}_{\boldsymbol{a}+\boldsymbol{1}_i}E = {}_{\boldsymbol{a}}E \otimes \mathcal{O}_X(D_i)$, where $\boldsymbol{1}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th component.
(3) ${}_{\boldsymbol{a}+\boldsymbol{\epsilon}}E = {}_{\boldsymbol{a}}E$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.
(4) The set of *weights* $\boldsymbol{a}$ such that ${}_{\boldsymbol{a}}E / {}_{\boldsymbol{a}-\boldsymbol{\epsilon}}E \neq 0$ is discrete in $\mathbb{R}^l$ for any vector $\boldsymbol{\epsilon} = (\epsilon, \ldots, \epsilon)$ with $0 < \epsilon \ll 1$.

A weight is normalized if it lies in $[0, 1)^l$. Denote ${}_{\boldsymbol{0}}E$ by $^{\diamond}E$, where $\boldsymbol{0} = (0, \ldots, 0)$. Note that the parabolic structure of $(E, {}_{\boldsymbol{a}}E, \theta)$ is uniquely determined by the filtration for weights lying in $[0, 1)^l$. A *parabolic bundle* on (X, D) consists of a vector bundle E on X with a parabolic structure, such that as a filtered bundle. When the parabolic sheaf only has a single weight $\boldsymbol{0}$, we say that it has *trivial parabolic structure.*

Definition 1.8. A *parabolic Higgs bundle* on (X, D) is a parabolic bundle $(E, {}_{\boldsymbol{a}}E, \theta)$ together with $\mathcal{O}_X$ linear map

$$\theta : {}^{\diamond}E \longrightarrow \Omega_X^1(\log D) \otimes {}^{\diamond}E$$

such that

$$\theta \wedge \theta = 0$$

and

$$\theta({}_{\boldsymbol{a}}E) \subseteq \Omega_X^1(\log D) \otimes {}_{\boldsymbol{a}}E,$$

for $\boldsymbol{a} \in [0, 1)^l$.

Throughout this paper, we mainly consider parabolic Higgs bundles with trivial parabolic structures on log pairs (X, D). In this case, it is equivalent to consider *log Higgs bundles* (E, θ) over (X, D), namely, E is a holomorphic vector bundle on X, and $\theta : E \longrightarrow E \otimes \Omega_X^1(\log D)$ with $\theta \wedge \theta = 0$.

A natural class of parabolic Higgs bundles comes from prolongations of tame harmonic bundle, which is discussed in the following section.

1.3. **Prolongation by an increased order.** By the work of Simpson [Sim90] and Mochizuki [Moc02, Moc07], there is a natural parabolic Higgs bundle induced by a tame harmonic bundle (E, θ, h). Let us recall their constructions.

We recall some notions in [Moc07, §2.2.1]. Let (X, D) be the pair in subsection 1.2. Let E be a holomorphic vector bundle with a $\mathscr{C}^\infty$ hermitian metric h over $X - D$.

Let U be an open subset of X, which is admissible with respect to D. For any section $\sigma \in \Gamma(U - D, E|_{U-D})$, let $|\sigma|_h$ denote the norm function of σ with respect to the metric h. We denote $|\sigma|_h \in \mathcal{O}(\prod_{i=1}^{\ell} |z_i|^{-b_i})$ if there exists a positive number C such that $|\sigma|_h \leq C \cdot \prod_{i=1}^{\ell} |z_i|^{-b_i}$. For any $\boldsymbol{b} \in \mathbb{R}^\ell$, say $-\mathrm{ord}(\sigma) \leq \boldsymbol{b}$ means the following:

$$|\sigma|_h = \mathcal{O}(\prod_{i=1}^{\ell} |z_i|^{-b_i - \varepsilon})$$

for any real number $\varepsilon > 0$. For any $\boldsymbol{b}$, the sheaf ${}_{\boldsymbol{b}}E$ is defined as follows:

$$\Gamma(U - D, {}_{\boldsymbol{b}}E) := \{\sigma \in \Gamma(U - D, E|_{U-D}) \mid -\mathrm{ord}(\sigma) \leq \boldsymbol{b}\}. \tag{1.3.1}$$

The sheaf ${}_{\boldsymbol{b}}E$ is called the prolongment of E by an increasing order $\boldsymbol{b}$. In particular,we use the notation ${}^{\diamond}E$ in the case $\boldsymbol{b} = (0, \ldots, 0)$.

According to Simpson [Sim90, Theorem 2] and Mochizuki [Moc07, Theorem 8.58], the above prolongation gives a parabolic Higgs bundles, especially θ preserves the filtration.

Theorem 1.9 (Simpson, Mochizuki). *Let (X, D) be a complex manifold X with a simple normal crossing divisor D. If (E, θ, h) is a tame harmonic bundle on $X - D$, then the corresponding filtration ${}_{\boldsymbol{b}}E$ according to the increasing order in the prolongment of E defines a parabolic bundle $(E, {}_{\boldsymbol{b}}E, \theta)$ on (X, D).* □

In this case, we say the harmonic metric is *adapted to* the parabolic structure of $(E, {}_{\boldsymbol{b}}E, \theta)$.

1.4. **Slope stability.** Let (X, ω) be a compact Kähler manifold of dimension n and let D be a simple normal crossing divisor on X. Let (E, θ) be a log Higgs bundle on (X, D). Let α be a big and nef cohomology $(1, 1)$-class on X. For any torsion free coherent sheaf F, its *degree with respect to* α is defined by $\deg_\alpha(F) := c_1(F) \cdot \alpha^{n-1}$, and its *slope with respect to* α is defined by

$$\mu_\alpha(F) := \frac{\deg_\alpha(F)}{\mathrm{rank}\, F}.$$

Consider a log Higgs bundle (E, θ) on (X, D). A *Higgs sub-sheaf* is a saturated coherent torsion free subsheaf $E' \subset E$ so that $\theta(E') \subset E' \otimes \Omega_X(\log D)$. We say (E, θ) is *μ_α-stable* if for Higgs sub-sheaf E' of E, with $0 < \mathrm{rank}\, E' < \mathrm{rank}\, E$, the condition

$$\mu_\alpha(E') < \mu_\alpha(E)$$

is satisfied. (E, θ) is *μ_α-polystable* if it is a direct sum of μ_α-stable log Higgs bundles with the same slope.

When $\alpha = \{\omega\}$ where ω is a Kähler form on X, we write μ_ω instead of μ_α. When $\alpha = c_1(L)$ for some ample line bundle L on X, we use the notation μ_L instead of μ_α.

By Simpson [Sim90], there is a $\mathbb{C}^*$-action on log Higgs bundles (E, θ) defined by $(E, t\theta)$ for any $t \in \mathbb{C}^*$. It follows from the definition that, if (E, θ) is μ_ω-stable, then $(E, t\theta)$ is also μ_ω-stable for any $t \in \mathbb{C}^*$.

The following celebrated *Simpson correspondence for tame harmonic bundles* proved by Mochizuki [Moc06] is a crucial ingredient in this paper.

Theorem 1.10 (Mochizuki). *Let (X, D) be a projective log pair endowed with an ample polarization L. A log Higgs bundle (E, θ) on (X, D) is μ_L-polystable with $\int_X c_1(E) \cdot c_1(L)^{\dim X - 1} = \int_X ch_2(E) \cdot c_1(L)^{\dim X - 2} = 0$ if and only if there is a harmonic metric h for $(E|_{X-D}, \theta|_{X-D})$ which is adapted to the trivial parabolic structure. When (E, θ) is moreover stable, such a harmonic metric h is unique up to some positive constant multiplication.*

Let us mention that in [Biq97] Biquard has proved a stronger theorem when the divisor D in Theorem 1.10 is smooth.

The poly-stability is also preserved under tensor product and dual by Mochizuki [Moc19, Proposition 4.10].

Theorem 1.11 (Mochizuki). *Let (X, D) be a projective log pair endowed with an ample polarization L. Let (E, θ) be a μ_L-polystable log Higgs bundle on (X, D). Then the tensor product $T^{a,b}(E, \theta)$ is still a μ_L-polystable log Higgs bundle for $a, b \in \mathbb{Z}_{\geq 0}$. Here $T^{a,b}E := Hom(E^{\otimes a}, E^{\otimes b})$ with $T^{a,b}\theta$ the induced Higgs field.*

1.5. **Simpson-Mochizuki correspondence for systems of log Hodge bundles.** A typical and important class of log Higgs bundle is the *system of log Hodge bundles.* In this subsection, we shall apply Theorem 1.10 to prove the *Simpson-Mochizuki correspondence for systems of log Hodge bundles.*

Definition 1.12 (System of log Hodge bundles). Let (E, θ) be a log Higgs bundle on a log pair (X, D). We say that (E, θ) is a *system of log Hodge bundles* if there is a decomposition of E into holomorphic vector bundles $E := \oplus_{p+q=w} E^{p,q}$ such that

$$\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X(\log D).$$

When $D = \varnothing$, such (E, θ) is called a *system of Hodge bundles.* A system of log Hodge bundles is μ_ω-(poly)stable if it is μ_ω-(poly)stable in the sense of log Higgs bundles.

Definition 1.13 (Hodge metric). Let $(E := \oplus_{p+q=w} E^{p,q}, \theta)$ be a system of Hodge bundles on a complex manifold X. A hermitian metric h for E is called a *Hodge metric* if h is harmonic, and it is a direct sum of metrics on the bundles $E^{p,q}$.

By Simpson [Sim88], a system of Hodge bundles equipped with a Hodge metric is equivalent to a *complex variation of Hodge structures.* He then established his correspondence for Hodge bundles over compact Kähler manifolds as follows.

Theorem 1.14 ([Sim88, Proposition 8.1]). *Suppose (X, ω) is a compact Kähler manifold. A Hodge bundle $(E := \oplus_{p+q=w} E^{p,q}, \theta)$ with $c_1(E) = 0$ and $c_2(E) \cdot [\omega]^{\dim X - 2} = 0$ is μ_ω-polystable if and only if it admits a Hodge metric.*

In the rest of this subsection, we will extend Theorem 1.14 to the log setting.

Let us first recall that, by Simpson [Sim90], a characterization of log Hodge bundles is the fixed point of $\mathbb{C}^*$-action. The automorphism of E obtained by multiplication by t^p on $E^{p,q}$ gives an isomorphism between (E, θ) and $(E, t\theta)$. The converse holds as follows.

Lemma 1.15 ([Sim90, Lemma 4.1] & [Sim92, Theorem 8]). *Let (E, θ) be a log Higgs bundle on a log pair (X, D). If $(E, \theta) \simeq (E, t\theta)$ for some $t \in \mathbb{C}^*$ which is not a root of unity, then (E, θ) has a structure of system of log Hodge bundles.*

Let us state and prove the main result in this subsection.

Proposition 1.16. *Let (X, D) be a projective log pair. Let $(E, \theta) = (\oplus_{p+q=w} E^{p,q}, \theta)$ be a system of log Hodge bundles on (X, D) which is μ_L-polystable with $\int_X c_1(E) \cdot c_1(L)^{\dim X - 1} = \int_X ch_2(E) \cdot c_1(L)^{\dim X - 2} = 0$. Then there is a decomposition $(E, \theta) = \oplus_{i \in I}(E_i, \theta_i)$ where each (E_i, θ_i) is μ_L-stable system of log Hodge bundles so that there is a Hodge metric h_i (unique up to a positive multiplication) for $(E_i|_{X-D}, \theta_i|_{X-D})$ which is adapted to the trivial parabolic structure of (E_i, θ_i).*

Proof. Let us first prove the proposition when (E, θ) is stable. By [Moc06, Theorem 9.1 & Propositions 5.1-5.3], there is a harmonic metrics h for $(E|_{X-D}, \theta|_{X-D})$ which is adapted to the trivial parabolic structure, and such a harmonic metric is unique up to a positive constant multiplication. We introduce automorphism $f_t : E \to E$ of E parametrized by $t \in U(1)$, defined by

$$f_t\Big(\sum_{p+q=w} e^{p,q}\Big) = \sum_{p+q=w} t^p e^{p,q}. \tag{1.5.1}$$

for every $e^{p,q} \in E^{p,q}$. Then $f_t : (E, \theta) \to (E, t\theta)$ is an isomorphism since $t\theta \circ f_t = f_t \circ \theta$. Hence by the uniqueness of harmonic metrics, there is a function $\lambda(t) : U(1) \to \mathbb{R}^+$ such that

$$f_t^* h = \lambda(t) \cdot h.$$

For every $e^{p,q} \in E^{p,q}$, one has

$$\lambda(t) \cdot h(e^{p,q}, e^{p,q}) = f_t^* h(e^{p,q}, e^{p,q}) = h(f_t(e^{p,q}), f_t(e^{p,q})) = |t^p|^2 h(e^{p,q}, e^{p,q}) = h(e^{p,q}, e^{p,q})$$

Hence $\lambda(t) \equiv 1$ for $t \in U(1)$, namely $f_t^* h = h$. On the other hand,

$$h(e^{p,q}, e^{r,s}) = f_t^* h(e^{p,q}, e^{r,s}) = h(f_t(e^{p,q}), f_t(e^{r,s})) = t^p t^{-r} h(e^{p,q}, e^{r,s})$$

for any $t \in U(1)$. Therefore, $h(e^{p,q}, e^{r,s}) = 0$ if $p \neq r$. Hence h is a direct sum of hermitian metrics for $E^{p,q}$, namely h is a Hodge metric. The proposition is proved if (E, θ) is stable.

Let us prove the general cases. By [Moc06, Corollary 3.11 & Theorem 9.1 & Propositions 5.1-5.3], there is a *canonical and unique* decomposition $(E, \theta) = \oplus_{i \in I} (E_i, \theta_i) \otimes \mathbb{C}^{p_i}$ where I is a finite set and harmonic metrics h_i for $(E_i|_{X-D}, \theta_i|_{X-D})$ which is adapted to the trivial parabolic structure so that (E_i, θ_i) is a μ_L-stable log Higgs bundle. By the above arguments, it suffices to prove that each (E_i, θ_i) is system of log Hodge bundles. Since (E, θ) is a system of log Hodge bundles, $(E, t\theta)$ is isomorphic to (E, θ) for any $t \in U(1)$. We have the following decomposition $(E, t\theta) = \oplus_i (E_i, t\theta_i) \otimes \mathbb{C}^{p_i}$. Note that $(E_i, t\theta_i)$ is still μ_L-stable. By the uniqueness of the decomposition, $(E_i, t\theta_i) \simeq (E_{i_t}, \theta_{i_t})$ for some $i_t \in I$. Since I is a finite set, there exists t_1, t_2 so that t_1/t_2 is not a root of unity and $i_{t_1} = i_{t_2}$. In other words, $(E_i, t_1\theta_i) \simeq (E_i, t_2\theta_i)$. By Lemma 1.15, $(E_i, t_1\theta_i)$ is a system of log Hodge bundles, and so is (E_i, θ_i). Hence (E, θ) is a direct sum of μ_L-stable system of log Hodge bundles (E_i, θ_i), and each $(E_i|_{X-D}, \theta_i|_{X-D})$ admits a Hodge metric h_i adapted to the trivial parabolic structure. The proposition is proved. □

2. Principal system of log Hodge bundles

In this section, we will extend Simpson's *principal system of log Hodge bundles* in [Sim88, §8] to the log setting. We will provide all necessary proofs for the claims for completeness sake. Let us mention that most results in this section follows from [Sim88, §8 & §9] with minor changes.

Let G_0 be a real connected algebraic group which is semi-simple with its Lie algebra denoted by $\mathfrak{g}_0$. Let G be its complexification with its Lie algebra denoted by $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_0 + \sqrt{-1}\mathfrak{g}_0$. G_0 is called a *Hodge group* if the following conditions hold.

- The Lie algebra $\mathfrak{g}$ of G admits a Hodge structure of weight 0, namely, one has a decomposition
$$\mathfrak{g} = \oplus \mathfrak{g}^{p,-p}$$
so that $[\mathfrak{g}^{p,-p}, \mathfrak{g}^{q,-q}] \subset \mathfrak{g}^{p+q,-p-q}$.
- If $\overline{\bullet}$ denotes the complex conjugation with respect to $\mathfrak{g}_0$, then $\overline{\mathfrak{g}^{p,-p}} = \mathfrak{g}^{-p,p}$.
- The form

$$h_{\mathfrak{g}}(U, V) := (-1)^{p+1} Tr(ad_U ad_{\bar{V}}) \quad \text{for} \quad U, V \in \mathfrak{g}^{p,-p} \tag{2.0.1}$$

is a positively definite hermitian metric for $\mathfrak{g}$.

let $K_0 \subset G_0$ be the Lie subgroup of G_0 so that its Lie algebra $\mathfrak{k}_0$ is $\mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$. Let $K \subset G$ (resp. $\mathfrak{k}$) be the complexification of K_0 (resp. $\mathfrak{k}_0$), and thus the Lie algebra of K is $\mathfrak{k}$. Then the restriction of the Killing form of $\mathfrak{g}_0$ on $\mathfrak{k}_0$ is positively definite, and thus K_0 is a compact real Lie group.

In the rest of the paper, we shall use the above notations without recalling their meanings.

The following concrete example of the Hodge group will be used in this paper, especially in the proof of Theorem A.

Example 2.1. Consider the a direct sum of $\mathbb{C}$-vector spaces

$$V = \oplus_{i+j=w} V^{i,j}$$

Denote by $r_i := \mathrm{rank} V^{i,j}$, and $r := \mathrm{rank}\, V$. Fix a hermitian metric $h = \oplus_{i+j=w} h_i$ for V where h_i is a hermitian metric for $V^{i,j}$. We take a sesquilinear form $Q(u,v) := (\sqrt{-1})^{i-j} h(u,v)$ for $u,v \in V^{i,j}$. Define $G_0 := PU(V,Q) = PU(p_0,q_0)$, where $p_0 := \sum_{i\ odd} r_i$ and $q_0 := \sum_{i\ even} r_i$. We shall show that G_0 is a *Hodge group*.

First we note that the complexification of G_0 is $G := PGL(V) \simeq PGL(r,\mathbb{C})$. Then the Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(V) \simeq \mathfrak{sl}(r,\mathbb{C})$, and the Lie algebra of G_0 is $\mathfrak{g}_0 = \mathfrak{su}(p_0,q_0)$. Let us define the *Hodge decomposition* as follows:

$$\mathfrak{g}^{p,-p} = \oplus_i \mathrm{Hom}(V^{i,j}, V^{i+p,j-p}) \cap \mathfrak{sl}(V).$$

Then $\mathfrak{g} = \oplus \mathfrak{g}^{p,-p}$. One can check that $\overline{\mathfrak{g}^{p,-p}} = \mathfrak{g}^{-p,p}$, where the conjugate is taken with respect to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$.

Let K be the subgroup of G which fix each $V^{i,j}$. Then $K = P(\prod_{i+j=w} GL(V^{i,j}))$, and its Lie algebra is $\mathfrak{k} = \mathfrak{g}^{0,0}$. Define $K_0 := K \cap G_0 = P(\prod_{i+j=w} U(V^{i,j}, h_i))$, whose Lie algebra is $\mathfrak{k}_0 = \mathfrak{g}^{0,0} \cap \mathfrak{g}_0$.

More precisely, if we fix a unitary frame $e_1, \ldots, e_{p_0}$ for $(\oplus_{i odd} V^{i,j}, \oplus_{i\ odd} h_i)$ and a unitary frame $f_1, \ldots, f_{q_0}$ for $(\oplus_{i\ even} V^{i,j}, \oplus_{i odd} h_i)$, elements in $\mathfrak{g}_0$ can be expressed as the ones in $M(r \times r, \mathbb{C})$ with the form

$$\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$$

where $A \in \mathfrak{u}(p_0)$ and $B \in \mathfrak{u}(q_0)$ so that $Tr(A) + Tr(B) = 0$. Note that the Killing form

$$Tr(ad_u ad_v) = 2r Tr(uv),$$

if we consider u, v as elements in $\mathfrak{sl}(r,\mathbb{C})$. Moreover, for $u \in \mathfrak{g}^{p,-p}$, one can show that

$$\bar{u} = \begin{cases} -u^* & \text{if } p \text{ is even} \\ u^* & \text{if } p \text{ is odd.} \end{cases}$$

where u^* denotes the conjugate transpose of u. Hence the hermitian metric $h_{\mathfrak{g}}$ defined in (2.0.1) can be simply expressed as

$$h_{\mathfrak{g}}(u,v) = 2r Tr(uv^*)$$

once we consider u, v as elements in $\mathfrak{sl}(r,\mathbb{C})$. In other words, for the natural inclusion $\iota : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, one has $h_{\mathfrak{g}} = 2r \cdot \iota^* h_{End(V)}$, where $h_{End(V)}$ is the hermitian metric on $\mathrm{End}(V)$ induced by h_V. This fact is an important ingredient in the proof of Theorem A.

Let us generalize Simpson's definition of principal system of Hodge bundles in [Sim88, §8] to the log setting as follows.

Definition 2.2 (Principal system of log Hodge bundles)**.** A *principal system of log Hodge bundles* on a log pair (X,D) is a pair (P,τ), where P is a holomorphic K-fiber bundle endowed with a holomorphic map

$$\tau : T_X(-\log D) \to P \times_K \mathfrak{g}^{-1,1}$$

such that $[\tau(u), \tau(v)] = 0$. A *metric* for $P|_{X-D}$ is a reduction $P_H \subset P|_{X-D}$ whose structure group is K_0. Let d_H be the Chern connection for P_H. Define $\bar{\tau}_H$ to be the complex conjugate of $\tau|_{X-D}$ with respect to the reduction P_H. Then

$$\bar{\tau}_H \in \mathscr{C}^\infty(X-D, (P_H \times_{K_0} \mathfrak{g}^{1,-1}) \otimes \Omega^{0,1}_{X-D}).$$

Set

$$D_H := d_H + \tau|_{X-D} + \overline{\tau}_H, \tag{2.0.2}$$

which is a connection on the smooth G_0-bundle $P_H \times_{K_0} G_0$. Such triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ is called a *principal variation of Hodge structures* over $X - D$ of Hodge group G_0, if the induced connection D_H in (2.0.2) is *flat*, namely the curvature of D_H is zero.

