Cayley-Hamilton Decomposition and Spectral Asymmetry

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Juin 2004

IHES/M/04/01

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Abstract. In this paper we derive Cayley-Hamilton decompositions, along some of their consequences, for compact operators and closed operators with compact resolvent on a (separable) Hilbert space. In particular, we make use these decompositions to give a spectral interpretation of a projector found by Wodzicki to encode the spectral asymmetry of elliptic ΨDO's on a compact manifold. As another application we get a convenient definition of the partial inverse of a closed operator with compact resolvent. Finally, we work out the results of this paper in the examples of an elliptic ΨDO on a compact manifold and of an elliptic ΨDO on a spectral triple (i.e. on a noncommutative manifold in the sense of Connes's noncommutative geometry).

1. Introduction

In this paper are derived Cayley-Hamilton decompositions, along some of their consequences, for compact operators and closed operators with compact resolvent on a separable Hilbert space. This extends the Cayley-Hamilton decomposition in finite dimension and generalizes the spectral representations of normal compact operators and normal closed operators with compact resolvent (see also [RN], [GK], [DS1], [DS2], [Ri], [Ma] for related results).

In fact, given a compact operator T on a separable Hilbert space $\mathcal{H}$ its Cayley-Hamilton decomposition is a simple consequence of the results of Riesz-Nagy [RN] and Gohberg-Krein [GK] about the characteristic subspaces and characteristic projectors associated to the non-zero eigenvalues of T. Recall that given an eigenvalue $\lambda \in \operatorname{Sp} T \setminus 0$ the characteristic space $E_\lambda(T)$ and the characteristic projector $\Pi_\lambda(T)$ are given by the formulas,

$$E_\lambda(T) = \cup_{k\geq 1} \ker(T-\lambda)^k \quad \text{and} \quad \Pi_\lambda(T) = \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}} (T-\xi)^{-1} d\xi, \tag{1.1}$$

where $\Gamma_{(\lambda)}$ is a small circle about λ which isolates λ from the rest of the spectrum. As it turns out $\Pi_\lambda(T)$ is a finite rank projector which projects onto $E_\lambda(T)$ and along $E_{\bar{\lambda}}(T^*)^\perp$, so that $E_\lambda(T)$ has finite dimension (*cf.* [GK] and Section 3). Here we show that we can also associate a characteristic data to the spectral value $\lambda = 0$ by letting

$$\Pi_0(T) = \lim_{r\to 0^+} \frac{-1}{2i\pi}\int_{|\xi|=r} (T-\xi)^{-1} d\xi \qquad \text{and} \qquad E_0(T) = \operatorname{im} \Pi_0(T), \tag{1.2}$$

where the limit converges in $\mathcal{L}(\mathcal{H})_s$, i.e. in $\mathcal{L}(\mathcal{H})$ equipped with its strong topology. This construction allows us to complete the Cayley-Hamilton decomposition of T, for we obtain:

Theorem 1.1. *The family $(\Pi_\lambda(T))_{\lambda\in\operatorname{Sp} T}$ is an orthogonal family of projectors and we have*

$$\mathcal{H} = \overline{\dot{+}_{\lambda\in\operatorname{Sp} T} E_\lambda(T)} \quad \textit{and} \quad \sum_{\lambda\in\operatorname{Sp} T} \Pi_\lambda(T) = 1, \tag{1.3}$$

where $\dot{+}$ denotes the algebraic direct sum, the subspace $E_0(T)$ may be empty and the series converges in $\mathcal{L}(\mathcal{H})_s$.

2000 *Mathematics Subject Classification.* Primary 58C40, 58J40; Secondary 58J42.
Key words and phrases. Spectral theory, pseudodifferential operators, noncommutative geometry.

Next, we can similarly get a Cayley-Hamilton decomposition for a closed operator T with compact resolvent on $\mathcal{H}$. In this setting the characteristic data associated to $\lambda = 0$ in the compact case correspond to the characteristic data associated to "the infinity of $\mathbb{C}$" and given by

$$\Pi_\infty(T) = \lim_{R\to\infty} \frac{1}{2i\pi} \int_{|\xi|=R} \xi^{-1} T(T-\xi)^{-1} d\xi \qquad \text{and} \qquad E_\infty(T) = \operatorname{im} \Pi_\infty(T), \tag{1.4}$$

with limit taken in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$. Then we get:

Theorem 1.2. *The family $(\Pi_\lambda(T))_{\lambda\in\operatorname{Sp}T\cup\{\infty\}}$ is an orthogonal family of projectors and we have*

$$\mathcal{H} = \overline{\dot{+}_{\lambda\in\operatorname{Sp}T\cup\{\infty\}} E_\lambda(T)} \quad \textit{and} \quad \sum_{\lambda\in\operatorname{Sp}T\cup\{\infty\}} \Pi_\lambda(T) = 1, \tag{1.5}$$

where the subspace $E_\infty(T)$ may be empty and the series converges in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$.

An important application of Theorem 1.2 is related to the results of Wodzicki ([Wo1]–[Wo4]; see also [Po1]) on the spectral asymmetry of elliptic pseudodifferential operators (in short ΨDO's). Recall that the study of the spectral asymmetry of elliptic ΨDO's was initiated (in the selfadjoint case) by Atiyah-Patodi-Singer ([APS1], [APS2]) in terms of the eta function $\eta(P;s)$ and the eta invariant $\eta(P)$ of the operator, in connection with their index theorem on manifolds with boundary. Motivated by an observation of Shubin [Sh, p. 114] Wodzicki ([Wo1]–[Wo4]) took a different point of view and looked at the difference $\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)$ of zeta functions coming from two different spectral cuts $L_\theta = \{\arg\lambda = \theta\}$ and $L_{\theta'} = \{\arg\lambda = \theta'\}$ with $0 \leq \theta < \theta' < 2\pi$ (here $\zeta_\theta(P;s) = \operatorname{Tr} P_\theta^{-s}$ where $(P_\theta^s)_{s\in\mathbb{C}}$ is the family of complex power associated to the cutting L_θ as defined in [Se]). As it turns out we have the following equalities of meromorphic functions,

$$\zeta_\theta(P;s) - \zeta_{\theta'}(P;s) = (1 - e^{-2i\pi s}) \operatorname{Tr} \Pi_{\theta,\theta'}(P) P_\theta^{-s}, \quad s \in \mathbb{C}, \tag{1.6}$$

where $\Pi_{\theta,\theta'}(P)$ is the ΨDO projector given by

$$\Pi_{\theta,\theta'}(P) = \frac{1}{2i\pi} \int_{\Gamma_{\theta,\theta',r}} \xi^{-1} P(P-\xi)^{-1} d\xi, \tag{1.7}$$

$$\Gamma_{\theta,\theta,r} = \{\rho e^{i\theta}; \infty < \rho \leq r\} \cup \{re^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \leq \rho \leq \infty\},$$

where r is small enough so that no non-zero eigenvalue of T lies in the disc $|\lambda| \leq r$. As a consequence for any integer $k \in \mathbb{Z}$ we get

$$\operatorname{ord} P. \lim_{s\to k} (\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = i\pi \operatorname{Res} \Pi_{\theta,\theta'}(P) P^{-k}, \tag{1.8}$$

where Res denotes the Wodzicki-Guillemin noncommutative residue trace ([Wo5], [Gu]).

One of the main motivations of this paper is to make use of the Cayley-Hamilton decomposition provided by Theorem 1.2 to give a spectral interpretation of Wodzicki's projector $\Pi_{\theta,\theta'}(T)$, in the general setting of a closed operator with compact resolvent on $\mathcal{H}$. Indeed, letting $\Lambda_{\theta,\theta'}$ denote the open angular sector $\{\theta < \arg\lambda < \theta'\}$ we first prove that we can define a characteristic projector and a characteristic subspace associated to the "infinity of $\Lambda_{\theta,\theta'}$" by letting

$$\Pi_{\theta,\theta',\infty}(T) = \lim_{R\to\infty} \frac{1}{2i\pi} \int_{\Gamma_{\theta,\theta',R}} \xi^{-1} T(T-\xi)^{-1} d\xi \quad \text{and} \quad E_{\theta,\theta',\infty}(T) = \operatorname{im} \Pi_{\theta,\theta',\infty}(T), \tag{1.9}$$

$$\Gamma_{\theta,\theta,r} = \{\rho e^{i\theta}; \infty < \rho \leq r\} \cup \{re^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \leq \rho \leq \infty\},$$

where the limit is shown to exist in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$. Then we get the following spectral interpretation of the projector $\Pi_{\theta,\theta'}(T)$.

Theorem 1.3. *1) The family* $\{\Pi_{\theta,\theta',\infty}(T)\} \cup \{\Pi_\lambda(T)\}_{\lambda\in \operatorname{Sp} T\cap\Lambda_{\theta,\theta'}}$ *is an orthogonal family of projectors and in* $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ *we have*

$$\Pi_{\theta,\theta'}(T) = \Pi_{\theta,\theta',\infty}(T) + \sum_{\lambda\in\operatorname{Sp} T\cap\Lambda_{\theta,\theta'}} \Pi_\lambda(T). \tag{1.10}$$

2) The projector $\Pi_{\theta,\theta'}(T)$ *projects onto* $E_{\theta,\theta',\infty}(T) \dot{+} (\overline{\dot{+}_{\lambda\in\operatorname{Sp} T\cap\Lambda_{\theta,\theta'}} E_\lambda(T)})$ *and along the subspace* $E_0(T) \dot{+} E_{\theta',\theta+2\pi,\infty}(T) \dot{+} (\overline{\dot{+}_{\lambda\in\operatorname{Sp} T\cap\Lambda_{\theta',\theta+2\pi}} E_\lambda(T)})$.

As a consequence of Theorem 1.3 we can show that $\Pi_{\theta,\theta'}(T)$ has finite rank if, and only if, T has at most finitely many eigenvalues in $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta',\infty}(T)$ is zero (Proposition 4.7).

Another consequence of the Cayley-Hamilton decomposition of a closed operator T with compact resolvent case is that we can define of the partial inverse of T. More precisely, it can be shown that the characteristic subspace,

$$E_{\bar{\mathbb{C}}\backslash 0}(T) = \overline{\dot{+}_{\lambda\in(\operatorname{Sp} T\cup\infty)\backslash 0} E_\lambda(T)}, \tag{1.11}$$

is globally invariant by T and on there T induces an invertible operator (Lemma 5.2). Therefore, we can define the partial inverse T^{-1} as the bounded operator that vanishes on $E_0(T)$ and inverts T on $E_{\bar{\mathbb{C}}\backslash 0}(T)$. This definition extends that of the inverse in the invertible case and the usual definition of the partial inverse in the normal case. Moreover the definition is compatible with the operations of taking adjoints and integer powers and connects nicely with Seeley's construction of complex powers (*cf.* Proposition 5.6 and Proposition 5.7).

Now, it is interesting to look at the previous results in the example of an elliptic ΨDO-operator $P: C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ of order $m>0$ and acting on the sections of a Hermitian vector bundle $\mathcal{E}$ over a compact Riemannian manifold M^n. In addition we assume that the spectrum of P is not the whole complex plane, which insures us that P has a compact resolvent. As a consequence of the ellipticity of P we see that for any $\lambda \in \operatorname{Sp} P$ the characteristic space $E_\lambda(P)$ is contained in $C^\infty(M,\mathcal{E})$ and the projector $\Pi_\lambda(P)$ is smoothing (*cf.* Section 6).

Moreover, the projector $\Pi_\infty(P)$ induces (for $s>0$) and extends to (for $s<0$) a unique continuous endomorphism of the Sobolev space $L^2_s(M,\mathcal{E})$, hence it induces (resp. extends to) a continuous endomorphism of $C^\infty(M,\mathcal{E})$ (resp. $\mathcal{D}'(M,\mathcal{E})$) (Lemma 6.2). Thus, we have a characteristic space $E_\infty(P)$ on each Sobolev space $L^2(M,\mathcal{E})$, $s\in\mathbb{R}$, which is given by

$$E^{(s)}_\infty(P) = \Pi_\infty(P)(L^2_s(M,\mathcal{E})), \tag{1.12}$$

Furthermore, this definition can be extended to $C^\infty(M,\mathcal{E})$ and $\mathcal{D}'(M,\mathcal{E})$ by letting

$$E^{(\infty)}_\infty(P) = \Pi_\infty(P)(C^\infty(M,\mathcal{E})) \quad \text{and} \quad E^{(-\infty)}_\infty(P) = \Pi_\infty(P)(\mathcal{D}'(M,\mathcal{E})). \tag{1.13}$$

Then we have the following version of Theorem 1.2.

Theorem 1.4. *1) Let* $s\in\mathbb{R}$. *Then we have*

$$L^2_s(M,\mathcal{E}) = E^{(s)}_\infty(P) \dot{+} (\overline{\dot{+}_{\lambda\in\operatorname{Sp} P} E_\lambda(P)}) \qquad \textit{and} \qquad \sum_{\lambda\in\operatorname{Sp} P\cup\{\infty\}} \Pi_\lambda(P) = 1, \tag{1.14}$$

where the series converges in $\mathcal{L}(L^2_s(M,\mathcal{E}))_s$.

