Conditional distributions for quantum systems

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Conditional distributions, as defined by the Markov category framework, are studied in the setting of matrix algebras (quantum systems). Their construction as linear unital maps are obtained via a categorical Bayesian inversion procedure. Simple criteria establishing when such linear maps are positive are obtained. Several examples are provided, including the standard EPR scenario, where the EPR correlations are reproduced in a purely compositional (categorical) manner. A comparison between the Bayes map and the Petz recovery map is provided, illustrating some key differences.

Keywords. Bayes, inference, Markov category, operator system, positive map, quantum information theory, quantum probability, recovery map

Contents

1 Introduction

There is a one-to-one correspondence between stochastic maps (conditional probabilities) on finite sets and positive unital maps on finite-dimensional commutative C^*-algebras. This correspondence is made precise categorically via a stochastic variant of Gelfand duality [14, 19]. Hence, any concept described categorically at the level of stochastic maps can be instantiated on arbitrary (not necessarily commutative) C^*-algebras. In particular, the notion of Bayesian inversion, disintegrations, and conditioning have been formulated categorically [5, 6, 8, 9, 12, 13, 15, 23], and the first two have been explored in the setting of finite-dimensional C^*-algebras in [23, 24] through a generalization of Markov categories [5, 13] to their quantum variants [21]. However, conditioning in this setting remains unexplored, as far as I am aware.[1]

The purpose of the present work is to begin the systematic study of quantum conditionals as positive maps between finite-dimensional C^*-algebras. Although this goal is not fully realized here, we are content with achieving it on bi-partite systems of matrix algebras equipped with states whose marginals are faithful. Even though this sounds quite restrictive, it already includes many cases of interest, including the fully entangled EPR state on a two qubit system [4, 10]. The case of multi-partite states, non-faithful marginals, and more general hybrid classical-quantum systems will be addressed in future work.

[1] The conditioning in [15] does not use the multiplication map in its formulation of conditioning in the quantum setting.

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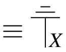
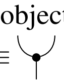
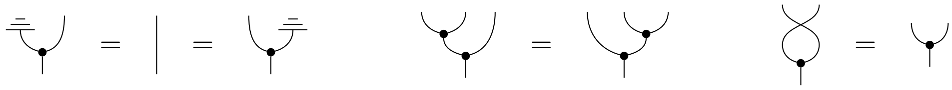
In this work, we use category theory to *define* what we mean by quantum conditionals. Then, we prove a purely categorical theorem indicating how one can *construct* quantum conditionals through the usage of *Bayes maps* (whose definition is motivated by categorical probability theory). We then implement this construction in the setting of matrix algebras. In general, the resulting conditional does not define a positive map. As such, we find necessary and sufficient conditions for conditionals to be positive. A positive conditional need not be *completely* positive, and EPR provides an example illustrating this point. We end by introducing the *conditional domain*, which is the largest operator system for which a conditional is positive (in the Heisenberg picture). Typically, this operator system is not a C^*-subalgebra. Examples are provided throughout.

2 Quantum Markov categories

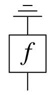
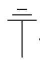
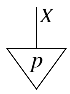
This section briefly reviews the abstract theory of quantum CD and Markov categories [21], which are generalizations of CD and Markov categories [5, 13]. String diagrams are reviewed in these mentioned papers, but see [27] for a more thorough exposition. Time will always go up the page. The composition will go up the page for definitions and the example **FinStoch**, while the composition will go down the page for C^*-algebra maps (in the Heisenberg picture).

Definition 2.1. A ***classical CD category*** is a symmetric monoidal category $(\mathscr{M},\otimes,I)$, with $\otimes$ the tensor product and I the unit (associators and unitors are excluded from the notation), and where each object X in $\mathscr{M}$ is equipped with morphisms $!_X \equiv$  $: X \to I$, called the ***discarder/grounding***, and $\Delta_X \equiv$  : $X \to X \otimes X$, called the ***copy/duplicate***, all satisfying the following conditions



(2.2)

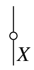
(2.3)

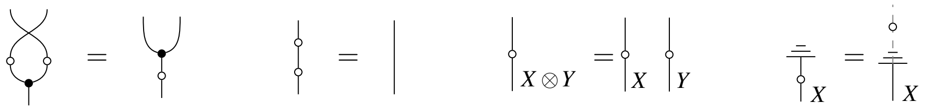
expressed using string diagrams. A ***classical Markov category*** is a classical CD category for which every morphism $X \xrightarrow{f} Y$ is ***unital***, i.e. natural with respect to  in the sense that  $=$ . A ***state*** on X is a morphism $I \xrightarrow{p} X$, which is drawn as .

Example 2.4. Let **FinStoch** be the category whose objects are finite sets and where a morphism $X \xrightarrow{f} Y$ is a stochastic map/conditional probability[2] from X to Y, which, by definition, assigns to each element $x \in X$ a probability measure f_x on Y, whose value on y is written as f_{yx}. The composite of a composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ is defined by the Chapman–Kolmogorov equation $(g \circ f)_{zx} := \sum_{y \in Y} g_{zy} f_{yx}$. The tensor product is the cartesian product of sets and the product $X \times X' \xrightarrow{f \times f'} Y \times Y'$ of stochastic maps $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$, and is given by $(f \times f')_{(y,y')(x,x')} := f_{yx} f_{y'x'}$. The tensor unit is the single element set, often denoted by $\{\bullet\}$. Functions are special kinds of stochastic maps whose probability measures are $\{0,1\}$-valued. In particular, the maps Δ_X and $!_X$ are the stochastic maps associated to the functions $\Delta_X(x) := (x,x)$ and $!_X(x) = \bullet$. A state on X encodes a probability measure on X.

[2]The reader will notice that the $\rightsquigarrow$ notation is not used in this article, unlike in our earlier works [19, 21, 23, 24]. The reason is because we do not need to emphasize the distinction between deterministic maps and stochastic maps in this work.

The conditions described in (2.2) suggest that classical Markov categories cannot be extended to the quantum setting due to the universal no-broadcasting theorem [2, 18]. However, there is a way around these conditions by working with a larger class of morphisms, adding an even and odd grading for morphisms, and substituting the commutativity condition for another closely-related condition [21].

