On Kontsevich Generalizations of Tian-Todorov Theorem and Applications

Lucian IONESCU

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Novembre 2020

IHES/M/20/12

ON KONTSEVICH GENERALIZATIONS OF TIAN-TODOROV THEOREM AND APPLICATIONS

LUCIAN M. IONESCU

Abstract. Kontsevich recently generalized Tian-Todorov Theorem regarding the structure of the Kuranish space of deformations of a Kahler manifold with trivial canonical bundle. An alternative proof was given using a general result regarding the smoothness of moduli space of formal deformations, based on BV-algebra resolutions. From this, various other generalizations ensue and a conjecture relating the dimension of the tangent space of formal deformations and the first non-trivial Hodge number $h(n-1,1)$.

Additional details are provided, together with a proposed explanation regarding the above conjecture. Related considerations regarding mirror symmetry and motives of Calabi-Yau manifolds are included, based on the idea of complexifying TQFTs, modeled after Chow pure motives.

Contents

Date: November 13, 2020.
Key words and phrases. Calabi-Yau, Kuranishi space, BV-Algebras.

1. Introduction

Kontsevich provides an alternative proof of Tian-Todorov Theorem (TTT for short) in [1], §1, p.1, regarding the smoothness of the moduli space of deformations of a compact Kahler manifolds with $c_1(X) = 0$, using a much more general result which can be proved based on the BV-algebra formalism [1].

This approach provides not only a quick proof ([1] §3), but also suggests to look "upstream" to the characterization of the moduli space deformations in general, conform with Prop. 2.2 loc. cit. p.2., providing as a corollary several generalizations.

A review of the BV-algebra tool to investigate formal moduli spaces is included, having as a secondary goal obtaining additional applications.

The significance of these results to the theory of motives of CY manifolds will be also discussed, starting from the corresponding section §2 from [1].

2. Generalizations of Tian-Todorov Theorem

The original proof of Tian-Todorov Theorem employed transcendental methods [10, 11]. Subsequently pure algebraic proofs were provided [4, 3]. In [1] Kontsevich uses a general smoothness result regarding moduli spaces of formal deformations, to derive a short proof and several generalizations.

The prominent result, beyond the smoothness property, is the coincidence between the dimension of the moduli space and first non-trivial Hodge number $h(n-1,1)$. This led to a natural question regarding "abstract" Hodge structures of "algebraic-geometric origin", which can be thought off as the *Inverse TT-Problem.*

In this article a generalization of Lie Theory for polarized Hodges structures is proposed, relating the linear and non-linear objects, addressing the question from [1].

2.1. Review of Tian-Todorov Theorem.

Tian-Todorov Theorem states that the moduli space of Calabi-Yau manifolds is smooth. From [1], we recall the following:

Theorem 2.1. *Let X be a compact Kahler manifold of dimension n, with $c_1(X) = 0 \in Pic(X)$. Then:*

(i) The Kuranishi space of deformations of complex structures on X is smooth of dimension $h^{n-1,1}(X) := rkH^{n-1,1}(X)$;

(ii) Manifold X with deformed complex structure is again Kahler, with $c_1(X) = 0$. Similarly, if X is projective and $\omega \in H^2(X, Z)$ is an ample class, then the Kuranish space of deformations of X with polarization ω is also smooth, of primitive cohomology dimension $rkH^{n-1,1}_{prim}$.

(iii) Moreover, any choice of splitting of the Hodge filtration on $H^n(X)$, respectively of $H^n_{prim}(X)$, defines an analytic affine structure on the Kuranishy space:

$$Def_X : Art_k \to Sets \quad pro-represented\ by\ k[[x_1, ..., x_n]],\ n = h^{n-1,1}(X).$$

[1] TTT generalizes an earlier result by Bogomolov; an exposition of the this theorem can be found in [3], and an algebraic proof, in [4].

