Superconnection in the spin factor approach to particle physics

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Abstract

The notion of superconnection devised by Quillen in 1985 and used in gauge-Higgs field theory in the 1990's is applied to the spin factors (finite-dimensional euclidean Jordan algebras) recently considered as representing the finite quantum geometry of one generation of fermions in the Standard Model of particle physics.

Contents

1 Introduction **3**

2 Internal symmetry and superconnection **5**

3 Fermionic oscillators. Particle subspace **8**

4 Higgs potential and bosonic Lagrangian **10**

5 Outlook **13**

References

1 Introduction

It is natural to expect that the finite spectrum of fundamental particles of matter corresponds to representations of a finite-dimensional algebra of quantum observables endowed with some further structure. On the basis of the spectral theory needed for quantum mechanics, these finite-dimensional algebras of quantum observables have been identified as the finite-dimensional euclidean (or formally real) Jordan algebras [23], [24] and have been classified [25]. These algebras are the quantum analogues of the finite-dimensional algebras of real functions, it is convenient to consider them as algebras of "real functions" on virtual "finite quantum spaces". We will use freely this analogy by refering to "the finite quantum space" corresponding to a finite-dimensional euclidean Jordan algebra. Any finite-dimensional euclidean Jordan algebra has a unit and is the direct sum of a finite number of simple ideals and the simple finite-dimensional euclidean Jordan algebras fall into 3 classes :

1. The hermitian $n \times n$-matrices $J_n^1 = \mathcal{H}_n(\mathbb{R})$, $J_n^2 = \mathcal{H}_n(\mathbb{C})$ and $J_n^4 = \mathcal{H}_n(\mathbb{H})$ over the reals, the complexes and the quaternions, for $n \geq 3$ and $\mathbb{R}$ $(= \mathcal{H}_1(\mathbb{R}) = \mathcal{H}_1(\mathbb{C}) = \mathcal{H}_1(\mathbb{H}))$.

2. The spin factors $J_2^n = JSpin_{n+1}$ $(n \geq 1)$.

3. The exceptional Jordan algebra of hermitian 3×3-matrices $J_3^8 = \mathcal{H}_3(\mathbb{O})$ over the octonions.

The Jordan algebra J_3^8 is exceptional in the sense that it cannot be realized as a subspace of an associative algebra stable under the symmetrized product [1]. The classes 1 and 2 contain only special (i.e. non exceptional) Jordan algebras. However there is an important difference between Class 1 and Class 2. Namely the Jordan algebras which belong to Class 1 are the real subspaces of all hermitian elements of associative $*$-algebras while in the case of Class 2, the spin factors $JSpin_n$ are only Jordan subalgebras of the Jordan algebras of all hermitian elements of the Clifford algebras $C\ell(n,0)$ or of their complex $*$-algebra versions $C\ell_n$ which are strict (and small) Jordan subalgebras except for the cases $n=2$ and $n=3$ where $JSpin_2 = \mathcal{H}_2(\mathbb{R})$ with $C\ell(2,0) = M_2(\mathbb{R})$ and $JSpin_3 = \mathcal{H}_2(\mathbb{C})$ with $C\ell(3,0) = M_2(\mathbb{C})$ and in the case $n=5$ where one has $JSpin_5 = \mathcal{H}_2(\mathbb{H})$ but then $C\ell(5,0) = M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$. Thus, in general, there is a non trivial envelop $\tilde{J}_2^n$ of $J_2^n = JSpin_{n+1}$ which is the euclidean Jordan algebra of hermitian elements of $C\ell(n+1,0)$ or of its complex $*$-algebra version $C\ell_{n+1}$ (see Section 4 of [16]).

The Jordan algebra approach to the finite quantum geometry of particle physics models was originally developed [14], [16] in the context of the exceptional Jordan algebra $J_3^8 = \mathcal{H}_3(\mathbb{O})$. It was realized in [33], [32], [16] that the quantum geometry of one generation is captured by a special Jordan algebra – the 10-dimensional spin factor

$$J_2^8 = JSpin_9 = \mathcal{H}_2(\mathbb{O})(\subset J_3^8)\,, \tag{1.1}$$

i.e. the 2×2 hermitian matrices with octonionic entries. The gauge symmetry group of the Standard Model (SM) of particle physics,

$$G_{\rm SM} = S(U(3) \times U(2)) = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6} \tag{1.2}$$

is the subgroup of the automorphism group Spin(9) of J_2^8 that preserves the splitting

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3 \tag{1.3}$$

and acts $\mathbb{C}$-linearly on $\mathbb{C}^3$.

The splitting (1.3) is preserved by the subgroup $SU(3)$ of the automorphism group G_2 of the octonions which was identified long ago to the colour symmetry of quarks by Gürsey and Günaydin [19], [20]. From the point of view of physics (1.3) corresponds to the quark-lepton symmetry. Conversely it was shown in [14] that the unitarity and the unimodularity of $SU(3)$ lead directly to a unital algebra structure on $\mathbb{C} \oplus \mathbb{C}^3$ which is isomorphic to $\mathbb{O}$ as real algebra, $SU(3)$ being then the group of $\mathbb{C}$-linear automorphisms. In other words, this associates the quark-lepton symmetry to the unimodularity of the colour group and selects the euclidean Jordan algebras $J_2^8 = \mathcal{H}_2(\mathbb{O})$ and $J_3^8 = \mathcal{H}_3(\mathbb{O})$ endowed with their automorphisms preserving the splitting (1.3), (notice that $\mathcal{H}_1(\mathbb{O}) = \mathbb{R}$ and that the $\mathcal{H}_n(\mathbb{O})$ for $n \geq 4$ are not Jordan algebras).

The resulting characterization of $G_{\rm SM}$ was recently commented in [26] where the action of Spin(9) on a pair of octonions (that spans the spinor representation $\mathbf{16} \simeq \mathbb{O}^2$ of Spin(9) and appear in the 27-dimensional algebra J_3^8) is exploited. A Jordan algebra modification of Connes' non-commutative geometry approach to the SM,[8], [7] is developed in [5].