Remark 2.3 (Period map of principal variation of Hodge structures). By Simpson [Sim88, p. 900], for a principal variation of Hodge structures (P, τ, P_H) on a complex manifold X, one can also define its *period map* as follows. Denote by $\pi : \tilde{X} \to X$ the universal cover of X. Set $(\tilde{P} := \pi^* P, \tilde{\tau} := \pi^*\tau, \tilde{P}_H := \pi^* P_H)$, which is a principal variation of Hodge structures on the simply connected complex manifold $\tilde{X}$. The flat connection D_H thus induces a flat trivialization $\tilde{P}_H \times_{K_0} G_0 \simeq \tilde{X} \times G_0$. Denote by $\phi : \tilde{P}_H \to G_0$ the composition of the inclusion $\tilde{P}_H \subset \tilde{P}_H \times_{K_0} G_0 \simeq \tilde{X} \times G_0$ and the projection $\tilde{X} \times G_0 \to G_0$. It induces a map

$$\begin{aligned} f : \tilde{X} &\to {}^{G_0}\!/\!{}_{K_0} =: \mathscr{D} \\ \tilde{x} &\mapsto \phi(e_x) \cdot K_0 \quad \forall e_x \in \tilde{P}_{H,\tilde{x}}. \end{aligned} \tag{2.0.3}$$

Alternatively, we view $G_0 \to \mathscr{D}$ as a principal K_0-fiber bundle over $\mathscr{D}$, and its pull-back on $\tilde{X}$ via f is nothing but the principal K_0-fiber bundle $\tilde{P}_H$ by our definition of f. Hence the complexified differential of f is

$$df^{\mathbb{C}} : T^{\mathbb{C}}_{\tilde{X}} \to f^* T^{\mathbb{C}}_{\mathscr{D}} \simeq f^*(G_0 \times_{K_0} \oplus_{p\neq 0} \mathfrak{g}^{p,-p}) = \tilde{P}_H \times_{K_0} \oplus_{p\neq 0} \mathfrak{g}^{p,-p}$$

One can prove that $df^{\mathbb{C}} = \tilde{\tau} + \overline{\tilde{\tau}}_H$, where $\overline{\tilde{\tau}}_H$ is the conjugate of $\tilde{\tau}$ with respect to $\tilde{P}_H$. Hence the restriction of $df^{\mathbb{C}}$ to the holomorphic tangent bundle $T_{\tilde{X}}$ is $\tilde{\tau}$, which is a holomorphic map since the holomorphic tangent bundle of $\mathscr{D}$ is $T_{\mathscr{D}} \simeq G_0 \times_{K_0} \oplus_{p<0} \mathfrak{g}^{p,-p}$. In conclusion, f is a holomorphic map, which is called the *period map* associated to the principal variation of Hodge structures (P, τ, P_H), whose differential is given by $df = \tilde{\tau}$.

The uniformization is related by Hodge group of Hermitian type.

Definition 2.4 ([Sim88, §9])**.** A Hodge group G_0 is called *Hermitian type* if the Hodge decomposition $\mathfrak{g}$ of the Lie algebra of G is

$$\mathfrak{g} = \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$$

and that G_0 has no compact factor. In this case, $K_0 \subset G_0$ is the maximal compact subgroup and $\mathscr{D} := {}^{G_0}\!/\!{}_{K_0}$ is a Hermitian symmetric space of the non-compact type.

We also have to generalize the definition of *uniformizing bundle* by Simpson [Sim88, §9] to the log setting.

Definition 2.5 (Uniformizing bundle)**.** Let G_0 be a Hodge group of Hermitian type. A *uniformizing bundle* on a log pair (X, D) is a principal system of log Hodge bundles (P, τ) such that $\tau : T_X(-\log D) \xrightarrow{\simeq} P \times_K \mathfrak{g}^{-1,1}$ is an isomorphism. A *uniformizing variation of Hodge structures* is a uniformizing bundle on a complex manifold X together with a flat metric $P_H \subset P$.

Remark 2.6 (Uniformization via uniformizing bundles). It follows from Definition 2.5 that, for a uniformizing variation of Hodge structures over a complex manifold X, the period map $f : \tilde{X} \to \mathscr{D}$ defined in (2.0.3) is locally biholomorphic. Note that the metric reduction P_H induces a hermitian metric h_H on $P\times_K \mathfrak{g}^{-1,1} \simeq P_H \times_{K_0} \mathfrak{g}^{-1,1}$ defined by

$$h_H\big((p,u),(p,v)\big) := Tr(ad_u \circ ad_{\bar{v}}) \tag{2.0.4}$$

for any $p\in P_H$ and $u,v\in \mathfrak{g}^{-1,1}$. Note that for any $k\in K_0$, one has

$$\begin{aligned} h_H\big((pk, Ad_{k^{-1}}u),(pk,Ad_{k^{-1}}v)\big) &= Tr(ad_{Ad_{k^{-1}}u}\circ ad_{\overline{Ad_{k^{-1}}v}}) \\ &= Tr(ad_{Ad_{k^{-1}}u}\circ ad_{Ad_{k^{-1}}\bar{v}}) \\ &= Tr(Ad_{k^{-1}}\circ ad_u \circ Ad_k \circ Ad_{k^{-1}}\circ ad_{\bar{v}}\circ Ad_k) \\ &= h_H\big((p,u),(p,v)\big). \end{aligned}$$

By the equivalence relation $(p,u)\sim (pk, Ad_{k^{-1}}u)$, the metric h_H is thus well-defined. For the period domain $\mathscr{D}$ which is a hermitian symmetric space, one can also define the hermitian metric $h_{\mathscr{D}}$ for $T_{\mathscr{D}}\simeq G_0\times_{K_0}\mathfrak{g}^{-1,1}$ in a similar way. Since $\tilde{P}_H = f^*G_0$ when we consider $G_0\to\mathscr{D}$ as a principal K_0-fiber bundle, one has

$$\pi^*\tau^*h_H = f^*h_{\mathscr{D}}. \tag{2.0.5}$$

In other words, $f:(\tilde{X}, h_{\tilde{X}} := \pi^*\tau^*h_H)\to(\mathscr{D},h_{\mathscr{D}})$ is a *local isometry*. Hence for the action of $\pi_1(X)$ on $\tilde{X}$, the metric $h_{\tilde{X}}$ is invariant under this $\pi_1(X)$-action. If τ^*h_H is a complete metric, so is $\pi^*\tau^*h_H$. By [Cha06, Theorem IV.1.2], $f:\tilde{X}\to\mathscr{D}$ is a covering map, which is moreover a biholomorphism since $\tilde{X}$ and $\mathscr{D}$ are both simply connected. In other words, X is uniformized by the hermitian symmetric space $\mathscr{D}$.

One can construct systems of log Hodge bundles from principal ones via Hodge representations.

Definition 2.7 ([Sim88, p. 900])**.** Let $(V=\oplus_{p+q=w}V^{p,q}, h_V)$ be a polarized Hodge structure. A *Hodge representation* of G_0 is a complex representation $\rho: G\to GL(V)$ satisfying the following conditions.

- The action of $\mathfrak{g}$ is compatible with Hodge type, and such that K_0 preserves Hodge type. In other words,
$$d\rho(\mathfrak{g}^{r,-r})(V^{p,q})\subset V^{p+r,q-r}$$
and $\rho(K_0)(V^{p,q})\subset V^{p,q}$.[1]
- The sesquilinear form Q defined by
$$Q(u,v) := (\sqrt{-1})^{p-q}h_V(u,v) \quad \text{for} \quad u,v\in V^{p,q} \tag{2.0.6}$$
is G_0 invariant. Namely, one has $\rho(G_0)\subset U(V,Q)$.

Example 2.8. For the Hodge group G_0, $(\mathfrak{g}=\oplus_p\mathfrak{g}^{p,-p}, h_{\mathfrak{g}})$ is a polarized Hodge structure of weight 0, where $h_{\mathfrak{g}}$ is the polarization defined in (2.0.1) via the Killing form. One can easily check that the adjoint representation $Ad: G\to GL(\mathfrak{g})$ is a Hodge representation for this polarized Hodge structure. Since G is a semi-simple Lie group, the differential $d(Ad):\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ is injective. When the center of G is trivial, then Ad is faithful.

A principal system of log Hodge bundles together with a Hodge representation induces a system of log Hodge bundles as follows.

[1] As remarked by Simpson [Sim88], this is not automatic if K_0 is not connected. However, in Example 2.1, K_0 is always connected, and thus such condition will be superfluous in that case.

Lemma 2.9. *If $\rho : G \to GL(V)$ is a Hodge representation of the Hodge group G_0 and (P, τ) is a principal system of log Hodge bundles on the log pair (X, D), then $(E := P \times_K V, \theta := d\rho(\tau))$ is a system of log Hodge bundles. A polarization h_V for V together with a metric P_H for $P|_{X-D}$ give a metric h_E on the system of Hodge bundles $(E, \theta)|_{X-D}$ over $X - D$. When $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures over $X - D$, $(E|_{X-D}, \theta|_{X-D}, h_E)$ gives rise to a complex variation of Hodge structures.*

Proof. By Definition 2.7, one has $\rho(K)(V^{p,q}) \subset V^{p,q}$. Hence $E := P \times_K V$ admits a decomposition of holomorphic vector bundles $E = \oplus_{p+q=w} E^{p,q}$ with $E^{p,q} := P \times_K V^{p,q}$. Let us define $\theta := d\rho(\tau)$. Since $\tau : T_X(-\log D) \to P \times_K \mathfrak{g}^{-1,1}$ satisfies $[\tau(u), \tau(v)] = 0$, and $d\rho(g^{-1,1})(V^{p,q}) \subset V^{p-1,q+1}$, one thus has $\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X(\log D)$, with $\theta \wedge \theta = 0$. Hence (E, θ) is a system of log Hodge bundles.

Let us now prove that $\rho|_{K_0} : K_0 \to GL(V)$ has image on $U(V, h_V)$. Since $\rho(K)(V^{p,q}) \subset V^{p,q}$, one thus has

$$\rho(K) \subset \prod_{p+q=w} GL(V^{p,q}).$$

Since the sesquilinear form Q in (2.0.6) is G_0 invariant, one thus has

$$\rho(G_0) = U(V, Q).$$

Hence

$$\rho(K_0) \subset \rho(G_0 \cap K) \subset \prod_{p+q=w} U(V^{p,q}, h_{p,q}) \subset U(V, h_V). \tag{2.0.7}$$

Note that $E = P \times_K V \simeq P_H \times_{K_0} V$. We define the hermitian metric h_E for E by setting

$$h_E((p, u), (p, v)) := h_V(u, v)$$

for any $p \in P_H$ and for any $u, v \in V$. Since $\rho(K_0) \subset U(V, h_V)$, one can check as Remark 2.6 that h is well-defined.

If $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$, the connection $D_H := d_H + \tau + \bar{\tau}_H$ is flat. By construction, the connection $D_{h_E} := d_{h_E} + \theta + \bar{\theta}_{h_E}$ for $E|_{X-D}$ is also flat, where d_{h_E} is the Chern connection for the metrized vector bundle (E, h_E), and $\bar{\theta}_{h_E}$ is the conjugate of θ with respect to h_E. Indeed, d_{h_E} is naturally induced by d_H, and $\theta = d\rho(\tau)$, $\bar{\theta}_{h_E} = d\rho(\bar{\tau}_H)$. By [Sim88, p. 898], the triple $(E|_{X-D}, \theta|_{X-D}, h_E)$ gives rise to a complex variation of Hodge structures on $X - D$. □

Conversely, one can associate a system of log Hodge bundles with a principal one as follows. The following result shall be applied in the proof of Theorem A.

Proposition 2.10. *Let $(E, \theta) = (\oplus_{p+q=w} E^{p,q}, \theta)$ be a system of log Hodge bundles on a log pair (X, D). Then there is a principal system of log Hodge bundles (P, τ) with the structure group K associated to (E, θ), where K is the semi-simple Lie group in Example 2.1. Moreover, any hermitian metric $h := \oplus_{p+q=w} h_p$ for $E|_{X-D}$ gives rise to a metric reduction P_H for $P|_{X-D}$ with the structure group K_0 defined in Example 2.1.*

Proof. We shall adopt the same notions as those in Example 2.1. Denote by $r_p := \operatorname{rank} E^{p,q}$, $r := \sum_{p+q=w} r_p$ and set $\ell_i := \sum_{p \geq i} r_i$. We consider the following frame bundle $\tilde{P}$. The fiber of $\tilde{P}$ over a point x is the set of all ordered bases $e_1, \ldots, e_r$ (or say frames) for E_x such that $e_{\ell_p - r_p + 1}, \ldots, e_{\ell_p}$ is a basis for $E^{p,q}_x$. The structure group of $\tilde{P}$ is thus $\prod_p GL(r_p, \mathbb{C})$, which is the subgroup of $GL(r, \mathbb{C})$. $\tilde{P}$ can be equipped with the holomorphic structure induced by E. Consider the homomorphism $f : GL(r, \mathbb{C}) \to PGL(r, \mathbb{C}) =: G$, and set $K = P\big(\prod_p GL(r_p, \mathbb{C})\big)$ to be the image of $\prod_p GL(r_p, \mathbb{C})$ under

f. Set P to be the holomorphic K-fiber bundle obtained by extending the structure group of $\prod_p GL(r_p, \mathbb{C})$ using f.

Note that $P \times_K \mathfrak{g}^{-1,1} = \oplus_{i+j=w} \mathrm{Hom}(E^{i,j}, E^{i-1,j+1})$. Let us define $\tau := \theta$. The pair (P, τ) is a principal system of log Hodge bundles on the log pair (X, D).

Recall that the metric h for the Hodge bundle $(E, \theta)|_{X-D}$ is a direct sum $h = \oplus_{p+q=w} h_p$. We take a sesquilinear form Q of E defined by $Q(u, v) := (\sqrt{-1})^{p-q} h(u, v)$ for $u, v \in E^{p,q}$. We take $\tilde{P}_H$ to be a reduction of $\tilde{P}|_{X-D}$ consisting of unitary frames with respect to Q. In other words, The fiber of $\tilde{P}$ over a point x is the set of frames $e_1, \ldots, e_r$ for E_x such that $e_{\ell_p - r_p + 1}, \ldots, e_{\ell_p}$ is an orthonormal basis for $(E_x^{p,q}, h_p)$. Hence the structure group of $\tilde{P}_H$ is $\tilde{K}_0 := \prod_{p+q=w} U(r_p)$. Define $K_0 := P\big(\prod_{p+q=w} U(r_p)\big)$, which is the image $f(\tilde{K}_0)$. Set P_H to be the smooth principal K_0-fiber bundle on $X - D$ obtained by extending the structure group of $\tilde{P}_H$ using $f : K \to K_0$. Then $P_H \subset P_{X-D}$ is also a metric reduction. The Hodge group G_0 will be $PU(p_0, q_0)$ where $p_0 := \sum_{p \text{ even}} r_p$ and $q_0 := \sum_{p \text{ odd}} r_p$, and $G := PGL(r, \mathbb{C})$ is the complexification of G_0. The proposition is proved. □

3. Tannakian consideration

In this section, we shall state and prove the *Simpson-Mochizuki correspondence for principal systems of log Hodge bundles over projective log pairs.* Its proof is based on Proposition 1.16 together with some Tannakian considerations in [Sim90, Moc06, Mau15].

Theorem 3.1. *Let (X, D) be a projective log pair endowed with an ample polarization L. Let (P, τ) be a principal system of log Hodge bundles on (X, D), and let ρ be any* faithful *Hodge representation $\rho : G \hookrightarrow GL(V)$ for some polarized Hodge structure $(V = \oplus_{i+j=w} V^{i,j}, h_V)$. If the system of log Hodge bundles $(E := P \times_K V, \theta := d\rho(\tau))$ defined in Lemma 2.9 is μ_L-polystable with $\int_X ch_2(E) \cdot c_1(L)^{\dim X - 2} = 0$, then there exists a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a* principal variation of Hodge structures *on $X - D$. Moreover, such P_H together with the polarization h_V for V gives rise to a Hodge metric h for $(E, \theta)|_{X-D}$ (defined in Lemma 2.9) which is adapted to the trivial parabolic structure of (E, θ).*

Proof. We first prove that $(E, \theta)|_{X-D}$ admits a Hodge metric h over $(E, \theta)|_{X-D}$ which is adapted to the trivial parabolic structure of (E, θ). Since K is a complex semi-simple Lie group, the Hodge representation $\rho' : K \to GL(\det V)$ induced by ρ has image contained in $SL(\det V) = 1$. Hence ρ' is trivial. Note that $\det E = P \times_K \det V$, which is thus a trivial line bundle on X. Hence $c_1(E) = 0$. Since we assume that (E, θ) is μ_L-polystable with $\int_X ch_2(E) \cdot c_1(L)^{\dim X - 2} = 0$, it follows from Proposition 1.16 that $(E, \theta)|_{X-D}$ admits a Hodge metric h over $(E, \theta)|_{X-D}$ which is adapted to the trivial parabolic structure of (E, θ).

Let us now recall some Tannakian arguments. The representation ρ induces a representation $\rho_{a,b} : G \to GL(T^{a,b}V)$ for any $a, b \in \mathbb{N}$, where $T^{a,b}V := \mathrm{Hom}(V^{\otimes a}, V^{\otimes b})$. Since ρ is faithful, we can consider K as a reductive algebraic subgroup of $GL(V)$. There is a one dimensional complex subspace $V_1 \in T^{a,b}V$ for some $(a, b) \in \mathbb{N}^2$ so that

$$K = \{g \in GL(V) \mid \rho_{a,b}(g)(V_1) = V_1\}. \tag{3.0.1}$$

Since K is reductive, there is a complementary subspace V_2 of $T^{a,b}V$ for V_1 which is invariant under K.

By Lemma 2.9, the Hodge representation $\rho_{a,b}$ and (P, τ) gives rise to a system of log Hodge bundles $(P \times_K T^{a,b}V, \theta^{a,b} := d\rho_{a,b}(\tau))$ over (X, D), which is nothing but

$T^{a,b}(E,\theta)$. Recall that $\rho_{a,b}(K)(V_1) = V_1$ and $\rho_{a,b}(K)(V_2) = V_2$. Consider the log Higgs bundles $(E_1, \theta_1) := (P \times_K V_1, d\rho_{a,b}(\tau))$ and $(E_2, \theta_2) := (P \times_K V_1, d\rho_{a,b}(\tau))$ over (X, D).

Note that $T^{a,b}(E,\theta) = (E_1,\theta_1) \oplus (E_2,\theta_2)$. By Theorem 1.10, $T^{a,b}(E,\theta)$ is μ_L-polystable with $\int_X c_1(T^{a,b}(E)) \cdot c_1(L)^{\dim X-1} = 0$ with respect to an arbitrary polarization L. Since $c_1(T^{a,b}(E)) = c_1(E_1) + c_1(E_2)$, by the polystability of $T^{a,b}(E,\theta)$, we conclude that (E_1,θ_1) and (E_2,θ_2) are both μ_L-polystable. By Proposition 1.16, each $(E_i|_{X-D}, \theta_i|_{X-D})$ admits a harmonic metric h_i which is adapted to the trivial parabolic structure of (E_i,θ_i). Moreover, h coincides with $h_1 \oplus h_2$ up to some obvious ambiguity.

In the rest of the proof, any object which appears is restricted over $X - D$. Let us first enlarge the structure group of P by defining $P_{GL(V)} := P \times_K GL(V)$ via the faithful representation $\rho|_K : K \to GL(V)$. This is the holomorphic principal (frame) bundle associated to E. We can consider $P = P \times_K K \subset P_{GL(V)}$ as a (metric) reduction of $P_{GL(V)}$. The metric h for E gives rise to a reduction $P_{U(E,h)}$ of $P_{GL(V)}$ with the structure group $U(V,h_V)$. Indeed, note that

$$E = P_{GL(V)} \times_{GL(V)} V$$

and thus the metric h for E induces a family of hermitian metrics h_e for V parametrized by $e \in P_{GL(V)}$. It has the obvious relation $h_{e\cdot g} = g^* h_e$ for any $g \in GL(V)$. We define

$$P_{U(E,h)} := \{e \in P_{GL(V)} \mid h_e = h_V\} \tag{3.0.2}$$

and it is obvious that if $e \in P_{U(E,h)}$, then $e \cdot g \in P_{U(E,h)}$ if and only if $g \in U(V,h_V)$. Hence the structure group of $P_{U(E,h)}$ is $U(V,h_V)$.

Let us define $P_H := P \cap P_{U(E,h)}$ whose structure group is $U(V,h_V) \cap K \supset K_0$ by (2.0.7). Since ρ is faithful, one has moreover $U(V,h_V) \cap K = K_0$. Indeed, this easily follows from that

$$K = \{\exp(\sqrt{-1}\eta)k \mid k \in K_0, \eta \in \mathfrak{k}_0 \subset \mathrm{Lie}(U(h,h_V))\}$$

and that

$$\sqrt{-1}\mathfrak{k}_0 \cap \mathrm{Lie}(U(h,h_V)) = \{0\}.$$

Obviously, if we follow Lemma 2.9 to define a new metric h' for E by setting

$$h'((p,u),(p,v)) := h_V(u,v)$$

for any $p \in P_H$ and for any $u, v \in V$, then

$$h' = h \tag{3.0.3}$$

by (3.0.2). We shall prove that $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$ following the elegant arguments in [Mau15, Proposition 3.7].

Let $A \in \mathscr{C}^\infty(P_{GL(V)}, T^*_{P_{GL(V)}} \otimes \mathfrak{gl}(V))$ be the Chern connection 1-form induced by the Chern connection d_h for (E,h). Since $T^{a,b}(E,h) = (E_1,h_1) \oplus (E_2,h_2)$, by (3.0.1), when we take a base point $p \in P \subset P_{GL(V)}$, the holonomy $Hol(p,\gamma)$ with respect to the connection A along any smooth loop γ based at $\pi(p)$ lies at $p \cdot K$, where we denote $\pi : P \to X$. Hence the restriction of A to P is 1-form with values in $\mathfrak{k}$. In other words, A is induced by a connection on P.