Proof 1. Smoothness. *The algebraic proofs focus on the smoothness claim, since a prior result of Grothendieck ensures that Def_X is pro-representable* [3], *p.2.*

In loc. cit. the T^1-lifting Theorem is used to infer smoothness; slightly more general techniques of Čech/cosimplicial dgLAs are used in [4], *to avoid the T^1-lifting Theorem, to prove the lack of obstructions in a slightly larger context.*

Dimension. *We will only recall the proof regarding the equality of dimensions in (i) and (iii). By classical deformation theory, the first Čech cohomology group $H^1(\mathcal{U}, \Theta_X)$ classifies first order deformations of X* [4], *p.1; alternatively, in therms of sheaf cohomology, which is isomorphic to Čech cohomology in our context:*

$$n = rk\ H^1(\mathcal{U}, \Theta_X) = rk\ H^1(X, T_X).$$

Recall Serre duality, a coherent sheaf cohomology analog of Poincare duality, for the tangent sheaf:

$$H^i(X, E) \times H^{n-i}(X, K_X \otimes T_X^*) \to H^n(X, K_X) \to k,$$

where the cup product followed by the natural trace is a perfect pairing. Alternativel, on compact complex manifolds, from Hodge and Dolbeault Theorems:

$$H^1(X, T_X) = H^{n-1}(X, K_X \otimes T_X^*)*,$$

Now if $c_1(X) = 0$ the canonical bundle K_X is trivial, and with $\Omega^1(X) = T_X^*$, we obtain the isomorphism relating the tangent space to the formal deformations to the "first" Hodge piece of top weight n:

$$H^{n-1}(X, \Omega^1(X)) \cong H^{n-1,1}(X) \cong H^{1,n-1}(X).$$

2.2. **BV-Algebra approach to smoothness.** Smoothness of the moduli space can be obtained from a formal affine structure on the moduli space using a resolution (deformation) of a BV-algebra structure associated to the dgLA controlling the corresponding deformation theory [2], Th. 4.18, p.92.

A brief recall following [1], Prop. 2.2, p.2, will allow to address the converse, and hence the main question from loc. cit.

2.2.1. *dgLAs from BV-Algebras.* Th original introduction of BV-algebras generalize BRST quantization in connection with gauge fixing in QFT. For such a purpose, a BV-algebra over C (A, d, Δ), consists of a unital super-algebra $(A, \cdot)$ endowed with two odd operators d and Δ, of orders at most one and at most two, satisfying the double complex condition $d^2 = \Delta^2 = 0, [d, \Delta] = 0$.

Kontsevich notes that $g = (\Pi A, [,]_\Delta)$, where:

$$[a, b]_\Delta = -\delta_H \Delta(a, b) = \Delta(ab) - \Delta(a)b - (-1)^{deg\ a} a\Delta(b)$$

is a naturally associated super Lie algebra, with d, Δ odd derivations.

Indeed, the Lie bracket is the curvature of the odd derivation Δ, and together with multiplication defines a *Gerstenhaber super-algebra.*

Hence it is natural to look at the associated formal moduli space, conform with the original theory of deformation of algebraic structures [17]. Then we have [1], Prop. 2.2, p.2:

Proposition 2.1. *Assume that* $H^\bullet(A[[u]], d + u\Delta)$ *is a free C[[u]]–module, where u is a formal even variable. Then the formal moduli space associated to the dgLA* $(g, d, [,]_\Delta)$ *is smooth: any trivialization of the above module structure gives flat coordinates on the moduli space.*

Recall that the curvature of the deformed differential $D = d + u\Delta$ (connection) is null on the formal deformation space (solutions of Maurer-Cartan equation):

$$\delta_H D = \delta_H(d) + u\delta_H(\Delta) = \delta_H d + [,]_\Delta == 0 \quad MC\ Eq.$$

Then the *tangent space to the deformation space* is $T(Def_g) = H^1(g)$ [18], and the moduli space $H^\bullet(g[[u]], D)$ is a trivial bundle (by assumption a free $C[[u]]$-module structure), with flat affine coordinates.