The aim of the present paper is to further develop the J_2^8 approach to the internal space of the Standard Model (for one generation) by using the notion of superconnection, introduced by Quillen et al. [29], [27] and applied to the Higgs mechanism within the Weinberg-Salam model in [9], [28], [30], [2].

As noted in [14] and elaborated in [26] one can similarly derive the electroweak subgroup $U(2)$ of the gauge group $G_{\rm SM}$ of the SM from the automorphism group Spin(5) of the spin factor $J_2^4 = \mathcal{H}_2(\mathbb{H})$:

$$\mathrm{Aut}(J_2^4) = \mathrm{Spin}(5) = U(2, \mathbb{H}) \, , \quad J_2^4 = \mathcal{H}_2(\mathbb{H}) \tag{1.4}$$

with the alternative (but non-associative) ring $\mathbb{O}$ of octonions substituted by the associative division algebra $\mathbb{H}$ of quaternions. The superconnection approach applies equally well to the "mini internal space" J_2^4 of the electroweak model of leptons which can thus serve as a simpler "toy model" for J_2^8.

We begin, in Sect. 2, by summarizing both our treatment of the euclidean extension

$$\widetilde{J}_2^8 := \mathcal{H}_{16}(\mathbb{C}) \oplus \mathcal{H}_{16}(\mathbb{C}) = J_{16}^2 \oplus J_{16}^2 \tag{1.5}$$

of J_2^8 (that admits an analogue $\widetilde{J}_2^4$ for J_2^4) and the quite natural realization of the notion of superconnection in it. In Sect. 3 we recall the fermionic oscillator realization of $C\ell(9,1)$ and characterize the 16-dimensional particle subalgebra $J(\mathcal{P})$ of $\widetilde{J}_2^8$. In Sect. 4 we introduce the Higgs potential allowing a symmetry breaking minimum and derive the mass matrix for the gauge fields. Section 5 is our temporary conclusion.

Our notations and conventions are the ones of [14] and [16] and of [31] in particular for the Clifford algebras and their "fermionic oscillator" (i.e. C.A.R.) representations for the even-dimensional case. Concerning the latter point, it should be mentioned that the representation of the Clifford algebra of an even-dimensional euclidean space as the algebra of canonical anti-commutation relations (C.A.R.) depends on the choice of a direction of simple spinor in the sense of Elie Cartan which is the corresponding direction of the Fock vacuum [11]. In fact the directions of simple spinors parametrize the isometric complex structures (see also [13] for a more general point of view). Finally it is worth noticing that Sections 3.2 and 3.3 of [14] and Section 2 of [31] contain motivated summaries of the Jordan-von Neumann-Wigner classification and that, in this respect [31] is a fairly complete reference. For Jordan algebras and Jordan modules our reference is [22] and for exceptional Lie groups see [34].

2 Internal symmetry and superconnection

As explained in Sect. 4 of [16] and in Sect. 2.2 of [31] the optimal euclidean extension of J_2^8 is the direct sum (1.5) of two Jordan algebras of complex hermitian 16×16 matrices. It contains, in particular, the hermitean generators $i\,\Gamma_{ab}$, $a, b = 0, 1, \ldots, 8$ of the derivation algebra $so(9)$ viewed as a sub Lie algebra of $so(9,1) \subset C\ell^0(9,1) \simeq C\ell(9,0)$, the (restricted) structure algebra of J_2^8. Choosing a basis $(e_0 = 1, e_1, \ldots, e_7)$ of octonion units we can think of J_2^8 as generated by the 2×2 hermitian octonionic matrices

$$\widehat{e}_a = \begin{pmatrix} 0 & e_a \\ e_a^* & 0 \end{pmatrix}, \quad a = 0, 1, \ldots, 7 \quad (e_0^* = e_0, e_j^* = -e_j \text{ for } j = 1, \ldots, 7),$$

$$\widehat{e}_8 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

We shall represent them by the products

$$\Gamma_{-1}\Gamma_a, \quad a = 0, 1, \ldots, 8, \quad [\Gamma_a, \Gamma_b]_+ := \Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\delta_{ab}\,,$$
$$[\Gamma_{-1}, \Gamma_a]_+ = 0\,, \quad \Gamma_{-1}^2 = -\mathbb{1} \Rightarrow (\Gamma_{-1}\Gamma_a)^2 = \mathbb{1}\,, \tag{2.2}$$

where Γ_α, $\alpha = -1, 0, \ldots, 8$, generate the Clifford algebra $C\ell(9,1)$. The Coxeter element $\omega_{9,1}$ of $C\ell(9,1)$ plays the role of chirality and commutes with $so(9,1)$:

$$\gamma := \omega_{9,1} = \Gamma_{-1}\Gamma_0\Gamma_1 \ldots \Gamma_7\Gamma_8\,,\ \gamma^2 = \mathbb{1}\,;\ [\gamma, \Gamma_{\alpha\beta}] = 0 \text{ for } \Gamma_{\alpha\beta} = \frac{1}{2}[\Gamma_\alpha, \Gamma_\beta]\,. \tag{2.3}$$

In a representation in which γ is diagonal the 32-dimensional Dirac spinor representation of $so(9,1)$, generated by $\Gamma_{\alpha\beta}$, is reduced:

$$\mathbf{32} = \mathbf{16}_L \oplus \mathbf{16}_R \,, \;\; (\gamma - 1)\mathbf{16}_L = 0 = (\gamma + 1)\mathbf{16}_R \,. \tag{2.4}$$

The $C\ell(9,1)$ generators anticommute with chirality and intertwine left and right chiral (Weyl) spinors

$$[\Gamma_\alpha, \gamma]_+ = 0 \,, \;\; \Gamma_\alpha : \mathbf{16}_{L,R} \to \mathbf{16}_{R,L} \,, \;\; \alpha = -1, 0, 1, \ldots, 8 \,. \tag{2.5}$$