Definition 2.5. A ***quantum CD category*** is a $\mathbb{Z}_2$-graded symmetric monoidal category $\mathscr{M}$,[3] and where each object X is equipped with an even discarder, an even copy map, and an odd ***involution*** $*_X \equiv$  : $X \to X$ satisfying the same conditions as a classical CD category, *except* the last condition in (2.2), and also satisfying the additional conditions



(2.6)

A ***quantum Markov category*** is a quantum CD category in which every morphism is unital.[4] A morphism $X \xrightarrow{f} Y$ is said to be ***$*$-preserving*** iff $f \circ *_X = *_Y \circ f$.

Example 2.7. From now on, all C^*-algebras will be assumed unital. Although the category of finite-dimensional C^*-algebras and *positive* unital maps (cf. Notation 3.11) does not form a quantum Markov category (essentially due to the no-broadcasting theorem), this category naturally embeds into a quantum Markov category, allowing the structure of the ambient quantum Markov category to be utilized [21]. Let $\mathbf{fdC^*\text{-}AlgU}_{\square}^{\mathrm{op}}$ be the category whose objects are finite-dimensional C^*-algebras (see [19, Section 2.3] for a review of C^*-algebras within a categorical setting).[5] For example, a matrix algebra will be written as $\mathscr{M}_n(\mathbb{C})$ indicating the C^*-algebra of complex $n \times n$ matrices. A morphism from $\mathscr{A}$ to $\mathscr{B}$ in $\mathbf{fdC^*\text{-}AlgU}_{\square}^{\mathrm{op}}$ is *either* a linear (even) or *conjugate-linear* (odd) unital map $\mathscr{B} \xrightarrow{F} \mathscr{A}$. Notice that the function goes backwards because of the superscript $^{\mathrm{op}}$. The tensor product (over $\mathbb{C}$) is the tensor product of finite-dimensional C^*-algebras, so that the unit is $\mathbb{C}$. The tensor product of linear maps is defined in the usual way, while the tensor product of conjugate-linear maps can be defined similarly [28, Section 9.2.1]. However, note that it does not make sense to define the tensor product of a linear map with a conjugate-linear one. The $*$ operation is the involution on a C^*-algebra, which is conjugate-linear. The copy map $\Delta_{\mathscr{A}}$ from $\mathscr{A}$ to $\mathscr{A} \otimes \mathscr{A}$ in $\mathbf{fdC^*\text{-}AlgU}_{\square}^{\mathrm{op}}$ is the multiplication map $\mathscr{A} \otimes \mathscr{A} \xrightarrow{\mu_{\mathscr{A}}} \mathscr{A}$ determined on elementary tensors by $A_1 \otimes A_2 \mapsto A_1 A_2$. The discard map from $\mathscr{A}$ to $\mathbb{C}$ in $\mathbf{fdC^*\text{-}AlgU}_{\square}^{\mathrm{op}}$ is defined to be the unit inclusion map $!_{\mathscr{A}} : \mathbb{C} \to \mathscr{A}$ sending $\lambda \in \mathbb{C}$ to $\lambda 1_{\mathscr{A}}$. A linear map $\mathscr{B} \xrightarrow{F} \mathscr{A}$ is $*$-preserving if and only if it sends self-adjoint elements in $\mathscr{B}$ to self-adjoint elements in $\mathscr{A}$. For convenience, we will drop the op and work directly with the unital maps on the algebras from now on.

Although we have introduced the categories **FinStoch** and $\mathbf{fdC^*\text{-}AlgU}_{\square}$, we will be more explicit and work mainly with matrix algebras, linear maps, and positive maps in our main results. The abstract setting will mainly be used in the next two sections to provide the general context.

[3] This means that there is a functor $\mathscr{M} \to \mathbb{B}\mathbb{Z}_2$ (where $\mathbb{B}\mathbb{Z}_2$ is the one object category whose set of morphisms equals $\mathbb{Z}_2 = \{0,1\}$ and whose composition is defined by addition modulo 2 in $\mathbb{Z}_2$) and a tensor product is defined for all objects and all morphisms of the *same* degree. Morphisms sent to 0/1 are called ***even/odd***. Note that the tensor product of morphisms of different degrees is not defined, but the collection of even morphisms is a symmetric monoidal category.

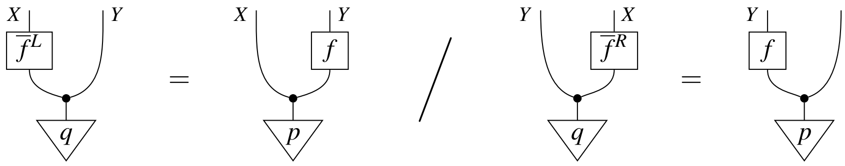
[4] Unitality is defined differently for odd morphisms. We exclude the details because we will not need this definition here.

[5] Every such finite-dimensional C^*-algebra is *-isomorphic to a finite direct sum of (square) matrix algebras [11, Theorem 5.5].

3 Bayes maps, conditionals, and a.e. equivalence

Here, we review two formulations of Bayes' theorem, which we express categorically. Throughout this section, $\mathscr{M}$ will denote either a classical or quantum Markov category and $\mathscr{C}$ will denote some (not necessarily monoidal) subcategory of $\mathscr{M}$. Furthermore, all morphisms will be even from now on.

Definition 3.1. Given states $I \xrightarrow{p} X$ and $I \xrightarrow{q} Y$, a ***state-preserving morphism*** $X \xrightarrow{f} Y$ (i.e. $q := f \circ p$) is written as a triple (f,p,q). A ***left/right Bayes map for*** (f,p,q) is a morphism $\overline{f}^L/\overline{f}^R : Y \to X$ in $\mathscr{M}$ such that

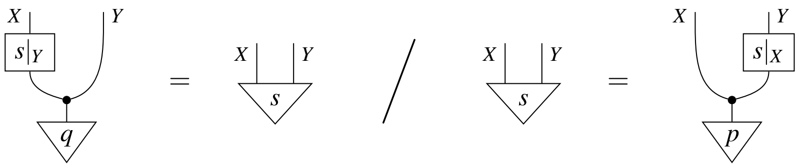


(3.2)

If all morphisms are in $\mathscr{C}$, then $\overline{f}^L/\overline{f}^R$ are said to be ***left/right Bayesian inverses of*** (f,p,q) (in $\mathscr{C}$).