2) The decomposition (6.3) holds in $C^\infty(M,\mathcal{E})$ *and in* $\mathcal{D}'(M,\mathcal{E})$ *provided that we replace* $E^{(s)}_\infty(P)$ *by* $E^{(\infty)}_\infty(P)$ *and* $E^{(-\infty)}_\infty(P)$ *respectively.*

Concerning the projector $\Pi_{\theta,\theta'}(P)$ we can consider rays L_θ and $L_{\theta'}$ with $0\leq\theta<\theta'<2\pi$ and such that L_θ and $L_{\theta'}$ are spectral cuts for both P and its principal symbol $p_m(x,\xi)$. Then results of Seeley [Se] (see also [Sh]) insures us that L_θ and $L_{\theta'}$ are rays of minimal growth for P, so that we may define the projector $\Pi_{\theta,\theta'}(P)$ as in (1.7). Furthermore, the parametric calculus developed by Seeley [Se] and Shubin [Sh] together with that fact $\Pi_{\theta,\theta'}(P)$ enables us to show that $\Pi_{\theta,\theta'}(P)$

either is a zero'th order ΨDO or is a smoothing operator (see [Wo4] and Proposition 6.6). In fact, we have:

Proposition 1.5. *The following are equivalent:*

(i) The principal symbol $p_m(x,\xi)$ *has no eigenvalue in* $\Lambda_{\theta,\theta'}$*;*

(ii) $\Pi_{\theta,\theta'}(P)$ *is a smoothing operator;*

(ii) $\operatorname{Sp} P \cap \Lambda_{\theta,\theta'}$ *is at most finite and* $\Pi_{\theta,\theta',\infty}(P) = 0$.

Moreover, in this context the partial inverse is a ΨDO of order $-m$ and, in fact, is even a parametrix for P (*cf.* Proposition 6.8).

The above results for an elliptic ΨDO on a compact manifold actually hold in many other contexts of ΨDO-algebras. To illustrate this fact we show that they even reach analogues in the framework of Connes' noncommutative geometry [Co]. Recall that in the latter framework a (noncommutative) manifold can be represented by a so-called spectral triple for which Connes-Moscovici [CM] built an algebra of ΨDO's. Then considering an elliptic ΨDO on an spectral triple enables us to get noncommutative analogues of Theorem 1.4 and Proposition 1.5 (see Theorem 7.10 and Proposition 7.11).

This paper is organized as follows. In Section 2 we derive the Cayley-Hamilton decomposition for compact operators and we deal with that of closed operators with compact resolvent in Section 3. The section 4 is devoted to the spectral interpretation of Wodzicki's projector. Finally, we look at the results in the examples of an elliptic ΨDO on a compact manifold (Section 6) and of an elliptic ΨDO on a spectral triple (Section 7).

2. Cayley-Hamilton decomposition of a compact operator

Throughout all this paper we let $\mathcal{H}$ denote a separable Hilbert space. In this section our aim is to derive a Cayley-Hamilton decomposition for a given compact operator T on $\mathcal{H}$. Recall that since T is compact its spectrum is a compact subset only containing the origin and, possibly, non-zero eigenvalues which have finite multiplicities and are isolated in $\operatorname{Sp} T$. In particular, $\operatorname{Sp} T$ is countable.

Following [RN] to any eigenvalue $\lambda \in \operatorname{Sp} T \setminus 0$ we can associate the characteristic subspace $E_\lambda(P)$ and characteristic projector $\Pi_\lambda(P)$ defined by

$$E_\lambda(T) = \cup_{k\geq 1} \ker(T-\lambda)^k \quad \text{and} \quad \Pi_\lambda(T) = \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}} (T-\xi)^{-1} d\xi, \tag{2.1}$$

where $\Gamma_{(\lambda)}$ is a small circle about λ which isolates λ from the rest of the spectrum.

Lemma 2.1 ([RN, Leçon 148]). *1) The family* $(\Pi_\lambda(T))_{\lambda\in \operatorname{Sp} T\setminus 0}$ *is an orthogonal family of projectors, i.e.* $\Pi_\lambda(T)\Pi_\mu(T) = \delta_{\lambda\mu}$ *for any* λ *and* μ *in* $\lambda \in \operatorname{Sp} T \setminus 0$.

2) For any $\lambda \in \operatorname{Sp} T \setminus 0$ *the projector* $\Pi_\lambda(T)$ *has finite rank.*

Proof. Let $\Gamma'_{(\lambda)}$ be a circle about λ with a smaller radius than that of $\Gamma_{(\lambda)}$. Then, using the identity,

$$(T-\xi)^{-1}(T-\eta)^{-1} = -(\eta-\xi)^{-1}(T-\xi)^{-1} - (\xi-\eta)^{-1}(T-\eta)^{-1}, \tag{2.2}$$

we deduce that $\Pi_\lambda(T)\Pi_\mu(T)$ is equal to

$$\begin{aligned} \Pi_\lambda(T)\Pi_\mu(T) &= \frac{-1}{4\pi^2}\int_{\Gamma'_{(\lambda)}}\int_{\Gamma_{(\mu)}} (T-\xi)^{-1}(T-\eta)^{-1} d\xi d\eta, \\ &= \frac{1}{4\pi^2}\int_{\Gamma'_{(\lambda)}} (T-\xi)^{-1}\Big(\int_{\Gamma_{(\mu)}} \frac{d\eta}{\eta-\xi}\Big) d\xi + \frac{1}{4\pi^2}\int_{\Gamma_{(\mu)}} (T-\eta)^{-1}\Big(\int_{\Gamma'_{(\lambda)}} \frac{d\xi}{\xi-\eta}\Big) d\eta. \end{aligned} \tag{2.3}$$

Thus, if $\lambda \neq \mu$ then $\Pi_\lambda(T)\Pi_\mu(T) = 0$, whereas for $\mu = \lambda$ we get

$$\Pi_\lambda(T)^2 = \frac{-1}{2i\pi}\int_{\Gamma'_{(\lambda)}} (T-\eta)^{-1} d\eta = \Pi_\lambda(T). \tag{2.4}$$

Hence $(\Pi_\lambda(P))_{\lambda\in\mathbb{C}}$ is an orthogonal family of projectors.

On the other hand, from the identities

$$(T-\xi)^{-1} = -\xi^{-1}T(T-\xi)^{-1} + \xi^{-1}(T-\xi)(T-\xi)^{-1} = -\xi^{-1}T(T-\xi)^{-1} + \xi^{-1}, \tag{2.5}$$

and the fact that $\Gamma_{(\lambda)}$ isolates λ from the rest of the spectrum, and so from the origin, we get

$$\Pi_\lambda(T) = \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}} \xi^{-1}T(T-\xi)^{-1}d\xi + \frac{1}{2i\pi}\int_{\Gamma_{(\lambda)}} \xi^{-1}d\xi = \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}} \xi^{-1}T(T-\xi)^{-1}d\xi. \tag{2.6}$$

Since T is compact it follows from this that $\Pi_\lambda(T)$ is compact. Therefore $\operatorname{im}\Pi_\lambda(T) = \ker(\Pi_\lambda(T)-1)$ has finite dimension. □

Lemma 2.2 ([GK, Sect. I.2]). *Let $\lambda \in \operatorname{Sp} T \setminus 0$. Then:*

1) $E_\lambda(T)$ has finite dimension and there is an integer $N \geq 1$ such that $E_\lambda(T) = \dim\ker(T-\lambda)^N$.

2) The subspace $E_{\bar\lambda}(T^*)^\perp$ is globally invariant by T and we have $\mathcal{H} = E_\lambda(T) \dotplus E_{\bar\lambda}(T^*)^\perp$.

3) The projector $\Pi_\lambda(T)$ projects onto $E_\lambda(T)$ and along $E_{\bar\lambda}(T^*)^\perp$.

Proof. Let $x \in \ker T^k$, $k \geq 1$, and set $S = T - \lambda$, so that $\Pi_\lambda(T) = \frac{-1}{2i\pi}\int_{\Gamma_{(0)}}(S-\xi)^{-1}$. Since for $\xi \neq 0$ we have

$$(S-\xi)\sum_{j=0}^{k-1}\xi^{-(j+1)}S^jx = S^kx - x = -x, \tag{2.7}$$

we see that $(S-\xi)^{-1}x = -\sum_{j=0}^{k-1}\xi^{-(j+1)}S^jx$. Thus,

$$\Pi_\lambda(T)x = \sum_{j=0}^{k-1}\frac{1}{2i\pi}\int_{\Gamma_{(0)}}\xi^{-(j+1)}S^jxd\xi = x, \tag{2.8}$$

that is x is in $\operatorname{im}\Pi_\lambda(T)$. Therefore $\operatorname{im}\Pi_\lambda(T)$ contains $\ker(T-\lambda)^k$ for any integer $k \geq 1$, hence contains $E_\lambda(T)$. As $\operatorname{im}\Pi_\lambda(T)$ has finite dimension it then follows that $E_\lambda(T)$ has finite dimension too. Since we have a flag of finite dimensional spaces $\ker(T-\lambda) \subset \ker(T-\lambda)^2 \subset \cdots \subset E_\lambda(T)$ the fact that $E_\lambda(T)$ has finite dimension implies that this flag is stationary, so that there exists an integer $N \geq 1$ such that $E_\lambda(T) = \ker(T-\lambda)^N$.

Now, as $\Pi_\lambda(T)$ commutes with T the subspaces $\ker\Pi_\lambda(T)$ and $\operatorname{im}\Pi_\lambda(T)$ are globally invariant by T. Moreover, if $x \in \operatorname{im}\Pi_\lambda(T)$ and $Tx = \mu x$ with $\mu \neq \lambda$ then $x = \Pi_\lambda(T)x = \frac{-1}{2i\pi}\int\frac{d\xi}{\mu-\xi}x = 0$. Thus $T_{|\operatorname{im}\Pi_\lambda(T)}$ has only the eigenvalue λ. As $\operatorname{im}\Pi_\lambda(T)$ is finite dimensional this implies that $T_{|\operatorname{im}\Pi_\lambda(T)} - \lambda$ is nilpotent, i.e. $\operatorname{im}\Pi_\lambda(T)$ is contained in $E_\lambda(T)$. Hence $\operatorname{im}\Pi_\lambda(T) = E_\lambda(T)$.

Finally, observe that we have

$$\Pi_\lambda(T)^* = \frac{1}{2i\pi}\int_{\overline{\Gamma_{(\lambda)}}}\frac{d\bar\xi}{T^*-\bar\xi} = \frac{-1}{2i\pi}\int_{\Gamma_{(\bar\lambda)}}\frac{d\xi}{T^*-\xi} = \Pi_{\bar\lambda}(T^*). \tag{2.9}$$

Thus $\ker\Pi_\lambda(T) = (\operatorname{im}\Pi_\lambda(T)^*)^\perp = (\operatorname{im}\Pi_{\bar\lambda}(T^*))^\perp = E_{\bar\lambda}(T^*)^\perp$. Hence $\mathcal{H} = E_\lambda(T) \dotplus E_{\bar\lambda}(T^*)^\perp$ and $\Pi_{\bar\lambda}(T)$ coincides with the projector onto $E_\lambda(T)$ and along $E_{\bar\lambda}(T^*)^\perp$. □

In the sequel we let $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ denote the space $\mathcal{L}(\mathcal{H})$ endowed with its strong topology, that is the simple convergence topology (note that as $\mathcal{H}$ is separable $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ is a Fréchet-Montel space).

Lemma 2.3. *Let $(\Pi_n)_{n\geq 1}$ be an orthogonal sequence of projectors. Then:*

1) The series $\sum_{n\geq 1}\Pi_n$ converges in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ to the projector onto $\overline{\dot{+}_{n\geq 1}\operatorname{im}\Pi_n}$ and along the subspace $\cap_{n\geq 1}\ker\Pi_n$.

2) Let $\Pi_0 = 1-\sum_{n\geq 1}\Pi_n$. Then $(\Pi_n)_{n\geq 0}$ is an orthogonal sequence of projectors and we have

$$\mathcal{H} = \overline{\dot{+}_{n\geq 0}\operatorname{im}\Pi_n} \qquad \text{and} \qquad \sum_{n\geq 0}\Pi_n = 1. \tag{2.10}$$

Proof. Set $E = \overline{\dot{+}_{n\geq 1}\operatorname{im}\Pi_n}$ and $F = \cap_{n\geq 1}\ker\Pi_n$ and for $N\geq 1$ let $P_N = \sum_{n\leq N}\Pi_n$. Since $(\Pi_n)_{n\geq 1}$ is an orthogonal sequence of projectors the operator P_N is the projector onto the subspace $E_N := \dot{+}_{n\leq N}\operatorname{im}\Pi_n$ and along $F_N := \cap_{n\leq N}\ker\Pi_n$. In particular, we have $\|P_N\|\leq 1$. Since $\mathcal{L}(H)_s$ is a Fréchet-Montel space it then follows that the sequence $(P_N)_{N\geq 1}$ is relatively compact in $\mathcal{L}(H)_s$.