On the other hand, by the definition of the Chern connection, A is also induced by a connection on $P_{U(E,h)}$. Since $\mathfrak{k}_0 = \mathfrak{k} \cap \mathrm{Lie}(U(V,h_V))$ where $\mathrm{Lie}(U(V,h_V))$ denotes the Lie algebra of $U(V,h_V)$, there is a connection $A_0 \in \mathscr{C}^\infty(P_H, T^*_{P_H} \otimes \mathfrak{k}_0)$ for the smooth principal K_0-fiber bundle $P_H := P_{U(E,h)} \cap P$ which induces A. A_0 is moreover the Chern connection with respect to the reduction P_H of P by the construction. Let us define

$F_H \in \mathscr{A}^{1,1}(P \times_K \mathfrak{g})$ to be the curvature of the connection $A_0 + \tau + \overline{\tau}_H$. Recall that one has $\theta = d\rho(\tau)$ and $\overline{\theta}_h = d\rho(\overline{\tau}_H)$. Hence

$$d\rho(F_H) = (d_h + \theta + \overline{\theta}_h)^2 = F_h(E) = 0 \tag{3.0.4}$$

where d_h is the Chern connection for (E, h). Since $\rho : G \to GL(V)$ is faithful, $d\rho : \mathfrak{g} \to \mathrm{End}(V)$ is thus injective. By (3.0.4) this implies that $F_H = 0$. In conclusion, $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$. □

As a consequence of Theorem 3.1, we can give a partial characterization of hermitian symmetric spaces, which partially extends Simpson's characterization of hermitian symmetric spaces [Sim88, Theorem 2] to the log setting.

Corollary 3.2. *Let (X, D) be a projective log pair endowed with an ample polarization L. Let (P, τ) be a principal system of log Hodge bundles on (X, D) with G* centerless. *Assume that the system of log Hodge bundle $(P \times_K \mathfrak{g}, d(Ad)(\tau))$ via the* faithful *Hodge representation $Ad : G \hookrightarrow GL(\mathfrak{g})$ in Example 2.8 is μ_L-polystable with $c_2(P \times_K \mathfrak{g}) = 0$. Then there is a metric reduction P_H for $P|_{X-D}$ so that the triple $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a* principal variation of Hodge structures *on $X - D$. When (P, τ) is further assumed to be a uniformizing bundle, the period map $f : \widetilde{X - D} \to G_0/K_0$ defined in (2.0.3) from the universal cover $\widetilde{X - D}$ of $X - D$ to the hermitian symmetric space G_0/K_0 is* locally biholomorphic.

We further conjecture that the above period map is moreover an isomorphism when (P, τ) is the uniformizing bundle, namely, the universal cover of $X - D$ is the hermitian symmetric space G_0/K_0.

4. Uniformization of quasi-projective manifolds by unit balls

This section is devoted to the proof of Theorem A. In § 4.2 we shall prove a basic result for the extension of plurisubharmonic functions. This lemma will be used in the proof of Theorem A. We shall also give an application of this fact in Hodge theory: we can give a much simpler proof of the negativity of kernel of Higgs fields for tame harmonic bundles originally proven by Brunebarbe [Bru17] (see also [Zuo00] for systems of log Hodge bundles). With all the tools developed above, we are able to prove Theorem A in § 4.3.

4.1. **Adaptedness to log order and acceptable metrics.** We recall some notions in [Moc07, §2.2.2]. Let X be a $\mathscr{C}^\infty$-manifold, and E be a $\mathscr{C}^\infty$-vector bundle with a hermitian metric h. Let $\mathbf{v} = (v_1, \ldots, v_r)$ be a $\mathscr{C}^\infty$-frame of E. We obtain the $H(r)$-valued function $H(h, \mathbf{v})$,whose (i, j)-component is given by $h(v_i, v_j)$.

Let us consider the case $X = \Delta^n$, and $D = \sum_{i=1}^{\ell} D_i$ with $D_i = (z_i = 0)$. We have the coordinate $(z_1, \ldots, z_n)$. Let h, E and $\mathbf{v}$ be as above.

A frame $\mathbf{v}$ is called *adapted up to log order*, if the following inequalities hold over $X - D$

$$C^{-1}(-\sum_{i=1}^{\ell} \log |z_i|)^{-M} \le H(h, \mathbf{v}) \le C(-\sum_{i=1}^{\ell} \log |z_i|)^{M}$$

for some positive numbers M and C.

Definition 4.1. Let (X, D) be a log pair, and let E be a holomorphic vector bundle on X. A hermitian metric h for $E|_{X-D}$ is *adapted to log order* if for any point $x \in D$, there is an admissible coordinate $(U; z_1, \ldots, z_n)$, a holomorphic frame $\mathbf{v}$ for $E|_U$ which is adapted up to log order.

Definition 4.2 (Acceptable metric)**.** Let (X, D) be a log pair and let (E, θ) be a log Higgs bundle over (X, D). We say that the metric h for $E|_{X-D}$ is acceptable at $p \in D$, if the following holds: there is an admissible coordinate $(U; z_1, \ldots, z_n)$ around p, so that the norm $|F_h|_{h,\omega_P} \leq C$ for some $C > 0$ over $U - D$. When (E, θ, h) is acceptable at any point p of D, it is called acceptable. Such triple (E, θ, h) is called an *acceptable bundle* on (X, D).

One can easily check that acceptable metrics and adaptedness to log order defined above are invariant under bimeromorphic transformations.

Lemma 4.3. *Let (X, D) be a log pair, and let $\mu : \tilde{X} \to X$ be a bimeromorphic morphism so that $\mu^{-1}(D) = \tilde{D}$. For a log Higgs bundle (E, θ) over (X, D), one can define a log Higgs bundle $(\tilde{E}, \tilde{\theta})$ on $(\tilde{X}, \tilde{D})$ by setting $\tilde{E} = \mu^* E$ and $\tilde{\theta}$ to be the composition*

$$\mu^* E \xrightarrow{\mu^*\theta} \mu^*(E \otimes \Omega^1_X(\log D)) \to \mu^* E \otimes \Omega^1_{\tilde{X}}(\log \tilde{D}).$$

If the metric h for $(E, \theta)|_{X-D}$ is acceptable or adapt to log order, so is the metric $\mu^* h$ for $(\tilde{E}, \tilde{\theta})|_{\tilde{X}-\tilde{D}}$.

Proof. Since this is a local statement, we work on the local models. Pick a point $\tilde{x} \in \tilde{D}$ with an admissible coordinate $(U; z_1, \ldots, z_n)$ with $\tilde{D} = (z_1 \cdots z_\ell = 0)$ locally and take an admissible coordinate $(V; y_1, \ldots, y_n)$ for $\mu(\tilde{x})$ with $D = (y_1 \cdots y_m = 0)$ such that $\mu(U) \Subset V$. Then for $i = 1, \ldots, m$, $\mu^* y_i = \prod_{j=1}^{\ell} z_j^{a_{ij}}$ with $a_{ij} \in \mathbb{Z}_{\geq 0}$ and $\sum_{j=1}^{\ell} a_{ij} > 0$. One has

$$\mu^* \log(-|y_i|^2) = \sum_{j=1}^{\ell} 2a_{ij} \log(-|z_j|^2).$$

Therefore, if h is adapted to log order, so is $\mu^* h$.

Let ω_1 and ω_2 be Poincaré metrics on U and V. One can easily show that

$$C\omega_1 \geq \mu^* \omega_2 \tag{4.1.1}$$

for some constant $C > 0$. Note that

$$\mu^* F_h(E) = F_{\tilde{h}}(\tilde{E})$$

Hence

$$|F_{\tilde{h}}(\tilde{E})|^2_{\tilde{h},\omega_1} = |\mu^* F_h(E)|^2_{\mu^* h,\omega_1} \leq \frac{1}{C} |\mu^* F_h(E)|^2_{\mu^* h,\mu^*\omega_2} = \mu^* \frac{1}{C} |F_h(E)|^2_{h,\omega_2}$$

In conlusion, if the metric h for $(E, \theta)|_{X-D}$ is acceptable, so is the metric $\mu^* h$ for $(\tilde{E}, \tilde{\theta})|_{\tilde{X}-\tilde{D}}$. □

4.2. **Extension of psh functions and negativity of kernel of Higgs fields.** In this subsection we shall prove a result on the extension of plurisubharmonic (psh for short) functions, which will be used in the proof of Theorem A and Proposition 5.6. As a byproduct, we give a very simple proof of the negativity of kernels of Higgs fields of tame harmonic bundles by Brunebarbe [Bru17, Theorem 1.3], which generalizes the earlier work by Zuo [Zuo00] for system of log Hodge bundles.

Lemma 4.4. *Let $X = \Delta^n$, and $D = \sum_{i=1}^{\ell} D_i$ with $D_i = (z_i = 0)$. Let φ be a psh function on X^*. We assume that for any $\delta > 0$, there is a positive constant C_δ so that*

$$\varphi(z) \leq \delta \sum_{j=1}^{\ell} (-\log |z_j|^2)) + C_\delta$$

on X^*. Then φ extends uniquely *to a psh function on X.*

Proof. Define $\varphi_\varepsilon := \varphi + \varepsilon \sum_{j=1}^{\ell}(\log |z_j|^2)$ for any $\varepsilon > 0$. Then for each $\varepsilon > 0$, φ_ε is locally bounded from above, which thus extends to a psh $\tilde{\varphi}_\varepsilon$ on the whole X by the well-known fact in pluripotential theory. By the maximum principle, for any $0 < r < 1$, there is a point $\xi_\varepsilon \in S(0,r) \times \cdots S(0,r)$ so that

$$\sup_{z\in\Delta(0,r)\times\cdots\times\Delta(0,r)} \varphi_\varepsilon(z) \leq \varphi_\varepsilon(\xi_\varepsilon) \leq \varphi(\xi_\varepsilon)$$

where $S(0,r) := \{z \in \Delta \mid |z| = r\}$. Note that the compact set $S(0,r) \times \cdots S(0,r)$ is contained in $X - D$. Since φ is psh on $X - D$, there exists $z_0 \in S(0,r) \times \cdots S(0,r)$ so that

$$\sup_{z\in S(0,r)\times\cdots S(0,r)} \varphi(z) \leq \varphi(z_0) < +\infty.$$

Hence φ_ε is *uniformly* locally bounded from above.

We define the *upper envelope*

$$\tilde{\varphi} := \sup_{\varepsilon>0} \tilde{\varphi}_\varepsilon,$$

and define the *upper semicontinuous regularization* of $\tilde{\varphi}$ by

$$\tilde{\varphi}^\star(x) := \lim_{\delta\to 0^+} \sup_{\mathbb{B}(x,\delta)} \tilde{\varphi}(z).$$

where $\mathbb{B}(x,\delta)$ is the unit ball of radius δ centered at x. Then by the well-known result in pluripotential theory [Dem12b, Chapter 1, Theorme 5.7], $\tilde{\varphi}^\star$ is a psh function on X. By our construction, $\tilde{\varphi}^\star(z) = \varphi(z)$ on $X - D$. This proves our result. □

A direct consequence of the above lemma is the following extension theorem of positive currents.

Lemma 4.5. *Let (X, D) be a log pair and let L be a line bundle on X. Assume that h is a smooth hermitian metric for $L|_{X-D}$, which is adapted to log order. Assume further that the curvature form $\sqrt{-1}R_h(L|_{X-D}) \geq 0$. Then h extends to a singular hermitian metric $\tilde{h}$ for L with zero Lelong numbers so that the curvature current $\sqrt{-1}R_{\tilde{h}}(L)$ is closed and positive. In particular, L is a nef line bundle.*

Let us show how to apply Lemma 4.4 to reprove the negativity of kernels of Higgs fields of tame harmonic bundles.

Theorem 4.6 (Brunebarbe)**.** *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let (E, θ, h) be a tame harmonic bundle on $X - D$, and let $({}^\diamond E, \theta)$ be the prolongation defined in § 1.3. Let $\mathcal{F}$ be any coherent torsion free subsheaf of ${}^\diamond E$ which lies in the kernel of the Higgs field $\theta : {}^\diamond E \to {}^\diamond E \otimes \Omega_X^1(\log D)$, namely $\theta(\mathcal{F}) = 0$. Then*

(i) *the singular hermitian metric $h|_{\mathcal{F}}$ for $\mathcal{F}$, is* semi-negatively curved *in the sense of [PT18, Definition 2.4.1].*
(ii) *The dual $\mathcal{F}^*$ of $\mathcal{F}$ is* weakly positive *over $X^\circ - D$ in the sense of Viehweg, where $X^\circ \subset X$ is the Zariski open set so that $\mathcal{F}|_{X^\circ} \to {}^\diamond E|_{X^\circ}$ is a subbundle.*
(iii) *If the harmonic metric h is adapted to log order and $\mathcal{F}$ is a subbundle of ${}^\diamond E$ so that $\theta(\mathcal{F}) = 0$, then the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)$ admits a singular hermitian metric g with zero Lelong numbers so that the curvature current $\sqrt{-1}R_g(\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)) \geq 0$; in particular, $\mathcal{F}^*$ is a nef vector bundle.*

Proof. By [PT18, Definition 2.4.1], it suffices to prove that for any open set U and any $s \in \mathcal{F}(U)$, $\log |s|_h^2$ extends to a psh function on U. Pick any point $x \in D$. By the definition of ${}^\diamond E$ in (1.3.1), for any $\delta > 0$, there are an admissible coordinate $(U; z_1, \ldots, z_n)$

centered at x, and a positive constant C_δ so that

$$\log|s|_h^2 \leq \delta \sum_{j=1}^{\ell}(-\log|z_j|^2)) + C_\delta$$

on $U - D$. Recall that $R_h(E) + [\theta, \overline{\theta}_h] = F_h(E) = 0$. We have

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial}\log|s|_h^2 &= -\frac{\sqrt{-1}\{R_h(E)s, s\}}{|s|_h^2} + \frac{\sqrt{-1}\{\partial_h s, \partial_h s\}}{|s|_h^2} - \sqrt{-1}\frac{\{\partial_h s, s\}}{|s|_h^2} \wedge \frac{\{s, \partial_h s\}}{|s|_h^2} \\
&\geq -\frac{\sqrt{-1}\{R_h(E)s, s\}}{|s|_h^2} \\
&= -\frac{\sqrt{-1}\{\theta s, \theta s\}}{|s|_h^2} - \frac{\sqrt{-1}\{\overline{\theta}_h s, \overline{\theta}_h s\}}{|s|_h^2} \\
&= -\frac{\sqrt{-1}\{\overline{\theta}_h s, \overline{\theta}_h s\}}{|s|_h^2} \geq 0.
\end{aligned}$$

over $X - D$. Hence $\log|s|_h^2$ is a psh function on $X - D$. By Lemma 4.4, we conclude that $\log|s|_h^2$ extends to a psh function on U. This proves that $(\mathcal{F}, h)$ is negatively curved in the sense of Păun-Takayama.

The metric h induces a negatively curved singular hermitian metric h_1 (in the sense of [PT18, Definition 2.2.1]) on the subbundle $\mathcal{F}|_{X^\circ}$. By Lemma 4.5, h_1 induces a singular metric g for the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*|_{X^\circ})}(1)$ so that $\sqrt{-1}R_g(\mathcal{O}_{\mathbb{P}(\mathcal{F}^*|_{X^\circ})}(1)) \geq 0$. Note that $X - X^\circ$ is a codimension at least two subvariety. The second statement then follows from Hörmander's L^2-techniques in [PT18, Proof of Theorem 2.5.2].

Let us prove the last statement. Since $\mathcal{F}$ is a subbundle of ${}^\diamond E$, one has $X^\circ = X$. Since h is assumed to be adapted to log order, the singular hermitian metric g for $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)$ thus has zero Lelong numbers everywhere. This implies the nefness of the vector bundle $\mathcal{F}^*$. □

Remark 4.7. In [Zuo00] Zuo proved the above statement when (E, θ, h) is moreover a system of log Hodge bundles with unipotent monodromies around the boundary (see also [FF17] for a refined result). Theorem 4.6 is proved by Brunebarbe in [Bru17]. Both their proofs made use of the monodromy filtration to obtain a precise estimate of the Hodge metric so that they can show that $\log|s|_h^2$ is locally bounded from above near D. Here we give a much more simplified proof which uses the very definitions of tame harmonic bundles and the prolongation of the tame harmonic bundles.

A special case of Theorem 4.6.(iii) comes from the complex variation of Hodge structures. For the complex variation of Hodge structures defined over $X - D$ with unipotent monodromies around D, the Hodge metric for the associated system of Hodge bundles is a harmonic metric which is adapted to log order by [CKS86] or [Moc02, Lemma 4.15]. Hence Theorem 4.6.(iii) also generalizes [FF17, Corollary 1.6], whose proof relies on the very delicate analysis by Kollár [Kol87].

4.3. **Characterization of non-compact ball quotient.** Let us state and prove our first main theorem in this paper.

Theorem 4.8. *Let X be an n-dimensional complex projective manifold and let D be a simple normal crossing divisor on X. Let L be an ample polarization on X. For the log Hodge bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ on (X, D) with θ defined in (0.1.1), we assume that* *it is μ_L-polystable. Then one has the following inequality*

$$\big(2c_2(\Omega_X^1(\log D)) - \frac{n}{n+1}c_1(\Omega_X^1(\log D))^2\big)\cdot c_1(L)^{n-2} \geq 0. \tag{4.3.1}$$

When the above equality holds,

(i) *if D is smooth, then $X - D \simeq \mathbb{B}^n/\Gamma$ for some torsion free lattice $\Gamma \subset PU(n,1)$ acting on $\mathbb{B}^n$. Moreover, X is the (unique) toroidal compactification of $\mathbb{B}^n/\Gamma$, and each connected component of D is the* smooth *quotient of an Abelian variety A by a finite group acting freely on A.*

(ii) *If D is not smooth, then the universal cover $\widetilde{X-D}$ of $X - D$ is* not *biholomorphic to $\mathbb{B}^n$, though there exists a holomorphic map $\widetilde{X-D} \to \mathbb{B}^n$ which is locally biholomorphic.*

In both cases, $K_X + D$ is big, nef and ample over $X - D$.

Proof. Denote the log Hodge bundle $(E,\theta) = (E^{1,0}\oplus E^{0,1},\theta)$ by

$$E^{1,0} := \Omega_X^1(\log D), \quad E^{0,1} := \mathcal{O}_X.$$

By [Moc06, Theorem 6.5] we have the following Bogomolov-Gieseker inequality for (E,θ)

$$\begin{aligned}\big(2c_2(\Omega_X^1(\log X)) - \frac{n}{n+1}c_1(\Omega_X^1(\log D))^2\big)\cdot c_1(L)^{n-2} &= \\ \big(2c_2(E) - \frac{\operatorname{rank} E - 1}{\operatorname{rank} E}c_1(E)^2\big)\cdot c_1(L)^{n-2} &\geq 0\end{aligned} \tag{4.3.2}$$

This shows the desired inequality (4.3.1).

The rest of the proof will be divided into three steps. In Step 1, we shall construct a uniformizing variation of Hodge structures on $X - D$ so that the corresponding period map defined in (2.0.3) induces a holomorphic map (so-called *period map* in Remark 2.6) from the universal cover of $X - D$ to $\mathbb{B}^n$ which is locally biholomorphic. By (2.0.5), this period map is moreover an *isometry* if we equip $X - D$ with hermitian metric induced by the Hodge metric. This proves Theorem 4.8.(ii). In Step two we will prove that, when D is smooth, the hermitian metric on $X - D$ induced by the Hodge metric is *complete*. Together with arguments in Remark 2.6, this proves that the above period map is indeed a biholomorphism. In Step three we shall prove Theorem 4.8.(ii) and the positivity of $K_X + D$.

Step 1. By Proposition 2.10, there is a *canonical* principal system of log Hodge bundles (P,τ) on (X,D) with the structure group $K = P(GL(V^{1,0})\times GL(V^{0,1}))$, and the Hodge group $G_0 = PU(n,1)$. Here $(V = V^{1,0}\oplus V^{0,1}, h_V)$ is a polarized Hodge structure with $\operatorname{rank} V^{1,0} = n$ and $\operatorname{rank} V^{0,1} = 1$. For the complexified group $G = PGL(V)$ of G_0, there is a faithful representation $\rho : G \to GL(V\otimes V^*)$, which is moreover a *Hodge representation* in the sense of Definition 2.7 when we equip $V\otimes V^*$ the induced polarized Hodge structure from $(V = V^{1,0}\oplus V^{0,1}, h_V)$.

By Lemma 2.9, such Hodge representation ρ induces a system of log Hodge bundles $(P\times_\rho(V\otimes V^*), d\rho(\tau))$ over (X,D). By our construction, this system of log Hodge bundle is nothing but $(\operatorname{End}(E),\theta_{End(E)})$. An easy computation shows that $c_1(\operatorname{End}(E)) = 0$, and

$$\begin{aligned}ch_2(\operatorname{End}(E)) &= -2\operatorname{rank} E\cdot c_2(E) + (\operatorname{rank} E - 1)c_1(E)^2 \\ &= nc_1^2(K_X+D) - 2(n+1)c_2(\Omega_X^1(\log D)) = 0\end{aligned}$$

since the equality in (4.3.2) holds by our assumption. Since we assume that (E,θ) is μ_L-polystable, by Theorem 1.11, $(\operatorname{End}(E),\theta_{End(E)})$ is also μ_L-polystable. We now apply Proposition 1.16 to find a Hodge metric h for the system of log Hodge bundle $(\mathrm{End}(E)|_{X-D}, \theta_{End(E)}|_{X-D})$ which is adapted to the trivial parabolic structure of $(\mathrm{End}(E), \theta_{End(E)})$. Using the Tannakian arguments in Theorem 3.1, we conclude that h induces a reduction P_H for $P|_{X-D}$ with the structure group $K_0 = P(U(n) \times U(1)) \simeq U(n)$, which is compatible with h such that $(P|_{X-D}, \tau|_{X-D}, P_H)$ is a principal variation of Hodge structures on $X - D$. Note that

$$T_X(-\log D) \xrightarrow{\tau} P \times_K \mathfrak{g}^{-1,1} = \mathrm{Hom}(E^{1,0}, E^{0,1}) \simeq \mathrm{Hom}(\Omega^1_X(\log D), \mathcal{O}_X)$$

is an isomorphism. Hence $(P|_{X-D}, \tau|_{X-D}, P_H)$ is moreover a *uniformizing variation of Hodge structures* over $X - D$ in the sense of Definition 2.5. By Remark 2.6, it gives rise to a holomorphic map, the so-called period map,

$$\widetilde{X - D} \to G_0 / K_0 = PU(n,1) / U(n) \simeq \mathbb{B}^n \tag{4.3.3}$$

defined in (2.0.3), which is locally *biholomorphic*. Here $\widetilde{X - D}$ is the *universal cover* of $X - D$.