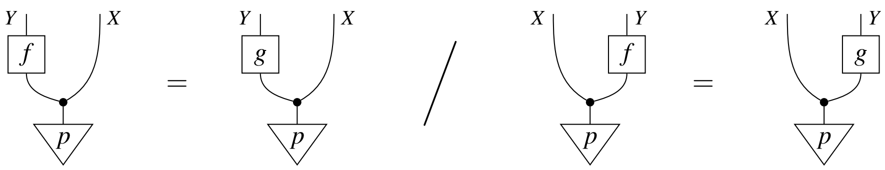
Bayes maps are automatically state-preserving. If all morphisms are $*$-preserving, then there is no distinction between left and right concepts (this is always the case in classical Markov categories [5,21]).

Definition 3.3. Let $I \xrightarrow{s} X \otimes Y$ be a state and let p and q denote its ***marginals*** $I \xrightarrow{s} X \otimes Y \xrightarrow{\pi_X} X$ and $I \xrightarrow{s} X \otimes Y \xrightarrow{\pi_Y} Y$, respectively. Here, π_X and π_Y are the ***projections***, which are defined as $\pi_X := \big(X \otimes Y \xrightarrow{\mathrm{id}_X \times !_Y} X \otimes I \cong X\big)$ and $\pi_Y := \big(X \otimes Y \xrightarrow{!_X \otimes \mathrm{id}_Y} I \otimes Y \cong Y\big)$. A ***conditional distribution of*** s ***given*** Y / X (or Y / X ***conditional*** for short) is a morphism $Y \xrightarrow{s|_Y} X$ / $X \xrightarrow{s|_X} Y$ such that



. (3.4)

Definition 3.5. Let X and Y be objects, let $I \xrightarrow{p} X$ be a state and let $f,g : X \to Y$ be morphisms. The morphism f is said to be ***left/right*** p***-a.e. equivalent to*** g iff



. (3.6)

All of these definitions are quite similar. Indeed, if $\overline{f}^L$ and $\overline{f}^R$ are left and right Bayes maps for some (f,p,q), then they are automatically left and right a.e. unique, respectively. Furthermore, the Y/X conditionals are also left/right a.e. unique. A.e. equivalence agrees with the standard measure-theoretic notion [5, Proposition 5.3 ,5.4] (as well as the C^*-algebraic one [23, Section 3.1], [21, Theorem 5.12]). With these preliminaries, Bayes' theorem can now be expressed in two different ways.

Theorem 3.7. *[Bayes' theorem via Bayesian inversion] Every triple (f,p,q) in* **FinStoch** *(a state-preserving $(X,p) \xrightarrow{f} (Y,q)$) admits a (necessarily a.e. unique) Bayesian inverse (in* **FinStoch***).*

Theorem 3.8. *[Bayes' theorem via conditional distributions] Every joint state $\{\bullet\} \xrightarrow{s} X \times Y$ in* **FinStoch** *admits both (necessarily a.e. unique) X and Y conditionals.*

Example 3.9. These two versions of Bayes' theorem are often expressed as the equations

$$p(x|y)p(y) = p(y|x)p(x) \qquad \text{and} \qquad p(x|y)p(y) = p(x,y) = p(y|x)p(x), \tag{3.10}$$

respectively. Although it seems as though the former is a special case of the latter,[6] notice that the input data for each definition is different. The first version has input data a morphism $X \xrightarrow{f} Y$ and a state $\{\bullet\} \xrightarrow{p} X$ (the state q on Y is obtained via composition). Meanwhile, the second version has input datum a state $\{\bullet\} \xrightarrow{s} X \times Y$. This distinction may seem pedantic, but it is crucial for generalizing to the non-commutative setting [24].

Notation 3.11. In what follows, if A is a matrix, then $A^\dagger$ denotes is conjugate transpose. A matrix $A \in \mathcal{M}_m(\mathbb{C})$ is ***positive*** iff it is ***self-adjoint*** ($A^\dagger = A$) and its eigenvalues are non-negative, equivalently $A = C^\dagger C$ for some $C \in \mathcal{M}_m(\mathbb{C})$. If $P \in \mathcal{M}_m(\mathbb{C})$ is an orthogonal projection (i.e. $P^\dagger P = P$), then $P^\perp := \mathbb{1}_m - P$ denotes its complement projection. The ***standard matrix units*** of $\mathcal{M}_m(\mathbb{C})$ will be denoted by $E_{ij}^{(m)}$ with $i,j \in \{1,\ldots,m\}$. They satisfy $E_{ij}^{(m)}E_{kl}^{(m)} = \delta_{jk}E_{il}^{(m)}$, where δ_{jk} is the Kronecker delta taking value 1 when $j = k$ and 0 otherwise. A linear map $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is ***positive*** (***completely positive***) iff it $(F \otimes \mathrm{id}_{\mathcal{M}_k(\mathbb{C})})$ sends positive matrices to positive matrices (for all $k \in \mathbb{N}$). In terms of the notation at the beginning of Section 3, $\mathcal{M} = \mathbf{fdC^{*}\text{-}AlgU}_{\square}$ and $\mathcal{C} = \mathbf{fdC^{*}\text{-}AlgPU}$ is the subcategory consisting of (linear) positive unital maps. If $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is linear, then F^* denotes its adjoint with respect to the ***Hilbert–Schmidt inner product*** on matrices, i.e. $F^* : \mathcal{M}_m(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ is the unique linear map satisfying $\mathrm{tr}(F^*(A)B) = \mathrm{tr}(AF(B))$ for all $A \in \mathcal{M}_m(\mathbb{C})$ and $B \in \mathcal{M}_n(\mathbb{C})$. An example that appears often is the Hilbert–Schmidt dual of the inclusion $\iota_{\mathcal{M}_m(\mathbb{C})} : \mathcal{M}_m(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ sending A to $A \otimes \mathbb{1}_n$, and is given by the CPU map $\mathrm{tr}_{\mathcal{M}_n(\mathbb{C})}$, which is called the ***partial trace***. Explicitly, $\mathrm{tr}_{\mathcal{M}_n(\mathbb{C})}$ is determined by its action on simple tensors, namely $\mathrm{tr}_{\mathcal{M}_n(\mathbb{C})}(A \otimes B) = \mathrm{tr}(B)A$, and it satisfies a partial form of cyclicity given by

$$\mathrm{tr}_{\mathcal{M}_n(\mathbb{C})}\big((A \otimes B)(\mathbb{1}_m \otimes C)\big) = \mathrm{tr}_{\mathcal{M}_n(\mathbb{C})}\big((\mathbb{1}_m \otimes C)(A \otimes B)\big) \tag{3.12}$$

for all inputs A,B,C.