Let $(P_{N_k})_{k\geq 1}$ be a subsequence which converges in $\mathcal{L}(H)_s$ to some operator P. Notice that we necessarily have $P^2=P$, so that P is a projector. Let us show that $E=\operatorname{im}P$ and $F=\ker P$.

First, as each projector P_N maps onto $E_N\subset E$ and E is closed we see that $\operatorname{im}P$ is contained in E. On the other hand, if $x\in E_M$ then $P_{N_k}x = x$ for $N_k\geq M$ and so $Px=x$, i.e. x is in $\operatorname{im}P$. Thus $\operatorname{im}P$ contains $E=\overline{\cap_{N\geq 1}E_N}$. Hence $E=\operatorname{im}\Pi$.

Let us now show that $F=\ker P$. If $x\in F$ then for any $n\geq 1$ we have $\Pi_n x = 0$, hence $Px = \lim P_{N_k}x = 0$. Thus F is contained in $\ker P$. Conversely, as $\Pi_n P_{N_k} = P_{N_k}$ for $n\leq N_k$ we deduce that $\Pi_n = \Pi_n P$ for any n. Therefore, if $Px=0$ then we have $\Pi_n x = \Pi_n Px = 0$. This shows that $\ker P$ is contained in F. Hence $F=\ker P$.

As a consequence of this we see that $\mathcal{H} = E\dot{+}F$ and that any limit point of the sequence $(P_N)_{N\geq 1}$ necessarily coincides with the projector Π onto E and along F. As mentioned above this sequence is relatively compact in the Fréchet space $\mathcal{L}(H)_s$, so the sequence converges to P in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$. This shows that $\sum_{n\geq 1}\Pi_n$ is convergent in n $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ with sum the projector Π.

Finally, let $\Pi_0 = 1-\sum_{n\geq 1}\Pi_n = 1-\Pi$. Then $\sum_{n\geq 0}\Pi_n = 1$ and Π_0 is a continuous projector with range $\ker\Pi = F$ and kernel $\operatorname{im}\Pi = E$. In particular, $\mathcal{H} = E\dot{+}\operatorname{im}\Pi_0 = \overline{\dot{+}_{n\geq 0}\operatorname{im}\Pi_n}$. This also implies that for any integer $n\geq 1$ we have $\operatorname{im}\Pi_n\subset\ker\Pi_0$ and $\operatorname{im}\Pi_0\subset\ker\Pi_n$, so that $\Pi_0\Pi_n = \Pi_n\Pi_0 = 0$. Hence $(\Pi_n)_{n\geq 0}$ is an orthogonal sequence of projectors. □

Applying this lemma to the family $(\Pi_\lambda(T))_{\lambda\in\operatorname{Sp}T\setminus 0}$ we see that the series $\sum_{\lambda\in\operatorname{Sp}T\setminus 0}\Pi_\lambda(T)$ converges to the projector onto $\overline{\dot{+}_{\lambda\in\operatorname{Sp}T\setminus 0}E_\lambda(T)}$ and along $\cap_{\lambda\in\operatorname{Sp}T\setminus 0}E_{\bar\lambda}(T^*)^\perp$. This will enable us to prove:

Lemma 2.4. *In $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ we have*

$$\lim_{r\to 0^+}\frac{-1}{2i\pi}\int_{|\xi|=r}(T-\xi)^{-1}d\xi = 1-\sum_{\lambda\in\operatorname{Sp}T\setminus 0}\Pi_\lambda(T). \tag{2.11}$$

Proof. Let $R_0=\sup\{|\lambda|;\ \lambda\in\operatorname{Sp}T\}$ and assume $0<r<R_0<R$. Then we have

$$\frac{-1}{2i\pi}\int_{|\xi|=r}(T-\xi)^{-1}d\xi = \frac{-1}{2i\pi}\int_{|\xi|=R}(T-\xi)^{-1}d\xi + \frac{1}{2i\pi}\int_{\Gamma(R,r)}(T-\xi)^{-1}d\xi, \tag{2.12}$$

where $\Gamma(R,r)$ denotes the contour $\{Re^{it};\ 0\leq t\leq 2\pi\}\cup\{re^{-it};\ 0\leq t\leq 2\pi\}$.

Notice that $\Gamma(R,r)$ is the oriented boundary of $C(R,r)=\{r\leq|\xi|\leq R\}$. Since $(T-\lambda)^{-1}$ is analytic near every point of the interior of $C(R,r)$, except near the finitely many of those that are eigenvalues of T, we get

$$\frac{1}{2i\pi}\int_{\Gamma(R,r)}(T-\xi)^{-1}d\xi = \sum_{\lambda\in\operatorname{Sp}T\cap C(R,r)}\frac{1}{2i\pi}\int_{\Gamma_{(\lambda)}}(T-\xi)^{-1}d\xi = -\sum_{\lambda\in\operatorname{Sp}T\setminus D(0,r)}\Pi_\lambda(T). \tag{2.13}$$

On the other hand, recall that $\frac{-1}{2i\pi}\int_{|\xi|=R}(T-\xi)^{-1}d\xi = 1$ for any $R > R_0$. Indeed, since $(T-\lambda)^{-1}$ is analytic outside $\overline{D(0,R_0)}$ the integral $\frac{-1}{2i\pi}\int_{|\xi|=R}(T-\xi)^{-1}d\xi$ does not depend on the value of R. However, as $\|(T-\lambda)\| = \mathrm{O}(|\lambda|^{-1})$ when λ becomes large, when $R\to\infty$ using (2.5) we get

$$\frac{-1}{2i\pi}\int_{|\xi|=R}(T-\xi)^{-1}d\xi = \frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}d\xi - \frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}T(T-\xi)^{-1}d\xi \to 1. \tag{2.14}$$

Combining this with (2.12) and (2.13) we get:

$$\frac{-1}{2i\pi}\int_{|\xi|=r}(T-\xi)^{-1}d\xi = 1 - \sum_{\lambda\in\mathrm{Sp}\,T\setminus D(0,r)}\Pi_\lambda(T). \tag{2.15}$$

Hence $\lim_{r\to 0^+}\frac{-1}{2i\pi}\int_{|\xi|=r}(T-\xi)^{-1}d\xi = 1 - \sum_{\lambda\in\mathrm{Sp}\,T\setminus 0}\Pi_\lambda(T)$ in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$. □

This lemma allows us to set-up the following.

Definition 2.5. *The characteristic space $E_0(T)$ and the characteristic projector $\Pi_0(T)$ associated to the spectral value $\lambda = 0$ are*

$$\Pi_0(T) = \lim_{r\to 0^+}\frac{-1}{2i\pi}\int_{|\xi|=r}(T-\xi)^{-1}d\xi \qquad \text{and} \qquad E_0(T) = \operatorname{im}\Pi_0(T). \tag{2.16}$$

These definitions are motivated by the lemma below.

Lemma 2.6. *The subspace $E_0(T)$ contains $\cup_{k\geq 1}\ker T^k$, is globally invariant by T and on there T induces a compact operator T_0 such that $\mathrm{Sp}\,T_0 = \{0\}$.*

Proof. First, using (2.7) and arguing as in (2.8) we see that for any $x\in\ker T^k$, $k\geq 1$, we have

$$\frac{-1}{2i\pi}\int_{|\xi|=r}(T-\xi)^{-1}x\,d\xi = x. \tag{2.17}$$

Taking the limit as $r\to 0^+$ then gives $\Pi_0(T)x = x$. Hence $E_0(T)$ contains $\cup_{k\geq 1}\ker T^k$.

Next, it follows from (2.1) that T commutes with $\Pi_0(T)$. Thus for any $x\in E_0(T)$ we have $\Pi_0(T)(Tx) = T\Pi_0(T)x = Tx$, i.e. Tx belongs to $E_0(T)$. This shows that $E_0(T)$ is globally invariant by T.

Now, let T_0 be the operator induced by T on $E_0(T)$. This is a compact operator, because $T_0 = \iota^* T\iota$, where ι denotes the canonical embedding of $E_0(T)$ into $\mathcal{H}$. Thus any non-zero spectral value λ of T_0 is an eigenvalue with eigenspace $\ker(T_0-\lambda)\subset E_\lambda(T)\cap E_0(T)$. However, as $\Pi_0(T) = 1 - \sum_{\mu\in\mathrm{Sp}\,T\setminus 0}\Pi_\lambda$ it follows from Lemma 2.4 that $\Pi_0(T)$ has kernel $\overline{\dot{+}_{\lambda\in\mathrm{Sp}\,T\setminus 0}\operatorname{im}\Pi_\lambda(T)} = \overline{\dot{+}_{\lambda\in\mathrm{Sp}\,T\setminus 0}E_\lambda(T)}$. Thus $E_\lambda(T)\cap E_0(T) = \{0\}$ for any $\lambda\in\mathrm{Sp}\,T\setminus 0$, so that T_0 cannot have a non-zero spectral value. Since $\mathrm{Sp}\,T_0$ cannot be empty it follows that the latter is just $\{0\}$. □

Now, as $\Pi_0(T) = 1 - \sum_{\lambda\in\mathrm{Sp}\,T\setminus 0}\Pi_\lambda(T)$, applying the 2nd part of Lemma 2.3 gives the Cayley-Hamilton decomposition below.

Theorem 2.7. *The family $(\Pi_\lambda(T))_{\lambda\in\mathrm{Sp}\,T}$ is an orthogonal family of projectors and we have*

$$\mathcal{H} = \overline{\dot{+}_{\lambda\in\mathrm{Sp}\,T}E_\lambda(T)} \quad \text{and} \quad \sum_{\lambda\in\mathrm{Sp}\,T}\Pi_\lambda(T) = 1, \tag{2.18}$$

where $\dot{+}$ denotes the algebraic direct sum, the subspace $E_0(T)$ may be trivial and the series converges in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$.

Finally, when T is normal, i.e. $T^*T = TT^*$, we can recover from Theorem 2.7 the well known fact that a compact normal operator diagonalizes in an orthonormal basis (e.g. [Ka, Thm. V.2.10]). Namely, we have:

Proposition 2.8. *Assume that T is normal. Then:*

1) We can write

$$\mathcal{H} = \bigoplus_{\lambda \in \operatorname{Sp} T} \ker(T - \lambda), \tag{2.19}$$

where $\oplus$ denotes the Hilbertian direct sum and $\ker T$ may be trivial.

2) For any $\lambda \in \operatorname{Sp} T$ the projector $\Pi_\lambda(T)$ is the orthogonal projector onto $\ker(T - \lambda)$.

Proof. Let $\lambda \in \operatorname{Sp} T \setminus 0$ and set $S = T - \lambda$. As S is normal we have $\ker S^* = \ker SS^* = \ker S^*S = \ker S$. Thus $\ker S^2 = S^{-1}(\ker S) = S^{-1}(\ker S^*) = \ker SS^* = \ker S^* = \ker S$. Similarly, we have $\ker S^{k+1} = \ker S$ for any integer $k \geq 0$. Hence $E_\lambda(T) = \ker(T - \lambda)$. Moreover, since $\ker S^* = \ker S$ we also get $E_{\bar{\lambda}}(T^*) = \ker(T - \lambda)^* = \ker(T - \lambda)$. Combining this with Lemma 2.2 we then deduce that $\Pi_\lambda(T)$ is the projector onto $\ker(T - \lambda)$ and along $\ker(T - \lambda)^\perp$, i.e. is the orthogonal projector onto $\ker(T - \lambda)$ and so is selfadjoint.

Let us now deal with the spectral value $\lambda = 0$. Since for any $\lambda \in \operatorname{Sp} T \setminus 0$ the projector $\Pi_\lambda(T)$ is selfadjoint it follows from Lemma 2.4 that $\Pi_0(T)$ is selfadjoint as well, so is the orthogonal projector onto $E_0(T)$. On the other hand, as $\Pi_\lambda(T)^* = \Pi_{\bar{\lambda}}(T^*)$ for any $\lambda \in \operatorname{Sp} T \setminus 0$ we also get

$$\Pi_0(T)^* = 1 - \sum_{\lambda \in \operatorname{Sp} T \setminus 0} \Pi_\lambda(T)^* = 1 - \sum_{\lambda \in \operatorname{Sp} T \setminus 0} \Pi_{\bar{\lambda}}(T^*) = \Pi_0(T^*). \tag{2.20}$$

Therefore we have $\Pi_0(T) = \Pi_0(T^*)$. Hence $E_0(T^*) = E_0(T)$.

Next, from Lemma 2.6 we know that $E_0(T)$ is globally invariant by T and on there T induces a compact operator T_0 such that $\operatorname{Sp} T_0 = \{0\}$. Since $E_0(T^*) = E_0(T)$ we similarly see that $E_0(T)$ is globally invariant by T^* and on there T^* induces an operator which is necessarily the adjoint of T_0. Thus for any $x \in E_0(T)$ we have $T_0^* T_0 x = T^*Tx = TT^*x = T_0 T_0^* x$, that is T_0 is a normal operator.