In Haag's approach [21] to quantum field theory the algebra of observables is a subalgebra of gauge invariant elements (with respect to the unbroken gauge symmetry) of a larger *field algebra*. The finite-dimensional (internal space) counterpart of the field algebra is $\mathbb{Z}_2$-graded with odd part anticommuting with γ, generated by Γ_α which will give room to the *Higgs* (scalar) *fields*, and an even part, commuting with γ generated by the 45 hermitean matrices

$$\Gamma_{-1a}, i\Gamma_{ab} \in so(10, \mathbb{C})(\supset so(9,1)) \,, \;\; a, b = 0, 1, \ldots, 7, 8, \tag{2.6}$$

associated, in particular, with the gauge fields.

We proceed to identifying the symmetry generators and a complete set of commuting observables. Singling out $e_7 \in \mathbb{O}$ as the imaginary unit preserved by $SU(3)$ we can write the decomposition (1.3) in the form (cf. Appendix):

$$\mathbb{O} \ni x = z + Z \,, \;\; z = x^0 + x^7 e_7 \,, \;\; Z = Z^1 e_1 + Z^2 e_2 + Z^4 e_4 \,,$$
$$Z^j = x^j + x^{3j (\mathrm{mod}\, 7)} e_7 \,, \;\; j = 1, 2, 4, \tag{2.7}$$

where we have used the octonionic multiplication rules of [3]

$$e_i \, e_{i+1} = e_{i+3 (\mathrm{mod}\, 7)} (= -e_{i+1} \, e_i) \,, \;\; i = 1, 2, \ldots, 7 \,. \tag{2.8}$$

We then identify the Pati-Salam subalgebra

$$su(2)_L \oplus su(2)_R \oplus su(4) \subset so(10)$$

by setting

$$su(4) \simeq so(6) = \mathrm{Span}\{\Gamma_{jk}, \;\; j, k = 1, 2, \ldots, 6\} \,,$$
$$su(2) \oplus su(2) \simeq so(4) = \mathrm{Span}\{\Gamma_{\alpha\beta}, \;\; \alpha\beta = -1, 0, 7, 8\} \,. \tag{2.9}$$

In particular, we choose a basis of $su(3) \oplus u(1)$ invariant commuting observables as

$$2I_3^L = \frac{1}{2}(\Gamma_{8-1} - i\Gamma_{07}) \,, \;\; 2I_3^R = -\frac{1}{2}(\Gamma_{8-1} + i\Gamma_{07}) \Rightarrow I_3^L I_3^R = 0 \,, \tag{2.10}$$

$$B - L = \frac{i}{3}(\Gamma_{13} + \Gamma_{26} + \Gamma_{45}) \,, \tag{2.11}$$

B and L being the baryon and the lepton numbers. The colour gauge Lie algebra $su(3)_c$ then appears as the commutant of $B-L$ in $su(4)$. The weak hypercharge Y and the electric charge Q are expressed as:

$$Y = B - L + 2I_3^R\,,\ Q = I_3^L + \frac{1}{2}Y = \frac{1}{2}(B-L) - \frac{i}{2}\Gamma_{07}\,. \tag{2.12}$$

The left and right isospins take values 0 and 1/2 so that $2I_3^L$ and $2I_3^R$ satisfy

$$(2I_3^X)^3 = 2I_3^X \quad \text{for} \quad X = L, R \Rightarrow P_1 := (2I_3^L)^2 = P_1^2 = \mathbb{I} - (2I_3^R)^2\,. \tag{2.13}$$

(P_1 being the $SU(2)_L$ invariant projector on the states of weak isospin 1/2.)

We can write the (skew hermitian) matrix valued gauge field 1-form

$$\widehat{A} = dx^\mu A_\mu^s X_s = i\widehat{W} + i\widehat{B} + i\widehat{G} \tag{2.14}$$

where $s = 1, \ldots, 12 = \dim G_{\rm SM}$ and X_s are suitable linear combination of the matrices (2.6); the three terms $\widehat{W}, \widehat{B}, \widehat{G}$ correspond to the subalgebras $su(2)_L$, $u(1)_Y$, $su(3)_c$, respectively, of the Lie algebra

$$\mathcal{G}_{\rm SM} = su(2)_L \oplus u(1)_Y \oplus su(3)_c \tag{2.15}$$

of the gauge group $G_{\rm SM}$ (1.2); they will be displayed explicitly in Sect. 3 below.

We shall interrupt for a moment our exposition in order to summarize, for reader's convenience, the notion of a superconnection on the example of the gauge group $U(n)$ acting on the exterior algebra $\bigwedge \mathbb{C}^n$ as worked out in [30]. We shall identify the $\mathbb{Z}_2$ grading of $\bigwedge \mathbb{C}^n$ with chirality, assuming (arbitrarily) that $\bigwedge^0 \mathbb{C}^n$ is right chiral (i.e. has negative chitality) and denote by $A^\pm$ left and right chiral propjections of the $U(n)$ connection $\hat{A}$. We then define the $U(n)$ *superconnection* 1-form on $T^*M \otimes \bigwedge \mathbb{C}^n$ by

$$\mathbb{D} = d + \widehat{A} + \widehat{\Phi}\,,\ \widehat{A} = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix},\ \widehat{\Phi} = \begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix} \tag{2.16}$$

where $d = dx^\mu \partial_\mu$ and the two by two block matrix has $2^{n-1} \times 2^{n-1}$ dimensional blocks. The $\mathbb{Z}_2$ grading of 1-forms is the combined grading of fields (in which A_μ^+ and A_μ^- are even and $\widehat{\Phi}$, ϕ^*, ϕ are odd) and of differential forms (in which dx^μ is odd, $dx^\mu \wedge dx^\nu$ is even, etc.). Thus the superconnection $\mathbb{D}$ is odd. The corresponding curvature form is obtained using the $\mathbb{Z}_2$ graded commutator:

$$\mathbb{F} = \widehat{F} + \mathbb{D}\,\widehat{\Phi}\,, \quad \widehat{F} = DA\,,\ \mathbb{D}\,\widehat{\Phi} = [\mathbb{D}, \widehat{\Phi}]_+ \tag{2.17}$$

where $D\widehat{A} = dx^\mu \wedge dx^\nu \begin{pmatrix} F_{\mu\nu}^+ & 0 \\ 0 & F_{\mu\nu}^- \end{pmatrix}$, $F_{\mu\nu}^\pm = \partial_\mu A_\nu^\pm - \partial_\nu A_\mu^\pm$, while

$$[\mathbb{D}, \widehat{\Phi}]_+ = \widehat{\Phi}^2 + \begin{pmatrix} 0 & (D\phi)^* \\ D\phi & 0 \end{pmatrix},\ D\phi = D^-\phi + \phi D^+ = dx^\mu((\partial_\mu + A_\mu^-)\phi - \phi A_\mu^+),$$

$$(D\phi)^* = dx^\mu((\partial_\mu + A_\mu^+)\phi^* - \phi^* A_\mu^-)\,. \tag{2.18}$$

In the last two equations we have used the anticommutativity of $\phi^{(*)}$ and dx^μ. We observe that the above construction works once one has the notion of chirality which allows to define "the Higgs" as a matrix valued chirality changing scalar field. Remarkably, embedding our $\tilde{J}_2^8$ model into $C\ell(9,1)$ provides a natural notion of chirality, Eq. (2.3), such that the operator

$$\widehat{\Phi} = \phi^\alpha \Gamma_\alpha \tag{2.19}$$

is chirality changing. For $\gamma = \sigma_3 \otimes \mathbf{1}_{16}$ the matrices (2.18) are reproduced.

3 Fermionic oscillators. Particle subspace

We shall use the following fermionic oscillator's representation of $C\ell(10,\mathbb{C})$ (cf. [17], [32], [31]):

$$2a_0 = \Gamma_0 + i\Gamma_7\,,\ 2a_j = \Gamma_1 + i\Gamma_3 j(mod 7), j = 1,2,4, 2a_8 = \Gamma_8 + \Gamma_{-1}$$

$$(2a_0^* = \Gamma_0 - i\Gamma_7, 2a_1^* = \Gamma_1 - i\Gamma_3, ..., 2a_8^* = \Gamma_8 - \Gamma_{-1}),$$
$$[a_\mu, a_\nu]_+ = 0\,, [a_\mu, a_\nu^*]_+ = 2\delta_{\mu\nu}\,. \tag{3.1}$$

The basic fermions and antifermions are given by the primitive idempotents of the abelian (unital) algebra generated by the Cartan subalgebra of the (complexified) $so(9,1)$. It is spanned by the idempotents

$$\pi_\nu = a_\nu a_\nu^* (= \pi_\nu^2), \pi_\nu' = a_\nu^* a_\nu = 1 - \pi_\nu\ (\pi_\nu \pi_\nu' = 0), \nu = 0,1,2,4,8. \tag{3.2}$$

They belong to the euclidean extension $\tilde{J}_2^8$ (1.5) of the octonionic spin factor J_2^8. The *symmetry subalgebra* (respecting the quark lepton splitting (1.3)) of the (complexified $so(9,1)$) is the rank five extension

$$\mathfrak{g}_{ext} = u(2) \oplus u(3),\ u(2) = Span\{a_\alpha^* a_\beta, \alpha, \beta = 0, 8\},$$
$$u(3) = Span\{a_j^* a_k, j,k = 1,2,4\}, \tag{3.3}$$

of the gauge Lie algebra $\mathfrak{g}_{SM} = s(u(2) \oplus u(3))$ of the SM. In particular, the (left) electroweak $su(2)_L$ symmetry generators,

$$I_+^L = a_8^* a_0,\ I_-^L = a_0^* a_8,\ 2I_3^L = [I_+^L, I_-^L] = \pi_8' - \pi_0', \tag{3.4}$$

are complemented by

$$2I_3^R = \pi_8 - \pi_0',\ B - L = \frac{1}{3} \sum_{j=1,2,4} [a_j^*, a_j] = \frac{1}{3} \sum_j (\pi_j' - \pi_j) \tag{3.5}$$

(cf. (2.10) (2.11)). The $u(1)$ centre of $\mathfrak{g}_{SM}$ is spanned by the hypercharge

$$Y = B - L + 2I_3^R = \frac{2}{3}(\pi_1' + \pi_2' + \pi_4') - \pi_0' - \pi_8', \tag{3.6}$$

the linear combination of $B-L$ and $2I_3^R$ that annihilates the right chiral (sterile) neutrino:

$$(\nu_R) := |\nu_R >< \nu_R| = \pi_0\pi_1\pi_2\pi_4\pi_8 \Rightarrow Y(\nu_R) = 0. \tag{3.7}$$

A general problem in theories with configuration space of the form $\mathcal{C}(M)\otimes\mathcal{F}$, the product of the commutative algebra of smooth functions on a spin manifold M with a finite dimensional (not necessarily commutative or associative) algebra $\mathcal{F}$, first encountered in the better developed noncommutative geometry approach [19], [7], is the problem of fermion doubling (or rather quadrupling) [18], recently tackled in [4]. In order to avoid (or reduce) the problem one can simply restrict attention to the 16 dimensional *particle subalgebra*

$$J(\mathcal{P}) = \mathcal{H}_8^L(\mathbb{C}) \oplus \mathcal{H}_8^R(\mathbb{C}) \tag{3.8}$$

of the Jordan algebra (1.5). The projector $\mathcal{P}$ on the particle subspace can be written as the sum of projectors ℓ and q on the lepton and the quark subspaces:

$$\mathcal{P} = \ell + q (= \mathcal{P}^2),\ \bar{\mathcal{P}}(= 1 - \mathcal{P}) = \bar{\ell} + \bar{q},\ \ell = \pi_1\pi_2\pi_4\ (L = \ell - \bar{\ell}),$$
$$\bar{\ell} = \pi_1'\pi_2'\pi_4',\ \ q = \sum_{j=1,2,4} U_j\bar{\ell} = \pi_1\pi_2'\pi_4' + \pi_1'\pi_2\pi_4' + \pi_1'\pi_2'\pi_4. \tag{3.9}$$