Example 3.13. Let $\mathcal{A} := \mathcal{M}_m(\mathbb{C})$ and $\mathcal{B} := \mathcal{M}_n(\mathbb{C})$ be two matrix algebras. Let $\omega = \mathrm{tr}(\rho\ \cdot\ )$ and $\xi = \mathrm{tr}(\sigma\ \cdot\ )$ be states on $\mathcal{A}$ and $\mathcal{B}$, respectively, with respective density matrices. Let $\mathcal{B} \xrightarrow{F} \mathcal{A}$ be a unital linear map. If σ is positive definite (so that the state ξ is faithful), then there are unique left and right Bayes maps for (F,ω,ξ). They are respectively given by

$$\overline{F}^L(A) := \sigma^{-1}F^*(\rho A) \qquad \text{and} \qquad \overline{F}^R(A) := F^*(A\rho)\sigma^{-1} \tag{3.14}$$

for all $A \in \mathcal{A}$. If F is $*$-preserving, demanding that these two functions be equal[7] is equivalent to demanding that there is a $*$-preserving Bayes map $\overline{F}$. In this case, its explicit formula is given by (see [24] for a proof)

$$\overline{F}(A) = \sqrt{\sigma^{-1}}F^*\left(\sqrt{\rho}A\sqrt{\rho}\right)\sqrt{\sigma^{-1}}. \tag{3.15}$$

Hence, if F is positive unital (PU) or completely positive unital (CPU), then so is $\overline{F}$. The reader will notice that (3.15) is the formula for the ***Petz recovery map*** [1,3,16,25,26]. However, we will later see that the Petz recovery map is distinct from the Bayes map in general. The difference between the Petz recovery map and the Bayes map is more pronounced in the case that σ is *not* positive definite (so that ξ is not faithful), though the details of this will not be discussed here (but see [22,24]).

Before using this example, we first need to explain how conditionals can be constructed using Bayes maps more abstractly. Afterwards, we will look at several examples by combining the two results.

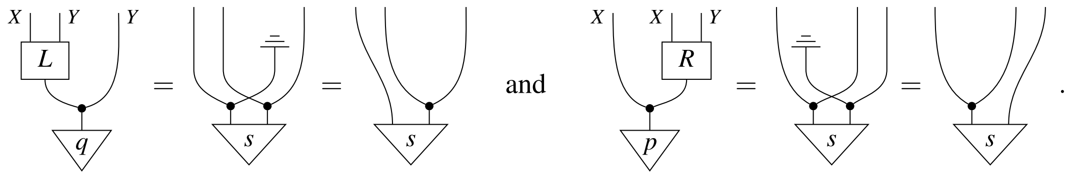
[6] This is especially due to the abusive notation of using p for all mathematical objects.

[7] Note that a.e. equivalence now reduces to equality since ξ is faithful.

4 Constructing conditionals using Bayes maps

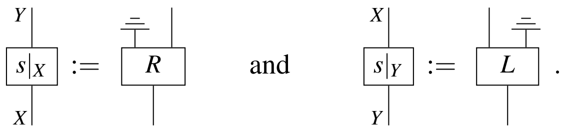
Theorem 4.1. *Let $I \xrightarrow{s} X \otimes Y$ be a joint state with marginals $I \xrightarrow{p} X$ and $I \xrightarrow{q} Y$. Let $Y \xrightarrow{L} X \otimes Y$ and $X \xrightarrow{R} X \otimes Y$ be left and right Bayes maps for (π_Y, s, q) and (π_X, s, p), respectively. Then the composites $s|_X := \big(X \xrightarrow{R} X \otimes Y \xrightarrow{\pi_Y} Y\big)$ and $s|_Y := \big(Y \xrightarrow{L} X \otimes Y \xrightarrow{\pi_X} X\big)$ are X and Y conditionals of s, respectively.*

Proof. By assumption



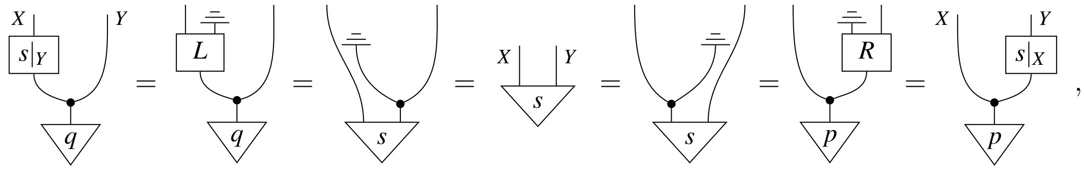
(4.2)

The definitions of $s|_X$ and $s|_Y$ are drawn as



(4.3)

From this, we immediately obtain



(4.4)

which is the desired conclusion. ■

This theorem, together with the left/right a.e. uniqueness of left/right Bayes maps, is useful because it allows us to write down explicit formulas for conditionals in the quantum setting, at least up to the supports of the states. For the remainder of this work, we will focus on applying this to matrix algebras, rather than arbitrary finite-dimensional C^*-algebras.

Corollary 4.5. *Set $\mathscr{A} := \mathscr{M}_m(\mathbb{C})$ and $\mathscr{B} := \mathscr{M}_n(\mathbb{C})$. Let $\zeta \equiv \mathrm{tr}(\nu\,\cdot\,)$ be a state on $\mathscr{A} \otimes \mathscr{B}$ (with density matrix ν) whose marginals on $\mathscr{A}$ and $\mathscr{B}$ are given by $\zeta \circ \iota_{\mathscr{A}} =: \omega \equiv \mathrm{tr}(\rho\,\cdot\,)$ and $\zeta \circ \iota_{\mathscr{B}} =: \xi \equiv \mathrm{tr}(\sigma\,\cdot\,)$, respectively. Suppose that ρ and σ are invertible. Then there are unique conditionals $\mathscr{B} \xrightarrow{F := \zeta|_{\mathscr{A}}} \mathscr{A}$ and $\mathscr{A} \xrightarrow{G := \zeta|_{\mathscr{B}}} \mathscr{B}$ given by*

$$F(B) = \mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B)\nu\big)\rho^{-1} \qquad \textit{and} \qquad G(A) = \sigma^{-1}\mathrm{tr}_{\mathscr{A}}\big(\nu(A \otimes \mathbb{1}_n)\big). \tag{4.6}$$