Now, as T_0 is normal we have $\|T_0\| = \sup\{|\lambda|;\ \lambda \in \operatorname{Sp} T_0\}$ (e.g. [Ka, p. 55]). Since $\operatorname{Sp} T_0 = \{0\}$ it follows that $T_0 = 0$, that is $E_0(T)$ is contained in $\ker T$. However, we know from Lemma 2.6 that $\ker T$ is contained in $E_0(T)$. Thus $E_0(T) = \ker T$ and so $\Pi_0(T)$ is the orthogonal projector onto $E_0(T) = \ker T$.

Finally, as $E_\lambda(T) = \ker(T - \lambda)$ for any $\lambda \in \operatorname{Sp} T$, using Theorem 2.7 we get:

$$\mathcal{H} = \overline{\dot{+} \ker(T - \lambda)}. \tag{2.21}$$

Moreover, Theorem 2.7 also tells us that $\Pi_\lambda(T)\Pi_\mu(T) = 0$ when $\lambda \neq \mu$, that is $\operatorname{im} \Pi_\mu(T) = \ker(T - \mu)$ is contained in $\ker \Pi_\lambda(T) = \ker(T - \lambda)^\perp$. Therefore, the decomposition (2.21) is that of a Hilbertian direct sum. $\square$

3. Cayley-Hamilton decomposition of a closed operator with compact resolvent

Since in the previous section we got a Cayley-Hamilton decomposition for compact operators we can similarly get a Cayley-Hamilton decomposition for any closed operator T on $\mathcal{H}$ with compact resolvent, i.e. there exists $\lambda_0 \in \mathbb{C} \setminus \operatorname{Sp} T$ such that $(T - \lambda_0)^{-1}$ is compact. Recall that we then have a bijection between the spectrum of T and $[\operatorname{Sp}(T - \lambda_0)^{-1}] \setminus 0$. Indeed, as for any $\lambda \neq \lambda_0$ we have

$$T - \lambda = -(T - \lambda_0)[(T - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1}](\lambda - \lambda_0), \tag{3.1}$$

we see that $T - \lambda$ is invertible if, and only if, $(T - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1}$ is invertible. Thus, λ is in $\operatorname{Sp} T$ if, and only if, $(\lambda - \lambda_0)^{-1}$ is in $[\operatorname{Sp}(T - \lambda_0)^{-1}] \setminus 0$.

In fact, from (3.1) we also deduce that

$$\ker(T - \lambda) = \ker[(T - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1}] \quad \text{for any } \lambda \in \operatorname{Sp} T, \tag{3.2}$$

whereas for any $\lambda \in \mathbb{C} \setminus \operatorname{Sp} T$ we have

$$(T - \lambda)^{-1} = -(T - \lambda_0)^{-1}[(T - \lambda_0)^{-1} - (\lambda - \lambda_0)^{-1}]^{-1}(\lambda - \lambda_0)^{-1}. \tag{3.3}$$

Therefore, the spectrum of T is a (possibly empty) discrete subset of isolated eigenvalues with finite multiplicities and, for any $\lambda \in \mathbb{C} \setminus \operatorname{Sp} T$, the resolvent $(T-\lambda)^{-1}$ is compact. In particular, to any eigenvalue $\lambda \in \operatorname{Sp} T$ we can associate the characteristic subspace $E_\lambda(T)$ and the characteristic projector $\Pi_\lambda(T)$ defined as in (2.1), i.e.

$$E_\lambda(T) = \cup_{k\geq 1} \ker(T-\lambda)^k \quad \text{and} \quad \Pi_\lambda(T) = \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}} (T-\xi)^{-1} d\xi, \tag{3.4}$$

where $\Gamma_{(\lambda)}$ is a small circle about λ which isolates λ from the rest of the spectrum.

Lemma 3.1. *1) For any $\lambda \in \operatorname{Sp} T$ the subspace $E_\lambda(T)$ has finite dimension and there is an integer $N \geq 1$ such that $E_\lambda(T) = \dim \ker(T-\lambda)^N$.*

2) The family $(\Pi_\lambda)_{\lambda\in \operatorname{Sp} T}$ is an orthogonal family of projectors.

3) For any $\lambda \in \operatorname{Sp} T$ the projector $\Pi_\lambda(T)$ maps continuously to $\cap \operatorname{dom} T^k$ and coincides with the projector onto $E_\lambda(T)$ and along $E_{\bar\lambda}(T^*)^\perp$, hence has finite rank.

Proof. Without any loss of generality we may assume that 0 is not in the spectrum of T. Then the inverse T^{-1} exists and is compact and in view of (3.3) for any integer $k \geq 1$ and for any $\lambda \neq 0$ we have $(T-\lambda)^k = -\lambda^k T^k (T-\lambda^{-1})^k$, so that $\ker(T-\lambda)^k = \ker(T^{-1}-\lambda^{-1})^k$. Thus,

$$E_\lambda(T) = E_{\lambda^{-1}}(T^{-1}) \quad \text{for any } \lambda \in \operatorname{Sp} T. \tag{3.5}$$

Therefore, the first assertion of the lemma follows from Lemma 2.2.

Next, the second assertion can be proven along the same lines as that of proof in the compact case (i.e. as in the proof of the 2nd part of Lemma 2.1). In fact, for any $\lambda \in \operatorname{Sp} T$ we have $\Pi_\lambda(T) = \Pi_{\lambda^{-1}}(T^{-1})$. To see this we use (3.3) to get

$$\Pi_\lambda(T) = \frac{1}{2i\pi}\int_{\Gamma_{(\lambda)}} T^{-1}(T^{-1}-\xi^{-1})^{-1}\xi^{-1} d\xi = \frac{-1}{2i\pi}\int_{\Gamma^{-1}_{(\lambda^{-1})}} T^{-1}(T^{-1}-\xi)^{-1}\xi^{-1} d\xi. \tag{3.6}$$

Then making use of (2.2) and noticing that $\Gamma^{-1}_{(\lambda)}$ bounds a domain about λ^{-1} which isolates it from the rest of the spectrum of T^{-1}, hence from the origin, we obtain

$$\Pi_\lambda(T) = \frac{-1}{2i\pi}\int_{\Gamma^{-1}_{(\lambda^{-1})}} (T^{-1}-\xi)^{-1} d\xi + \frac{1}{2i\pi}\int_{\Gamma^{-1}_{(\lambda)}} \xi^{-1} d\xi = \Pi_{\lambda^{-1}}(T^{-1}). \tag{3.7}$$

Combining this with Lemma 2.2 we deduce that $\Pi_\lambda(T)$ is the projection onto $E_{\lambda^{-1}}(T^{-1}) = E_\lambda(T)$ and along $E_{\bar\lambda^{-1}}((T^{-1})^*)^\perp = E_{\bar\lambda^{-1}}((T^*)^{-1})^\perp = E_{\bar\lambda}(T^*)^\perp$.

Finally, for any integer $k = 1, 2, \ldots$ successive integrations by parts in the integral in (3.4) give

$$\Pi_\lambda(T) = \frac{(-1)^k}{2i\pi}\int_{\Gamma_{(\lambda)}} \xi^k (T-\xi)^{-(k+1)} d\xi. \tag{3.8}$$

Since $(T-\xi)^{-(k+1)}$ is an element of $\mathcal{L}(\mathcal{H}, \operatorname{dom} T^{k+1})$, and as such depends continuously on ξ, it follows that $\Pi_\lambda(T)$ maps continuously into $\operatorname{dom} T^k$ for any integer k, hence maps continuously to $\cap \operatorname{dom} T^k$. □

Next, as $(\Pi_\lambda)_{\lambda\in\operatorname{Sp} T}$ is an orthogonal family of projectors it follows from Lemma 2.3 that the series $\sum_{\lambda\in\operatorname{Sp} T} \Pi_\lambda(T)$ converges to a projector with range $\overline{\dotplus_{\lambda\in\operatorname{Sp} T} E_\lambda(T)}$.

Lemma 3.2. *In $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ we have*

$$\lim_{R\to\infty} \frac{1}{2i\pi}\int_{|\xi|=R} \xi^{-1} T(T-\xi)^{-1} d\xi = 1 - \sum_{\lambda\in\operatorname{Sp} T} \Pi_\lambda(T). \tag{3.9}$$

Proof. Using (2.5) and arguing as in (2.6) we get

$$\begin{aligned}\frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}T(T-\xi)^{-1}d\xi &= \frac{1}{2i\pi}\int_{|\xi|=R}(T-\xi)^{-1}d\xi + \frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}d\xi,\\ &= -\sum_{\lambda\in \operatorname{Sp} P\cap D(0,R)}\Pi_\lambda(T)+1.\end{aligned} \tag{3.10}$$

Hence $\lim_{R\to\infty}\frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}T(T-\xi)^{-1}d\xi = 1-\sum_{\lambda\in\operatorname{Sp}T}\Pi_\lambda(T)$ in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$. □

We can now set-up the following.

Definition 3.3. *The characteristic subspace $E_\infty(T)$ and the characteristic projector $\Pi_\infty(T)$ associated to the "infinity" of $\mathbb{C}$ are*

$$\Pi_\infty(T) = \lim_{R\to\infty}\frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}T(T-\xi)^{-1}d\xi \qquad \text{and} \qquad E_\infty(T)=\operatorname{im}\Pi_\infty(T). \tag{3.11}$$

Lemma 3.4. *1) We have $\Pi_\infty(T)=\Pi_0((T-\lambda_0)^{-1})$ and $E_\infty(T)=E_0((T-\lambda_0)^{-1})$ for any λ_0 outside $\operatorname{Sp}T$.*

2) The subspace $E_\infty(T)$ is globally invariant by T (i.e. T maps $\operatorname{dom}T\cap E_\infty(T)$ to $E_\infty(T)$) and on there T induces a closed operator T_∞ with compact resolvent and empty spectrum.

Proof. First, notice that the value of the sum $\sum_{\lambda\in\operatorname{Sp}T}\Pi_\lambda(T)$ is unaffected by a translation change $T\to T-\lambda_0$. Since by Lemma 3.2 we have $\Pi_\infty(T)=1-\sum_{\lambda\in\operatorname{Sp}T}\Pi_\lambda(T)$ we then see that $\Pi_\infty(T)=\Pi_\infty(T-\lambda_0)$ for any $\lambda_0\in\mathbb{C}$.

Now, let $\lambda_0\in\mathbb{C}\setminus\operatorname{Sp}T$ and let us show that $E_\infty(T)=E_0((T-\lambda_0)^{-1})$. Thanks to the remark above, possibly by replacing T by $T+\lambda_0$, we may assume $\lambda_0=0$. Then by (3.7) we have $\Pi_\lambda(T)=\Pi_{\lambda^{-1}}(T^{-1})$ for any $\lambda\in\operatorname{Sp}T$. Thus $\Pi_\infty(T)=1-\sum_{\lambda\in\operatorname{Sp}T}P_{\lambda^{-1}}(T^{-1})=\Pi_0(T^{-1})$. Incidentally, $E_\infty(T)=\operatorname{im}\Pi_\infty(T)=\operatorname{im}\Pi_0(T^{-1})=E_0(T^{-1})$. The first assertion is thus proved.

Next, for proving the second statement we may assume that 0 is not the spectrum of T. Then, we have $E_0(T^{-1})=E_\infty(T)$, so that $E_\infty(T)$ is globally invariant by T^{-1} by Lemma 3.2. Since T^{-1} maps to $\operatorname{dom}T$ we then see that it maps $E_\infty(T)$ to $E_\infty(T)\cap\operatorname{dom}T$. Thus T maps $E_\infty(T)\cap\operatorname{dom}T$ to $E_\infty(T)$, i.e. $E_\infty(T)$ is globally invariant by T.

Finally, let T_∞ be the operator with domain $\operatorname{dom}T\cap E_\infty(T)$ induced by T on $E_\infty(T)$. Recall that by Lemma 2.6 the operator T_0^{-1} induced by T^{-1} on $E_0(T^{-1})=E_\infty(T)$ is compact and is such that $\operatorname{Sp}T_0^{-1}=\{0\}$. This operator inverts T_∞, so the latter has a compact resolvent and its spectrum is in bijection with $(\operatorname{Sp}T_0^{-1})\setminus 0=\emptyset$, hence is empty. □

Now, as $\Pi_\infty(T)=1-\sum_{\lambda\in\operatorname{Sp}T}\Pi_\lambda(T)$ applying the second part of Lemma 2.3 yields the Cayley-Hamilton decomposition below.

Theorem 3.5. *The family $(\Pi_\lambda(T))_{\lambda\in\operatorname{Sp}T\cup\{\infty\}}$ is an orthogonal family of projectors and we have*

$$\mathcal{H}=\overline{\dot{+}_{\lambda\in\operatorname{Sp}T\cup\{\infty\}}E_\lambda(T)} \quad \text{and} \quad \sum_{\lambda\in\operatorname{Sp}T\cup\{\infty\}}\Pi_\lambda(T)=1, \tag{3.12}$$

where the subspace $E_\infty(T)$ may be empty and the series converges in $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$.