Note that the reduction P_H together with the hermitian metric $h_{\mathfrak{g}}$ in (2.0.1) gives rise to a natural metric h_H over $P \times_K \mathfrak{g}^{-1,1}|_{X-D}$ defined in (2.0.4). By Remark 2.6 again, if the pull back $\tau^* h_H$ is a *complete metric* on $X - D$, then $X - D$ is uniformized by $G_0 / K_0 = PU(n,1) / U(n)$ which is the complex unit ball of dimension n, denoted by $\mathbb{B}^n$. The rest of the proof is devoted to show the completeness of $\tau^* h_H$.

From the following commutative diagram

$$\begin{tikzcd}
G = PGL(V) \arrow[dr, "\rho"] & \\
GL(V) \arrow[u, "p"] \arrow[r, "Ad"] & GL(\mathfrak{gl}(V))
\end{tikzcd}$$

and the fact that $\mathfrak{sl}(V)$ is invariant under Ad_g for any $g \in GL(V)$, we conclude that $\mathfrak{g} = \mathfrak{sl}(V)$ is an invariant subspace under $\rho(g)$ for any $g \in G$. Hence for the adjoint representation

$$G \xrightarrow{Ad} GL(\mathfrak{g}) = GL(\mathfrak{sl}(V)),$$

one has

$$\rho(g)|_{\mathfrak{g}} = Ad_g \in GL(\mathfrak{g}).$$

Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(E^{1,0}, E^{0,1}) \arrow[r, hook] \arrow[d, equal] \arrow[rr, bend left, "j"] & \mathrm{End}(E)^{\perp} \arrow[r, hook] \arrow[d, equal] & \mathrm{End}(E) \arrow[d, equal] \\
P \times_K \mathfrak{g}^{-1,1} \arrow[r, hook] & P \times_K \mathfrak{g} \arrow[r, hook] & P \times_\rho \mathfrak{gl}(V)
\end{tikzcd} \tag{4.3.4}$$

where $\mathrm{End}(E)^{\perp}$ is the trace-free subbundle of $\mathrm{End}(E)$.

It follows from (3.0.3) that the Hodge metric h for $(\mathrm{End}(E)|_{X-D}, \theta_{End(E)}|_{X-D}) \simeq (P \times_\rho (V \otimes V^*), d\rho(\tau))$ can be redefined via the reduction P_H together with the hermitian metric $h_{End(V)}$ of $\mathrm{End}(V)$ induced by (V, h_V) as in (2.0.4). Recall that in Example 2.1, for the natural inclusion $\iota : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, one has $h_{\mathfrak{g}} = 2(n+1) \cdot \iota^* h_{End(V)}$. By (4.3.4), one has

$$2(n+1) j^* h = h_H,$$

where we recall that h_H is the metric over $P \times_K \mathfrak{g}^{-1,1}|_{X-D}$ induced by the reduction P_H together with the hermitian metric $h_{\mathfrak{g}}$ in (2.0.1). It now suffices to show that $\tau^* h$ is complete if we want to prove that $X - D$ is uniformized by $\mathbb{B}^n$. In next step, we will apply similar ideas by Simpson [Sim90, Corollary 4.2] to prove this. Note that until now we made no assumption on the smoothness of D.

Step 2. Throughout Step 2, we will assume that D is smooth. Consider now the log Higgs bundle $(\mathcal{E}, \eta) := (\mathrm{End}(E), \theta_{End(E)})$. We first mention that the above Hodge metric h for $(\mathcal{E}, \eta)|_{X-D}$ is adapted to log order in the sense of Definition 4.1. Indeed, it follows from [Moc02, Corollary 4.9] that the eigenvalues of monodromies of the flat connection $D := \partial_h + \bar{\partial} + \eta + \overline{\eta}_h$ around the divisor D are 1. By the "weak" norm estimate in [Moc02, Lemma 4.15], we conclude that h is adapted to log order[2].

We first give an estimate for $\tau^* h$. For any point $x \in D$, consider an admissible coordinates $(U; z_1, \ldots, z_n)$ centered at x as Definition 1.3 so that $D \cap U = (z_1 = 0)$. To distinguish the sections of Higgs bundles and log forms, we write $e_1 := d \log z_1$ and $e_i = dz_i$ for $i = 2, \ldots, n$. Denote by $e_0 = 1$ the constant section of $\mathcal{O}_X$. Let us introduce a new metric $\tilde{h}$ on $(E, \theta)|_{U^*}$ as follows.

$$\begin{aligned} |e_1|^2_{\tilde{h}} &:= (-\log |z_1|^2) \\ \langle e_i, e_j \rangle_{\tilde{h}} &:= 0 \quad \text{for} \quad i \neq j; \\ |e_i|^2_{\tilde{h}} &:= 1 \quad \text{for} \quad i = 2, \ldots, n; \\ |e_0|^2_{\tilde{h}} &:= (-\log |z_1|^2)^{-1} \end{aligned}$$

Within this basis $\{e_1, \ldots, e_n, e_0\}$, θ can be expressed as

$$\theta = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ d \log z_1 & \cdots & dz_n & 0 \end{bmatrix}$$

Denote by $H := (h_{ij})_{0 \leq i,j \leq n}$ the metric matrix of $\tilde{h}$ with respect to the above basis. One has

$$\overline{\theta}_h = \overline{H}^{-1} \theta^* \overline{H} = \begin{bmatrix} 0 & \cdots & 0 & h_{11}^{-1} h_{00} \frac{d\bar{z}_1}{\bar{z}_1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & h_{nn}^{-1} h_{00} d\bar{z}_n \\ 0 & \cdots & 0 & 0 \end{bmatrix} \tag{4.3.5}$$

[2]Indeed, a strong norm estimate has already been obtained by Cattani-Kaplan-Schmid in [CKS86]. Here we only need to know that h is adapted to log order, which is a bit easier to obtain using Andreotti-Vesentini type results by Simpson [Sim90] and Mochizuki [Moc02, Lemma 4.15].

Hence for $2 \leq i \leq j \leq n$, one has

$$
\begin{aligned}
[\theta, \overline{\theta}_h]_{11} &= h_{11}^{-1} h_{00} \frac{d\bar{z}_1}{\bar{z}_1} \wedge \frac{dz_1}{z_1} \\
[\theta, \overline{\theta}_h]_{ij} &= h_{ii}^{-1} h_{00} d\bar{z}_i \wedge dz_j \\
[\theta, \overline{\theta}_h]_{i1} &= h_{ii}^{-1} h_{00} d\bar{z}_i \wedge \frac{dz_1}{z_1} \\
[\theta, \overline{\theta}_h]_{1i} &= h_{11}^{-1} h_{00} \frac{d\bar{z}_1}{\bar{z}_1} \wedge dz_i \\
[\theta, \overline{\theta}_h]_{00} &= h_{11}^{-1} h_{00} \frac{dz_1}{z_1} \wedge \frac{d\bar{z}_1}{\bar{z}_1} + \sum_{i=2}^{n} h_{ii}^{-1} h_{00} dz_i \wedge d\bar{z}_i.
\end{aligned}
$$

Write $F_{\tilde{h}}(E) := F_{\tilde{h}}(E)_{kj} \otimes e_j^* \otimes e_k$. Then for $i, j = 2, \ldots, n$, one has

$$
\begin{aligned}
F_{\tilde{h}}(E)_{11} &= F_{\tilde{h}}(E)_{10} = F_{\tilde{h}}(E)_{01} = F_{\tilde{h}}(E)_{0i} = F_{\tilde{h}}(E)_{j0} = 0 \\
F_{\tilde{h}}(E)_{ij} &= (-\log |z_1|^2)^{-1} d\bar{z}_i \wedge dz_j \\
F_{\tilde{h}}(E)_{1i} &= \frac{1}{(-\log |z_1|^2)^2 \bar{z}_1} d\bar{z}_1 \wedge dz_i \\
F_{\tilde{h}}(E)_{i1} &= \frac{1}{(-\log |z_1|^2) z_1} d\bar{z}_i \wedge dz_1 \\
F_{\tilde{h}}(E)_{00} &= \sum_{i=2}^{n} (-\log |z_1|^2)^{-1} dz_i \wedge d\bar{z}_i.
\end{aligned}
$$

In conclusion, there is a constant $C_1 > 0$ so that one has

$$
|F_{\tilde{h}}(E)|^2_{h,\omega_e} = \sum_{0 \leq j,k \leq n} |F_{\tilde{h}}(E)_{kj} \otimes e_j^* \otimes e_k|^2_{h,\omega_e} \leq \frac{C_1}{(-\log |z_1|^2)^3 |z_1|^2} \tag{4.3.6}
$$

over $U^*(\frac{1}{2})$ (notation defined in Definition 1.3), where $\omega_e = \sqrt{-1} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ denotes the Euclidean metric on U^*.

We abusively denote by $\tilde{h}$ the induced metric on $(\mathcal{E}, \eta)|_{U^*} := (\mathrm{End}(E), \theta_{End(E)})|_{U^*}$, which is adapted to log order on $(U, D \cap U)$ in the sense of Definition 4.1 by our construction. Then

$$
\begin{aligned}
F_{\tilde{h}}(\mathcal{E}) &= F_{\tilde{h}}(E) \otimes \mathbb{1}_{E^*} + \mathbb{1}_E \otimes F_{\tilde{h}^*}(E^*) \\
&= F_{\tilde{h}}(E) \otimes \mathbb{1}_{E^*} - \mathbb{1}_E \otimes F_{\tilde{h}}(E)^\dagger
\end{aligned}
$$

where $F_{\tilde{h}}(E)^\dagger$ is the transpose of $F_{\tilde{h}}(E)$. Hence

$$
F_{\tilde{h}}(\mathcal{E})(e_i \otimes e_j^*) = \sum_{k,\ell} (\delta_{j\ell} F_{\tilde{h}}(E)_{ik} - \delta_{ik} F_{\tilde{h}}(E)_{\ell j})(e_k \otimes e_\ell^*)
$$

for $0 \leq i, j, k, \ell \leq n$. It then follows from (4.3.6) that

$$
|F_{\tilde{h}}(\mathcal{E})|^2_{h,\omega_e} \leq \frac{C_2}{(-\log |z_1|^2)^3 |z_1|^2} \tag{4.3.7}
$$

over $U^*(\frac{1}{2})$ for some constant $C_2 > 0$. Consider the identity map s for $\mathcal{E}$, which can be seen as a holomorphic section of $\mathrm{End}(\mathcal{E}, \mathcal{E})$. We denote by $(\mathcal{F}, \Phi) := (\mathrm{End}(\mathcal{E}, \mathcal{E}), \eta_{End(\mathcal{E})})$

the induced Higgs bundle by $(\mathcal{E},\eta)$. Note that for any section e of $\mathcal{E}$, one has

$$\begin{aligned}0&=(\bar\partial_{\mathcal{E}}+\eta)(s(e))-s\big((\bar\partial_{\mathcal{E}}+\eta)(e)\big)\\&=\big((\bar\partial_{\mathcal{F}}+\Phi)(s)\big)(e)\\&=\Phi(s)(e).\end{aligned}$$

Hence

$$\Phi(s)=0. \tag{4.3.8}$$

We equip $\mathcal{F}|_{U^*}$ with the metric $h_{\mathcal{F}}:=\tilde h\otimes h^*$, where h is the harmonic metric constructed in Step one. Note that

$$\begin{aligned}F_{h_{\mathcal{F}}}(\mathcal{F})&=F_{\tilde h}(\mathcal{E})\otimes \mathbb{1}_{\mathcal{E}^*}+\mathbb{1}_{\mathcal{E}}\otimes F_{h^*}(\mathcal{E}^*)\\&=F_{\tilde h}(\mathcal{E})\otimes \mathbb{1}_{E^*}\end{aligned}$$

By (4.3.6), there is a constant $C_0>0$ so that one has

$$|F_{h_{\mathcal{F}}}(\mathcal{F})|_{h_{\mathcal{F}},\omega_e}\leq \frac{C_0}{(-\log|z_1|^2)^{\frac32}|z_1|} \tag{4.3.9}$$

over $U^*(\frac12)$. Then

$$\begin{aligned}\sqrt{-1}\partial\bar\partial\log|s|^2_{h_{\mathcal{F}}}&=-\frac{\sqrt{-1}\{R_{h_{\mathcal{F}}}s,s\}}{|s|^2_{h_{\mathcal{F}}}}+\frac{\sqrt{-1}\{\partial_{h_{\mathcal{F}}}s,\partial_{h_{\mathcal{F}}}s\}}{|s|^2_{h_{\mathcal{F}}}}-\sqrt{-1}\frac{\{\partial_{h_{\mathcal{F}}}s,s\}}{|s|^2_{h_{\mathcal{F}}}}\wedge\frac{\{s,\partial_{h_{\mathcal{F}}}s\}}{|s|^2_{h_{\mathcal{F}}}}\\&\geq-\frac{\sqrt{-1}\{R_{h_{\mathcal{F}}}s,s\}}{|s|^2_{h_{\mathcal{F}}}}\\&=-\frac{\sqrt{-1}\{\Phi s,\Phi s\}}{|s|^2_{h_{\mathcal{F}}}}-\frac{\sqrt{-1}\{\overline{\Phi}_{h_{\mathcal{F}}}s,\overline{\Phi}_{h_{\mathcal{F}}}s\}}{|s|^2_{h_{\mathcal{F}}}}+\frac{\sqrt{-1}\{F_{h_{\mathcal{F}}}(\mathcal{F})s,s\}}{|s|^2_{h_{\mathcal{F}}}}\\&=-\frac{\sqrt{-1}\{\overline{\Phi}_{h_{\mathcal{F}}}s,\overline{\Phi}_{h_{\mathcal{F}}}s\}}{|s|^2_{h_{\mathcal{F}}}}+\frac{\sqrt{-1}\{F_{h_{\mathcal{F}}}(\mathcal{F})s,s\}}{|s|^2_{h_{\mathcal{F}}}}\\&\geq\frac{\sqrt{-1}\{F_{h_{\mathcal{F}}}(\mathcal{F})s,s\}}{|s|^2_{h_{\mathcal{F}}}}.\end{aligned}$$

Here the second inequality is due to Cauchy-Schwarz inequality, and the fourth one follows from (4.3.8). For any $\xi=(\xi_2,\ldots,\xi_n)$ with $0\leq\xi_2,\ldots,\xi_n\leq\frac12$, we define a smooth function f_ξ over Δ^* parametrized by ξ by

$$f_\xi(z_1):=\log|s|^2_{h_{\mathcal{F}}}(z_1,\xi_2,\ldots,\xi_n).$$

Then the above inequality together with (4.3.9) implies that

$$\Delta f_\xi\geq-|F_{h_{\mathcal{F}}}(\mathcal{F})|\geq-\frac{C_0}{(-\log|z_1|^2)^{\frac32}|z_1|}=:\varphi$$

where C_0 is some uniform constant which does not depend on ξ. Note that

$$\|\varphi\|_{L^2}:=\int_{0<|z_1|<\frac12}|\varphi(z_1)|^2dz_1d\bar z_1<C_4 \tag{4.3.10}$$

for some constant $C_4 > 0$. For any fixed $0 \leq \xi_2, \ldots, \xi_n \leq \frac{1}{2}$, consider the Dirichlet problem

$$\begin{cases} \phi = f_\xi & \text{on} \quad \{z_1 \mid |z_1| = \frac{1}{2}\} \\ \Delta\phi = \varphi & \text{on} \quad \{z_1 \mid 0 < |z_1| < \frac{1}{2}\} \end{cases} \tag{4.3.11}$$

By (4.3.10) and the elliptic estimate, one has

$$\sup_{0<|z_1|<\frac{1}{2}} |\phi(z_1)| \leq C_5(\|\varphi\|_{L^2} + \sup_{|z_1|=\frac{1}{2}} f_\xi). \tag{4.3.12}$$

over $\{z_1 \mid 0 < |z_1| < \frac{1}{2}\}$ for some uniform constant C_5 which does not depending on ξ. Hence $\Delta(f_\xi - \phi) \geq 0$ over $\{z_1 \mid 0 < |z_1| < \frac{1}{2}\}$. Since both h and $\tilde{h}$ are adapted to log order, so is $h_{\mathcal{F}}$. Hence there is a constant $C_6 > 0$ so that

$$\log |s|^2_{h_{\mathcal{F}}} \leq C_6 \log(-\sum_{i=1}^{\ell} \log |z_i|)$$

over $U^*(\frac{1}{2})$. By Lemma 4.4, we conclude that $f_\xi - \phi$ extends to a subharmonic function on $\{z_1 \mid |z_1| < \frac{1}{2}\}$. Note that $f_\xi(z_1) - \phi(z_1) = 0$ when $|z_1| = \frac{1}{2}$. Hence by maximum principle,

$$f_\xi(z_1) \leq \phi(z_1)$$

for any $0 < |z_1| < \frac{1}{2}$. Let

$$C_7 := \sup_{|z_1|=\frac{1}{2}, 0\leq\xi_2,\ldots,\xi_n\leq\frac{1}{2}} f_\xi(z_1)$$

which is finite. By (4.3.10) and (4.3.12), we have

$$\sup_{0<|z_1|<\frac{1}{2}, 0\leq z_2,\ldots,z_n\leq\frac{1}{2}} \log |s|^2_{h_{\mathcal{F}}}(z_1, \ldots, z_n) \leq C_5(C_4 + C_7).$$

This implies that $h \geq C_8 \cdot \tilde{h}$ over $U^*(\frac{1}{2})$ for some constant $C_8 > 0$. By (4.3.7), one has

$$|F_{\tilde{h}^*}(\mathcal{E}^*)|^2_{h^*,\omega_e} \leq \frac{C_0}{(-\log |z_1|^2)^3 |z_1|^2}.$$

Hence if we use the metric $h \otimes \tilde{h}^*$ for $\mathcal{F}$ and do the same proof, we can prove that $h \leq C_9 \cdot \tilde{h}$ over $U^*(\frac{1}{2})$ for some constant $C_9 > 0$. Therefore, $\tilde{h}$ and h are *mutually bounded* on $U^*(\frac{1}{2})$. By

$$\tau(z_1 \frac{\partial}{\partial z_1}) = e_1^* \otimes e_0 \tag{4.3.13}$$

$$\tau(\frac{\partial}{\partial z_j}) = e_j^* \otimes e_0 \quad \text{for} \quad j = 2, \ldots, n, \tag{4.3.14}$$

we obtain the norm estimate for the metric

$$\tau^* h \sim \tau^* \tilde{h} = \frac{\sqrt{-1} dz_1 \wedge d\bar{z}_1}{|z_1|^2 (\log |z_1|^2)^2} + \sum_{k=2}^{n} \frac{\sqrt{-1} dz_k \wedge d\bar{z}_k}{-\log |z_1|^2} \tag{4.3.15}$$

Though $\tau^* h$ is strictly less than the Poincaré metric near D, one can easily prove that it is still a *complete metric*. Therefore, the hermitian metric $\tau^* h_H = 2(n+1) \cdot \tau^* h$ on $X - D$ is also complete. Based on Remark 2.6, we conclude that $X - D$ is uniformized by the complex unit ball of dimension n, namely, there is a torsion free lattice $\Gamma \subset PU(n, 1)$ so that $X - D \simeq \mathbb{B}^n/\Gamma$. Since h is adapted to log order, by (4.3.13) and (4.3.14), the canonical Kähler-Einstein metric $\omega := \tau^* h$ for $T_X(-\log D)|_U$ is also adapted to log order. It follows from Theorem A.8 that X is the unique toroidal compactification for the non-compact ball quotient $\mathbb{B}^n/\Gamma$. We accomplish the proof of Theorem 4.8.(i).

Step 3. Assume now D is not smooth. By (4.3.3), the period map $\widetilde{X-D} \to \mathbb{B}^n$ is locally biholomorphic. Assume by contradiction that it is an isomorphism. As discussed above, the canonical Kähler-Einstein metric $\omega := \tau^* h$ for $T_X(-\log D)|_U$ is adapted to log order. It follows from Theorem A.8 that D cannot be singular. The contradiction is obtained, and thus the period map is not a uniformizing mapping. We proved Theorem 4.8.(ii).

Let us show that $K_X + D$ is big, nef and ample over $X - D$. Note that the metric $\det \omega^{-1}$ for $(K_X + D)|_U$ is adapted to log order, and that

$$R_{\det \omega^{-1}}((K_X + D)|_U) = (n+1)\omega.$$

By Lemma 4.5, the hermitian metric $\det \omega^{-1}$ extends to a singular hermitian metric h_{K_X+D} for K_X+D with zero Lelong numbers. Hence K_X+D is nef. Since $\sqrt{-1}R_{h_{K_X+D}}(K_X+D) > 0$ on $X - D$, $K_X + D$ is thus big and ample over $X - D$. We finish the proof of the theorem. □

Remark 4.9. Note that the asymptotic behavior of the metric (4.3.15) is exactly the same as that of the Kähler-Einstein metric for the ball quotient near the boundary of its toroidal compactification (see [Mok12, eq. (8) on p. 338]). This is indeed the hint for our construction of $\tilde{h}$.

Remark 4.10. We expect that Theorem 4.8.(ii) cannot happen. This is the case when $\dim X = 2$. Indeed, when the Miyaoka-Yau type equality in (0.1.2) holds, together with the conclusion that $K_X + D$ is big, nef and ample over $X - D$ in Theorem 4.8, it follows from [Kob85] that $X - D$ is uniformized by $\mathbb{B}^2$, which is a contradiction to Theorem 4.8.(ii).

5. Higgs bundles associated to non-compact ball quotients

In this section, we will prove Theorem B. §§ 5.1 and 5.2 are technical preliminaries. In § 5.3 we prove that a log Higgs bundle (E, θ) on a compact Kähler log pair is slope polystable with respect to some polarization by big and nef cohomology $(1,1)$-class, if (E, θ) admits a Hermitian-Yang-Mills metric with "mild singularity" near the boundary divisor. In § 5.4 we use the Bergman metric for quotients of complex unit balls by torsion free lattices to construct such Hermitian-Yang-Mills metric. This proves Theorem B.

5.1. **Notions of positivity for curvature tensors.** We recall some notions of positivity for Higgs bundles in [DH19, §1.3].