The Hilbert–Schmidt duals of these maps are given by

$$F^*(A) = \mathrm{tr}_{\mathscr{A}}\big(\nu(\rho^{-1}A \otimes \mathbb{1}_n)\big) \qquad \textit{and} \qquad G^*(B) = \mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B\sigma^{-1})\nu\big). \tag{4.7}$$

Proof. The first claim follows from Theorem 4.1 and Example 3.13. For instance, $F(B) = \iota_{\mathscr{A}}^*\big((1_{\mathscr{A}} \otimes B)\nu\big)\rho^{-1} = \mathrm{tr}_{\mathscr{B}}\big((1_{\mathscr{A}} \otimes B)\nu\big)\rho^{-1}$. The second claim follows from the definition of the Hilbert–Schmidt inner product and the cyclic properties of the trace. ■

Are the conditionals F and G in Corollary 4.5 positive maps? Let's look at some examples.

Example 4.8. In the notation of Corollary 4.5, take $m = n = 2$ and take ν to be Bohm's EPR density matrix $\nu := \frac{1}{2}\left[\begin{smallmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{smallmatrix}\right]$ corresponding to the pure state $\frac{1}{\sqrt{2}}\big(|\uparrow\rangle\otimes|\downarrow\rangle - |\downarrow\rangle\otimes|\uparrow\rangle\big)$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are just $e_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ expressed in Dirac notation, [4, 10]. Then the marginal density matrices ρ and σ both equal $\frac{1}{2}\mathbb{1}_2$. Since this is invertible, the conclusions of Corollary 4.5 apply. Hence,

$$F\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)=\mathrm{tr}_{\mathscr{B}}\left(\begin{bmatrix}a&b&0&0\\c&d&0&0\\0&0&a&b\\0&0&c&d\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix}\right)=\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}^T\begin{bmatrix}0&-1\\1&0\end{bmatrix}, \tag{4.9}$$

which shows that F is PU, but not CPU. The same formula is obtained for G. It is worth comparing this expression to the one obtained by using the Petz recovery map instead of the Bayes map. The Petz recovery map $\mathfrak{R} : \mathscr{A}\otimes\mathscr{B} \to \mathscr{A}$ associated to the inclusion $\iota_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}\otimes\mathscr{B}$ and the state ζ on $\mathscr{A}\otimes\mathscr{B}$ is given by

$$\mathfrak{R}(A\otimes B) = \sqrt{\rho^{-1}}\,\mathrm{tr}_{\mathscr{B}}\left(\sqrt{\nu}(A\otimes B)\sqrt{\nu}\right)\sqrt{\rho^{-1}} = 4\,\mathrm{tr}_{\mathscr{B}}\left(\nu(A\otimes B)\nu\right), \tag{4.10}$$

where we have used the fact that $\rho = \frac{1}{2}\mathbb{1}_2$ and $\nu^2 = \nu$ (because ν is a rank 1 density matrix), so that $\sqrt{\nu} = \nu$. Precomposing $\mathfrak{R}$ with the inclusion gives $F' := \mathfrak{R}\circ\iota_{\mathscr{B}}$, which acts as

$$F'\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = \frac{1}{2}\mathrm{tr}_{\mathscr{B}}\left(\begin{bmatrix}0&0&0&0\\0&a+d&-a-d&0\\0&-a-d&a+d&0\\0&0&0&0\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}a+d&0\\0&a+d\end{bmatrix} = \frac{\mathrm{tr}\left(\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\right)}{2}\mathbb{1}_2. \tag{4.11}$$

Notice that the map F', obtained using the Petz recovery map, is actually CPU, unlike the conditional F we obtained in (4.9). However, which one of these two maps recovers the standard EPR correlations?

Suppose that Alice (represented by $\mathscr{A}$) obtains new evidence in the form of a state $\varphi = \langle\uparrow|\cdot|\uparrow\rangle$ (for example, suppose that she set up an apparatus to measure the spin and obtained the result spin up). Then by applying the maps F and F' to these states via pullback,[8] Alice infers that Bob (represented by $\mathscr{B}$) would obtain the state on $\mathscr{B}$ given by

$$(\varphi\circ F)\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = d = \left\langle\downarrow\middle|\begin{bmatrix}a&b\\c&d\end{bmatrix}\middle|\downarrow\right\rangle \quad\text{and}\quad (\varphi\circ F')\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = \frac{\mathrm{tr}\left(\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]\right)}{2}. \tag{4.12}$$

The first map shows that the spin up state for Alice changes to the spin down state for Bob once the map F is applied. On the other hand, the map F' always gives the totally mixed state for Bob. This indicates that F is a more suitable inference map describing the EPR correlations, since F' loses all the entanglement (more precisely, it is an entanglement breaking channel). Note that analogous conclusions hold if the evidence Alice has is the spin in any direction: F will predict the opposite spin for Bob while F' still predicts the totally mixed state. More details relating this to Bayesian updating will be presented elsewhere [22].

The fact that F, and not F', reproduced the EPR correlations suggests that it has its merits and deserves further study (an alternative derivation of the EPR correlations is done via conditional density matrices in [17, Section V.A.3]). Example 4.8 also shows that a joint state can have positive conditionals that are not necessarily CPU. But do the conditionals always need to be positive? The next example shows that the answers to this question is no.

[8]One could equivalently obtain the Hilbert–Schmidt duals and act on the associated density matrices in the Schrödinger picture.