In the same way as in the compact setting when T is normal, i.e. $T^*T=TT^*$ with equality of domains, we get:

Proposition 3.6. *Assume that T is normal. Then:*

1) We have

$$\mathcal{H}=\bigoplus_{\lambda\in\operatorname{Sp}T}\ker(T-\lambda), \tag{3.13}$$

where ⊕ denotes here the Hilbertian direct sum.

2) For any $\lambda \in \operatorname{Sp} T$ *the projector* $\Pi_\lambda(T)$ *is the orthogonal projector onto* $\ker(T-\lambda)$*, whereas* $\Pi_\infty(T)=0$ *and* $E_\infty(T)$ *is trivial.*

4. Cayley-Hamilton decomposition and spectral asymmetry

An important application of the Cayley-Hamilton decomposition provided by Theorem 3.5 is related to the results of Wodzicki ([Wo1]–[Wo4]; see also [Po1]) on the spectral asymmetry of elliptic ΨDO's. Recall that the study of the spectral asymmetry of elliptic ΨDO's was initiated (in the selfadjoint case) by Atiyah-Patodi-Singer ([APS1], [APS2]) in connection with their index theorem on manifolds with boundary. In [Wo1]–[Wo4] looked at the problem in terms of the difference of the zeta functions coming from different cuts exhibited a ΨDO projector which encodes this difference (*cf.* the formulas (1.6)–(1.8) of the introduction).

In this section we shall give a spectral interpretation of this projector. In fact, the definition of the projector of Wodzicki makes sense in a purely Hilbertian setting as follows.

Let T be a closed operator on $\mathcal{H}$ such that T has compact resolvent. Recall that a ray $L_\theta = \{\arg\lambda=\theta\}$ is said to be of minimal growth for T if it does not contain any eigenvalue of T and if as $\lambda\in L_\theta$ becomes large we have

$$\|(T-\lambda)^{-1}\| = \mathrm{O}(|\lambda|^{-1}). \tag{4.1}$$

In fact, the minimal growth condition is an open condition thanks to the lemma below.

Lemma 4.1. *If* L_θ *is a ray of minimal growth then there exists an open angular sector* Λ *containing* L_θ *such that* Λ *does not contain any eigenvalue of* T *and for some constant* $C_\Lambda>0$ *we have*

$$\|(T-\lambda)^{-1}\| \leq C_\Lambda|\lambda|^{-1}, \qquad \lambda\in\Lambda\setminus D(0,1). \tag{4.2}$$

In particular, any ray contained in Λ *is a ray of minimal growth for* T.

Proof. First, since L_θ is a ray of minimal growth there exists $C_\theta>1/4$ such that

$$\|(T-\lambda)^{-1}\| \leq C_\theta|\lambda|^{-1}, \qquad \lambda\in L_\theta\setminus D(0,1). \tag{4.3}$$

Second, let $\mu\in\mathbb{C}\setminus D(0,1)$ and write
$mu=\rho e^{i\theta'}$. Then set $\lambda=\rho e^{i\theta}\in L_\theta$. Notice that

$$T-\mu=(T-\lambda)[1-(\lambda-\mu)(T-\lambda)^{-1}]. \tag{4.4}$$

Let $R=(\lambda-\mu)(T-\lambda)^{-1}$ and observe that, as λ and μ have same modulus, we have

$$|\lambda-\mu|=|\lambda||e^{i\theta}-e^{i\theta'}|=|\lambda|\,|e^{i\frac{\theta-\theta'}{2}}-e^{-i\frac{\theta-\theta'}{2}}|=2|\lambda|\,|\sin(\theta-\theta')|. \tag{4.5}$$

As λ is in L_θ we can make use of (4.2) to get

$$\|R\|\leq|\lambda-\mu|\,\|(T-\lambda)^{-1}\|\leq 2C_\theta|\sin(\theta-\theta')|. \tag{4.6}$$

Set $\delta=\sin^{-1}(1/4C)$ and consider the angular sector $\Lambda=\{|\arg\mu-\theta|<\delta\}$. Then thanks to (4.6) for any $\mu\in\Lambda$ such that $|\mu|\geq 1$ we have $\|R\|\leq 1/2$, so that $1-R$ is invertible and we have $\|(1-R)^{-1}\|\leq\sum_{j\geq 0}2^{-j}=2$. Combining this with (4.4) we see that $T-\mu$ is invertible and has inverse $(1-R)^{-1}(T-\mu)^{-1}$, in such way that

$$\|(T-\mu)^{-1}\|\leq\|(1-R)^{-1}\|\|(T-\lambda)^{-1}\|\leq 2C_\theta|\lambda|^{-1}\leq 2C_\theta|\lambda|^{-1}. \tag{4.7}$$

In particular, there is no eigenvalue in Λ with modulus ≥ 1. As there are at most finitely many eigenvalues of modulus <1 it follows that, possibly by reducing the aperture of Λ, we get an open angular sector about L_θ containing no eigenvalue such that the resolvent $(P-\lambda)^{-1}$ satisfies the estimate (4.3) on $\Lambda\setminus D(0,1)$. □

Now, let us assume that $L_\theta = \{\arg \lambda = \theta\}$ and $L_{\theta'} = \{\arg \lambda = \theta'\}$ are rays of minimal growth for T such that $\theta < \theta' < \theta + 2\pi$. Then following Wodzicki ([Wo3], [Wo4]) we define

$$\Pi_{\theta,\theta'}(T) = \frac{1}{2i\pi} \int_{\Gamma_{\theta,\theta',r}} \xi^{-1} T(T-\xi)^{-1} d\xi, \tag{4.8}$$

$$\Gamma_{\theta,\theta,r} = \{\rho e^{i\theta}; \infty < \rho \leq r\} \cup \{re^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \leq \rho \leq \infty\},$$

where r is small enough so that no non-zero eigenvalue of T lies in the disc $|\lambda| \leq r$.

Lemma 4.2 ([Wo3], [Wo4]). *The operator $\Pi_{\theta,\theta'}(T)$ is a projector.*

Proof. By Lemma 4.1 there exist open angular sectors Λ_θ and $\Lambda_{\theta'}$ about L_θ and $L_{\theta'}$ and such that there is no eigenvalue in $\Lambda_\theta \cup \Lambda_{\theta'}$ and the resolvent satisfies the estimate (4.3) on $\Lambda_\theta \setminus D(0,1)$ and $\Lambda_{\theta'} \setminus D(0,1)$. Therefore, in the formula (4.8) we can replace $\Gamma_{\theta_1,\theta_1',r_1}$ with $\theta < \theta_1 < \theta_1' < \theta'$ and $r_1 > r$ without changing the value of the integral as soon as θ_1, θ_1' and r_1 are close enough to θ, θ' and r. Then we have

$$\Pi_{\theta,\theta'}(T)\Pi_{\theta,\theta'}(T) = \frac{-1}{4\pi} \int_{\Gamma_{\theta,\theta',r}} \int_{\Gamma_{\theta_1,\theta_1',r_1}} T(T-\xi)^{-1} T(T-\eta)^{-1} d\xi d\eta. \tag{4.9}$$

Thus, using (2.2) and arguing as in (2.4) we see that $\Pi_{\theta,\theta'}(T)^2$ is equal to

$$\frac{-1}{4\pi} \int_{\Gamma_{\theta,\theta',r}} T(T-\xi)^{-1} \int_{\Gamma_{\theta_1,\theta_1',r_1}} \frac{\eta^{-1} d\eta}{\xi - \eta} d\xi + \frac{1}{4\pi} \int_{\Gamma_{\theta_1,\theta_1',r_1}} \eta^{-1} T(T-\eta)^{-1} \int_{\Gamma_{\theta,\theta',r}} \frac{d\xi}{\xi - \eta} d\eta = \Pi_{\theta,\theta'}(T),$$

that is $\Pi_{\theta,\theta'}(T)$ is a projector. ☐

We shall now understand $\Pi_{\theta,\theta'}(T)$ in terms of the Cayley-Hamilton decomposition of T. To this end we let $\Lambda_{\theta,\theta'}$ denote the open angular sector $\{\theta < \arg \lambda < \theta'\}$.

Lemma 4.3. *In $\mathcal{L}(\mathcal{H})_{\mathrm{s}}$ we have*

$$\lim_{R \to \infty} \int_{\Gamma_{\theta,\theta',R}} \xi^{-1} T(T-\xi)^{-1} d\xi = 1 - \sum_{\lambda \in \operatorname{Sp} T \cap \Lambda_{\theta,\theta'}} \Pi_\lambda(T). \tag{4.10}$$

Proof. For $r < \operatorname{dist}(0, \operatorname{Sp} T \setminus 0) < R$ we have

$$\Pi_{\theta,\theta'}(T) = \frac{1}{2i\pi} \int_{\Gamma_{\theta,\theta',R}} \xi^{-1} T(T-\xi)^{-1} d\xi + \frac{1}{2i\pi} \int_{\Gamma_{\theta,\theta',r,R}} \xi^{-1} T(T-\xi)^{-1} d\xi, \tag{4.11}$$

where $\Gamma_{\theta,\theta',r,R} = \{\rho e^{i\theta}; R \geq \rho \geq r\} \cup \{re^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \leq \rho \leq R\} \cup \{re^{it}; \theta' \geq t \geq \theta\}$. As in (2.6) and in (3.10) we have

$$\frac{1}{2i\pi} \int_\Gamma \xi^{-1} T(T-\xi)^{-1} d\xi = -\sum_{\lambda \in \operatorname{Sp} T \cap \Lambda_{\theta,\theta'} \cap D(0,R)} \Pi_\lambda(T). \tag{4.12}$$

Thus $\lim_{R\to\infty} \int_{\Gamma_{\theta,\theta',R}} \xi^{-1} T(T-\xi)^{-1} d\xi = \Pi_{\theta,\theta'}(T) - \sum_{\lambda \in \operatorname{Sp} T \cap \Lambda_{\theta,\theta'}} \Pi_\lambda(T)$. ☐

Definition 4.4. *The characteristic space $E_{\theta,\theta',\infty}(T)$ and the characteristic projector $\Pi_{\theta,\theta',\infty}(T)$ associated to the "infinity" of $\Lambda_{\theta,\theta'}$ are*

$$\Pi_{\theta,\theta',\infty}(T) = \lim_{R \to \infty} \frac{1}{2i\pi} \int_{\Gamma_{\theta,\theta',R}} \xi^{-1} T(T-\xi)^{-1} d\xi \quad \text{and} \quad E_{\theta,\theta',\infty}(T) = \operatorname{im} \Pi_{\theta,\theta',\infty}(T). \tag{4.13}$$

Lemma 4.5. *The projector $\Pi_{\theta,\theta',\infty}(T)$ is orthogonal to $\Pi_{\theta',\theta+2\pi,\infty}(T)$ in such way that*

$$\Pi_\infty(T) = \Pi_{\theta,\theta',\infty}(T) + \Pi_{\theta',\theta+2\pi,\infty}(T) \quad \text{and} \quad E_\infty(T) = E_{\theta,\theta',\infty}(T) \dotplus E_{\theta',\theta+2\pi,\infty}(T). \tag{4.14}$$

Proof. First, the operator $\Pi_{\theta,\theta',\infty}(T)+\Pi_{\theta',\theta+2\pi,\infty}(T)$ is equal to

$$\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_{\theta,\theta',R}\cup\Gamma_{\theta',\theta+2\pi,R}}\xi^{-1}T(T-\xi)^{-1}d\xi=\lim_{R\to\infty}\frac{1}{2i\pi}\int_{|\xi|=R}\xi^{-1}T(T-\xi)^{-1}d\xi=\Pi_{\infty}(T). \tag{4.15}$$

On the other hand, $\Pi_{\theta,\theta',\infty}(T)\Pi_{\theta',\theta+2\pi,\infty}(T)$ and $\Pi_{\theta',\theta+2\pi,\infty}(T)\Pi_{\theta,\theta',\infty}(T)$ are both equal to

$$\lim_{R\to\infty}\frac{-1}{4\pi^2}\int_{\Gamma_{\theta_1,\theta_1',R}}\int_{\Gamma_{\theta',\theta+2\pi,R}}\xi^{-1}T(T-\xi)^{-1}\eta^{-1}T(T-\eta)^{-1}d\xi d\eta, \tag{4.16}$$

where θ_1 and θ_1' are close enough to θ and θ' and such that $\theta<\theta_1<\theta_1'<\theta'$. However, using (2.2) we see that on the domain of T we have

$$\begin{aligned}\int_{\Gamma_{\theta_1,\theta_1',R}}\int_{\Gamma_{\theta',\theta+2\pi,R}}&\xi^{-1}T(T-\xi)^{-1}\eta^{-1}T(T-\eta)^{-1}d\xi d\eta,\\ &=(\int_{\Gamma_{\theta_1,\theta_1',R}}\xi^{-1}T(T-\xi)^{-1}(\int_{\Gamma_{\theta',\theta+2\pi,R}}\eta^{-1}(\xi-\eta)^{-1}d\eta)d\xi)T\\ &\quad+(\int_{\Gamma_{\theta',\theta+2\pi,R}}\eta^{-1}T(T-\eta)^{-1}(\int_{\Gamma_{\theta_1,\theta_1',R}}\xi^{-1}(\eta-\xi)^{-1}d\xi)d\eta)T.\end{aligned} \tag{4.17}$$

Since as $R\to\infty$ the integrals of the r.h.s. have limits 0 we then see that $\Pi_{\theta,\theta',\infty}(T)\Pi_{\theta',\theta+2\pi,\infty}(T)$ and $\Pi_{\theta',\theta+2\pi,\infty}(T)\Pi_{\theta,\theta',\infty}(T)$ are both zero on $\operatorname{dom} T$, so are zero on $\mathcal{H}$ since $\operatorname{dom} T$ is dense. As $\Pi_\infty(T)=\Pi_{\theta,\theta',\infty}(T)+\Pi_{\theta',\theta+2\pi,\infty}(T)$ it follows that $E_\infty(T)=E_{\theta,\theta',\infty}(T)\dotplus E_{\theta',\theta+2\pi,\infty}(T)$. □

We are now ready to prove the main result of this section.