Here $U_j = U(a_j^*, a_j)$ is the (polarized) *quadratic Jordan operator* (see Eq. (3.24) of [31] and references cited there):

$$U_\nu X := a_\nu^* X a_\nu + a_\nu X a_\nu^*. \tag{3.10}$$

The gauge invariant states of the subalgebra $J(\mathcal{P})$ are uniquely characterized by the eigenvalues of $2I_3^L$ (2.10) and Y (2.12). In particular, the chirality γ in $J(P)$ is determined by anyone of these quantum numbers:

$$\gamma + (-1)^{2I_3^L} = 0 = \gamma + (-1)^{3Y}\,. \tag{3.11}$$

Conversely, Eq. (3.11) determines the subalgebra $J(\mathcal{P})$. The orthogonal projector $\mathcal{P} : J_{16}^2 \oplus J_{16}^2 \to J(\mathcal{P})$ is given by:

$$\mathcal{P}(= \ell + q) = \frac{1}{2}(1 - \gamma(-1)^{2I_3^L}) = \frac{1}{2}(1 - \gamma(-1)^{3Y})\,. \tag{3.12}$$

The $SU(2)_L$-invariant projectors in $J(\mathcal{P})$ are determined by the eigenvalues of Y. For the left chiral particles for which $P_1 = (2I_3^L)^2 = 1$ (cf. (2.13)) Y takes two values, -1 and $\frac{1}{3}$, of multiplicity two and six, respectively. In $\mathcal{H}_8^R(\mathbb{C})$, for $P_1 = 0$, the hypercharge takes four eigenvalues: two nondegenerate $Y = 0, -2$ and two others, $Y = \frac{4}{3}, -\frac{2}{3}$ of multiplicity three each. We note that for the electroweak model (based on the Jordan algebra J_2^4 (1.4)) - with only leptons present - the trace of the hypercharge in the left and the right particle space is -2, so that only their difference, the *supertrace*, vanishes (as emphasized in [9]). By contrast, in the full SM the trace of Y vanishes in $\mathcal{H}_8^L$ and $\mathcal{H}_8^R$, separately.

The expression (3.12) for $\mathcal{P}$ together with the anticommutativity of $a_j^{(*)} (= a_j$ or $a_j^*)$ with γ and their left isospin independence for $j = 1, 2, 4$ implies that their projection on $J(\mathcal{P})$ vanishes:

$$a_j^{(*)}\gamma = -\gamma a_j^{(*)},\ [I_3^L, a_j^{(*)}] = 0 \Rightarrow \mathcal{P} a_j^{(*)} \mathcal{P} = 0,\ j = 1, 2, 4. \tag{3.13}$$

Thus, the Higgs field in the particle subspace can be written in the form:

$$\hat{\Phi}(x) = \bar{\phi}_0 a_0 + \phi_0 a_0^* + \bar{\phi}_8 a_8 + \phi_8 a_8^*. \tag{3.14}$$

Then the connection and the curvature form, the counterparts of (2.16) (2.17), can be written simply as:

$$\mathbb{D} = d + \hat{A} + \hat{\Phi},\ \mathbb{D}^2 = \hat{F} + d\hat{\Phi} + dx^\mu [A_\mu, \hat{\Phi}] + \hat{\Phi}^2,\ \hat{F} = d\hat{A} + \hat{A}^2. \tag{3.15}$$

4 Higgs potential and bosonic Lagrangian

The bosonic action in a gauge theory is defined as the trace of (half of) the square of the curvature. In order to account for symmetry breaking we shall replace $\hat{\Phi}^4$ by a more general fourth order expression, invariant with respect to the unbroken gauge symmetry with Lie algebra

$$su(3)_c \oplus u(1)_Y \oplus u(1)_L \subset \mathfrak{g}_{SM},\ u(1)_L = Span\{I_3^L = Q - \frac{1}{2}Y\}. \tag{4.1}$$

(This extends the procedure adopted in [30] where one subtracts from $\hat{\Phi}^2$ a general $U(n)$ invariant operator). We shall write the Higgs potential as:

$$\begin{aligned} V(\phi) &= \frac{1}{2} tr \left(\hat{\Phi}(\kappa P_1 + P_1')\hat{\Phi} - m^2 (P_1 + \kappa P_1') \right)^2 + \lambda \phi_0 \bar{\phi}_0 \phi_8 \bar{\phi}_8 \\ &= \frac{1}{2} (\phi\bar{\phi} - m^2)^2 tr(P_1 + \kappa^2 P_1') + \lambda \phi_0 \bar{\phi}_0 \phi_8 \bar{\phi}_8,\ \ m, \kappa, \lambda > 0. \end{aligned} \tag{4.2}$$

Here we have used the relations (cf. (2.13)):

$$\begin{gathered} P_1' := 1 - P_1 = (2I_3^R)^2\ (P_1 P_1' = 0, P_1 + P_1' = 1), \\ \hat{\Phi} P_1 \hat{\Phi} = \phi\bar{\phi} P_1',\ \hat{\Phi} P_1' \hat{\Phi} = \phi\bar{\phi} P_1,\ \ \phi\bar{\phi} = \phi_0 \bar{\phi}_0 + \phi_8 \bar{\phi}_8. \end{gathered} \tag{4.3}$$

It is the last, fourth order, term in (4.2) that breaks the $U(2)$ electroweak symmetry to $U(1) \times U(1)$ (the independent change of phases of ϕ_0, ϕ_8).