Example 4.13. Set $\mathscr{A} := \mathscr{M}_2(\mathbb{C})$ and $\mathscr{B} := \mathscr{M}_2(\mathbb{C})$. A general pure state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ is of the form $|\Psi\rangle = c_{\uparrow\uparrow}|\uparrow\uparrow\rangle + c_{\uparrow\downarrow}|\uparrow\downarrow\rangle + c_{\downarrow\uparrow}|\downarrow\uparrow\rangle + c_{\downarrow\downarrow}|\downarrow\downarrow\rangle$, where $c_{\uparrow\uparrow}, c_{\uparrow\downarrow}, c_{\downarrow\uparrow}, c_{\downarrow\downarrow} \in \mathbb{C}$ satisfy $|c_{\uparrow\uparrow}|^2 + |c_{\uparrow\downarrow}|^2 + |c_{\downarrow\uparrow}|^2 + |c_{\downarrow\downarrow}|^2 = 1$ (here $|\uparrow\downarrow\rangle = |\uparrow\rangle \otimes |\downarrow\rangle$ and similarly for the other vectors). Given $p \in (0,1)$, set $q := 1-p$ and let $\nu := |\Psi\rangle\langle\Psi|$ be the density matrix in $\mathscr{A} \otimes \mathscr{B} \cong \mathscr{M}_4(\mathbb{C})$ associated to the pure state with $c_{\uparrow\uparrow} = \sqrt{\frac{p}{2}}$, $c_{\uparrow\downarrow} = \sqrt{\frac{q}{2}}$, $c_{\downarrow\uparrow} = -\sqrt{\frac{p}{2}}$, and $c_{\downarrow\downarrow} = \sqrt{\frac{q}{2}}$. Then

$$\nu = \frac{1}{2}\begin{bmatrix} p & \sqrt{pq} & -p & \sqrt{pq} \\ \sqrt{pq} & q & -\sqrt{pq} & q \\ -p & -\sqrt{pq} & p & -\sqrt{pq} \\ \sqrt{pq} & q & -\sqrt{pq} & q \end{bmatrix}, \quad \rho = \frac{1}{2}\begin{bmatrix} 1 & q-p \\ q-p & 1 \end{bmatrix}, \quad \text{and} \quad \sigma = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}. \tag{4.14}$$

Thus, $\rho^{-1} = \frac{1}{2pq}\left[\begin{smallmatrix} 1 & p-q \\ p-q & 1 \end{smallmatrix}\right]$ and $\sigma^{-1} = \left[\begin{smallmatrix} p^{-1} & 0 \\ 0 & q^{-1} \end{smallmatrix}\right]$. Using Corollary 4.5, one obtains the explicit formulas

$$F\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \frac{1}{2}\begin{bmatrix} a+d+\frac{pc+qb}{\sqrt{pq}} & d-a+\frac{pc-qb}{\sqrt{pq}} \\ d-a+\frac{qb-pc}{\sqrt{pq}} & a+d-\frac{pc+qb}{\sqrt{pq}} \end{bmatrix}, \; G^*\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \frac{1}{2}\begin{bmatrix} a+d+\frac{pb+qc}{\sqrt{pq}} & d-a+\frac{qc-pb}{\sqrt{pq}} \\ d-a+\frac{pb-qc}{\sqrt{pq}} & a+d-\frac{pb+qc}{\sqrt{pq}} \end{bmatrix}, \tag{4.15}$$

$$G\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \frac{1}{2}\begin{bmatrix} a-b-c+d & \sqrt{\frac{q}{p}}(a-b+c-d) \\ \sqrt{\frac{p}{q}}(a+b-c-d) & a+b+c+d \end{bmatrix}, \quad \text{and} \tag{4.16}$$

$$F^*\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = \frac{1}{2}\begin{bmatrix} a-b-c+d & \sqrt{\frac{p}{q}}(a-b+c-d) \\ \sqrt{\frac{q}{p}}(a+b-c-d) & a+b+c+d \end{bmatrix}. \tag{4.17}$$

If $p = q = \frac{1}{2}$, then all of these maps are positive. Indeed, given a positive matrix of the form $C := \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\left[\begin{smallmatrix} \overline{a} & \overline{b} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \overline{a}a & a\overline{b} \\ b\overline{a} & b\overline{b} \end{smallmatrix}\right]$, F and G send this matrix to[9]

$$F(C) = \frac{1}{2}\begin{bmatrix} \overline{a}+\overline{b} \\ \overline{b}-\overline{a} \end{bmatrix}\begin{bmatrix} a+b & b-a \end{bmatrix} \quad \text{and} \quad G(C) = \frac{1}{2}\begin{bmatrix} \overline{a}-\overline{b} \\ \overline{a}+\overline{b} \end{bmatrix}\begin{bmatrix} a-b & a+b \end{bmatrix} \quad \text{when } p = q = \frac{1}{2}. \tag{4.18}$$

However, when $p \neq \frac{1}{2}$, then neither F nor G are positive. In fact, neither F nor G are $*$-preserving, which is a necessary condition for positivity. We will come back to this in the next section.

5 Positive conditionals

The expressions for conditionals in Corollary 4.5 have two disadvantages. First, they are only partially defined on the supports.[10] Second, they need not be positive maps. A necessary condition for positivity is $*$-preservation, so we will first analyze when conditionals are $*$-preserving.

[9] All positive matrices are non-negative sums of matrices of this type. Hence, proving $F(C)$ and $G(C)$ are positive is sufficient to proving that F and G are positive, respectively. Proving F and G are positive is also equivalent to proving F^* and G^* are positive.

[10] We have not discussed this aspect here because we assumed the marginals are invertible. See [24] for more details regarding supports and their role in Bayesian inversion. The analogous situation for conditionals is part of ongoing work.

Lemma 5.1. *Let $\mathscr{A} := \mathscr{M}_m(\mathbb{C}), \mathscr{B} := \mathscr{M}_n(\mathbb{C})$ be the algebras and $\zeta = \mathrm{tr}(\nu\,\cdot\,), \omega = \mathrm{tr}(\rho\,\cdot\,)$, and $\xi = \mathrm{tr}(\sigma\,\cdot\,)$ the states as defined in Corollary 4.5. Then $*$-preserving conditionals $\mathscr{B} \xrightarrow{F:=\zeta|_{\mathscr{A}}} \mathscr{A}$ and $\mathscr{A} \xrightarrow{G:=\zeta|_{\mathscr{B}}} \mathscr{B}$ respectively exist if and only if*

$$\big[\rho, \mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big)\big] = 0 \quad \forall\, B \in \mathscr{B}, \quad i.e. \quad \Big[\rho, \mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes E^{(n)}_{kl})\big)\Big] = 0 \quad \forall\, k,l \in \{1,\dots,n\}, \tag{5.2}$$

and

$$\big[\sigma, \mathrm{tr}_{\mathscr{A}}\big(\nu(A \otimes \mathbb{1}_n)\big)\big] = 0 \quad \forall\, A \in \mathscr{A}, \quad i.e. \quad \Big[\sigma, \mathrm{tr}_{\mathscr{A}}\big(\nu(E^{(m)}_{ij} \otimes \mathbb{1}_n)\big)\Big] = 0 \quad \forall\, i,j \in \{1,\dots,m\}, \tag{5.3}$$

respectively.