Let (E, θ) be a Higgs bundle endowed with a smooth metric h. For any $x \in X$, let $e_1, \ldots, e_r$ be a frame of E at x, and let $e^1, \ldots, e^r$ be its dual in E^*. Let $z_1, \ldots, z_n$ be a local coordinate centered at x. We write

$$F_h(E) = R_h(E) + [\theta, \overline{\theta}_h] = R^{\beta}_{j\bar{k}\alpha} dz_j \wedge d\bar{z}_k \otimes e^{\alpha} \otimes e_{\beta}$$

Set $R_{j\bar{k}\alpha\bar{\beta}} := h_{\gamma\bar{\beta}} R^{\gamma}_{j\bar{k}\alpha}$, where $h_{\gamma\bar{\beta}} = h(e_\gamma, e_\beta)$. $F_h(E)$ is called *Nakano semi-positive* at x if

$$\sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\bar{\beta}} u^{j\alpha} \overline{u^{k\beta}} \geq 0$$

for any $u = \sum_{j,\alpha} u^{j\alpha} \frac{\partial}{\partial z_j} \otimes e_\alpha \in (T_X^{1,0} \otimes E)_x$. (E, θ, h) is called Nakano semipositive if $F_h(E)$ is Nakano semi-positive at every $x \in X$. When $\theta = 0$, this reduces to the same positivity concepts in [Dem12b, Chapter VII, §6] for vector bundles.

We write

$$F_h(E) \geq_{\mathrm{Nak}} \lambda(\omega \otimes \mathbb{1}_E) \quad \text{ for } \lambda \in \mathbb{R}$$

if

$$\sum_{j,k,\alpha,\beta} (R_{j\bar{k}\alpha\bar{\beta}} - \lambda \omega_{j\bar{k}} h_{\alpha\bar{\beta}})(x) u^{j\alpha} \overline{u^{k\beta}} \geq 0$$

for any $x \in X$ and any $u = \sum_{j,\alpha} u^{j\alpha} \frac{\partial}{\partial z_j} \otimes e_\alpha \in (T_X^{1,0} \otimes E)_x$.

Let us recall the following lemma in [DH19, Lemma 1.8].

Lemma 5.1. *Let (E, θ, h) be a Higgs bundle on a Kähler manifold (X, ω). If there is a positive constant C so that $|F_h(x)|_{h,\omega} \leq C$ for any $x \in X$, then*

$$C\omega \otimes \mathbb{1}_E \geq_{Nak} F_h \geq_{Nak} -C\omega \otimes \mathbb{1}_E$$

The following easy fact in [DH19, Lemma 1.9] will be useful in this paper.

Lemma 5.2. *Let (E_1, θ_2, h_1) and (E_2, θ_2, h_2) are two metrized Higgs bundles over a Kähler manifold (X, ω) such that $|F_{h_1}(x)|_{h_1,\omega} \leq C_1$ and $|F_{h_2}(x)|_{h_2,\omega} \leq C_2$ for all $x \in X$. Then for the hermitian vector bundle $(E_1 \otimes E_2, h_1 h_2)$, one has*

$$|F_{h_1 \otimes h_2}(x)|_{h_1 \otimes h_2,\omega} \leq \sqrt{2r_2 C_1^2 + 2r_1 C_2^2}$$

for all $x \in X$. Here $r_i := \mathrm{rank} E_i$.

5.2. **Some pluripotential theories.** In this subsection we recall some results of deep pluripotential theories in [BEGZ10, Gue14]. The results in this subsection will be used in the proof of Proposition 5.6. Let us first recall the definitions of big or nef cohomology $(1, 1)$-classes in [Dem12a, §6].

Definition 5.3. Let (X, ω) be a compact Kähler manifold. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a cohomology $(1, 1)$-class of X. The class α is *nef* (numerically eventual free) if for any $\varepsilon > 0$, there is a smooth closed $(1, 1)$-form $\eta_\varepsilon \in \alpha$ so that $\eta_\varepsilon \geq -\varepsilon\omega$. The class α is *big* if there is a closed positive $(1, 1)$-current $T \in \alpha$ so that $T \geq \delta\omega$ for some $\delta > 0$. Such a current T will be called a *Kähler current.*

Let X be a complex manifold of dimension n and let $U \subset X$ be a Zariski open set of X. Pick a smooth hermitian form ω on X. For any smooth differential form η of degree p on U so that

$$\int_U |\eta|_\omega \wedge \omega^n < +\infty,$$

one can *trivially* extend η to *a current T_η on X of degree $n - p$* by setting

$$\langle T_\eta, u \rangle := \int_U \eta \wedge u \tag{5.2.1}$$

where u is the any *test form* of degree p which has compact support. In general, T_η might not be closed even if η is closed.

Let (X, ω) be a compact Kähler manifold of dimension n. Let $\alpha_1, \ldots, \alpha_n$ be big cohomology classes. Let $T_i \in \alpha_i$ be positive closed $(1, 1)$-currents whose local potential is locally bounded outside a closed analytic subvariety of X (a particular case of *small* *unbounded locus* of [BEGZ10, Definition 1.2]). In this celebrated work by Boucksom-Eyssidieux-Guedj-Zariahi [BEGZ10], they defined non-pluripolar product for these currents

$$\langle T_1 \wedge \cdots \wedge T_p \rangle$$

which is a closed positive (p, p)-current, and does not charge on any closed proper analytic subsets. Therefore, if we assume further that T_i is smooth over $X - A$ where A is a closed analytic subvariety of X, then $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is nothing but the trivial extension of the (p, p)-form $(T_1 \wedge \cdots \wedge T_p)|_{X-A}$ to X.

Following [BEGZ10, Definition 1.21], for a big class α, a positive $(1, 1)$-current $T \in \alpha$ has *full Monge-Ampère mass* if

$$\int_X \langle T_i^n \rangle = \mathrm{Vol}(\alpha).$$

The set of such positive currents in α with full Monge-Ampère mass is denoted by $\mathcal{E}(\alpha)$. We will not recall the definition of the *volume of big classes* by Boucksom in [Bou02]. We just mention that when the class α is big and nef, one has

$$\mathrm{Vol}(\alpha) = \alpha^n.$$

The following lemma will be used in § 5.3.

Lemma 5.4. *Let (X, ω) be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let S be a closed positive $(1, 1)$-current on X so that $S|_{X-D}$ is a smooth $(1, 1)$-form over $X - D$ which is strictly positive at one point and has at most* Poincaré growth *near D. Then the cohomology class $\alpha := \{S\}$ is big and nef, and $S \in \mathcal{E}(\alpha)$.*

Proof. Let T be the Kähler current on X constructed in Remark 1.5. Since $T|_{X-D}$ has at most Poincaré growth near D, there exists a constant $C_1 > 0$ so that

$$C_1 T - S \geq 0.$$

Pick any point $x \in D$. Then there exists some admissible coordinates $(U; z_1, \ldots, z_n)$ centered at x so that the local potential φ of S satisfies that

$$\varphi \geq -C_1 \log(-\prod_{i=1}^{\ell} \log |z_1|^2) - C_2$$

for some constant $C_2 > 0$. Hence S has zero Lelong numbers everywhere and thus α is nef. Since S is strictly positive at one point on $X - D$, it is big by [Bou02]. It follows from [Gue14, Proposition 2.3] that $S \in \mathcal{E}(\alpha)$. The lemma is proved. □

Let us recall an important theorem in [BEGZ10].

Theorem 5.5 ([BEGZ10, Corollary 2.15])**.** *Let (X, ω) be a compact Kähler manifold of dimension n. Let $\alpha_1, \ldots, \alpha_n$ be big and nef classes on X. For $T_i \in \mathcal{E}(\alpha_i)$ which are all smooth outside a closed proper analytic subset A, one has*

$$\int_{X-A} T_1 \wedge \cdots \wedge T_n = \int_X \langle T_1 \wedge \cdots \wedge T_n \rangle = \alpha_1 \cdots \alpha_n.$$

5.3. **Hermitian-Yang-Mills metric and stability.** Let (X, ω) be a compact Kähler manifold and let D be a simple normal crossing divisor on X. As we mentioned in § 0.4, for applications of birational geometry, one usually considers more general polarization by big and nef line bundles. In this subsection, we will prove that a log Higgs bundle (E, θ) on (X, D) is μ_α-polystable if $(E, \theta)|_{X-D}$ admits a Hermitian-Yang-Mills metric whose growth at infinity is "mild", where α is certain big and nef cohomology class. When $\dim X = 1$ or $D = \varnothing$ and the polarization is Kähler, this has been proved by Simpson [Sim88,Sim90]. As we have seen in Theorem 1.10, when X is projective and both the first and second Chern classes of E vanish and the polarization is an ample line bundle, this result has been proved by Mochizuki.

We start with the following technical result, which is strongly inspired by the deep result of Guenancia [Gue16, Proposition 3.8].

Proposition 5.6. *Let (X, ω_0) be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let (E, θ) be a log Higgs bundle on (X, D). Let α be a big and nef cohomology $(1, 1)$-class containing a positive closed $(1, 1)$-current $\omega \in \alpha$ so that $\omega|_{X-D}$ is a smooth Kähler form and has at most Poincaré growth near D. Assume that there is a hermitian metric h for $(E, \theta)|_{X-D}$ which is adapted to log order (in the sense of Definition 4.1) and is acceptable (in the sense of Definition 4.2). Then for any saturated Higgs subsheaf $G \subset E$, one has*

$$c_1(G) \cdot \alpha^{n-1} = \int_{X-D-Z} Tr(\sqrt{-1} R_{h_G}(G)) \wedge \omega^{n-1} \tag{5.3.1}$$

where Z is the analytic subvariety of codimension at least two so that $G|_{X-Z} \subset E|_{X-Z}$ is a subbundle, and h_G is the metric on G induced by h.

Proof. By Remark 1.5, one can construct a *Kähler current*

$$T := \omega_0 - \sqrt{-1}\partial\bar{\partial} \log(-\prod_{i=1}^{\ell} \log |\varepsilon \cdot \sigma_i|^2_{h_i}), \tag{5.3.2}$$

over X, whose restriction on $X - D$ is a complete Kähler form ω_P, which has the same Poincaré growth near D. Here σ_i is the section $H^0(X, \mathcal{O}_X(D_i))$ defining D_i, and h_i is some smooth metric for the line bundle $\mathcal{O}_X(D_i)$. Since we assume that h is acceptable, (after rescaling T by multiplying a constant) one thus has

$$|F_h(E)|_{h, \omega_P} \leq 1.$$

By Lemma 5.1, one has

$$-\mathbb{1} \otimes \omega_P \leq_{Nak} F_h(E) \leq_{Nak} \mathbb{1} \otimes \omega_P$$

over $X - D$.

We first consider the case that G is an invertible saturated subsheaf of E which is invariant under θ. Then the metric h of E induces a *singular hermitian metric* h_G for G defined on the whole X, which is smooth on on $X^\circ := X - D - Z$. The curvature current $\sqrt{-1} R_{h_G}(G)$ is a closed $(1, 1)$-current on $X - D$, which is a smooth $(1, 1)$-form on X°. Define by $\pi : E|_{X^\circ} \to G|_{X^\circ}$ the orthogonal projection with respect to h and $\pi^\perp : E|_{X^\circ} \to G^\perp|_{X^\circ}$ the projection to its orthogonal complement. By the Chern-Weil formula (see for example [Sim88, Lemma 2.3]), over X°, we have

$$R_{h_G}(G) = F_{h_G}(G) = F_h(E)|_G + \overline{\beta}_h \wedge \beta - \varphi \wedge \overline{\varphi}_h \tag{5.3.3}$$

where $F_h(E)|_G$ is the orthogonal projection of $F_h(E)$ on $\mathrm{Hom}(G, G)|_{X^\circ} = \mathcal{O}_{X^\circ}$, and $\beta \in \mathscr{A}^{1,0}(X^\circ, \mathrm{Hom}(G, G^\perp))$ is the second fundamental form, and $\varphi \in \mathscr{A}^{1,0}(X^\circ, \mathrm{Hom}(G^\perp, G))$ is equal to $\theta|_{G^\perp}$. Hence $\sqrt{-1} R_{h_G}(G) \leq \sqrt{-1} F_h(E)|_G$.

For any local frame e of $G|_{X^\circ}$, note that

$$|e|_h^2 \cdot \sqrt{-1}F_h(E)|_G = \langle \sqrt{-1}F_h(E)(e), e\rangle_h \leq \langle \mathbb{1} \otimes \omega_P e, e\rangle_h = |e|_h^2 \cdot \omega_P$$

Hence $\sqrt{-1}F_h(E)|_G - \omega_P$ is a semi-negative $(1,1)$-form on X°, and thus over X° one has

$$-\sqrt{-1}R_{h_G}(G) + T \geq \omega_P - \sqrt{-1}F_h(E)|_G \geq 0$$

Since we assume that (E,h) is adapted to log order, $(G^{-1}|_{X-Z}, h_G^{-1}|_{X-Z})$ is thus adapted to log order for the log pair $(X-Z, D-Z)$. By Lemma 4.5 and (5.3.2), $-\sqrt{-1}R_{h_G}(G)+T$ extends to a closed positive $(1,1)$-current on $X-Z$. Since Z is of codimension at least two, a standard fact in pluripotential theory shows that $-\sqrt{-1}R_{h_G}(G) + T$ extends to a positive closed $(1,1)$-current on the whole X.

Denote by $s \in H^0(X, E \otimes G^{-1})$ the section defining the inclusion $G \to E$. We fix a smooth hermitian metric h_0 for G and we define a function $H := |s|^2_{h \cdot h_0^{-1}} = h_G \cdot h_0^{-1}$ on $X - D$. Then

$$\sqrt{-1}\partial\bar{\partial}\log H = \sqrt{-1}R_{h_0}(G) - \sqrt{-1}R_{h_G}(G). \tag{5.3.4}$$

Hence there is a constant $C_0 > 0$ so that

$$\sqrt{-1}\partial\bar{\partial}\log H + C_0 T \geq T. \tag{5.3.5}$$

By Lemma 5.4, $\omega \in \mathcal{E}(\alpha)$. Since $\sqrt{-1}R_{h_0}(G)$ is a smooth $(1,1)$-form on X, it follows from Theorem 5.5 that

$$\int_{X^\circ} \sqrt{-1}R_{h_0}(G) \wedge \omega^{n-1} = c_1(G) \cdot \alpha^{n-1}.$$

To prove (5.3.1), by (5.3.4) and the above equality it suffices to prove that

$$\int_{X^\circ} \sqrt{-1}\partial\bar{\partial}\log H \wedge \omega^{n-1} = 0. \tag{5.3.6}$$

We will pursue the ideas in [Gue16, Proposition 3.8] to prove this equality.

Let us take a log resolution $\mu : \tilde{X} \to X$ of the ideal sheaf $\mathscr{I}$ defined by $s \in H^0(X, E \otimes G^{-1})$, with $\mathcal{O}_{\tilde{X}}(-A) = \mu^*\mathscr{I}$ and $\tilde{D} := \mu^{-1}(D)$ a simple normal crossing divisor. Let us denote by $(\tilde{E}, \tilde{\theta})$ the induced log Higgs bundle on $(\tilde{X}, \tilde{D})$ by pulling back (E, θ) via μ. Then the metric $\tilde{h} := \mu^* h$ for $(\tilde{E}, \tilde{\theta})|_{\tilde{X}-\tilde{D}}$ is also adapted to log order and acceptable by Lemma 4.3.

Note that $\mathrm{Supp}(\mathcal{O}_X/\mathscr{I}) = Z$. Write $\tilde{G} := \mu^* G$. There is a nowhere vanishing section

$$\tilde{s} \in H^0(\tilde{X}, \tilde{E} \otimes \tilde{G}^{-1} \otimes \mathcal{O}_{\tilde{X}}(-A))$$

so that $\mu^* s = \tilde{s} \cdot \sigma_A$, where σ_A is the canonical section in $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(A))$ which defines the effective exceptional divisor A.

Fix a Kähler form $\tilde{\omega}$ on $\tilde{X}$, as Remark 1.5 we construct another Kähler current

$$\tilde{T} := \tilde{\omega} - \sqrt{-1}\partial\bar{\partial}\log(-\prod_{i=1}^{m} \log|\varepsilon \cdot \tilde{\sigma}_i|^2_{\tilde{h}_i}), \tag{5.3.7}$$

over $\tilde{X}$, whose restriction on $\tilde{X} - \tilde{D}$ is a complete Kähler form, which has the same Poincaré growth near $\tilde{D}$. Here $\tilde{\sigma}_i$ is the section $H^0(X, \mathcal{O}_X(\tilde{D}_i))$ defining $\tilde{D}_i$, and $\tilde{h}_i$ is some smooth metric for the line bundle $\mathcal{O}_{\tilde{X}}(\tilde{D}_i)$.

Let us fix a smooth hermitian metric h_A for $\mathcal{O}_{\tilde{X}}(A)$. Write $\tilde{H} := |\tilde{s}|^2_{\tilde{h}\cdot\mu^*h_0^{-1}\cdot h_A^{-1}}$. Since $\tilde{h}$ is adapted to log order and $\tilde{s}$ is nowhere vanishing, there is a constant $C_1, C_2 > 0$ so that

$$\log \tilde{H} \geq C_1\varphi_P - C_2, \tag{5.3.8}$$

where we denote by

$$\varphi_P := -\log(-\prod_{i=1}^{\ell} \log|\varepsilon\cdot\tilde{\sigma}_i|^2_{\tilde{h}_i}).$$

Since $\tilde{h} := \mu^*h$ for $(\tilde{E},\tilde{\theta})|_{\tilde{X}-\tilde{D}}$ is acceptable, by same arguments as those for (5.3.5), one can show that

$$\sqrt{-1}\partial\bar{\partial}\log\tilde{H} + C_3\tilde{T} \geq \tilde{T}$$

over $\tilde{X}-\tilde{D}$ for some constant $C_3 > 0$. Note that the local potential of $\sqrt{-1}\partial\bar{\partial}\log\tilde{H}+C_3\tilde{T}$ is bounded from below by $(C_1+C_3)\varphi_P$ according to (5.3.8). By [Gue14, Proposition 2.3], one has

$$\sqrt{-1}\partial\bar{\partial}\log\tilde{H} + C_3\tilde{T} \in \mathcal{E}(\{C_3\tilde{T}\}).$$

It follows from (4.1.1) that $\mu^*\omega \leq C_4\tilde{T}$ for some constant $C_4 > 0$. By Lemma 5.4 again, $\mu^*\omega \in \mathcal{E}(\mu^*\alpha)$. Hence by Theorem 5.5 one has

$$\int_{\mu^{-1}(X^\circ)} (\sqrt{-1}\partial\bar{\partial}\log\tilde{H} + C_3\tilde{T}) \wedge \mu^*\omega^{n-1} = \{C_3\tilde{T}\}\cdot\mu^*\alpha^{n-1}.$$

Recall that $\tilde{T} \in \mathcal{E}(\{\tilde{T}\})$ by Lemma 5.4. Hence

$$\int_{\mu^{-1}(X^\circ)} C_3\tilde{T}\wedge\mu^*\omega^{n-1} = \{C_3\tilde{T}\}\cdot\mu^*\alpha^{n-1}.$$

One thus has

$$\int_{\mu^{-1}(X^\circ)} \sqrt{-1}\partial\bar{\partial}\log\tilde{H}\wedge\mu^*\omega^{n-1} = 0. \tag{5.3.9}$$

Note that over $\tilde{X}-\tilde{D}$, one has

$$\sqrt{-1}\partial\bar{\partial}\log\tilde{H} + [A] - \sqrt{-1}R_{h_A}(A) = \mu^*\sqrt{-1}\partial\bar{\partial}\log H$$

where $[A]$ is the current of integration of A. Hence over $\mu^{-1}(X^\circ) \simeq X^\circ$, one has

$$\sqrt{-1}\partial\bar{\partial}\log\tilde{H} - \sqrt{-1}R_{h_A}(A) = \mu^*\sqrt{-1}\partial\bar{\partial}\log H. \tag{5.3.10}$$

By Theorem 5.5 again,

$$\int_{\mu^{-1}(X^\circ)} \sqrt{-1}R_{h_A}(A)\wedge\mu^*\omega^{n-1} = c_1(A)\cdot\mu^*\alpha^{n-1} = 0, \tag{5.3.11}$$

where the last equality follows from the fact that A is μ-exceptional. (5.3.9), (5.3.10) together with (5.3.11) shows the desired equality (5.3.6). We finish the proof of (5.3.1) when $\operatorname{rank} G = 1$.

Assume that $\operatorname{rank} G = r$. We replace (E,θ,h) by the wedge product $(\tilde{E},\tilde{\theta},\tilde{h}) := \Lambda^r(E,\theta,h)$. By Lemma 5.2, the induced metric $\tilde{h}$ is also acceptable and one can easily check that it is also adapted to log order. Note that $\det G$ is also invariant under $\tilde{\theta}$, and that

$$\det G \to \Lambda^r E.$$

We then reduce the general cases to rank 1 cases. The proposition is thus proved. ◻

Let us state and prove the main result in this section.

Theorem 5.7. *Let X be a compact Kähler manifold and let D be a simple normal crossing divisor on X. Let α be a big and nef cohomology $(1,1)$-class containing a positive closed $(1,1)$-current $\omega \in \alpha$ so that $\omega|_{X-D}$ is a smooth Kähler form and has at most Poincaré growth near D. Let (E,θ) be a log Higgs bundle on (X,D). Assume that there is a hermitian metric h on $(E,\theta)|_{X-D}$ such that*

- *it is adapted to log order (in the sense of Definition 4.1);*
- *it is acceptable (in the sense of Definition 4.2);*
- *it is* Hermitian-Yang-Mills*:*
$$\Lambda_\omega F_h(E)^\perp = 0.$$

Then (E,θ) is μ_α-polystable.

Proof. We shall use the same notations as those in Proposition 5.6. Let G be any saturated Higgs-subsheaf $G \subset E$, and denote by Z the analytic subvariety of codimension at least two so that $G|_{X-Z} \subset E|_{X-Z}$ is a subbundle. By the Chern-Weil formula again, over $X^\circ := X - Z - D$ we have

$$\begin{aligned}\Lambda_\omega F_{h_G}(G) &= \Lambda_\omega F_h(E)|_G + \Lambda_\omega(\overline{\beta}_h \wedge \beta - \varphi \wedge \overline{\varphi}_h)\\ &= \Lambda_\omega F_h^\perp(E)|_G + \frac{\Lambda_\omega TrF_h(E)}{\operatorname{rank} E} \otimes \mathbb{1}_G + \Lambda_\omega(\overline{\beta}_h \wedge \beta - \varphi \wedge \overline{\varphi}_h)\\ &= \frac{\Lambda_\omega Tr(F_h(E))}{\operatorname{rank} E} \otimes \mathbb{1}_G + \Lambda_\omega(\overline{\beta}_h \wedge \beta - \varphi \wedge \overline{\varphi}_h).\end{aligned}$$

where $\beta \in \mathscr{A}^{1,0}(X^\circ, \operatorname{Hom}(G, G^\perp))$ is the second fundamental form of G in E with respect to the metric h, and $\varphi \in \mathscr{A}^{1,0}(X^\circ, \operatorname{Hom}(G^\perp, G))$ is equal to $\theta|_{G^\perp}$.