We shall write the bosonic Lagrangian of the SM in the form:

$$\begin{aligned} L(A, \phi) &= -\frac{1}{4} \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{2} \mathrm{tr}(\partial_\mu \phi \partial^\mu \phi) + \\ &+ \frac{1}{2} \mathrm{tr}[A_\mu, \phi][A^\mu, \phi] + V(\phi), \end{aligned} \tag{4.4}$$

where A_μ is the total gauge field of the SM:

$$A_\mu = i(W_\mu^+ I_+ + W_\mu^- I_- + W_\mu^3 I_3 + NB_\mu Y + \frac{1}{2}\sum_{s=0}^{8}\sum_{i,j=1,2,4} G_\mu^s a_i^* \lambda_s^{ij} a_j), \quad (4.5)$$

W_μ and B_μ are an $SU(2)_L$ triplet and singlet, respectively, G_μ is the gluon ($SU(3)_c$) octet, λ_s are the su(3) Gell-Mann matrices such that $tr(\lambda_s\lambda_t) = 2\delta_{st}$. The normalization constant N is determined from the condition that I_3 and NY are equally normalized in $J(\mathcal{P})$:

$$tr(I_3^L)^2 (= \frac{1}{2}(1+3)) = 2 = tr(NY)^2 = \frac{40}{3}N^2 \Rightarrow N^2 = \frac{3}{20}. \quad (4.6)$$

Here we have used the calculation: $trY^2 = 1\times 2 + \frac{1}{9}\times 6 + 4 + \frac{4}{9}\times 3 + \frac{16}{9}\times 3 = \frac{40}{3}$. Clearly, the value of N depends on the spectrum of fundamental fermions. For the leptonic (electroweak) model one has a smaller ratio, $N^2 = \frac{1}{12}$. We shall see that the resulting N^2 gives the value of the computed Weinberg angle.

We will obtain the (quadratic) mass form for the electroweak gauge fields,

$$\mathcal{Q}(W,B) := -\frac{1}{2}tr[W^+ I_+^L + W^- I_-^L + W^3 I_3^L + NBY, \phi]^2, \quad (4.7)$$

by noting that $[G,\phi] = 0$ and substituting in the third term of the Lagrangian (4.4) the components of $\phi(x)$ by constant values which minimize $V(\phi)$:

$$|\phi_\alpha|^2 = \rho_\alpha,\ \alpha = 0, 8,\ \rho_0 + \rho_8 = m^2,\ \rho_0\rho_8 = 0. \quad (4.8)$$

In writing down (4.7) (and later) we are omitting the (contracted) vector index μ of the gauge fields. Taking further into account the relations

$$[W^+ a_8^* a_0 + W^- a_0^* a_8, \phi]^2 = [W^+(\phi_0 a_8^* - \bar{\phi}_8 a_0), W^-(\phi_8 a_0^* - \bar{\phi}_0 a_8)]_+, \quad (4.9)$$

$$\begin{aligned}[W_3 I_3 + NBY, \phi]^2 = &\frac{1}{4}(W_3 + 2NB)^2(\phi_0 a_0^* - \bar{\phi}_0 a_0)^2 \\ &+ \frac{1}{4}(W_3 - 2NB)^2(\phi_8 a_8^* - \bar{\phi}_8 a_8)^2, \end{aligned} \quad (4.10)$$

and inserting the values (4.8) of ϕ_0, ϕ_8 that minimize the potential, we find

$$\begin{aligned}\mathcal{Q}(W,B) = &\frac{1}{4}tr\{(\rho_0+\rho_8)(W^+W^- + W^-W^+) \\ &+ \frac{1}{2}\left(\rho_0(W_3+2NB)^2 + \rho_8(W_3-2NB)^2\right)\} \\ = 4m^2 &\left(W^+W^- + W^-W^+ + \frac{1}{2}(W_3^2 + 4N^2B^2) + 2NBW_3\,\varepsilon\right), \\ &\varepsilon = \varepsilon(\rho_0,\rho_8) = \frac{\rho_0 - \rho_8}{\rho_0+\rho_8} = \pm 1. \end{aligned} \quad (4.11)$$

Eq. (4.11) tells us that the parameter $2m$ appears as the mass of the charged, $W^{\pm}$, bosons. The mixing matrix for the neutral gauge bosons W_3 and B,

$$\begin{pmatrix} 1 & 2N\varepsilon \\ 2N\varepsilon & 4N^2 \end{pmatrix},$$

has determinant 0 for $\varepsilon^2 = 1$ as ensured by the last equation (4.8). This implies the existence of a zero mass photon. The physical neutral gauge fields A^γ and the Z-boson diagonalize the mixing matrix by a rotation on the Weinberg angle:

$$A^\gamma = cB - \varepsilon s W_3, , \; Z = \varepsilon s B + c W_3,$$

$$c^2 = \cos^2\theta_w = \frac{1}{1+4N^2} = \frac{5}{8}, \; s^2 = \sin^2\theta_w = \frac{4N^2}{1+4N^2} = \frac{3}{8}, \tag{4.12}$$

for $4N^2 = \frac{3}{5}$, (4.6). The relations (4.12) just reflect the fermion spectrum:

$$tg^2\theta_w = 4N^2 = \frac{tr_{J(\mathcal{P})}(2I_3^L)^2}{tr_{J(\mathcal{P})}Y^2} (= \frac{3}{5}). \tag{4.13}$$

No wonder that the same result is derived in grand unified theories. For $4N^2 = \frac{1}{3}$, the value in the leptonic model based on J_2^4, we would have reproduced the result $s^2 = \frac{1}{4}$ of [30] (also obtained in [9] and earlier, under different premises, in work of Neeman and Fairley, cited in [30]).