This result yields a shorter proof of Tian-Todorov Theorem [1], Prop. 2.2, p.2.

Proof 2. Apply the above result to the BV-algebra obtained from the Dolbeault complex valued in the Lie algebra of polyvector fields:

$$A_X := (\Gamma(X, \Omega^{0,\bullet} \otimes_{\mathcal{O}(X)} \wedge^\bullet T_X), \bar\partial, Div)$$

with Daulbeault differential $\bar\partial$ as d and divergence operator with respect to the volume as Δ. This is a free $C[[u]]$-module by the $\partial\bar\partial$-Lemma: combining Dolbeault complex (with its Hodge structure, to be used later on) and the Lie algebra of polyvector fields (Examples 1.9 and 1.10 [18], p.3). □

2.3. **Generalizations.** Certain generalizations are obtained [2], §4.3.3, recalled in [1] §3, based on the above more general result on moduli spaces, using BV-algebra formalism.

The specific cases (1)-(5) considered loc. cit. are a kind of *relative version* of Tian-Todorov Theorem, for a "broken Calabi-Yau variety"(X, D) with a divisor D with normal crossings, such that the Calabi-Yau condition holds "modulo boundary"; see [1] for the technical details. The technique in the proof applies also to certain non-compact Calabi-Yau fibered over a smooth curve $uv = f(x, y)$: flat coordinates on the moduli space are obtained from a splitting of the Hodge filtration (loc. cit. §4, p.4).

But then it is natural to ask how general is this coincidence between the dimension of the moduli space of deformations and the "first non-trivial" Hodge number $h(n-1, 1)$, in the non-compact or when singularities occur.

Kontsevich generalizations provide such examples when the proof based on BV-algebra still applies.

For singular varieties, Grothendieck's generalizations of Serre duality, more specifically *coherent duality* Serre-Grothendieck-Verdier duality [13] and *Grothendieck's relative point of view* (GRPoV) towards considering *variations* [14], as in the *Theory of Period Domains*, may be enough to extend the already existing proofs of the Tian-Todorov Theorem, including the equality of dimensions (compatibility with Hodge structure).

In what follows we will formalize a framework for addressing the abstract-algebraic framework for Hodge structures, in the spirit of Lie Theory, identifying the linear algebraic structure corresponding to the actual formal deformation and the two ways correspondence.

2.4. **A Lie Theory analog of Hodge Theory.** We will need a more formal approach in order to address Kontsevich question regarding the dimension of the moduli space of deformations [1], p.5; we will regard this as a type of "Inverse Problem".

2.4.1. *Lie vs. Hodge Theory: a comparison.* Recall the Lie paradigm, completed by Killing and Cartan together with the general structure via *Levi Decomposition*, as a guiding analogy:

$$\underline{Linear\ fibration} \quad \overset{Def.\ Theory}{\longrightarrow} \quad \underline{Top-Diff.}$$

$Lie\ algebra$	$Lie\ Group$	$Moduli\ Bundle$ $(GRPoV)$
$Levi\ Dec.$	$Solv. \to g$	$H.\ Str. \to X_b$
	$\downarrow$ $short\ exact\ seq.$	$\downarrow$ $Fibration$
$Root\ System$	$SS-Lie\ alg.$	$Period\ Domain$

The idea is to take a polarized linear Hodge structure (H_Z, Q, h):

$$Q : H_Z \times H_Z \to Z,\ h : C^\times \to GL(V_C),\ V_C := H_Z \otimes_Z C,$$

where h is a weight k algebraic representation on the complexification of a lattice with a polarization isomorphism $Q : H_Z \to H_Z^*$ [15], as an analog of the semisimple Lie algebra above, described by a root system analog to the lattice and polarization, yielding a highest weight representation of weight k (Killing-Cartan-Weyl theory). Then define the analog of a Lie group obtained from a Lie algebra via exponential, using instead the deformation functor, where the exponential is replaced by the Maurer-Cartan differential (graded) equation, which yields formal solutions as 1-parameters of objects in a given tangent direction.