Hence

$$\begin{aligned}\int_{X^\circ} Tr(\sqrt{-1}F_{h_G}(G)) \wedge \omega^{n-1} &= \int_{X^\circ} Tr(\Lambda_\omega \sqrt{-1}F_{h_G}(G))\frac{\omega^n}{n}\\ &= \int_{X^\circ} \frac{\operatorname{rank} G}{\operatorname{rank} E}\Lambda_\omega Tr(\sqrt{-1}F_h(E))\frac{\omega^n}{n}\\ &+ Tr\Lambda_\omega(\sqrt{-1}\overline{\beta}_h \wedge \beta - \sqrt{-1}\varphi \wedge \overline{\varphi}_h)\frac{\omega^n}{n}\\ &= \int_{X^\circ} \frac{\operatorname{rank} G}{\operatorname{rank} E} Tr(\sqrt{-1}F_h(E)) \wedge \omega^{n-1} - (|\beta|_h^2 + |\varphi|_h^2)\frac{\omega^n}{n}\end{aligned}$$

By Proposition 5.6 together with the above inequality, one concludes the slope inequality

$$\mu_\alpha(G) \le \mu_\alpha(E)$$

and the equality holds if and only if $\beta \equiv 0$ and $\varphi \equiv 0$. We shall prove that if the above slope equality holds, G is a *sub-Higgs bundle* of E, and we have the decomposition

$$(E,\theta) = (G, \theta|_G) \oplus (F, \theta_F)$$

where (F, θ_F) is another sub-Higgs bundle of E.

Set $\operatorname{rank} E = r$ and $\operatorname{rank} G = m$. We first prove that G is a subbundle of E. It is equivalent to show that $\det G \to \Lambda^r E$ is a subbundle, and we thus reduce the problem to the case that $\operatorname{rank} G = 1$. Assume that $\mu_\alpha(G) = \mu_\alpha(E)$ and thus $\beta \equiv 0$ and $\varphi \equiv 0$. By (5.3.3), over X° one has

$$\sqrt{-1}R_{h_G}(G) = \sqrt{-1}F_h(E)|_G \ge -T|_{X^\circ}, \tag{5.3.12}$$

where T is the Kähler current defined in (5.3.2). By Lemma 4.5, $\sqrt{-1}R_{h_G}(G) + T$ extends to a closed positive $(1,1)$-current on t $X - Z$, and thus to the whole X.

Assume now $x_0 \in X$ is a point where $(E/G)_{x_0}$ is not locally free. Take a local holomorphic frame e of G on some open neighborhood $(U; z_1, \ldots, z_n)$ of x, and a holomorphic frame $e_1, \ldots, e_r$ of E. Then $e = \sum_{i=1}^{r} f_i(x) e_i$, where $f_i \in \mathcal{O}(U_i)$ so that $f_1(x_0) = \cdots = f_r(x_0) = 0$. By the asssumption that h is adapted to log order, one concludes that

$$\log |e|_h^2 \leq C_1 \log(|z_1|^2 + \cdots + |z_n|^2) + C_2 \log(-\log(\prod_{i=1}^{\ell} |z|_i^2)) \tag{5.3.13}$$

for some positive constants C_1 and C_2. On the other hand, by (5.3.12) on U we have

$$\sqrt{-1}\partial\bar{\partial} \log |e|_h^2 = -\sqrt{-1} R_{h_G}(G) \leq T.$$

By the construction of T, we conclude that

$$\log |e|_h^2 \geq C_3 \log(-\log(\prod_{i=1}^{\ell} |z|_i^2)) + C_4,$$

for some $C_3 > 0$ and $C_4 < 0$. This contradicts with (5.3.13). Hence we conclude that when the slope equality holds, G is a subbundle of E.

We now find the desired decomposition of (E, θ). By the above argument, when the slope equality holds, $(G, \theta|_G)$ is a Higgs subbundle of (E, θ) (not assumed to be rank 1 now), and $\beta \equiv 0$ and $\varphi \equiv 0$. This means that the orthogonal projection $\pi : E|_{X-D} \to G|_{X-D}$ is holomorphic, that $G^\perp$ is a holomorphic subbundle of $E|_{X-D}$, and that

$$(E, \theta)|_{X-D} = (G, \theta|_G)|_{X-D} \oplus (G^\perp, \theta|_{G^\perp}). \tag{5.3.14}$$

We shall prove that π extends to a morphism $\tilde{\pi} : E \to G$ so that $\pi \circ \iota = \mathbb{1}$. For any point $x_0 \in D$, we pick an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x_0 and a holomorphic fame $(e_1, \ldots, e_r)$ for $E|_U$ adapted to log order so that $(e_1, \ldots, e_m)$ is a holomorphic fame for $G|_U$. Write $\pi(e_j|_{X-D}) = \sum_{i=1}^{r} f_i(x) e_i$, where $f_i(x) \in \mathcal{O}(U - D)$. For $j = 1, \ldots, m$, one has $\pi(e_j|_{X-D}) = e_j$ and it extends naturally. For $j > m$, over $U^* = U - D$ one has

$$C(-\log(\prod_{i=1}^{\ell} |z|_i^2))^M \geq |e_j|_h^2 \geq |\pi(e_j)|_h^2 \geq H_{ij} |f_i||f_j|$$

for some $C, M > 0$, where $H_{ij} := h(e_i, e_j)$ with $(H_{ij})_{1 \leq i,j \leq r}$ adapted to log order. Hence each $|f_i|$ is locally bounded from above on U, and it thus extends to a holomorphic function on U. We conclude that π extends to a morphism $\tilde{\pi} : E \to G$, whose rank is constant and $\tilde{\pi} \circ \iota = \mathbb{1}$, where $\iota : G \to E$ denotes the inclusion. Let us define by $F := \ker \tilde{\pi}$, which is a subbundle of E so that $E = G \oplus F$. Note that $F|_{X-D} = G^\perp$. By (5.3.14) together with the continuity propery we conclude that F is a sub-Higgs bundle of (E, θ), and that $(E, \theta) = (G, \theta|_G) \oplus (F, \theta|_F)$. Since $h|_G$ (resp. $h|_F$) is a Hermitian-Yang-Mills metric for $(G, \theta|_G)$ (resp. $(F, \theta|_F)$) satisfying the three conditions in the theorem, we can argue in the same way as above to decompose $(G, \theta|_G)$ and $(F, \theta|_F)$ further to show that (E, θ) is a direct sum of μ_α-stable log Higgs bundles with the same slope. Hence (E, θ) is μ_α-polystable. We prove the theorem. □

5.4. **Application to toroidal compactification of ball quotient.** Let $\Gamma \in PU(n, 1)$ be a torsion free lattice, and let $\mathbb{B}^n/\Gamma$ be the associated ball quotient. By the work of Baily-Borel, Siu-Yau and Mok [Mok12], $\mathbb{B}^n/\Gamma$ has a unique structure of a quasi-projective complex algebraic variety (see for example [BU20, Theorem 3.1.12]). When the parabolic subgroups of Γ are unipotent, by the work of Ash et al. [AMRT10] and

Mok [Mok12, Theorem 1], ${}^{\mathbb{B}^n}\!/_{\Gamma}$ admits a *unique* smooth toroidal compactification, which we denote by X. Let us denote by $D := X - {}^{\mathbb{B}^n}\!/_{\Gamma}$ the boundary divisor, which is a disjoint union of abelian varieties. Let g_B be the Bergman metric for $\mathbb{B}^n$, which is complete, invariant under $PU(n,1)$ and has constant holomorphic sectional curvature -1. Hence it descends to a metric ω on $X - D$. If we consider ω as a metric for $T_X(-\log D)|_{X-D}$, by [To93, Proposition 2.1] it is *good* in the sense of Mumford [Mum77, Section 1]. Therefore, by for any $k \geq 1$, it follows from [Mum77, Theorem 1.4] that the trivial extension of the Chern form $c_k(T_{X-D},\omega)$ onto X defines a (k,k)-current $[c_k(T_{X-D},\omega)]$ on X, which represents the cohomology class $c_k(T_X(-\log D)) \in H^{k,k}(X)$. Let us first prove (0.1.3), which is indeed an easy computation.

For any $x_0 \in X - D$, we take a normal coordinate system $(z_1,\ldots,z_n)$ centered at x_0 so that

$$\omega = \sqrt{-1}\sum_{1\leq \ell,m\leq n}\delta_{\ell m}dz_\ell\wedge d\bar{z}_m - \sum_{j,k,\ell,m}c_{jk\ell m}z_j\bar{z}_k + O(|z|^3)$$

where $c_{jk\ell m}$ is the coefficients of the Chern curvature tensor

$$R_\omega(T_X) = \sum_{j,k,\ell,m}c_{jk\ell m}dz_j\wedge d\bar{z}_k\otimes\Big(\frac{\partial}{\partial z_\ell}\Big)^*\otimes\frac{\partial}{\partial z_m}.$$

By [Mok89, p. 177], one has

$$c_{jk\ell m}(x_0) = -(\delta_{jk}\delta_{\ell m}+\delta_{jm}\delta_{k\ell}). \tag{5.4.1}$$

Hence

$$\begin{aligned} c_1(T_{X-D},\omega)|_{x_0} &= -\frac{i}{2\pi}(n+1)\omega|_{x_0} \\ c_2(T_{X-D},\omega)|_{x_0} &= \frac{tr(R_\omega(T_{X-D})\wedge R_\omega(T_{X-D})) - tr(R_\omega(T_{X-D}))^2}{8\pi^2} \\ &= \frac{(n+1)\omega\wedge\omega|_{x_0} - (n+1)^2\omega\wedge\omega|_{x_0}}{8\pi^2} \end{aligned}$$

This implies that

$$nc_1(T_{X-D},\omega)^2 - 2(n+1)c_2(T_{X-D},\omega) \equiv 0.$$

We thus conclude that the Chern classes $c_k(\Omega^1_X(\log D))$ satisfies

$$nc_1(\Omega^1_X(\log D))^2 - 2(n+1)c_2(\Omega^1_X(\log D)) = 0.$$

Hence (0.1.3) in Theorem B holds.

For the log Hodge bundle $(E,\theta) = (E^{1,0}\oplus E^{0,1},\theta)$, given by

$$E^{1,0} := \Omega^1_X(\log D), \quad E^{0,1} := \mathcal{O}_X$$

with the Higgs field θ defined in (0.1.1), we shall prove that it is μ_α-polystable for the big and nef polarization α in Theorem 5.7. We equipped $(E^{1,0}\oplus E^{0,1})|_{X-D}$ with the metric

$$h := \omega^{-1}\oplus h_c \tag{5.4.2}$$

where h_c is the canonical metric on $\mathcal{O}_{X-D}$ so that $|1|_{h_c} = 1$. Recall that the curvature $F_h(E)$ of the connection $D_h := d_h + \theta + \overline{\theta}_h$ is

$$F_h(E) = R_h(E) + [\theta,\overline{\theta}_h],$$

where $R_h(E)$ is the Chern curvature of (E,h). Let us now compute $F_h(E)$, which is also an easy exercise.

To distinguish the sections of Higgs bundles and forms, we write $e_i := dz_i$, and denote by $e_0 = 1$ the constant section of $\mathcal{O}_X$. Hence $(e_0, e_1, \ldots, e_n)$ is an orthonormal basis at x_0 with respect to the metric h, and

$$\theta(e_0) = 0, \quad \theta(e_i) = e_0 \otimes dz_i \quad \text{for} \quad i = 1, \ldots, n.$$

Moreover,

$$\bar{\theta}_h(e_0|_{x_0}) = \sum_{j=1}^{n} e_j|_{x_0} \otimes d\bar{z}_j; \quad \bar{\theta}_h(e_i) = 0 \quad \text{for} \quad i = 1, \ldots, n$$

Then one has

$$R_h(E) = -c_{jkm\ell} dz_j \wedge d\bar{z}_k \otimes (e_\ell)^* \otimes e_m.$$

By (5.4.1), for $i = 1, \ldots, n$,

$$\begin{aligned}\sqrt{-1}F_h(E)(e_i|_{x_0}) &= -\sum_{j,k,m} \sqrt{-1}c_{jkmi} dz_j \wedge d\bar{z}_k \otimes e_m|_{x_0} + \sum_k \sqrt{-1} d\bar{z}_k \wedge dz_i \otimes e_k|_{x_0} \\ &= \sum_j \sqrt{-1} dz_j \wedge d\bar{z}_j \otimes e_i|_{x_0} + \sum_k \sqrt{-1} dz_i \wedge d\bar{z}_k \otimes e_k|_{x_0} \\ &\quad + \sum_k \sqrt{-1} d\bar{z}_k \wedge dz_i \otimes e_k|_{x_0} = \omega \otimes e_i|_{x_0}.\end{aligned}$$

Also,

$$\sqrt{-1}F_h(E)(e_0|_{x_0}) = \sqrt{-1}\theta \wedge \bar{\theta}_h(e_0|_{x_0}) = \omega \otimes e_0|_{x_0}$$

In conclusion, one has

$$\sqrt{-1}F_h(E) = \omega \otimes \mathbb{1},$$

In particular, h is a Hermitian-Yang-Mills metric for $(E, \theta)|_{X-D}$. We shall show that it satisfies the three conditions in Theorem 5.7. Indeed, we only have to check the first two conditions since $\sqrt{-1}F_h(E)^{\perp} \equiv 0$.

We first note that ω has at most Poincaré growth near D in the sense of Definition 1.4. Indeed, this follows easily from the Ahlfors-Schwarz lemma (see for example [Nad89, Lemma 2.1]) since the holomorphic sectional curvature of ω is -1. Hence for any admissible coordinate system $(U; z_1, \ldots, z_n)$ as in Definition 1.3, one has $|F_h(E)|_{h,\omega_P} \le C$, where ω_P is the Poincaré metric on U^*.

By the following result, we see that h is adapted to log order.

Lemma 5.8 ([Mok12, eq. (8) on p. 338])**.** *Let (X, D) be as above. Then for any $x \in D$, there is an admissible coordinate $(U; z_1, \ldots, z_n)$ at x so that the frame $z_1\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{n-1}}, \frac{\partial}{\partial z_n}$ is adapted to log order (in the sense of § 4.1) with respect to the above metric ω.*

Therefore, the metric h for $(E, \theta)|_{X-D}$ satisfies the three conditions in Theorem 5.7. In conclusion, (E, θ) is μ_α-polystable for the big and nef class α in Theorem 5.7

To finish the proof of Theorem B, we have to show that $c_1(K_X + D)$ can be made as a polarization in Theorem 5.7, which follows from the following result.

Lemma 5.9 ([Mok12, Proposition 1])**.** *The Kähler form $\frac{(n+1)}{2\pi}\omega$ on $X - D$ defined above extends to a closed positive $(1,1)$-current $\varpi \in c_1(K_X + D)$ with zero Lelong numbers. In particular, $K_X + D$ is big and nef.*

Let us provide a quick proof here for completeness sake.

Proof of Lemma 5.9. Note that the volume form ω^n defined a metric h_v for $(K_X + D)|_{X-D}$, which is adapted to log order by Lemma 5.8. By (5.4.1), one has

$$Ric(\omega) = -(n+1)\omega.$$

Hence $\sqrt{-1}R_{h_v}((K_X + D)|_{X-D}) = (n+1)\omega$. By Lemma 4.5, h_v extends to a singular metric $\tilde{h}_v$ for $K_X + D$ so that its curvature current $\sqrt{-1}R_{\tilde{h}_v}(K_X + D)$ is positive. The Lelong number of $\sqrt{-1}R_{\tilde{h}_v}(K_X+D)$ is zero everywhere since $\tilde{h}_v$ is adapted to log order. This shows that $K_X + D$ is big and nef, which is ample over $X - D$. □

6. Conjugate non-compact ball quotient

As an application of Theorems A and B, we shall prove that the conjugate of non-compact ball quotient under an automorphism of $\mathbb{C}$ is still a ball quotient. It was proved by Kazhdan [Kaz83] for arithmetic lattice, and by Mok-Yeung [MY93] and Baldi-Ullmo [BU20] for non-arithmetic lattice. The cocompact case can be easily proved using the Miyaoka-Yau inequality in [Yau78].

Let us make the following conventions for this section. Let X be a complex projective variety with $X_{\rm alg}$ the corresponding algebraic variety over $\mathbb{C}$. For any coherent sheaf $\mathcal{E}$ on X, denote by $\mathcal{E}_{\rm alg}$ the corresponding coherent sheaf on $X_{\rm alg}$. Conversely, for any coherent sheaf $\mathcal{E}_{\rm alg}$ on $X_{\rm alg}$, we denote by $\mathcal{E}$ the corresponding coherent sheaf on X.

Proof of Corollary C. We first assume that parabolic subgroups of Γ are unipotent. By [Mok12, Theorem 1], there is a toroidal compactification $\overline{X}$ for the ball quotient $X := \mathbb{B}^n/\Gamma$, so that $D := \overline{X} - X$ is a smooth divisor. Moreover, $\overline{X}$ is projective, whose algebraic structure is unique, denoted by $\overline{X}_{\rm alg}$. By Grothendieck's comparison theorem (see *e.g.* [CS14, Theorem 11.1.2]), there is a canonical isomorphism

$$\varphi : H^i(\overline{X}_{\rm alg}) \xrightarrow{\sim} H^i(\overline{X}, \mathbb{C}). \tag{6.0.1}$$

Consider the conjugate variety $\overline{X}^\sigma_{\rm alg}$ by the Cartesian diagram

$$\begin{array}{ccc} \overline{X}^\sigma_{\rm alg} & \xrightarrow{\sigma^{-1}} & \overline{X}_{\rm alg} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\sigma^*} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Then $D^\sigma_{\rm alg} := \sigma^{-1}(D_{\rm alg})$ is also a smooth divisor on the smooth projective variety $\overline{X}^\sigma_{\rm alg}$. Denote by $(\overline{X}^\sigma, D^\sigma)$ the analytification of $(\overline{X}^\sigma_{\rm alg}, D^\sigma_{\rm alg})$. We are going to show that the projective log pair $(\overline{X}^\sigma, D^\sigma)$ satisfies all the conditions in Theorem A.

We set up the notations in what follows. For a coherent sheaf $\mathcal{F}_{\rm alg}$ on $\overline{X}_{\rm alg}$, we denote by $\mathcal{F}^\sigma_{\rm alg} := (\sigma^{-1})^*\mathcal{F}_{\rm alg}$, whose analytification is denoted by $\mathcal{F}^\sigma$.

Fix an ample line bundle $L_{\rm alg}$ on $\overline{X}_{\rm alg}$. Then L^σ is an ample line bundle over $\overline{X}^\sigma$. By [CS14, p. 473] σ^{-1} induces natural isomorphism

$$(\sigma^{-1})^* : H^i(\overline{X}_{\rm alg}) \xrightarrow{\sim} H^i(\overline{X}^\sigma_{\rm alg}). \tag{6.0.2}$$

and

$$(\sigma^{-1})^*\Omega^i_{\overline{X}_{\rm alg}}(\log D_{\rm alg}) \xrightarrow{\sim} \Omega^i_{\overline{X}^\sigma_{\rm alg}}(\log D^\sigma_{\rm alg}). \tag{6.0.3}$$

Moreover, for any vector bundle $E_{\rm alg}$ on $\overline{X}_{\rm alg}$, one has

$$\varphi(c_k(E_{\rm alg})) = c_k(E) \tag{6.0.4}$$

and

$$(\sigma^{-1})^*(c_k(E_{\rm alg})) = c_k(E^\sigma_{\rm alg}).$$

By (0.1.3) in Theorem B, one has

$$2c_2(\Omega^1_{\overline{X}}(\log D)) - \frac{n}{n+1}c_1(\Omega^1_{\overline{X}}(\log D))^2 = 0. \tag{6.0.5}$$

It then follows from (6.0.3) and (6.0.4) that

$$2c_2(\Omega^1_{\overline{X}^\sigma}(\log D^\sigma)) - \frac{n}{n+1}c_1(\Omega^1_{\overline{X}^\sigma}(\log D^\sigma))^2 = 0. \tag{6.0.6}$$

By Theorem B, the log Higgs bundle $(E,\theta) := (\Omega^1_{\overline{X}}(\log D) \oplus \mathcal{O}_{\overline{X}}, \theta)$ defined as (0.1.1) is μ_L-polystable. By (6.0.3), its conjugate via σ is the log Higgs bundle $(E^\sigma, \theta^\sigma) := (\Omega^1_{\overline{X}^\sigma}(\log D^\sigma) \oplus \mathcal{O}_{\overline{X}^\sigma}, \theta^\sigma)$, where θ^σ is defined as (0.1.1). Let $\mathcal{F} \subset E^\sigma$ be any saturated coherent Higgs sub-sheaf. Then $\mathcal{F}^{\sigma^{-1}}$ is a Higgs subsheaf of (E,θ). Note that we always have the slope inequality $\mu_L(\mathcal{F}^{\sigma^{-1}}) \leq \mu_L(E)$, and the equality holds if and only if $(\mathcal{F}^{\sigma^{-1}}, \theta|_{\mathcal{F}^{\sigma^{-1}}})$ is a direct summand of (E,θ). It then follows from (6.0.3) and (6.0.4) that

$$\mu_{L^\sigma}(\mathcal{F}) = \mu_L(\mathcal{F}^{\sigma^{-1}}) \leq \mu_L(E) = \mu_{L^\sigma}(E^\sigma). \tag{6.0.7}$$

Note that the conjugate of $(\mathcal{F}^{\sigma^{-1}})^\sigma = \mathcal{F}$ for $\sigma \circ \sigma^{-1} = \mathbb{1}$. We thus conclude that, when the equality (6.0.7) holds, $(\mathcal{F}, \theta^\sigma|_{\mathcal{F}})$ is a direct summand of $(E^\sigma, \theta^\sigma)$. Hence the log Higgs bundle $(E^\sigma, \theta^\sigma)$ is μ_{L^σ}-polystable.

In conclusion, the projective log pair $(\overline{X}^\sigma, D^\sigma)$ satisfies all the conditions in Theorem A. Applying Theorem A, we conclude that the universal cover of $X^\sigma = \overline{X}^\sigma - D^\sigma$ is also the complex unit ball $\mathbb{B}^n$. This proves the corollary when parabolic subgroups of Γ are unipotent.