The constant κ in $V(\phi)$ (4.2) does not appear in the mass matrix for the gauge bosons. It does affect, however, the mass square of the Higgs mass identified as the coefficient $8m^2(1+\kappa^2)$ to $\phi\bar{\phi}$ in the quadratic term of $V(\phi)$ giving

$$m_H^2 = 2(1+\kappa^2)m_w^2. \tag{4.14}$$

This allows to accomodate the observed relation $16m_h \approx 25m_w$ for $\kappa \lessapprox \frac{1}{2}$.

We end with two remarks placing our result in a more familiar context.

1. The Lagrangian (4.4) involves no coupling constants. A way to introduce the gauge coupling g of the charged W-bosons and the gluons consists in replacing $L(A, \phi)$ (4.4) by $\frac{1}{g^2} L(gA, g\phi)$, a scaling that preserves the kinetic (and, more generally, the quadratic) term (cf. [30]); we then identify (a multiple of) g with the W and G gauge coupling. The couplings g' of the Z boson and e of the photon A^γ are determined by g and the Weinberg angle:

$$g' = g \,\mathrm{tg}\,\theta_w, \qquad e^2 = g^2 \sin^2\theta_w, \tag{4.15}$$

yielding in our case $g^2 = \frac{5}{3} g'^2 = \frac{8}{3} e^2$.

2. Our calculation (as well as that of [30] and in the work cited there) is classical, corresponding to a tree quantum field theoretic approximation. According to the renormalization group analysis the coupling constants $g, g', \ldots$ depend on the energy scale (or the momentum transfer – a dependance now confirmed experimentally). Our argument, or a similar one in a grand unified theory, is believed to be exact at "unification scale" (at inaccessibly high energy – up to $10^{15} - 10^{16}\, GeV$). The measured value of $\sin^2\theta_w$ is 0.2312 (at momentum transfer $91.4\,\frac{GeV}{c}$). The value $\sin^2\theta_w = \frac{1}{4}$ based on the $U(2)$ electroweak theory is, in fact, closer to it than the value 3/8 computed for the full SM.

5 Outlook

The fact that the euclidean extensions of the spin factors J_2^4 and J_2^8 are related to the "structure Clifford algebras" $C\ell(5,1)$ and $C\ell(9,1)$ makes the superconnection approach of [29], [27], adopted by physicists and neatly formulated in [30], particularly natural. The generators Γ_a of $C\ell(4n+1,1)$ $(n=1,2)$ anticommute with the chirality operator $\gamma = \omega_{4n+1,1}$ and intertwine between the (internal symmetry counterpart of) left and right chiral fermions. This begs to identify the (multicomponent) scalar field

$$\widehat{\Phi}(x) = \sum_a \phi^a(x)\Gamma_a\,, \quad \text{or rather} \quad \mathcal{P}\widehat{\Phi}(x)\mathcal{P} \tag{5.1}$$

where $\mathcal{P}$ projects on the particle subspace (excluding antiparticles) with the matrix valued odd part of the superconnection associated with the Higgs field. The detailed explicit calculation of Sects. 3, 4 aimed to demonstrate the accessibility and the relative simplicity of this approach.

Let us make some comments on the description of the theory of fundamental particles of matter for one generation of the Standard Model given here. One has an internal quantum space which corresponds to the Jordan algebra $J_2^8 = JSpin_9$ of hermitian 2×2-matrices over $\mathbb{O}$ acted by the subgroup of automorphisms preserving the splitting $\mathbb{O} = \mathbb{C}\oplus\mathbb{C}^3$ which is the subgroup G_{SM} (1.2) of $Aut(J_2^8) = Spin(9,0)$. One also has an external classical space which corresponds to the algebra $\mathcal{C}(M)$ of real functions on spacetime acted by the subgroup of automorphisms preserving the Minkowskian structure which is the Poincaré group. Particles are then described by modules over J_2^8 and $\mathcal{C}(M)$ respectively that is by the Clifford algebra $C\ell_9$ or its hermitian part for the internal structure and by the module $\mathcal{S}$ of sections of the (Weyl) spin bundles for the external structure. These modules being equivariant respectively by G_{SM} and by the Poincaré group. Here, we have taken into account $\mathcal{C}_9\times\mathcal{S}$ as a module over $\mathcal{C}(M)$ and investigated the corresponding (super-)gauge theory. In a sense this is not so natural. Indeed, from the very beginning $\mathcal{C}_9\times\mathcal{S}$ is a module over the Jordan algebra $J_2^8\times\mathcal{C}(M) = \mathcal{C}(M,J_2^8)$. In [14] differential calculi over general Jordan algebras and a corresponding theory of connections over Jordan modules have been defined, which has been further developed in [6]. Thus it would be more natural to write an action for the theory of (super-)connections over the Jordan algebra $\mathcal{C}(M,J_2^8)$ (cf. the approach of [15], [10] and [12]). If one does that, a lot of supplementary scalar fields appear, namely the components of the connection in the quantum directions (i.e. over the part J_2^8). It is an open problem to classify these fields and to analyse their relevance for physics.