Proof. By Corollary 4.5, the formulas for F and G are uniquely determined. These linear maps are $*$-preserving if and only if $F(B^\dagger) = F(B)^\dagger$ and $G(A^\dagger) = G(A)^\dagger$ for all inputs, or equivalently $F(B^\dagger)^\dagger = F(B)$ and $G(A^\dagger)^\dagger = G(A)$ for all inputs. We begin with F and assume F is $*$-preserving. Then,

$$\mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B)\nu\big)\rho^{-1} = F(B) = F(B^\dagger)^\dagger = \Big(\mathrm{tr}_{\mathscr{B}}\Big(\big(\mathbb{1}_m \otimes B^\dagger\big)\nu\Big)\rho^{-1}\Big)^\dagger = \rho^{-1}\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big). \tag{5.4}$$

Multiplying both sides by ρ and using the properties of the partial trace, this is equivalent to

$$\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big)\rho = \rho\,\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big). \tag{5.5}$$

Since every B can be expressed as a linear combination $B = \sum_{k,l} B_{kl} E^{(n)}_{kl}$, this is equivalent to the second condition in (5.2). By a similar calculation, if (5.2) holds, then F is $*$-preserving. An analogous argument shows that G is $*$-preserving if and only if (5.3) holds. ■

Example 5.6. One can also use Lemma 5.1 to prove that the maps F and G in Example 4.13 are not $*$-preserving if $p \neq \frac{1}{2}$.

Lemma 5.1 provides a necessary condition for positive conditionals to exist. Are they sufficient? Namely, if a conditional is $*$-preserving, is it necessarily positive? The reason we ask this question is because this is (perhaps surprisingly) true for Bayes maps on matrix algebras (when one of the density matrices has full support). In fact, the $*$-preserving condition is strong enough to imply *complete positivity* for Bayes maps [24]. Based on Example 4.8, we so far know that $*$-preservation is *not* strong enough to imply complete positivity (or even Schwarz-positivity) for conditionals, so it is natural to ask about positivity alone. In the following theorem, we settle this question.

Theorem 5.7. *In the notation of Lemma 5.1, positive conditionals $\mathscr{B} \xrightarrow{F:=\zeta|_{\mathscr{A}}} \mathscr{A}$ and $\mathscr{A} \xrightarrow{G:=\zeta|_{\mathscr{B}}} \mathscr{B}$ exist if and only if Equations (5.2) and (5.3) hold, respectively (i.e. if and only if $*$-preserving conditionals exist).*

Proof. It suffices to prove the claim for F. If F is positive, then it is automatically $*$-preserving, which is where Lemma 5.1 applies. Conversely, suppose that F is $*$-preserving. Then by Corollary 4.5, Equation (5.2), the properties of the partial trace, and the functional calculus[11]

$$F(B^\dagger B) = \mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B^\dagger B)\nu\big)\rho^{-1} = \sqrt{\rho^{-1}}\,\mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B)\sqrt{\nu}\sqrt{\nu}(\mathbb{1}_m \otimes B^\dagger)\big)\sqrt{\rho^{-1}}. \tag{5.8}$$

Since the right-hand-side of this expression is manifestly positive, F is positive. ■

[11] Since ρ commutes with $\mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B)\nu\big)$ for every B, any function of ρ commutes with $\mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B)\nu\big)$ as well (see "The Functional Calculus" series in [20]).

Even if the $*$-preserving conditions do not hold for *all* elements in the domain algebras, we can always find maximal *subspaces* on which F and G *are* positive.

Definition 5.9. In the notation of Lemma 5.1, set

$$\mathscr{A}_{\rho^c} := \big\{A \in \mathscr{A} \ : \ [\rho, A] = 0\big\} \quad \text{and} \quad \mathscr{B}_{\sigma^c} := \big\{B \in \mathscr{B} \ : \ [\sigma, B] = 0\big\} \tag{5.10}$$

to be the ***commutants*** of $\{\rho\}$ and $\{\sigma\}$ inside $\mathscr{A}$ and $\mathscr{B}$, respectively. Set

$$\mathscr{B}_\nu := \Big\{B \in \mathscr{B} \ : \ \mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big) \in \mathscr{A}_{\rho^c}\Big\} \quad \text{and} \quad \mathscr{A}_\nu := \Big\{A \in \mathscr{A} \ : \ \mathrm{tr}_{\mathscr{A}}\big((A \otimes \mathbb{1}_n)\nu\big) \in \mathscr{B}_{\sigma^c}\Big\} \tag{5.11}$$

to be the ***conditional domains*** of ν inside $\mathscr{B}$ and $\mathscr{A}$, respectively. A ***(concrete) operator system*** inside $\mathscr{M}_k(\mathbb{C})$ is a (complex) vector subspace $\mathscr{O} \subseteq \mathscr{M}_k(\mathbb{C})$ such that $\mathbb{1}_k \in \mathscr{O}$ and $A \in \mathscr{O}$ implies $A^\dagger \in \mathscr{O}$.

Lemma 5.12. *In the notation of Definition 5.9, $\mathscr{B}_\nu$ and $\mathscr{A}_\nu$ are operator systems.*

Proof. It suffices to prove this for $\mathscr{B}_\nu$. First, $\mathscr{B}_\nu$ is a subspace by linearity. Second, $\mathrm{tr}_{\mathscr{B}}(\nu) = \rho$ and $\rho \in \mathscr{A}_{\rho^c}$ imply $\mathbb{1}_n \in \mathscr{B}_\nu$. Third, if $B \in \mathscr{B}_\nu$, then

$$\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B^\dagger)\big) = \big(\mathrm{tr}_{\mathscr{B}}\big((\mathbb{1}_m \otimes B)\nu\big)\big)^\dagger = \big(\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big)\big)^\dagger. \tag{5.13}$$

Since $\mathscr{A}_{\rho^c}$ is a $*$-algebra $\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big) \in \mathscr{A}_{\rho^c}$ implies $\big(\mathrm{tr}_{\mathscr{B}}\big(\nu(\mathbb{1}_m \otimes B)\big)\big)^\dagger \in \mathscr{A}_{\rho^c}$. Hence, $B^\dagger \in \mathscr{B}_\nu$ by (5.13). Thus $\mathscr{B}_\nu$ is an operator system. ∎

In this way, although one might not be able to condition on the full algebra to obtain a positive map, one might be able to condition on an operator system inside that algebra.