Here we apply the intuitive picture from manifolds and tangent spaces to categories of objects, consistent with GRPoV, e.g. variations of complex structures or Hodge structures. We then inspect the resulting moduli space of formal deformations, as if coming from an actual algebraic-geometric object, like the case of a Calabi-Yau manifold.

In the converse direction, the formal moduli space is an analog of a homogeneous space, or torsor, and can be represented as a flag variety, providing the fibration over a period domain as a flag variety, or Hodge structure via the correspondence between homogeneous spaces relative to a parabolic subgroup G/P, and associated flag varieties $V_0 \subset ... \subset V_n$. This will provide the algebraic-geometric object, e.g. variations of a Kähler manifold over a period domain, and then taking sheaf cohomology to obtain the variation of Hodge structures.

2.4.2. *Is there a "Converse" to Tian-Todorov Theorem?* In other words, we will observe the following principles:

1) Algebraic-geometric objects are bundles of algebraic structures and variations are analogous to geometric curves in Differential Geometry; moreover Deformation Theory is a broad generalization of Lie Theory [16];

2) The Hodge structures coming from sheaf cohomology correspond to representations of flag varieties, via Deformation Theory.

3) The Inverse Problem: from a polarized Hodge structure, as if of "algebraic-geometric origin", i.e. viewed as a highest weight orbit of a flag variety (projective homogeneous variety), construct the Moduli Space over a Period Domain , via Formal Deformation Theory: $Cl_Q(V_C; h)$ as a deformation of exterior algebra and Grassmann flag variety, *as if* controlled by the Kontsevich's dgLA of polyvector fields;

4) Prove that these two constructions are inverse to one another:

a) From polarized H.S. to Formal Moduli Space satisfies TTT conclusion;

b) TTT: given an algebraic-geometric manifold, e.g. Kahler / Calabi-Yau, so that some form of generalized Serre duality enables the equality of dimensions, the smoothness of moduli formal space holds when the algebraic fiberwise condition is an "open" condition, via a proof using the BV-algebra resolution (geometric quantisation in disguise: BRST formalism).

To place ourselves in the context from Prop. 2.2, i.e. a unital BV-algebra $(A, m, 1, d, \Delta)$ with associated Lie algebra $g = (\Pi A, [,]_\Delta$, consider a polarized lattice (H_Z, Q) and the Clifford algebra $Cl_Q(H_C)$ of the complexification, with its associated Hodge structure (coming from complexification), as the underlying algebra of the BV algebra. Include the exterior algebra $\wedge^\bullet H_C$ as the degenerate case ("trivial deformation"), with $Q = 0$.

This provide the Dolbeault "half" of the BV-algebra, except for the Dolbeault differential capturing the manifold's complex structure.

To play the role of the polyvector fields, consider $\wedge^\bullet g$ where the Lie algebra elements are formally "vector fields", i.e. "infinitesimal" generators of MC solutions.

Now we need to glue them over the same "Grothendieck cover" to get the analog of a coherent sheaf of a variety. A natural approach is via Rees construction, interpreting modules over the Cliffordd algebra in the spirit of Swan's Theorem, as playing the role of a bundle.

This will be investigated elsewhere, returning to Kontssevich implicit question.

3. Complexification of TQFTs and Applications

The role of Hodge structures go beyond the Calabi-Yau motives, as addressed in the previous section. The following direction of generalization will need a separate treatment, to be detailed elsewhere.

In [5] it was explained how Chow motives can be viewed as embedded cobordisms with cohomology playing the role of a TQFT; we even expect an analog of Whitney Theorem for manifolds to hold.

In the other direction, this invites to generalize Hodge structure mechanism to TQFTs. The original category of cobordisms consisted of real manifolds[7], where the bulk-boundary dimension step is always one.

3.1. **Complex TQFTs and Periods.** Now it is natural to complexify such a theory[2] and consider complex manifolds instead, with the cobordisms category consisting of complex manifolds and divisors, as in the theory of abstract periods from [6].