Appendix: The splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ and the associated $\mathbb{Z}_3$-symmetry

The splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ corresponds to the choice of an imaginary unit $\mathbf{i} \in \mathbb{O}$ which plays the role of the complex imaginary $i \in \mathbb{C}$. One can then write an octonion $x \in \mathbb{O}$ as

$$x = z + \sum_k Z^k \mathbf{e}_k = z + Z$$

where z and the Z^k are elements of $\mathbb{C} = \mathbb{R} + \mathbf{i}\mathbb{R} (\subset \mathbb{O})$ and where $(\mathbf{e}_k)$ is the canonical basis of $\mathbb{C}^3$, $k \in \{1, 2, 3\}$. One recovers the product of $\mathbb{O}$ by setting

$$\begin{cases} \mathbf{i}^2 = -1 \\ \mathbf{i}\mathbf{e}_k = -\mathbf{e}_k \mathbf{i} \\ \mathbf{e}_k \mathbf{e}_\ell = -\delta_{k\ell} 1 + \sum_m \varepsilon_{k\ell m} \mathbf{e}_m \end{cases}$$

i.e. the $\mathbf{e}_k$ generate a quaternionic subalgebra. The subgroup of $G_2 = \mathrm{Aut}(\mathbb{O})$ which preserves $\mathbf{i} \in \mathbb{O}$ is isomorphic to $SU(3)$ and is identified in our picture to the colour group $SU(3)_c (\subset G_2)$ while the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ is identified to the quark-lepton symmetry, $\mathbb{C}^3$ for the quark and $\mathbb{C}$ for the lepton.

Following [34], let us consider the center $\mathbb{Z}_3$ of $SU(3)_c$, this is the subgroup of G_2 induced by the action w of $\mathbf{j} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i} \in \mathbb{O}$ on $x = z + Z \in \mathbb{O}$ as

$$w(x) = w(z + Z) = z + \mathbf{j}Z$$

where $Z = (Z^k) \in \mathbb{C}^3 \subset \mathbb{O}$ and $\mathbf{j}Z = (\mathbf{j}Z^k)$ is the diagonal action. Then, by construction $w \in G_2$ and the subgroup of G_2 which commutes with w is again $SU(3)_c \subset G_2$.

Consider the Jordan algebra $J_2^8 = JSpin_9 = \mathcal{H}_2(\mathbb{O})$ of the hermitian octonionic 2×2 matrices. The group of automorphisms of J_2^8 is the group $Spin(9)$ and the mapping

$$w_2 : \begin{pmatrix} \lambda_1 & x \\ \bar{x} & \lambda_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & w(x) \\ \overline{w(x)} & \lambda_2 \end{pmatrix}$$

defines an automorphism of J_2^8 which induces an action of $\mathbb{Z}_3$ on J_2^8. The subgroup of $\mathrm{Aut}(J_2^8) = Spin(9)$ which commutes with this action (i.e. with w_2) is the group G_{SM} defined by (1.2) which preserves the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ and the $\mathbb{C}$-linearity in $\mathbb{C}^3$.

Consider now the exceptional Jordan algebra $J_3^8 = \mathcal{H}_3(\mathbb{O})$, then the mapping

$$w_3 : \begin{pmatrix} \lambda_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \lambda_2 & x_1 \\ x_2 & \bar{x}_1 & \lambda_3 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 & w(x_3) & \overline{w(x_2)} \\ \overline{w(x_3)} & \lambda_2 & w(x_1) \\ w(x_2) & \overline{w(x_1)} & \lambda_3 \end{pmatrix}$$

defines an automorphism of J_3^8 (i.e. $w_3 \in F_4 = \mathrm{Aut}(J_3^8)$) and induces an action of $\mathbb{Z}_3$ on J_3^8. The subgroup of F_4 which commutes with this action (i.e. with w_3) is the subgroup of $\mathrm{Aut}(J_3^8) = F_4$ isomorphic to

$$SU(3) \times SU(3)/\mathbb{Z}_3$$

which preserves the splitting $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ and the $\mathbb{C}$-linearity in $\mathbb{C}^3$, [34]. This subgroup was denoted as $SU(3)_c \times SU(3)_{ew}/\mathbb{Z}_3$ in [16].

Warning : Our presentation of $\mathbb{O}$ at the beginning of this appendix is clearly related to the Cayley-Dickson construction applied to the transition from $\mathbb{H}$ to $\mathbb{O}$ by adding the "new" imaginary unit $\mathbf{i}$, but this $\mathbf{i} \in \mathbb{O}$ should not be confused with the complex number i involved in the complexification $C\ell_9$ of $C\ell(9,0)$ in [16] and in $C\ell(10,\mathbb{C})$ in Section 3.

Once one works in $\mathbb{O}$, it is much more natural to index a basis of the imaginary octonionic units by the field $\mathbb{Z}_7$ of the integers modulo 7. Among such a choice the choice of [3] is particularly nice since in the basis $(e_\alpha)_{\alpha\in\mathbb{Z}_7}$ of [3] the relations of $\mathbb{O}$ (i.e. the multiplication table of $\mathbb{O}$) are translational invariant

$$e_\alpha e_\beta = e_\gamma \Rightarrow e_{\alpha+1} e_{\beta+1} = e_{\gamma+1}$$

and invariant by the dilatation by 2, i.e.

$$e_\alpha e_\beta = e_\gamma \Rightarrow e_{2\alpha} e_{2\beta} = e_{2\gamma}$$

so that everything is fixed by setting $e_1 e_2 = e_4$ (which is then necessary for the consistence) and we stick to the above choice for $\mathbb{O}$. In such a basis e_7(" $= e_0$") has the particularity to be invariant by dilatation

$$e_{\alpha 7} = e_7,\ \forall \alpha \in \mathbb{Z}_7$$

and is unique under this condition since 7 is a prime number (i.e.$\mathbb{Z}_7$ is a field).

Since in our approach the splitting $\mathbb{O} = \mathbb{C}\oplus\mathbb{C}^3$ is fundamentally linked to the color symmetry of quarks and to the quark-lepton symmetry [14], it is natural to identify $\mathbf{i} \in \mathbb{O}$ as $\mathbf{i} = e_7$ (" $= e_0$") in this frame. This justifies our choice of notations all along our paper. The relation $\mathbf{i} = e_7$ must be supplemented by $\mathbf{e}_1 = e_1, \mathbf{e}_2 = e_2$ and $\mathbf{e}_3 = e_4$ to express the previous items in term of basis $(e_\alpha)_{\alpha\in\mathbb{Z}_7}$ of [3].

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