In the general case, there is a finite index subgroup $\Gamma' \subset \Gamma$ so that parabolic subgroups of Γ' are unipotent (see for example [BU20, §3.3]). Denote by $X := {\mathbb{B}^n}/{\Gamma}$ and $Y := {\mathbb{B}^n}/{\Gamma'}$. Recall that there are unique algbraic varieties $X_{\rm alg}$ and $Y_{\rm alg}$ whose analytifications are X and Y. The finite cover $Y \to X$ induces a finite étale surjective morphism $Y_{\rm alg} \to X_{\rm alg}$. Since the base change of an étale morphism is étale, we conclude that $Y^\sigma_{\rm alg} \to X^\sigma_{\rm alg}$ is also a finite étale surjective morphism. By the above result, Y^σ is the ball quotient. Since $Y^\sigma \to X^\sigma$ is a finite cover, X^σ is also the ball quotient. The corollary is proved. □

Appendix A. Metric rigidity for toroidal compactification of non-compact ball quotients

by Benoît Cadorel and Ya Deng

The main motivation of this appendix is to provide one building block for Theorem A. Our main result, Theorem A.8, says that there is no other smooth compactification for non-compact ball quotient than the toroidal one, so that the Bergman metric grows "mildly" near the boundary. Besides its own interests, this result is applied in this paper to show that

- the smoothness of D in Theorem A is necessary if one would like to characterize non-compact ball quotients;

- the “moreover”-statement of Theorem A: the projective log pair (X, D) is the toroidal compactification of a non-compact ball quotient.

A.1. **Toroidal compactifications of quotients by non-neat lattices.** In this section, we recall a well known way of constructing the toroidal compactifications of ball quotients in the case where the lattice has torsion at infinity. The reader will find more details about the natural orbifold structure on these compactifications in [Eys18]. For our purposes, the basic result given in Proposition A.1 will be sufficient.

Recall that we say that a lattice $\Gamma \subset PU(n,1)$ is *neat* (cf. [Bor69]) if for any $g \in \Gamma$, the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of g is torsion free. This implies that Γ is torsion free and that all parabolic elements of Γ are unipotent, so that the toroidal compactifications of ${}^{\mathbb{B}^n}\!/_{\Gamma}$ provided by [AMRT10, Mok12] are *smooth* (there is no "torsion at infinity").

Proposition A.1. *Let $\Gamma \subset PU(n,1)$ be a torsion free lattice, and let $\Gamma' \subset \Gamma$ be a finite index normal* neat *sublattice. Let $U = {}^{\mathbb{B}^n}\!/_{\Gamma}$, $U' = {}^{\mathbb{B}^n}\!/_{\Gamma'}$, and denote by X' the smooth toroidal compactification of $U' = {}^{\mathbb{B}^n}\!/_{\Gamma'}$ as constructed in [AMRT10, Mok12].*

Then the natural action of the finite group $G = {}^{\Gamma}\!/_{\Gamma'}$ on U' extends to X', and the quotient $X = {}^{X'}\!/_{G}$ is a normal projective space, with boundary $X - U$ made of quotient of abelian varieties by finite groups. Moreover, when Γ is arithmetic, X coincides with the toroidal compactification of U constructed in [AMRT10].

Remark A.2. By [Bor69, Proposition 17.4] in the arithmetic case, and [Bor63], or [Rag72, Theorem 6.11] in the general case, any lattice in $PU(n,1)$ admits a finite index neat sublattice.

Before explaining how to prove Proposition A.1, let us recall the construction of X' as it is defined in [Mok12] (see also [Cad16] for a similar discussion).

Each component D of $X' - U'$ is associated to a certain Γ'-orbit of points of $\partial\mathbb{B}^n$, whose points are called the Γ'*-rational boundary components* of ∂B^n (cf. [AMRT10, Chapter 3] or [Mok12, §1.3]). Let $b \in \partial\mathbb{B}^n$ be such a point, and let $N_b \subset PU(n,1)$ the stabilizer of b. This is a maximal parabolic real subgroup of $PU(n,1)$; let us denote by W_b its unipotent radical. This group can be written as an extension $1 \to U_b \to W_b \xrightarrow{\pi} A_b \to 1$, where $A_b \cong \mathbb{C}^{n-1}$, and $U_b \cong \mathbb{R}$ is the center of W_b. Let $L_b = {}^{N_b}\!/_{W_b}$. This reductive group can be embedded as a Levi subgroup in N_b, so that $N_b = W_b \cdot L_b$. Moreover, we have a decomposition $L_b = U(n-1) \times \mathbb{R}$, where the factor $U(n-1)$ corresponds to complex rotations around the axis $\mathbb{C}b$, and $\mathbb{R}$ corresponds to transvections of $\mathbb{B}^n$ along the axis $\mathbb{R}b$ (this description of W_b can be obtained e.g. by specializing the discussion of [BB66, Section 1.3] or [AMRT10, Section 4.2] to the case of the ball).

This Lie theoretic description of N_b can be understood more easily by expressing the action of the previous groups on the horoballs tangent to b. Let $(S_b^{(N)})_{N\geq 0}$ be the family of these horoballs. Each $S_b^{(N)} \subset \mathbb{B}^n$ can be described as an open subset in a Siegel domain of the third kind, as follows:

$$S_b^{(N)} \simeq \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid \operatorname{Im} z_n > ||z'||^2 + N\}. \tag{A.1.1}$$

We have $S_b^{(0)} = \mathbb{B}^n$, and when $b = (0, ..., 0, 1)$, the change of coordinates between the two descriptions of the ball is given by the *Cayley transform*

$$(w_1, ..., w_{n-1}, w_n) \in \mathbb{B}^n \mapsto (z', z_n) = \Big(\frac{w_1}{1-w_n}, ..., \frac{w_{n-1}}{1-w_n}, i\frac{1+w_n}{1-w_n}\Big) \in S^{(0)}_{(0,...,0,1)}.$$

The previous expression for $S_b^{(N)}$ can be used to give explicit formulas for the action of W_b and L_b on the ball. If $g \in W_b$, we can write $g = (s, a)$ accordingly to the decomposition $W_b \overset{sets}{=} U_b \times A_b$ ($U_b \cong \mathbb{R}$, $A_b \cong \mathbb{C}^{n-1}$), and we have, for any $(w', w_n) \in S_b^{(N)}$:

$$g \cdot (z', z_n) = (z' + a, z_n + i||a||^2 + 2i\overline{a} \cdot z' + s). \tag{A.1.2}$$

We check easily that $S_b^{(N)}$ is preserved by W_b. Also, for any $g \in L_b \simeq U(n-1) \times \mathbb{R}$, we can write $g = (r, t)$, and we then have

$$g \cdot (z', z_n) = (e^t(r \cdot z'), e^{2t} z_n). \tag{A.1.3}$$

Note that the element g above sends $S_b^{(N)}$ onto $S_b^{(e^{2t}N)}$.

We are now ready to describe the quotients of $S_b^{(N)}$ by the action of $\Gamma' \cap N_b$. Note first that since Γ' is neat, we have $\Gamma' \cap N_b \subset W_b$. Then, by the discussion above, we obtain a decomposition as *sets* $N_b \overset{sets}{=} (\mathbb{C}^{n-1} \times \mathbb{R}) \times (U(n-1) \times \mathbb{R})$, in which the elements of $\Gamma' \cap N_b$ can be written as $(a, t, \mathrm{Id}, 0)$. It also follows from [Mok12] that $\Gamma' \cap U_b = \mathbb{Z}\tau$ for some $\tau \in U_b \simeq \mathbb{R}$. This last fact permits to form the quotient $G_b^{(N)} = S_b^{(N)} / U_b \cap \Gamma'$; using (A.1.1), we can also express the latter quotient as an open subset of $\mathbb{C}^{n-1} \times \mathbb{C}^*$:

$$G_b^{(N)} = \{(w', w_n) \in \mathbb{C}^{n-1} \times \mathbb{C}^* \mid |w_n| e^{\frac{2\pi}{\tau}||w'||^2} < e^{-\frac{2\pi}{\tau}N}\},$$

and the quotient is then realized by the map $(z', z_n) \in S_b^{(N)} \to (z', e^{\frac{2i\pi}{\tau} z_n}) \in G_b^{(N)}$.

The group $\Lambda_b := \pi(\Gamma' \cap W_b) \subset \mathbb{C}^{n-1}$ is an abelian lattice of rank $2(n-1)$, which acts on $G_b^{(N)} \subset \mathbb{C}^{n-1} \times \mathbb{C}^*$ as

$$a \cdot (z', z_n) = (z' + a, e^{-\frac{2\pi}{\tau}||a||^2 - \frac{4\pi}{\tau}\overline{a} \cdot z'} z_n),$$

Clearly, the closure $\overline{G_b^{(N)}}$ in $\mathbb{C}^n$ is an open neighborhood of $\mathbb{C}^{n-1} \times \{0\}$. We can form the quotient

$$\Omega_b^{(N)} = \overline{G_b^{(N)}} / \Lambda_b$$

which is then isomorphic to a tubular neighborhood of the abelian variety $\mathbb{C}^{n-1} / \Lambda_b$ in some negative line bundle. Finally, the toroidal compactification X' can be obtained by glueing the open varieties $\Omega_b^{(N)}$ to U' (as b runs among a system of representatives of the rational boundary components, and N is chosen large enough for each cusp).

Our claims about X can be derived from the following lemma.

Lemma A.3. *Let $b \in \partial\mathbb{B}^n$ be a Γ'-rational boundary component, and let $g \in \Gamma$. Then the point $b' = g \cdot b$ is also Γ'-rational, and there exists $N, N' > 0$, for which g induces an isomorphism $S_b^{(N)} \xrightarrow{g} S_{b'}^{(N')}$, yielding in turn a unique compatible biholomorphism $\Omega_b^{(N)} \to \Omega_{b'}^{(N')}$.*

Proof. As Γ' is torsion free, a point $z \in \partial\mathbb{B}^n$ is Γ'-rational if and only if $W_b \cap \Gamma' \neq \{e\}$ (see [Mok12, §1.3]). Since g normalizes Γ', we have $g(W_b \cap \Gamma')g^{-1} \subset W_{b'} \cap \Gamma'$ so b' is Γ'-rational if b is.

As for our second claim, since the set of horoballs is preserved by the action of $PU(n, 1)$, we may find N, N' such that g induces a isomorphism $S_b^{(N)} \to S_{b'}^{(N)}$. Let (x', x_n) (resp. (y', y_n)) be standard coordinates on $S_b^{(N)}$ (resp. $S_b^{(N')}$) as in (A.1.1). It is always possible to choose the coordinates so that $(y', y_n) = (x', x_n) \circ u$ for some $u \in U(n)$ satisfying $u \cdot b' = b$. Then $ug \in N_b$, and the formulas (A.1.2) and (A.1.3) imply that $(x', x_n) \circ (ug)$ is an affine function of (x', x_n). Thus $(y', y_n) \circ g = f(x', x_n)$ for some *affine* map f.

Since g normalizes Γ', we have $g(\Gamma' \cap U_b)g^{-1} = \Gamma' \cap U_{b'}$, so the map $S_b^{(N)} \xrightarrow{g} S_b^{(N')}$ passes to the quotient to give a map $\widetilde{g} : G_b^{(N)} \to G_{b'}^{(N')}$. Using an explicit expression for the affine map f, we find an (*a priori* multivaluate) expression for $\widetilde{g}$ as

$$(z', z_n) \in G_b^{(N)} \overset{\widetilde{g}}{\mapsto} (A \cdot z' + u \log z_n + z'_0,\ C z_n^a e^{b \cdot z'}) \in G_{b'}^{(N')}$$

for some $A \in M_{n-1}(\mathbb{C})$, some vectors $u, b, z'_0 \in \mathbb{C}^{n-1}$ and $C, a \in \mathbb{C}$. Since the formula above must yield a well-defined, invertible map $G_b^{(N)} \to G_{b'}^{(N')}$, we must have $u = 0, a = 1$. This shows that $\widetilde{g}$ has unique holomorphic extension $\overline{G_b^{(N)}} \to \overline{G_{b'}^{(N')}}$. Finally, as g normalizes Γ', this map passes to the quotient by $\Lambda_b = \pi(\Gamma \cap W_b)$ (resp. $\Lambda'_b = \pi(\Gamma \cap W_{b'})$), which gives a uniquely defined biholomorphism $\Omega_b^{(N)} \to \Omega_{b'}^{(N')}$. □

Remark A.4. Note that it is easy to describe the action of the stabilizers of the boundary components of $X' - U'$. Assume indeed that $g \in \Gamma$ preserves one of the Γ'-rational boundary components $b \in \partial\mathbb{B}^n$. Then we can write $g = u \cdot d$, in the Levi decomposition $N_b = W_b \cdot L_b$, and further decompose $u = (s, a)$ (in $W_b \overset{sets}{=} U_b \times A_b$), and $d = (r, t)$ (in $L_b = U(n-1) \times \mathbb{R}$). Now, since $\Gamma' \subset \Gamma$ is of finite index, and since $\Gamma' \cap N_b \subset W_b$, the element d has finite order. This implies that $t = 0$, so d is simply a unitary rotation around the complex axis $\mathbb{C}b$.

It is now clear from the explicit formulas (A.1.2) and (A.1.3) that the action of g on $G_b^{(N)}$ can be described as

$$g \cdot (z', z_n) = (rz' + a, e^{-\frac{2\pi}{\tau}||a||^2 - \frac{4\pi}{\tau}\overline{a} \cdot (rz') + \frac{2i\pi}{\tau}s} z_n),$$

and this formula induces in turn a natural action on $\Omega_b^{(N)}$. We see in particular that g acts on the abelian variety $\mathbb{C}^{n-1}/\Lambda_b$ *via* an affine map, with linear part belonging to $U(n-1)$.

Going back to the proof of Proposition A.1, we see that Lemma A.3 permits to define a unique action of the quotient $G = \Gamma/\Gamma'$ on X' compatible with its natural action on U'. The complex projective space X can be defined as the quotient X'/G. The following lemma ends the proof of Proposition A.1, and clarifies the link with the construction of [AMRT10].

Lemma A.5. *The variety X defined above does not depend on the choice of Γ'. When the lattice Γ is arithmetic, X coincides with the toroidal compactification of U as constructed in [AMRT10].*

Proof. Let $\Gamma', \Gamma'' \subset \Gamma$ be two neat lattices of finite index. We want to show that the varieties constructed from Γ' and Γ'' are the same. Since $\Gamma \cap \Gamma'$ also has finite index in Γ, we may assume $\Gamma'' \subset \Gamma'$. The previous discussion shows that the action of two lattices $\Gamma'' \subset \Gamma'$ are compatible with each other on each open set $G_b^{(N)}$, which suffices to prove the first point. In general, we can also argue as follows.

For any arithmetic quotient of a hermitian symmetric space Ω/Γ, the construction of a toroidal compactification of [AMRT10] depends on a certain choice of Γ-*admissible polyhedra* for each rational boundary component (see [AMRT10, Definition 5.1]). In the case where $\Omega = \mathbb{B}^n$, since $\dim_{\mathbb{R}} U_b = 1$ for any $b \in \partial\mathbb{B}^n$, there is only one such possible choice (cf. [*loc. cit.*, Theorem 4.1.(2)]). Both claims then follow from the functoriality of toroidal compactifications (see [Har89, Lemma 2.6]), since "choices" of polyhedra admissible for two lattices $\Gamma' \subset \Gamma$ are thus automatically compatible with each other. □

Note that even though this construction of X is well adapted to our purposes, it should not be used to define X as an *orbifold*, as it has the drawback of producing artificial ramification orders along the boundary components of X. As explained in [Eys18], a better way of proceeding would be to construct directly open neighborhoods of the components of $X - U$ as *stacks*, before glueing them to U.

A.2. **Main results.** Let us first begin with the following lemma.

Lemma A.6. *Let Y be the toroidal compactification of the ball quotient $U := {}^{\mathbb{B}^n}\!/_{\Gamma}$ by a torsion free lattice $\Gamma \subset PU(n,1)$ whose parabolic isometries are all unipotent. Let X be another projective compactification of U, and assume one of the following:*

(a) X has at most quotient singularities,
(b) or, more generally, X has at most klt singularities.

Then the identity map of U extends to a birational morphism $f : X \to Y$.

Proof. The identity map of U extends to a birational map $f : X \dashrightarrow Y$. It suffices to show that f is regular. Assume by contradiction that f is not regular. One can take a resolution of indeterminacy $\mu : \tilde{X} \to X$ for f so that $\mu|_{\mu^{-1}(U)} : \mu^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism and

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\mu"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

By the rigidity result (see [Deb01, Chapter 3, Lemma 1.15]), there is at least one fiber $\mu^{-1}(z)$ with $z \in D$ which cannot be contracted by $\tilde{f}$. Clearly, we have $\tilde{f}(\mu^{-1}(z)) \subset Y - U$.

(1) If X has quotient singularities, [Kol93, Theorem 7.5] implies that every fiber of μ is simply connected. As $Y - U$ is a disjoint union of Abelian varieties A by [AMRT10, Mok12], the image of $\tilde{f} : \mu^{-1}(z) \to Y - U$ must be a point.
(2) If we assume only that X has klt singularities, we can use the work of Hacon-McKernan [HM07] which implies that every fiber of μ is rationally connected. In this case, $\widetilde{f}(\mu^{-1}(z))$ is also a point since abelian varieties do not contain rational curves.

This is a contradiction in both cases. □

Let us introduce a natural class of pairs under which our rigidity theorem will hold.

Definition A.7. Let (X, D) be a pair consisting of normal algebraic variety and a reduced divisor. We say that the pair (X, D) has *algebraic quotient singularities* if it admits a finite affine cover $(X_i)_{i \in I}$, such that each $(X_i, D \cap X_i)$ is the quotient of a smooth SNC pair (U_i, D_i) by a finite group G_i leaving D_i invariant.

We can now state our main result as follows.

Theorem A.8. *Let $U := {}^{\mathbb{B}^n}\!/_{\Gamma}$ be an n-dimensional ball quotient by a torsion free lattice $\Gamma \subset PU(n,1)$. Let X be a normal compactification of U, and let $D := X - U$. Assume one of the following:*

(1) D is a reduced divisor, and the pair (X, D) has algebraic quotient singularities;
(2) the variety X has at most klt singularities.

Let $D^{(1)} \subset D$ be the divisorial part of D. If the Kähler-Einstein metric ω for $T_X(-\log D^{(1)})|_U$ is adapted to log order near the generic point of any component of $D^{(1)}$, then (X, D) identifies with the toroidal compactification of U.

Remark A.9. (1) Note that if (X, D) has algebraic quotient singularities, then X is klt; however the proof in case (a) will not appeal to the difficult result of [HM07] which was used in Lemma A.6. Note also that for any lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, if X is the toroidal compactification of $U = {}^{\mathbb{B}^n}\!/\!_{\Gamma}$ described in Section A.1, then the pair $(X, X - U)$ has algebraic quotient singularities. This class of pairs seems then to be a natural setting for Theorem A.8 to hold.
(2) As an easy consequence of the case (b) above, we can remark that there is no klt compactification X of U such that $X - U$ has codimension ≥ 2.

Corollary A.10. *With the same assumptions as in Theorem A.8, if X is smooth and D has simple normal crossings, then D is in fact smooth, and each component is a* smooth *quotient of an abelian variety A by some finite group acting freely on A.*

Let us prove Theorem A.8. For the time being, we do not distinguish between our two hypotheses on X. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index so that all parabolic elements of Γ' are unipotent. Writing $U' := {}^{\mathbb{B}^n}\!/\!_{\Gamma'}$, this gives a finite étale surjective morphism $\mu_0 : U' \to U$.

Let X' be the normalization of X in the function field of U': this is a normal projective variety X' compactifying U' so that μ_0 extends to a (unique) finite surjective morphism $\mu : X' \to X$ (see e.g. [AHCG11, Chapter 12, §9]). Let us recall how to construct X' . We first take an arbitrary smooth projective compactification $\tilde{X}$ of U' so that μ_0 extends to a rational map $\tilde{\mu} : \tilde{X} \dashrightarrow X$. We then take a further blow-up $\tilde{X}' \to \tilde{X}$ so that its composition with $\tilde{\mu}$, denoted by $\mu' : \tilde{X}' \to X$, is a generically finite surjective morphism. Take a Stein factorization $\tilde{X}' \to X' \xrightarrow{\mu} X$ for μ'. Then $\mu : X' \to X$ is a finite surjective morphism with X' normal projective variety. One can check that such a morphism μ does not depend on the choice of $\tilde{X}$ and $\tilde{X}'$.

Lemma A.11. *The variety X' has one of the following types of singularities:*

(a) if the pair (X, D) has algebraic quotient singularities, then X' has algebraic quotient singularities ;
(b) if X has klt singularities, then X' also has klt singularities.

Proof. The case (b) is easy to settle, since klt singularities are preserved under finite surjective morphisms (see [KM98, Corollary 5.20]). Let us now deal with the case (a). Note that the statement is local on X, so since (X, D) has algebraic quotient singularities, we can assume that there exists a *finite* cover $\tau : Z \to X$ such that $E = \tau^{-1}(D)$ has simple normal crossings. In this setting, (X, D) is the quotient of (Z, E) by a finite groupoid $\mathcal{G}$ leaving E invariant. Let Z' be the normalization of the fiber product $Z \times_X X'$. We get a commutative diagram:

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "q"] & X' \arrow[d] \\ Z \arrow[r, "\tau"] & X \end{tikzcd}$$

The map $q : Z' \longrightarrow Z$ is a finite dominant morphism between normal varieties, with Z smooth. Moreover, it is étale above $Z - E$, where E is SNC. Hence, [Kol07, Theorem 2.23] implies that Z' has *abelian quotient* singularities. To conclude, remark that Lemma A.12 below implies that X' is the finite quotient of Z' by the groupoid $\mathcal{G}$. Indeed, with the notations of this lemma, it suffices to check that $R(Z')^{\mathcal{G}} = R(X')$. This can be seen easily from the identifications $R(Z)^{\mathcal{G}} = R(X)$ and $R(Z') = R(Z) \otimes_{R(X)} R(X')$. □

In the above proof, we made use of the following simple lemma, that we include for completeness.