Example 5.14. In terms of Example 4.13 and assuming $p \neq \frac{1}{2}$, one can show

$$\mathscr{A}_{\rho^c} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \ : \ a, b \in \mathbb{C} \right\} \subset \mathscr{A} \quad \text{and} \quad \mathscr{B}_{\sigma^c} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \ : \ a, d \in \mathbb{C} \right\} \subset \mathscr{B} \tag{5.15}$$

are the commutants. The conditional domains are given by

$$\mathscr{B}_\nu = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \ : \ a, d \in \mathbb{C} \right\} \subset \mathscr{B} \quad \text{and} \quad \mathscr{A}_\nu = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \ : \ a, b \in \mathbb{C} \right\} \subset \mathscr{A}. \tag{5.16}$$

Example 5.14 suggests that $\mathscr{B}_\nu$ and $\mathscr{A}_\nu$ are not only operator systems, but they might even be C^*-subalgebras. Is this always the case?

Example 5.17. The answer to this question is no, though the simplest counterexample I could currently find involves a 9×9 rank 2 density matrix with $\mathscr{A} = \mathscr{M}_3(\mathbb{C})$ and $\mathscr{B} = \mathscr{M}_3(\mathbb{C})$. Its expression is not particularly enlightening, so I have chosen to not record it here.[12]

Corollary 5.18. *In the notation of Lemma 5.1 and Definition 5.9, there exist conditionals $\mathscr{B} \xrightarrow{F} \mathscr{A}$ and $\mathscr{A} \xrightarrow{G} \mathscr{B}$ such that the restrictions $\mathscr{B}_\nu \hookrightarrow \mathscr{B} \xrightarrow{F} \mathscr{A}$ and $\mathscr{A}_\nu \hookrightarrow \mathscr{A} \xrightarrow{G} \mathscr{B}$ are positive unital maps from operator systems to C^*-algebras. In terms of the Hilbert–Schmidt duals (the Schrödinger picture), the restrictions $\mathscr{A}_{\rho^c} \hookrightarrow \mathscr{A} \xrightarrow{F^*} \mathscr{B}$ and $\mathscr{B}_{\sigma^c} \hookrightarrow \mathscr{B} \xrightarrow{G^*} \mathscr{A}$ are positive trace-preserving maps between C^*-algebras.*

[12] Also, I could not find a 4×4 density matrix ν for which the conditional domains are *not* C^*-subalgebras, and I suspect that this may always be the case. I hope to resolve this in future work.

Positivity of the Hilbert–Schmidt duals guarantees that density matrices living in the respective commutants always get sent to density matrices under the conditionals.

Example 5.19. In terms of Example 4.13 (see also Example 5.14), suppose that Bob (represented by $\mathscr{B}$) obtains new evidence in the form of a state $\varphi = \langle\uparrow|\cdot|\uparrow\rangle$. This is represented by the density matrix $|\uparrow\rangle\langle\uparrow|$, which is in $\mathscr{B}_{\sigma^c}$. The conditional G^* sends this density matrix to $\frac{1}{2}\left[\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right]$. In other words, with evidence $|\uparrow\rangle\langle\uparrow|$, Bob will infer that the state Alice receives is given by $\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. However, if Bob obtains new evidence that is represented by a density matrix *not* in $\mathscr{B}_{\rho^c}$, such as $\frac{1}{2}\left[\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right]$, then the image of this under G^* is given by $\frac{1}{2}\mathbb{1}_2 + \frac{1}{4\sqrt{pq}}\left[\begin{smallmatrix}-1 & p-q\\ q-p & 1\end{smallmatrix}\right]$. Although this is a matrix with nonnegative eigenvalues (they are 0 and 1), it is not a density matrix because it is not self-adjoint, unless $p = q = \frac{1}{2}$.

6 Discussion and future directions

The work presented here includes preliminary investigations on the structure of conditioning in the setting of hybrid classical-quantum systems (finite-dimensional C^*-algebras) from the Markov category perspective. We have focused only on matrix algebras and joint states for which the marginal density matrices are invertible. Our definitions are distinct from those of [15], which defines conditioning in terms of predicates and uses operations analogous to those used to define the Petz recovery map (similar constructions are done using the $Q_{1/2}$ calculus in [7, 17]). The root of the distinction between these two approaches comes from the fact that we use the multiplication map to formulate Bayes maps, even though it is not a positive map. By using quantum Markov categories [21], we have been able to construct conditioning in a way completely analogous to what is done in the classical theory, while still using the multiplication map, and then finding conditions for which the resulting maps are positive.

Some work in progress includes the extension of the results presented here to the case where the marginal density matrices are not invertible. Although this seems like an innocent generalization, this is where most of the intricate details occur when analyzing the case of disintegrations and Bayesian inversion in [23, 24]. It is also what accentuates the difference between Bayesian inverses and the Petz recovery map. Another crucial generalization is to the case of general finite-dimensional C^*-algebras, i.e. direct sums of matrix algebras, to include hybrid classical-quantum systems. Upon obtaining suitable necessary and sufficient conditions for positive conditionals to exist, one should show that these conditions are automatically satisfied for commutative algebras in such a way so that the conditional version of Bayes' theorem (Theorem 3.8) is reproduced (Theorem 3.7 was already reproduced in [24]).

I see many interesting future directions based on the ideas presented here. For example, what does the set of joint states admitting positive conditionals look like? What is the structure of conditionals for multi-partite (as opposed to bi-partite) states on quantum systems? For example, one can take three (classical) random variables and construct multiple conditionals for other purposes, such as defining conditional independence or constructing Markov chains. There are also theorems describing the consistency of successive conditioning in classical probability theory (for a viewpoint similar to the one presented here, see [13, Section 11], particularly Lemma 11.11, and the references therein). One wonders if such results still hold in the quantum setting from this perspective, at least on the conditional domains defined here. Another direction for future investigations is to obtain approximate versions of these results using distance measures between states, such as the fidelity or statistical distance.

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