Theorem 4.6. *1) The family $\{\Pi_{\theta,\theta',\infty}(T)\}\cup\{\Pi_\lambda(T)\}_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}$ is an orthogonal family of projectors and we have*

$$\Pi_{\theta,\theta'}(T)=\Pi_{\theta,\theta',\infty}(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(T). \tag{4.18}$$

2) The projector $\Pi_{\theta,\theta'}(T)$ projects onto $E_{\theta,\theta',\infty}(T)\dotplus(\overline{\dotplus_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}E_\lambda(T)})$ and along the subspace $E_0(T)\dotplus E_{\theta',\theta+2\pi,\infty}(T)\dotplus(\overline{\dotplus_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta',\theta+2\pi}}E_\lambda(T)})$.

Proof. First, by Lemma 4.3 we have $\Pi_{\theta,\theta'}(T)=\Pi_{\theta,\theta',\infty}(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(T)$. Second, recall that by Theorem 3.5 the family $(\Pi_\lambda(T))_{\lambda\in\operatorname{Sp}T\cup\{\infty\}}$ is an orthogonal family of projectors. Also, it follows from Lemma 4.5 that $\Pi_{\theta,\theta',\infty}(T)=\Pi_{\theta,\theta',\infty}(T)\Pi_\infty(T)=\Pi_\infty(T)\Pi_{\theta,\theta',\infty}(T)$. Thus, for any $\lambda\in\operatorname{Sp}T$ we have $\Pi_{\theta,\theta',\infty}(T)\Pi_\lambda(T)=\Pi_{\theta,\theta',\infty}(T)\Pi_\infty(T)\Pi_\lambda(T)=0$ and, similarly, $\Pi_\lambda(T)\Pi_{\theta,\theta',\infty}(T)=0$. Therefore, $\{\Pi_{\theta,\theta',\infty}(T)\}\cup\{\Pi_\lambda(T)\}_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}$ is an orthogonal family of projectors. As $\Pi_{\theta,\theta'}(T)=\Pi_{\theta,\theta',\infty}(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(T)$ it then follows from Lemma 2.3 that $\Pi_{\theta,\theta'}(T)$ has range $E_{\theta,\theta',\infty}(T)\dotplus(\overline{\dotplus_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}E_\lambda(T)})$.

It remains to determine the kernel of $\Pi_{\theta,\theta'}(T)$. In the same way as above the projector $\Pi_{\theta',\theta+2\pi}(T)$ is equal to $\Pi_{\theta',\theta+2\pi,\infty}(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta',\theta+2\pi}}\Pi_\lambda(T)$, has range $E_{\theta',\theta+2\pi,\infty}(T)\dotplus(\overline{\dotplus_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta',\theta+2\pi}}E_\lambda(T)})$ and we have $\Pi_{\theta',\theta+2\pi,\infty}(T)\Pi_\lambda(T)=\Pi_\lambda(T)\Pi_{\theta',\theta+2\pi,\infty}(T)=0$ for any $\lambda\in\operatorname{Sp}T$. Combining this with Theorem 3.5 and Lemma 4.5 we obtain:

$$(4.19)\quad \Pi_{\theta,\theta'}(T)+\Pi_{\theta',\theta+2\pi}(T)+\Pi_0(T),$$
$$=\Pi_{\theta,\theta',\infty}(T)+\Pi_{\theta',\theta+2\pi,\infty}(T)+\Pi_0(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta',\theta+2\pi}}\Pi_\lambda(T),$$
$$=\Pi_\infty(T)+\sum_{\lambda\in\operatorname{Sp}T}\Pi_\lambda(T)=1.$$

Thus $1-\Pi_{\theta,\theta'}(T)=\Pi_0(T)+\Pi_{\theta',\theta+2\pi}(T)$, so that the kernel of $\Pi_{\theta,\theta'}(T)$ coincides with the range of $\Pi_0(T)+\Pi_{\theta',\theta+2\pi}(T)$. Moreover, we have

$$(4.20)\qquad \Pi_0(T)\Pi_{\theta',\theta+2\pi}(T)=\Pi_0(T)\Pi_{\theta',\theta+2\pi,\infty}(T)+\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta',\theta+2\pi}}\Pi_0(T)\Pi_\lambda(T)=0,$$

and, similarly, $\Pi_{\theta',\theta+2\pi}(T)\Pi_0(T)=0$. Therefore, $\ker\Pi_{\theta,\theta'}(T)=\operatorname{im}\Pi_0(T)\dotplus\operatorname{im}\Pi_{\theta',\theta+2\pi}(T)$, which shows that $\Pi_{\theta,\theta'}(T)$ projects along $E_0(T)\dotplus E_{\theta',\theta+2\pi,\infty}(T)\dotplus(\overline{\dotplus_{\theta'<\arg\lambda<\theta+2\pi}E_\lambda(T)})$. □

Finally, we have:

Proposition 4.7. *The following are equivalent.*

(i) The rank of $\Pi_{\theta,\theta'}(T)$ *is finite.*

(ii) $\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}$ *is at most finite and we have* $\Pi_{\theta,\theta',\infty}(T)=0$.

Proof. First, suppose that $\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}$ is finite and that $\Pi_{\theta,\theta',\infty}(T)=0$. Then Theorem 4.6 tells us that $\Pi_{\theta,\theta'}(T)=\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(T)$. Since each projector $\Pi_\lambda(T)$, $\lambda\in\operatorname{Sp}T$, has finite rank by Lemma 3.1, it follows that $\Pi_{\theta,\theta'}(T)$ has finite rank too.

Conversely, assume that $\Pi_{\theta,\theta'}(T)$ has finite rank. By Theorem 4.6 the range of $\Pi_{\theta,\theta'}(T)$ contains $\dotplus_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}E_\lambda(T)$. However, each characteristic space $E_\lambda(T)$ is non-trivial, so if $\Pi_{\theta,\theta'}(T)$ has finite rank then $\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}$ must be at most finite.

Let us now show that $\Pi_{\theta,\theta',\infty}(T)=0$. Observe that it follows from its definition that $\Pi_{\theta,\theta',\infty}(T)$ commutes with T on its domain, so its range $E_{\theta,\theta',\infty}(T)$ is globally invariant by T. On the other hand, by Lemma 4.5 the subspace $E_{\theta,\theta',\infty}(T)$ is contained in $E_\infty(T)$, on which T induces an operator with empty spectrum. Thus T also induces an operator on $E_{\theta,\theta',\infty}(T)$ with empty spectrum. This readily implies that $E_{\theta,\theta',\infty}(T)$ is either trivial or infinite dimensional, for on a non-trivial space of finite dimension an operator cannot have an empty spectrum. However, $E_{\theta,\theta',\infty}(T)$ is contained in $E_\infty(T)$, which has finite dimension since this the range of Π_∞. Thus $E_{\theta,\theta',\infty}(T)$ must be trivial. Hence $\Pi_{\theta,\theta',\infty}(T)=0$. □

5. Partial inverse of a closed operator with compact resolvent

In this section we let T be a closed operator with compact resolvent on $\mathcal{H}$. Another consequence of the Cayley-Hamilton decomposition of T given by Theorem 3.5 is a definition of the partial inverse of T as follows.

Definition 5.1. *The characteristic subspace* $E_{\bar{\mathbb{C}}\setminus 0}(T)$ *is*

$$(5.1)\qquad E_{\bar{\mathbb{C}}\setminus 0}(T)=\overline{\dotplus_{\lambda\in(\operatorname{Sp}T\cup\{\infty\})\setminus 0}E_\lambda(T)}.$$

Lemma 5.2. *1) The subspace* $E_{\bar{\mathbb{C}}\setminus 0}(T)$ *is the kernel of* $\Pi_0(T)$ *and is globally invariant by* T.

2) On there $E_{\bar{\mathbb{C}}\setminus 0}(T)$ *the operator* T *induces an operator* $T_{\bar{\mathbb{C}}\setminus 0}$ *such that* $\operatorname{Sp}T_{\bar{\mathbb{C}}\setminus 0}=\operatorname{Sp}T\setminus 0$. *In particular,* $T_{\bar{\mathbb{C}}\setminus 0}$ *is invertible.*

Proof. First, we already know that $1-\Pi_0(T)$ has kernel $E_0(T)$. Moreover, from Theorem 3.5 we deduce that $\mathcal{H} = E_{\bar{\mathbb{C}}\setminus 0}(T) \dotplus E_0(T)$ and $1-\Pi_0(T) = \sum_{\lambda\in(\operatorname{Sp}T\cup\{\infty\})\setminus 0}\Pi_\lambda(T)$. Thus, by Lemma 2.3 the projector $1-\Pi_0(T)$ has range $E_{\bar{\mathbb{C}}\setminus 0}(T)$, i.e. $E_{\bar{\mathbb{C}}\setminus 0}(T)$ is the kernel of $\Pi_0(T)$.

Next, using (2.1) one sees that $\Pi_0(T)$ commutes with T on $\operatorname{dom} T$. Thus $E_{\bar{\mathbb{C}}\setminus 0}(T)$ is globally invariant by T. Therefore, under the decomposition $\mathcal{H} = E_{\bar{\mathbb{C}}\setminus 0}(T) \dotplus E_0(T)$ the operator T takes the form

$$T = \begin{pmatrix} T_{\bar{\mathbb{C}}\setminus 0} & 0 \\ 0 & T_0 \end{pmatrix}, \tag{5.2}$$

where T_0 denotes the operator T_0 induced by T on the finite dimensional space $E_0(T)$. Notice that T_0 must be nilpotent since by Lemma 3.1 we have $E_0(T) = \ker T^N$ for some integer $N \geq 1$. Therefore, $\operatorname{Sp} T = \operatorname{Sp} T_{\bar{\mathbb{C}}\setminus 0} \cup \{0\}$. However, 0 cannot be an eigenvalue of $T_{\bar{\mathbb{C}}\setminus 0}$, because $\ker T_{\bar{\mathbb{C}}\setminus 0} = \ker T \cap E_{\bar{\mathbb{C}}\setminus 0}(T) \subset E_0(T) \cap E_{\bar{\mathbb{C}}\setminus 0}(T) = \{0\}$. Thus $\operatorname{Sp} T_{\bar{\mathbb{C}}\setminus 0} = \operatorname{Sp} T \setminus 0$. □

Definition 5.3. *The partial inverse of T is the bounded operator on $\mathcal{H}$ denoted T^{-1} that is zero on $E_0(T)$ and inverts T on $E_{\bar{\mathbb{C}}\setminus 0}(T)$.*

The partial inverse satisfies the properties below.

Proposition 5.4. *1) We have:*

$$TT^{-1} = 1-\Pi_0(T) \qquad \text{and} \qquad T^{-1}T = 1-\Pi_0(T), \tag{5.3}$$

where the first equality holds on $\mathcal{H}$ and the second one on $\operatorname{dom} T$.

2) The spectrum of T^{-1} is equal to $(\operatorname{Sp} T\setminus 0)^{-1}\cup\{0\}$ *in such way that:*

$$\Pi_\lambda(T^{-1}) = \Pi_{\lambda^{-1}}(T) \quad \textit{and} \quad E_\lambda(T^{-1}) = E_{\lambda^{-1}}(T) \quad \textit{for any } \lambda\in \operatorname{Sp} T\setminus 0, \tag{5.4}$$

$$\Pi_0(T^{-1}) = \Pi_0(T)+\Pi_\infty(T) \quad \textit{and} \quad E_0(T^{-1}) = E_0(T)\dotplus E_\infty(T). \tag{5.5}$$

Proof. 1) First, as TT^{-1} is the identity on $E_{\bar{\mathbb{C}}\setminus 0}(T)$ and is zero on $E_0(T)$ it coincides with the projection onto the former and along the latter, i.e. $TT^{-1} = 1-\Pi_0(T)$. Similarly, the operator $T^{-1}T$ is equal to $1-\Pi_0(T)$ on $\operatorname{dom} T$.