Lemma A.12. *Let $f : M \longrightarrow N$ be a finite surjective morphism between two normal reduced schemes. Assume that M is acted upon by a finite groupoid $\mathcal{G}$, and that f is $\mathcal{G}$-invariant. Suppose in addition that $R(M)^{\mathcal{G}} = R(N)$, where $R(M), R(N)$ are the rings of rational functions on M, N. Then N is the quotient of M by $\mathcal{G}$.*

Proof. It suffices to show that $f_*(\mathcal{O}_M)^{\mathcal{G}} = \mathcal{O}_N$. This is a local statement on the base, so we may assume that $N = \operatorname{Spec} A$, $M = \operatorname{Spec} B$, and A is integral. We then have a finite extension $A \subset B$. Let $s \in B^{\mathcal{G}}$. Then $s \in R(B)^{\mathcal{G}} = R(A)$ by assumption. As the element s is finite over A, and A is integrally closed, this implies $s \in A$. This gives the result. □

Let Y' be the toroidal compactification of U', so that the boundary $A := Y' - U'$ is a smooth divisor.

Lemma A.13. *The identity map on U' extends as an isomorphism $f : X' \longrightarrow Y'$.*

Proof. By Lemma A.6 and Lemma A.11, the identity map of U' extends to a birational morphism $f : X' \longrightarrow Y'$ in case (a), or in the more general case (b). From now on, we will not distinguish between these two cases anymore.

Assume by contradiction that f is not an isomorphism. As Y' is smooth, it follows from [KM98, Corollary 2.63] that the exceptional set $\operatorname{Ex}(f)$ is of pure codimension one. Thus, the birational morphism f must contract at least one irreducible divisor, denoted by E, which must be an irreducible divisorial component of the boundary $D' := X' - U'$. Denote by D^{sing} the singular locus of D. Pick any point $x' \in \mu^{-1}(D - D^{\text{sing}}) \cap E$. Note that $x := \mu(x')$ belongs to the divisorial part $D^{(1)}$. Let us take an admissible coordinate chart $(\mathcal{V}; x_1, \ldots, x_n)$ centered at x with $(x_1 = 0) = \mathcal{V} \cap D$ so that the frame $(d \log x_1, dx_2, \ldots, dx_n)$ for $\Omega^1_X(\log D^{(1)})|_{\mathcal{V}}$ is adapted to log order with respect to the metric ω^{-1}. Let $\omega' := \mu^*\omega$, be the canonical Kähler Einstein metric on U'.

Lemma A.14 below shows that ω' is adapted to log-order for $T_{X'^\circ}(-\log E^\circ)$, where $X'^\circ := \mu^{-1}(X - D^{\text{sing}})$, and $E^\circ := X'^\circ \cap E$. We are going to derive a contradiction with the fact the E is contracted. Denote by A_1 a component of A so that $f(E) \subset A_1$. We can take admissible coordinates $(\mathcal{W}; z_1, \ldots, z_n)$ and $(\mathcal{U}; w_1, \ldots, w_n)$ centered at some well-chosen $x' \in E \cap X'^\circ$ and $y := f(x') \in A_1$ respectively so that $f(\mathcal{W}) \subset \mathcal{U}$, and $f|_E : E \longrightarrow f(E)$ is smooth at x'. Moreover, within these coordinates, $E \cap \mathcal{W} = D' \cap \mathcal{W} = \{z_1 = 0\}$, and $A_1 \cap \mathcal{U} = A \cap \mathcal{U} = \{w_1 = 0\}$. Denote by $(f_1(z), \ldots, f_n(z))$ the expression of f within these coordinates. Then if the admissible coordinates are chosen properly, one has

$$(f_1(z), \ldots, f_n(z)) = (z_1^{m_1} g_1(z), \ldots, z_1^{m_k} g_k(z), g_{k+1}, \ldots, g_n)$$

where $g_1(z), \ldots, g_k(z)$ are holomorphic functions defined on $\mathcal{W}$ so that $g_i(z) \neq 0$ and $m_i \geq 1$ for $i = 1, \ldots, k$. Since E is exceptional, one has $k \geq 2$. By the norm estimate in [Mok12, eq. (8) on p. 338], the Kähler-Einstein metric ω for $T_Y(-\log A)|_U$ is adapted to log order. More precisely, one has

$$|dw_2|^2_{\omega^{-1}} \sim (-\log|w_1|^2).$$

Since

$$f^* d\log w_2 = m_2 d\log z_1 + d\log g_2(w),$$

one thus has the following norm estimate

$$|d\log z_1|^2_{\omega'^{-1}} \geq \frac{1}{m_2^2}\mu^*|d\log w_2|^2_{\omega^{-1}} - \frac{1}{m_2^2}\mu^*|\frac{dg_2}{g_2}|^2_{\omega^{-1}} \geq \frac{C(-\log|z_1|^2)}{|z_1|^{2m_2}}$$

for some constants $C > 0$. Since $d\log z_1$ is a local nowhere vanishing section for $\Omega^1_{X'}(\log D')$, we conclude that the metric ω'^{-1} for $\Omega^1_{X'^\circ}(-\log D'^\circ)$ is *not* adapted to log order, and so is ω' for $T_{X'^\circ}(-\log D'^\circ)$.

The contradiction is obtained, which ends the proof of the lemma. □

Lemma A.14. *With the notations of the proof of Lemma A.13, the metric ω' is adapted to log-order for $T_{X'^\circ}(-\log E^\circ)$.*

Proof. Write $\mathcal{W} := \mu^{-1}(\mathcal{V})$. Since $\mu|_{\mathcal{W}-D'} : \mathcal{W} - D' \to \mathcal{V} - D$ is a finite unramified cover, the image of $(\mu|_{\mathcal{W}-D'})_*\big(\pi_1(\mathcal{W} - D')\big)$ is a subgroup of $\pi_1(\mathcal{W} - D) \simeq \mathbb{Z}$ index m. Set

$$\nu : \Delta^n \to \Delta^n$$
$$(z_1, \ldots, z_n) \mapsto (z_1^m, z_2, \ldots, z_n)$$

One thus has the following commutative diagram

$$\begin{tikzcd}
\Delta^* \times \Delta^{n-1} \arrow[r, "h^\circ"] \arrow[d, "\nu|_{\Delta^*\times\Delta^{n-1}}"] & \mathcal{W} \arrow[d, "\mu|_{\mathcal{W}}"] \\
\Delta^n \arrow[r, "\simeq"] & \mathcal{V}
\end{tikzcd}$$

so that $h^\circ_{\Delta^*\times\Delta^{n-1}} : \Delta^* \times \Delta^{n-1} \to \mathcal{W} \cap U'$ is an isomorphism. By the Riemann removable singularities theorem, h extends to a holomorphic map $h : \Delta^n \to \mathcal{W}$. One can easily check that h is surjective with finite fibers. Hence h is moreover biholomorphic. $(\mathcal{W}; z_1, \ldots, z_n; h)$ is therefore an admissible coordinate centered at x' with $(z_1 = 0) = \mathcal{W} \cap D'$ so that μ is expressed as ν within the admissible coordinates of $(\mathcal{W}; z_1, \ldots, z_n)$ and $(\mathcal{V}; x_1, \ldots, x_n)$. . Since

$$\mu^* d\log x_1 = m d\log z_1, \mu^* dx_2 = dz_2, \ldots, \mu^* dx_n = dz_n,$$

the frame $(d\log z_1, dz_2, \ldots, dz_n)$ for $\Omega^1_{X'}(\log D')|_{\mathcal{W}}$ is adapted to log order. This shows that the metric ω' is adapted to log order for $T_{X'^\circ}(-\log D'^\circ)$. □

We have shown that there is a finite surjective morphism

$$g : Y' \to X,$$

which identifies with the étale and surjective map $U' \to U$ over $X - D$.

We can now conclude the case discussed in Corollary A.10, where (X, D) is assumed to be a smooth log-pair. Since the irreducible components of $Y' - U'$ are connected, this implies right away that D must be smooth. Moreover, for each connected component A_i of A, there is a connected component D_j of D so that $g|_{A_i} : A_i \to D_j$ is a finite surjective morphism, which is also étale by the local description of μ given in the proof of Lemma A.14. Hence in this case, D_i is a smooth quotient of an abelian variety by the free action of some finite group G_i. This suffices to establish Corollary A.10.

The proof of Theorem A.8 will be complete with the following lemma.

Lemma A.15. *The variety X identifies with the quotient of Y' by the natural action of $G = {}^{\Gamma}\!/_{\Gamma'}$.*

This result comes right away from Lemma A.12, taking $M = Y'$, $N = X$, and $G = \mathcal{G}$. Remark that we have $R(Y')^G = R(U')^G = R(U) = R(X)$ since $U = {}^{U'}\!/_{G}$.

To conclude, it suffices to remark that Proposition A.1 claims that the toroidal compactification Y of U also identifies with the quotient ${}^{Y'}\!/_{G}$. Thus, there is an isomorphism $Y \cong X$ compatible with the identity on U. Theorem A.8 is proved.

References

[AHCG11] E. Arbarello, J.D. Harris, M. Cornalba, and P. Griffiths. *Geometry of Algebraic Curves: Volume II with a contribution by Joseph Daniel Harris*. Grundlehren der mathematischen Wissenschaften, (Springer Berlin Heidelberg2011). URL https://books.google.com.vc/books?id=_FAgwAEACAAJ. ↑ 44

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai. *Smooth compactifications of locally symmetric varieties. With the collaboration of Peter Scholze. 2nd ed.*, (Cambridge: Cambridge University Press2010), 2nd ed. ed. ↑ 35, 40, 42, 43

[BB66] W. L. Baily and A. Borel. "Compactification of Arithmetic Quotients of Bounded Symmetric Domains." *Annals of Mathematics* (1966) vol. 84: 442–528. ↑ 40

[BEGZ10] Sébastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. "Monge-Ampère equations in big cohomology classes." *Acta Math.* (2010) vol. 205 (2): 199–262. ↑ 29, 30

[Biq97] Olivier Biquard. "Fibrés de Higgs et connexions intégrables: Le cas logarithmique (diviseur lisse)." *Ann. Sci. Éc. Norm. Supér. (4)* (1997) vol. 30 (1): 41–96. ↑ 9

[Bor63] A Borel. "Compact Clifford-Klein forms of symmetric spaces." *Topology* (1963) vol. 2: 111–122. ↑ 40

[Bor69] A. Borel. *Introduction aux groupes arithmétiques*. Actualités scientifiques et industrielles, (Hermann1969). URL https://books.google.fr/books?id=MvPuAAAAMAAJ. ↑ 40

[Bou02] Sébastien Boucksom. "On the volume of a line bundle." *Int. J. Math.* (2002) vol. 13 (10): 1043–1063. ↑ 30

[Bru17] Yohan Brunebarbe. "Semi-positivity from Higgs bundles." *arXiv e-prints* (2017) arXiv:1707.08495. ↑ 3, 18, 19, 21

[BU20] Gregorio Baldi and Emmanuel Ullmo. "Special subvarieties of non-arithmetic ball quotients and Hodge Theory." *arXiv e-prints* (2020) arXiv:2005.03524. ↑ 3, 35, 38, 39

[Cad16] Benoit Cadorel. "Symmetric differentials on complex hyperbolic manifolds with cusps." *arXiv:1606.05470, to appear in J. Diff. Geom* (2016) . ↑ 40

[CG75] Maurizio Cornalba and Phillip Griffiths. "Analytic cycles and vector bundles on non-compact algebraic varieties." *Invent. Math.* (1975) vol. 28: 1–106. ↑ 7

[Cha06] Isaac Chavel. *Riemannian geometry. A modern introduction. 2nd ed.*, vol. 98, (Cambridge: Cambridge University Press2006), 2nd ed. ed. ↑ 14

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. "Degeneration of Hodge structures." *Ann. of Math. (2)* (1986) vol. 123 (3): 457–535. URL http://dx.doi.org/10.2307/1971333. ↑ 21, 24

[CP11] Frédéric Campana and Thomas Peternell. "Stabilité géométrique du fibré cotangent et du recouvrement universel d'une variété projective." *Bull. Soc. Math. Fr.* (2011) vol. 139 (1): 41–74. ↑ 5

[CP19] Frédéric Campana and Mihai Păun. "Foliations with positive slopes and birational stability of orbifold cotangent bundles." *Publ. Math., Inst. Hautes Étud. Sci.* (2019) vol. 129: 1–49. ↑ 5

[CS14] Franncois Charles and Christian Schnell. "Notes on absolute Hodge classes." In "Hodge theory. Based on lectures delivered at the summer school on Hodge theory and related topics, Trieste, Italy, June 14 – July 2, 2010," 469–530, (Princeton, NJ: Princeton University Press2014). ↑ 38

[CY80] Shiu-Yuen Cheng and Shing-Tung Yau. "On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation." *Commun. Pure Appl. Math.* (1980) vol. 33: 507–544. ↑ 3

[CY86] ———. "Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of SU(2,1)." Complex differential geometry and nonlinear differential equations, Proc. AMS-IMS-SIAM Joint Summer Res. Conf., Brunswick/Maine 1984, Contemp. Math. 49, 31-44 (1986). (1986). ↑ 3

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.*, (New York, NY: Springer2001). ↑ 43

[Dem12a] Jean-Pierre Demailly. *Analytic methods in algebraic geometry.*, vol. 1, (Somerville, MA: International Press; Beijing: Higher Education Press2012). ↑ 29

[Dem12b] ———. *Complex Analytic and Differential Geometry*, (online book2012). URL http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf. ↑ 20, 29

[DH19] Ya Deng and Feng Hao. "Vanishing theorem for tame harmonic bundles via L^2-cohomology." *arXiv e-prints* (2019) arXiv:1912.02586. ↑ 28, 29

[Eys18] Philippe Eyssidieux. "Orbifold Kähler Groups related to arithmetic complex hyperbolic lattices." *https://arxiv.org/abs/1805.00767* (2018) . ↑ 40, 43

[FF17] Osamu Fujino and Taro Fujisawa. "On semipositivity theorems." *arXiv e-prints* (2017) arXiv:1701.02039. ↑ 21

[GKP16] Daniel Greb, Stefan Kebekus, and Thomas Peternell. "Movable curves and semistable sheaves." *Int. Math. Res. Not.* (2016) vol. 2016 (2): 536–570. ↑ 5

[GKPT19a] Daniel Greb, Stefan Kebekus, Thomas Peternell, and Behrouz Taji. "Harmonic metrics on higgs sheaves and uniformization of varieties of general type." *Mathematische Annalen* (2019) 1–34. ↑ 3

[GKPT19b] ———. "The miyaoka-Yau inequality and uniformisation of canonical models." *Ann. Sci. Éc. Norm. Supér. (4)* (2019) vol. 52 (6): 1487–1535. ↑ 3

[GKT16] Daniel Greb, Stefan Kebekus, and Behrouz Taji. "Uniformisation of higher-dimensional minimal varieties." *arXiv e-prints* (2016) arXiv:1608.06644. ↑ 3

[GT16] Henri Guenancia and Behrouz Taji. "Orbifold Stability and Miyaoka-Yau Inequality for minimal pairs." *arXiv e-prints* (2016) arXiv:1611.05981. ↑ 3

[Gue14] Henri Guenancia. "Kähler-Einstein metrics with mixed Poincaré and cone singularities along a normal crossing divisor." *Ann. Inst. Fourier* (2014) vol. 64 (3): 1291–1330. ↑ 29, 30, 33

[Gue16] ———. "Semistability of the tangent sheaf of singular varieties." *Algebr. Geom.* (2016) vol. 3 (5): 508–542. ↑ 31, 32

[Har89] Michael Harris. "Functorial Properties of Toroidal Compactifications of Locally Symmetric Varieties." *Proceedings of the London Mathematical Society* (1989) vol. 1: 1–22. ↑ 43

[HM07] Christopher D. Hacon and James McKernan. "On Shokurov's rational connectedness conjecture." *Duke Math. J.* (2007) vol. 138 (1): 119–136. ↑ 43, 44

[Kaz83] David Kazhdan. "On arithmetic varieties. II." *Isr. J. Math.* (1983) vol. 44: 139–159. ↑ 3, 38

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti.*, vol. 134, (Cambridge: Cambridge University Press1998). ↑ 44, 45

[Kob84] Ryoichi Kobayashi. "Einstein-Kähler metrics on open algebraic surfaces of general type." *Tohoku Math. J. (2)* (1984) vol. 37: 43–77. ↑ 3

[Kob85] ———. "Einstein-Kähler V-metrics on open Satake V-surfaces with isolated quotient singularities." *Math. Ann.* (1985) vol. 272: 385–398. ↑ 3, 28

[Kol87] János Kollár. "Subadditivity of the Kodaira dimension: Fibers of general type." Algebraic geometry, Proc. Symp., Sendai/Jap. 1985, Adv. Stud. Pure Math. 10, 361-398 (1987). (1987). ↑ 21

[Kol93] ———. "Shafarevich maps and plurigenera of algebraic varieties." *Inventiones mathematicae* (1993) vol. 113: 177–215. ↑ 43

[Kol07] ———. *Lectures on Resolution of Singularities (AM-166)*, (Princeton University Press2007). ↑ 45

[Mau15] Julien Maubon. "Higgs bundles and representations of complex hyperbolic lattices." In Lizhen Ji, Athanase Papadopoulos, and Shing-Tung Yau, eds., "Handbook of Group Actions (vol. II)," Advanced Lectures in Mathematics Volume 32, 201–244, (International Press, Somerville, MA; Higher Education Press, Beijing2015). URL https://hal.archives-ouvertes.fr/hal-01275595. ↑ 16, 17

[Moc02] Takuro Mochizuki. "Asymptotic behaviour of tame nilpotent harmonic bundles with trivial parabolic structure." *J. Differential Geom.* (2002) vol. 62 (3): 351–559. URL http://projecteuclid.org/euclid.jdg/1090426286. ↑ 6, 8, 21, 24

[Moc06] ———. "Kobayashi-Hitchin correspondence for tame harmonic bundles and an application." *Astérisque* (2006) (309): viii+117. ↑ 2, 4, 5, 9, 10, 11, 16, 22

[Moc07] ———. "Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. I." *Mem. Amer. Math. Soc.* (2007) vol. 185 (869): xii+324. URL http://dx.doi.org/10.1090/memo/0869. ↑ 6, 8, 9, 18

[Moc19] ———. "Good wild harmonic bundles and good filtered Higgs bundles." *arXiv e-prints* (2019) arXiv:1902.08298v3. ↑ 9

[Mok89] Ngaiming Mok. *Metric rigidity theorems on Hermitian locally symmetric manifolds.*, vol. 6, (Singapore: World Scientific1989). ↑ 36

[Mok12] ———. "Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume." In "Perspectives in analysis, geometry, and topology. On the occasion of the 60th birthday of Oleg Viro. Based on the Marcus Wallenberg symposium on perspectives in analysis, geometry, and topology, Stockholm, Sweden, May 19–25, 2008," 331–354, (Basel: Birkhäuser2012). ↑ 28, 35, 36, 37, 38, 40, 41, 43, 46

[Mum77] David Mumford. "Hirzebruch's proportionality theorem in the non-compact case." *Invent. Math.* (1977) vol. 42: 239–272. ↑ 36

[MY93] Ngaiming Mok and Sai Kee Yeung. "Geometric realizations of uniformization of conjugates of Hermitian locally symmetric manifolds." In "Complex analysis and geometry," 253–270, (New York: Plenum Press1993). ↑ 3, 38

[Nad89] Alan Michael Nadel. "The nonexistence of certain level structures on abelian varieties over complex function fields." *Ann. Math. (2)* (1989) vol. 129 (1): 161–178. ↑ 37

[PT18] Mihai Păun and Shigeharu Takayama. "Positivity of twisted relative pluricanonical bundles and their direct images." *J. Algebr. Geom.* (2018) vol. 27 (2): 211–272. ↑ 20, 21

[Rag72] M. S. Raghunathan. *Discrete subgroups of Lie groups.* Results and Problems in Cell Differentiation, (Springer1972). ↑ 40

[Sim88] Carlos T. Simpson. "Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization." *J. Amer. Math. Soc.* (1988) vol. 1 (4): 867–918. URL http://dx.doi.org/10.2307/1990994. ↑ 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 18, 31

[Sim90] ———. "Harmonic bundles on noncompact curves." *J. Amer. Math. Soc.* (1990) vol. 3 (3): 713–770. URL http://dx.doi.org/10.2307/1990935. ↑ 4, 5, 6, 8, 9, 10, 16, 24, 31

[Sim92] ———. "Higgs bundles and local systems." *Inst. Hautes Études Sci. Publ. Math.* (1992) (75): 5–95. URL http://www.numdam.org/item?id=PMIHES_1992__75__5_0. ↑ 5, 6, 10

[Sim11] ———. "Local systems on proper algebraic V-manifolds." *Pure Appl. Math. Q.* (2011) vol. 7 (4): 1675–1759. ↑ 5

[To93] Wing-Keung To. "Total geodesy of proper holomorphic immersions between complex hyperbolic space forms of finite volume." *Math. Ann.* (1993) vol. 297 (1): 59–84. ↑ 36

[Tsu88] Hajime Tsuji. "A characterization of ball quotients with smooth boundary." *Duke Math. J.* (1988) vol. 57 (2): 537–553. ↑ 3

[TY87] Gang Tian and Shing-Tung Yau. "Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry." Mathematical aspects of string theory, Proc. Conf., San Diego/Calif. 1986, Adv. Ser. Math. Phys. 1, 574-629 (1987). (1987). ↑ 3

[Yau78] Shing-Tung Yau. "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I." *Commun. Pure Appl. Math.* (1978) vol. 31: 339–411. ↑ 3, 38

[Yau93] ———. "A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces." *Commun. Anal. Geom.* (1993) vol. 1 (3): 473–486. ↑ 3

[Zha09] Yuguang Zhang. "Miyaoka-Yau inequality for minimal projective manifolds of general type." *Proc. Am. Math. Soc.* (2009) vol. 137 (8): 2749–2754. ↑ 3

[Zuo00] Kang Zuo. "On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications." *Asian J. Math.* (2000) vol. 4 (1): 279–301. ↑ 3, 18, 19, 21

(Ya Deng) Institut des Hautes Études Scientifiques, Université Paris-Saclay, 35 route de Chartres, 91440, Bures-sur-Yvette, France
Email address: deng@ihes.fr
URL: https://www.ihes.fr/~deng

(Benoît Cadorel) Institut Élie Cartan de Lorraine, UMR 7502, Université de Lorraine, Site de Nancy, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex
Email address: benoit.cadorel@univ-lorraine.fr
URL: http://www.normalesup.org/~bcadorel/