For example, the $1D$-Complex TQFT (CTQFT for short), would have cobordisms the punctured Riemann surfaces $\mathcal{M}(g, n)$. A CTQFT is essentially a CFT, but with an eye on TQFTs and the other on motives ...

Now let us look for an analog of the Hodge decomposition.

In such a Kahler type of framework, with its additional "rotational capabilities" reminiscent of a Wick rotation, allows to exchange space and time dimensions, where here "time" refers to 1-parameter variations of a subvariety.

A familiar example is that of a semi-Riemannian space-time having a Cauchy hypersurface, and allowing to represent the total space as a fibration [8], p.?.

When applying this idea to a divisor ($n-1$ complex dimensional subvariety) in the total space of formal deformations, with the 1-parameter fibers due to gauge transformations, we expect to have the moduli space the base of a similar fibration. Indeed, in view of an analog of Hurwitz Theorem, the abelianization of Chow motives, as cobordisms modulo conjugation, should yield the hypercohomology.

$$Chow\ motives: \ (X, D, \gamma)/\sim \quad \overset{Ab}{\to} \quad H^1(X, \Omega^{n-1}).$$

Then the dimension of the tangent space to the moduli space of deformations should correspond to the $H^{n-1,1}$ piece in the Hodge decomposition:

$$h^{n-1,1} = rk\ H^{n-1,1}(X), \quad H^{p,q}(X, C) = H^q(X, \Omega^p).$$

[2]... after reading Arnold [9]; see the other examples.

3.2. **Example: Punctured Riemann Surfaces.** Consider the category of punctured Riemann surfaces (Σ, D) and holomorphic maps $f : (\Sigma, D_1) \to (\Sigma_2, D_2)$ as *cobordisms*. Then a $CTQFT$ is a functor associating to a point a vector space $F(*) = V$ and to such a morphism an operator $F(f) : V^{\otimes |D_1|} \to V^{|D_2|}$, where $|D_i|$ denotes the degree of the divisors (number of punctures).

This is a genuine theory of "vertex operators category". One could consider a global object summing over all the objects in the category, and then investigate the relation with VOAs.

4. Conclusions

The theory of Hodge structures in the context of algebraic varieties or sheaf theory, originally based on filtrations, leads to a theory of flag varieties. The categorical framework proposed by the author is based on Jordan-Holder 2-category structures, to be addressed elsewhere. The usual approach for the theory of periods as in [6] is based on torsors, essentially homogeneous spaces, and is linked to a flag varieties approach.

The Lie Theory analog of Hodge Theory proposed, via Clifford algebras, may be thought of as the alternative to the universal enveloping algebra approach to quantum groups; both are based on the Deformation Theory paradigm. While Lie-Cartan Theory makes use of root systems as a foundation for highest weight representation theory, the polarized Hodge structure is essentially a theory of projective homogeneous varieties (flag varieties), and highest weight vector orbit in projective representations of the corresponding group.

In the "non-linear-to-linear" direction, Tian-Todorov Theorem with its BV-algebra reinterpretation by Kontsevich provides half of the theory: from algebraic-geometric object and its variations in a given category, to the linear object belonging to Deformation Theory.

The question regarding the "Inverse Problem" suggested the other half, which was addressed by introducing the analog of Lie exponential relating the Lie algebra and the Lie group; here, from the dgLA with additional structure (BV-algebra related) and its formal moduli space, the deformation functor commutes with the Clifford construction, providing the Hodge structure of "algebraic-geometric origin".

The constructions in the two directions, once the proofs are consolidated, are expected to define an equivalence of categories.

The directions of research in the theory of Hodge structures, towards motivic algebraic-geometry, evolved in the theory of flag varieties concept; we prefer a Jordan-Holder structure categorical approach, an alternative line towards a categorical formulation of motives. Complexifying TQFTs in the sense of Arnold, yields a theory of representations of categories of cobordisms of complex varieties, with boundaries divisors. For example, Chow motives maybe thought off as "embedded cobordisms"; similarly, Kontsevich abstract periods [6], allows to define such functors. C TQFTs are comparable to Nori Motives approach via representations of categories of diagrams [12].