2) Let us first assume that T is invertible. Then the partial inverse is its actual inverse, so that we have $\operatorname{Sp} T^{-1} = (\operatorname{Sp} T)^{-1}\cup\{0\}$. Moreover, by (3.5) and (3.7) we have $\Pi_{\lambda^{-1}}(T^{-1}) = \Pi_\lambda(T)$ and $E_{\lambda^{-1}}(T^{-1}) = E_\lambda(T)$ for any $\lambda \in \operatorname{Sp} T^{-1}\setminus 0$. Also, by Lemma 3.4 we have $\Pi_0(T^{-1}) = \Pi_\infty(T)$ and $E_0(T) = E_\infty(T)$. Thus the second part of the proposition holds when T is invertible.

Let us now assume that 0 is an eigenvalue. Under the decomposition $\mathcal{H} = E_{\bar{\mathbb{C}}\setminus 0}(T)\dotplus E_0(T)$ the operators T and T^{-1} take the forms

$$T = \begin{pmatrix} T_{\bar{\mathbb{C}}\setminus 0} & 0 \\ 0 & T_0 \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} T^{-1}_{\bar{\mathbb{C}}\setminus 0} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.6}$$

where T_0 denotes the nilpotent operator induced by T on $E_0(T)$ (*cf.* proof of Lemma 5.2). Thus $\operatorname{Sp} T^{-1} = \operatorname{Sp} T_{\bar{\mathbb{C}}\setminus 0}$. Notice that from Lemma 5.2 we also have $\operatorname{Sp} T_{\bar{\mathbb{C}}\setminus 0} = \operatorname{Sp} T\setminus 0$, so that $T_{\bar{\mathbb{C}}\setminus 0}$ is invertible. In particular, the second part of the proposition is true for $T_{\bar{\mathbb{C}}\setminus 0}$. Hence $\operatorname{Sp} T^{-1} = (\operatorname{Sp} T_{\bar{\mathbb{C}}\setminus 0})^{-1}\cup\{0\} = (\operatorname{Sp} T\setminus 0)^{-1}\cup\{0\}$.

Furthermore, from (5.6) we deduce that for any $\lambda\in \operatorname{Sp} T^{-1}\setminus 0$ we have

$$\Pi_\lambda(T^{-1}) = \begin{pmatrix} \Pi_\lambda(T^{-1}_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi_{\lambda^{-1}}(T) = \begin{pmatrix} \Pi_{\lambda^{-1}}(T_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & \Pi_{\lambda^{-1}}(T_0) \end{pmatrix}. \tag{5.7}$$

Since, on the one hand, $\Pi_\lambda(T^{-1}_{\bar{\mathbb{C}}\setminus 0}) = \Pi_{\lambda^{-1}}(T_{\bar{\mathbb{C}}\setminus 0})$ because $T_{\bar{\mathbb{C}}\setminus 0}$ is invertible and, on the other hand, $\Pi_{\lambda^{-1}}(T_0) = 0$ because T_0 is nilpotent, it follows that $\Pi_\lambda(T^{-1}) = \Pi_{\lambda^{-1}}(T)$. Hence $E_\lambda(T^{-1}) = E_{\lambda^{-1}}(T)$.

On the other hand, using Definition 2.5 and Definition 3.3 we obtain

$$\Pi_0(T^{-1}) = \begin{pmatrix} \Pi_0(T^{-1}_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & \Pi_0(0) \end{pmatrix} = \begin{pmatrix} \Pi_0(T^{-1}_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.8}$$

$$\Pi_\infty(T) = \begin{pmatrix} \Pi_\infty(T_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & \Pi_\infty(T_0) \end{pmatrix} = \begin{pmatrix} \Pi_\infty(T_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & \Pi_\infty(T_0) \end{pmatrix}. \tag{5.9}$$

As $T_{\bar{\mathbb{C}}\setminus 0}$ is invertible we have $\Pi_0(T^{-1}_{\bar{\mathbb{C}}\setminus 0}) = \Pi_\infty(T_{\bar{\mathbb{C}}\setminus 0})$. Moreover, since $\Pi_0(T_0) = 1$ on $E_0(T_0)$, because $\operatorname{Sp} T_0 = \{0\}$ and $E_0(T)$ has finite dimension, we have $\Pi_\infty(T_0) = 0$. Therefore, we get

$$\Pi_0(T^{-1}) = \begin{pmatrix} \Pi_\infty(T_{\bar{\mathbb{C}}\setminus 0}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \Pi_\infty(T) + \Pi_0(T). \tag{5.10}$$

Then, as by Theorem 3.5 we have $\Pi_\infty(T)\Pi_0(T) = \Pi_0(T)\Pi_\infty(T) = 0$, it follows that $E_0(T^{-1})$ is equal to $E_0(T) \dotplus E_\infty(T)$. The proof is thus complete. □

Remark 5.5. In fact, if $S \in \mathcal{L}(\mathcal{H})$ vanishes on $E_0(T)$ and $ST = 1 - E_0(T)$ on $\operatorname{dom} T$ then S is the partial inverse of T. Indeed, we get $S(1 - \Pi_0(T)) = STT^{-1} = (1 - \Pi_0(T))T^{-1} = T^{-1}$, so that $S = T^{-1}$ on $E_{\bar{\mathbb{C}}\setminus 0}(T)$. As $S = 0 = T^{-1}$ on $E_0(T)$ it follows that $S = T^{-1}$ on $\mathcal{H}$.

Proposition 5.6. *We have*

$$(T^*)^{-1} = (T^{-1})^* \quad \text{and} \quad (T^k)^{-1} = (T^{-1})^k \quad \text{for any integer } k \geq 1. \tag{5.11}$$

Proof. First, it follows from the definition of T^{-1} that its range is contained in $E_{\bar{\mathbb{C}}\setminus 0}(T)$. Since by Lemma 5.2 the latter is the kernel of $\Pi_0(T)$ we deduce that $\Pi_0(T)T^{-1} = 0$. Taking adjoints we get $\Pi_0(T)^*(T^{-1})^* = 0$. Since by (2.2) we have $\Pi_0(T)^* = \Pi_0(T^*)$ we see that $\Pi_0(T^*)(T^{-1})^* = 0$. Hence $(T^{-1})^*$ vanishes on $E_0(T^*)$.

Moreover, since $\Pi_0(T)^* = \Pi_0(T^*)$ from the equality $TT^{-1} = 1 - \Pi_0(T)$ we see that, for any $x \in \operatorname{dom} T^*$ and any $y \in \mathcal{H}$, we have

$$\langle (T^{-1})^*T^*x, y\rangle = \langle x, TT^{-1}y\rangle = \langle x, (1 - \Pi_0(T))y\rangle = \langle (1 - \Pi_0(T^*))x, y\rangle. \tag{5.12}$$

Thus $(T^{-1})^*T^* = 1 - \Pi_0(T^*)$ on $\operatorname{dom} T^*$. Since T^* vanishes on $E_0(T^*)$ it then follows from Remark 5.5 that $(T^{-1})^* = (T^*)^{-1}$.

Now, let k be an integer ≥ 1 and let us show that $(T^k)^{-1} = (T^{-1})^k$. First, since $E_0(T^k) = E_0(T)$ we see that $(T^{-1})^k$ vanishes on $E_0(T^k)$.

Next, by Lemma 5.2 and Lemma 3.1 the subspaces $E_{\bar{\mathbb{C}}\setminus 0}(T)$ and $E_0(T^*)^\perp$ are both the kernel of $\Pi_0(T)$, so are equal. Similarly, we have $E_{\bar{\mathbb{C}}\setminus 0}(T^k) = E_0((T^k)^*)^\perp$. Thus $E_{\bar{\mathbb{C}}\setminus 0}(T^k) = E_0((T^*)^k)^\perp = E_0(T^*)^\perp = E_{\bar{\mathbb{C}}\setminus 0}(T)$. Therefore, $(T^{-1})^k$ inverts T^k on $E_{\bar{\mathbb{C}}\setminus 0}(T) = E_{\bar{\mathbb{C}}\setminus 0}(T^k)$. In view of the definition of the partial inverse it follows that $(T^k)^{-1} = (T^{-1})^k$. □

Finally, it is interesting to compare the previous definition of the partial inverse with Seeley's construction of complex powers in [Se] (see also [Sh, p. 88]). Assume that the ray $L_\theta = \{\arg\lambda = \theta\}$, $0 \leq \theta < 2\pi$, is of minimal growth for T. Then for $\Re s < 0$ Seeley defines the power T^s_θ as the bounded operator on $\mathcal{H}$ given by

$$T^s_\theta = \frac{1}{2i\pi}\int_{\Gamma_{\theta,r}} \xi^s (T - \xi)^{-1} d\xi, \tag{5.13}$$

$$\Gamma_{\theta,r} = \{\rho e^{i\theta}; \infty > \rho \geq r\} \cup \{re^{it}; \theta \leq t \leq \theta - 2\pi\} \cup \{\rho e^{i\theta}; r \leq \rho < \infty\},$$

where r is small enough so that no non-zero eigenvalue of T lies in the disc $|\lambda| \leq r$ and the power λ^s is defined by means of the continuous determination of the argument on $\mathbb{C} \setminus L_\theta$ with values in $(\theta - 2\pi, \theta)$. As it turns out we have the semi-group property,

$$T_\theta^{s_1} T_\theta^{s_2} = T_\theta^{s_1+s_2}, \quad \Re s_j < 0. \tag{5.14}$$

It is also shown by Seeley that if T is invertible then for any integer $k \geq 1$ the operator T_θ^{-k} is the inverse of T^k. It turns out that when T is not invertible this result is also true in terms of partial inverses, for we have:

Proposition 5.7. *1) For $\Re s < 0$ we have $(1 - \Pi_0(T))T_\theta^s = T_\theta^s(1 - \Pi_0(T)) = T_\theta^s$, i.e. T_θ^s vanishes on $E_0(T)$ and acts on $E_{\mathbb{C}\setminus 0}(T)$.*

2) For any integer $k \geq 1$ the power T_θ^{-k} is the partial inverse of T^k.

Proof. First, we can assume that in the definition (3.4) of $\Pi_0(T)$ the radius of the circle $\Gamma_{(0)}$ is $< r$. Then using (2.2) and arguing as in (2.3) we get:

$$\begin{aligned} T_\theta^s \Pi_0(T) = \Pi_0(T) T_\theta^s &= \frac{-1}{4\pi^2} \int_{\Gamma_{\theta,r}} \int_{\Gamma_{(0)}} \xi^s (T - \xi)^{-1} (T - \eta)^{-1} d\xi d\eta, \\ &= \frac{1}{4\pi^2} \int_{\Gamma_{(0)}} (T - \eta)^{-1} \Big(\int_{\Gamma_{\theta,r}} \frac{\xi^s d\xi}{\xi - \eta} \Big) d\eta + \frac{1}{4\pi^2} \int_{\Gamma_{(0)}} \xi^s (T - \xi)^{-1} \Big(\int_{\Gamma_{\theta,r}} \frac{d\eta}{\eta - \xi} \Big) d\xi. \end{aligned} \tag{5.15}$$

In the last line the first integral vanishes and we have $\int_{\Gamma_\theta} \frac{\xi^s d\xi}{\xi - \eta} = 0$ because the value of that integral is independent of $r > |\xi|$ and converges to 0 as $r \to \infty$. Thus $T_\theta^s \Pi_0(T) = \Pi_0(T) T_\theta^s = 0$ or, equivalently, $(1 - \Pi_0(T))T_\theta^s = T_\theta^s(1 - \Pi_0(T)) = T_\theta^s$. In particular, T_θ^s vanishes on $\operatorname{im} \Pi_0(T) = E_0(T)$ and acts on $\operatorname{im}(1 - \Pi_0(T)) = E_{\mathbb{C}\setminus 0}(T)$.

Now, let k be an integer ≥ 1 and let us show that T_θ^{-k} is the partial inverse of T^k. In fact, since from (5.11) we get $T_\theta^{-k} = (T_\theta^{-1})^k$ and by Proposition 5.6 the operator $(T^{-1})^k$ is the partial inverse of T^k, it is enough to prove the result for $k = 1$.

Observe that in the integral which defines T_θ^{-1} in (5.13) the two integrations along the ray L_θ cancel each other since -1 is an integer. Thus, on $\operatorname{dom} T$ we have

$$T_\theta^{-1} T = \frac{1}{2i\pi} \int_{|\xi| = r} \xi^{-1} (T - \xi)^{-1} d\xi. \tag{5.16}$$

Combining this with (3.10) and the fact that the origin is the only possible eigenvalue in the disc $|\lambda| \leq r$ on $\operatorname{dom} T$ we get

$$T_\theta^{-1} T = 1 - \sum_{\lambda \in \operatorname{Sp} T \cap D(0,r)} \Pi_\lambda(T) = 1 - \Pi_0(T). \tag{5.17}$$

Since we already know that T_θ^{-1} vanishes on $E_0(T)$ we then deduce from Remark 5.5 that T^{-1} is the partial inverse of T. The proof of the lemma is thus complete. □

6. Example 1: Elliptic ΨDO on a compact manifold

In this section given a compact Riemannian manifold M^n and a Hermitian vector bundle $\mathcal{E}$ over M we work out the results of the previous sections in the example of an elliptic ΨDO operator $P : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ of order $m > 0$. To this end we assume that the spectrum of P is not $\mathbb{C}$, which insures us that P has a compact resolvent on $L^2(M, \mathcal{E})$, since for any $\lambda \in \mathbb{C} \setminus \operatorname{Sp} P$ the resolvent $(P - \lambda)^{-1}$ maps continuously to the Sobolev space $L^2_m(M, \mathcal{E}) = \operatorname{dom}_{L^2} P$. In particular, the spectrum of P is a discrete subset of isolated eigenvalues, which allows us to define characteristic subspaces and characteristic projectors as in (3.1).