Looking for a pair of "Lie type" construction / reconstruction theorems, as above, between linear polarized Hodge structures and formal moduli spaces are an invitation to a comparison with the other duality theories and representability results (Pontryagin, Tannaka-Krein, Kashdan-Lusztig).

While related, the various approaches to a *Theory of Periods*, are complementary, with emphasis on Mathematics or Physics, and certainly author dependent; yet they aim to fulfill Grothendieck's vision for a *Theory of Motives*.

5. Acknowledgements

The author thanks IHES for the wonderful research conditions provided during this recent visit. He also thanks Prof. Maxim Kontsevich for providing such a motivating question and exciting topic of research.

The author also hopes the reader will choose to take the many loose ends as windows of opportunity for further developments.

References

[1] M. Kontsevich, Generalized Tian-Todorov Theorems, https://www.ihes.fr/ maxim/TEXTS/Kinosaki.pdf

[2] L. Katzarkov, M. Kontsevich and T. Pantev, Hodge theoretic aspects of Mirror symmetry, https://arxiv.org/pdf/0806.0107.pdf

[3] Daniel Litt, An exposition of the Tian-Todorov Theorem, https://www.daniellitt.com/expository-notes

[4] Donatella Iacono, Marco Manetti, An algebraic proof of Bogomolov-Tian-Todorov Theorem, https://arxiv.org/abs/0902.0732

[5] L. M. Ionescu, "Periods, Hodge structure and Motives", IHES research proposal, 2019; www.ilstu.edu/˜lmionesISUP/ISU-GP.html

[6] M. Kontsevich and D. Zagier, Periods, IHES 2001, https://www.maths.ed.ac.uk/˜v1ranick/papers/kontzagi.pdf

[7] M. Atiyah, TQFTs, Publications Mathematiques de l'IHES, 1988 http://www.numdam.org/article/PMIHES_1988__68__175_0.pdf

[8] B. O'Neil, Semi-Riemannian Geometry With Applications to Relativity", Pure and Applied Mathematics, Volume 103, 1st Edition, 1983.

[9] V. I. Arnold, "Symplectization Complexication and Mathematical Trinities", Fields Institute Communications 1999, Volume 24, pp.23-38; https://www.maths.ed.ac.uk/˜v1ranick/papers/arnold4.pdf

[10] G. Tian: Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. Mathematical Aspects of String Theory (San Diego, 1986), Adv. Ser. Math. Phys. 1, World Sci. Publishing, Singapore, (1987), 629-646.

[11] A.N. Todorov: The Weil-Petersson geometry of the moduli space of SU(n≥ 3) (Calabi-Yau) Manifolds I. Commun. Math. Phys., 126, (l989), 325-346; https://link.springer.com/article/10.1007/BF02125128

[12] Annette Huber and Stefan Müller-Stach, "Periods and Nori Motives", A Series of Modern Surveys in mathematics, Vol. 53, Springer, 2017.

[13] Wikipedia, "Coherent duality", https://en.wikipedia.org/wiki/Coherent_duality
[14] Wikipedia, "Grothendieck's relative point of view", https://en.wikipedia.org/wiki/Grothendieck%27s_relative_point_of_view
[15] L. M. Ionescu, "On Hodge Structures and Periods - Part I: Algebraic Aspects", 2020.
[16] L. M. Ionescu, "From Lie Theory to Deformation Theory", https://arxiv.org/abs/0704.2213
[17] Murray Gerstenhaber, "On the deformations of rings and algebras", 1962, https://www.jstor.org/stable/1970484?seq=1#metadata_info_tab_contents
[18] Marco Manetti, "Deformation theory via differential Lie algebras", 2005, math.AG/0507284.

Department of Mathematics, Illinois State University, IL 61790-4520
Email address: `lmiones@ilstu.edu`