Lemma 6.1. *For any $\lambda \in \operatorname{Sp} P$ the characteristic projector $\Pi_\lambda(P)$ is a smoothing operator and the characteristic subspace $E_\lambda(P)$ is contained in $C^\infty(M, \mathcal{E})$.*

Proof. By Lemma 3.1 the projector $\Pi_\lambda(P)$ maps continuously $L^2(M, \mathcal{E})$ to $\cap \operatorname{dom} P^k$, that is to $\cap L^2_{mk}(M, \mathcal{E}) = C^\infty(M, \mathcal{E})$ since P is elliptic. The same is true for $\Pi_{\bar\lambda}(P^*)$, so by duality we deduce that $\Pi_\lambda(P) = \Pi_{\bar\lambda}(P^*)^*$ extends to a continuous linear mapping from $\mathcal{D}'(M, \mathcal{E})$ to $L^2(M, \mathcal{E})$. As $\Pi_\lambda(P) = \Pi_\lambda(P)\Pi_\lambda(P)$ it follows that $\Pi_\lambda(P)$ extends to a continuous linear mapping from $\mathcal{D}'(M, \mathcal{E})$ to $C^\infty(M, \mathcal{E})$. Therefore, $\Pi_\lambda(P)$ is smoothing and its range $E_\lambda(P)$ is contained in $C^\infty(M, \mathcal{E})$. □

Thanks to this lemma we can from now on look at $\Pi_\lambda(P)$, $\lambda \in \operatorname{Sp} P$, as a continuous operator from $\mathcal{D}'(M, \mathcal{E})$ to $C^\infty(M, \mathcal{E})$.

Lemma 6.2. *For any $s > 0$ (resp. for any $s < 0$) the projector $\Pi_\infty(P)$ induces (resp. extends to) a (unique) continuous endomorphism of $L^2_s(M, \mathcal{E})$. Therefore, $\Pi_\infty(P)$ induces (resp. extends to) a unique continuous endomorphism of $C^\infty(M, \mathcal{E})$ (resp. $\mathcal{D}'(M, \mathcal{E})$).*

Proof. Let $s \in \mathbb{R}$ and let $P^{(s)}$ denote the closure of P in $L^2_s(M, \mathcal{E})$ with domain $L^2_{s+m}(M, \mathcal{E})$. Recall that $\operatorname{Sp} P^{(s)} = \operatorname{Sp} P$ and that for any $\lambda \in \operatorname{Sp} P$ we have $(P^{(s)} - \lambda)^{-1} = (P - \lambda)^{-1}$ on $L^2(M, \mathcal{E}) \cap L^2_s(M, \mathcal{E})$ (e.g. [Sh]). Therefore, in view of formula (3.4) we have $\Pi_\lambda(P^{(s)}) = \Pi_\lambda(P)$ on $L^2_s(M, \mathcal{E})$. Combining this with Lemma 3.4 we also see that $\Pi_\infty(P^{(s)}) = \Pi_\infty(P)$ on $L^2_s(M, \mathcal{E}) \cap L^2(M, \mathcal{E})$. Thus $\Pi_\infty(P)$ induces (when $s > 0$) or extends to (when $s < 0$) a unique continuous endomorphism of $L^2_s(M, \mathcal{E})$. □

This result allows us to set-up the following.

Definition 6.3. *For any $s \in \mathbb{R}$ we let*

$$E^{(s)}_\infty(P) = \Pi_\infty(P)(L^2_s(M, \mathcal{E})). \tag{6.1}$$

This definition is extended to $C^\infty(M, \mathcal{E})$ and $\mathcal{D}'(M, \mathcal{E})$ by letting

$$E^{(\infty)}_\infty(P) = \Pi_\infty(P)(C^\infty(M, \mathcal{E})) \quad \text{and} \quad E^{(-\infty)}_\infty(P) = \Pi_\infty(P)(\mathcal{D}'(M, \mathcal{E})). \tag{6.2}$$

We are now ready to prove:

Proposition 6.4. *Let $s \in \mathbb{R}$. Then:*

$$L^2_s(M, \mathcal{E}) = E^{(s)}_\infty(P) \dot{+} \left(\overline{\dot{+}_{\lambda \in \operatorname{Sp} P} E_\lambda(P)}\right) \quad \textit{and} \quad \sum_{\lambda \in \operatorname{Sp} P \cup \{\infty\}} \Pi_\lambda(P) = 1, \tag{6.3}$$

where the closure is taken with respect to the topology of $L^2_s(M, \mathcal{E})$ and the series converges with respect to the strong topology of $\mathcal{L}(L^2_s(M, \mathcal{E}))$.

Proof. Let $P^{(s)}$ denote the closure of P in $L^2_s(M, \mathcal{E})$. In the course of the proof of Lemma 6.2 we have seen that for any λ in $\operatorname{Sp} P^{(s)} = \operatorname{Sp} P$ we have $\Pi_\lambda(P^{(s)}) = \Pi_\lambda(P)$ on $L^2_s(M, \mathcal{E})$. Since $E_\lambda(P) \subset C^\infty(M, \mathcal{E})$ it follows that $E_\lambda(P^{(s)}) = \Pi_\lambda(P)(L^2_s(M, \mathcal{E}) \cap E_\lambda(P)) = E_\lambda(P)$. We have also seen there that $\Pi_\infty(P)$ gives rise to a unique continuous endomorphism on $L^2_s(M, \mathcal{E})$ which coincides with $\Pi_\infty(P^{(s)})$. Thus $E_\infty(P^{(s)}) = \Pi_\infty(L^2_s(M, \mathcal{E})) = E^{(s)}_\infty(P)$. Bearing all this in mind and applying Theorem 3.5 to $P^{(s)}$ we then obtain

$$L^2_s(M, \mathcal{E}) = E^{(s)}_\infty(P) \dot{+} \left(\overline{\dot{+}_{\lambda \in \operatorname{Sp} P} E_\lambda(P)}\right) \quad \text{and} \quad \sum_{\lambda \in \operatorname{Sp} P \cup \{\infty\}} \Pi_\lambda(P) = 1, \tag{6.4}$$

where the closure is taken in $L^2_s(M, \mathcal{E})$ and the series converges in $\mathcal{L}(L^2_s(M, \mathcal{E}))_{\mathrm{s}}$. □

From this we immediately deduce:

Corollary 6.5. *The Cayley-Hamilton decomposition (6.3) holds in $C^\infty(M, \mathcal{E})$ and in $\mathcal{D}'(M, \mathcal{E})$ provided that we replace $E^{(s)}_\infty(P)$ by $E^{(\infty)}_\infty(P)$ and $E^{(-\infty)}_\infty(P)$ respectively.*

Now, a special case where the spectrum of P is not $\mathbb{C}$ occurs when the principal symbol $p_m(x,\xi)$ of P has a spectral cut, i.e. there exists a ray $L_\theta = \{\arg\lambda = \theta\}$ such that $p_m(x,\xi) - \lambda$ is invertible for any $\lambda \in L_\theta$. Indeed, since the cospshere bundle S^*M of M is compact this implies there exists an open conical neighborhood Λ of L_θ such that Λ is spectral cut for $p_m(x,\xi)$. Then P admits an asymptotic resolvent as a parametrix for $P-\lambda$ in a suitable class $\Psi^{-m}(M,\mathcal{E};\Lambda)$ of ΨDO's with parameter (e.g. [Se], [Sh], [GS], [Po2]). Using this asymptotic resolvent one can show that, on any cone $\Lambda' \subset \Lambda$ such that $\overline{\Lambda'} \setminus 0 \subset \Lambda$, there are at most finitely many eigenvalues of P. Moreover, for R large enough there exists $C_{\Lambda' R} > 0$ such that

$$\|(P-\lambda)^{-1}\|_{\mathcal{L}(L^2(M,\mathcal{E}))} \leq C_{\Lambda' R}|\lambda|^{-1} \qquad \lambda \in \Lambda' \setminus D(0,R). \tag{6.5}$$

Thus there are infinitely many rays $L_\theta = \{\arg\lambda = \theta\}$ contained in the spectral cut Λ which are not through an eigenvalue of P. In particular the spectrum of P is not $\mathbb{C}$.

Bearing this in mind let $L_\theta = \{\arg\lambda = \theta\}$ and $L_{\theta'} = \{\arg\lambda = \theta'\}$ be spectral cuts for P and its principal symbol $p_m(x,\xi)$ with $\theta < \theta' < \theta + 2\pi$. Then let $\Lambda_{\theta,\theta'}$ denote the open angular sector $\{\theta < \arg\lambda < \theta'\}$ and consider the projector $\Pi_{\theta,\theta'}(P)$ defined as in (4.8). Then we can prove the following (see also [Wo4]).

Proposition 6.6. *There are only two possibilities for the projector $\Pi_{\theta,\theta'}(P)$: either $p_m(x,\xi)$ has some eigenvalues in $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta'}(P)$ is a zero'th order ΨDO with principal symbol $\Pi_{\theta,\theta'}(p_m(x,\xi))$ or $p_m(x,\xi)$ has no eigenvalue in $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta'}(P)$ is a smoothing operator.*

Proof. Recall that $\Pi_{\theta,\theta'}(P)$ is given by the formula,

$$\Pi_{\theta,\theta'}(P) = \frac{1}{2i\pi}\int_{\Gamma_{\theta,\theta',r}} \xi^{-1}P(P-\xi)^{-1}d\xi, \tag{6.6}$$

$$\Gamma_{\theta,\theta,r} = \{\rho e^{i\theta}; \infty < \rho \leq r\} \cup \{re^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \geq \rho \leq \infty\},$$

where r is small enough so that no non-zero eigenvalue of P lie in the disc $|\lambda| \leq r$. Notice that $P(P-\lambda)^{-1}$ is a zero'th order parametric ΨDO with principal symbol $p_m(p_m-\lambda)^{-1}$. Thus, standard arguments (e.g. [Se, Thm. 3], [Sh, Thm. 11.2]) show that the integral in (6.6) defines a ΨDO of order ≤ 0 with zero'th order symbol

$$\pi_0(x,\xi) = \frac{1}{2i\pi}\int_{\delta_{(x,\xi)}\Gamma_{\theta,\theta'}} \zeta^{-1}p_m(x,\xi)(p_m(x,\xi)-\zeta)^{-1}d\zeta = \Pi_{\theta,\theta'}(p_m(x,\xi)), \tag{6.7}$$

where $\delta_{(x,\xi)}$ is small enough so that no eigenvalue of $p_m(x,\xi)$ lies in the disc $|\lambda| \leq \delta_{(x,\xi)}r$. Since a ΨDO projector either has order ≥ 0 or is smoothing we then deduce that either $\Pi_{\theta,\theta'}(p_m(x,\xi))$ is not zero and $\Pi_{\theta,\theta'}(P)$ is a zero'th order ΨDO with principal symbol $\Pi_{\theta,\theta'}(p_m(x,\xi))$ or $\Pi_{\theta,\theta'}(p_m(x,\xi))$ vanishes everywhere and $\Pi_{\theta,\theta'}(P)$ is smoothing.

Now, if $p_m(x,\xi)$ has no eigenvalue in $\Lambda_{\theta,\theta'}$ then the integral (6.7) vanishes and so $\Pi_{\theta,\theta'}(p_m(x,\xi))$ is zero. On the other hand, if $p_m(x,\xi)$ has one eigenvalue λ in $\Lambda_{\theta,\theta'}$ then the range of $\Pi_{\theta,\theta'}(p_m(x,\xi))$ at least contains the non-trivial space $E_\lambda(p_m(x,\xi))$, so that $\Pi_{\theta,\theta'}(p_m(x,\xi))$ is not zero. Therefore, we see that either $p_m(x,\xi)$ has some eigenvalues in $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta'}(P)$ is a zero'th order ΨDO with principal symbol $\Pi_{\theta,\theta'}(p_m(x,\xi))$, or $p_m(x,\xi)$ has no eigenvalue in $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta'}(P)$ is a smoothing operator. □

Combining this with Proposition 4.7 this will prove:

Proposition 6.7. *Let $\Lambda_{\theta,\theta'}$ denote the open angular sector $\{\theta < \arg\lambda < \theta'\}$. Then the following are equivalent:*

(i) The principal symbol $p_m(x,\xi)$ has no eigenvalue in $\Lambda_{\theta,\theta'}$;

(ii) The projector $\Pi_{\theta,\theta'}(P)$ is a smoothing operator;

(ii) P has at most finitely many eigenvalues in $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta',\infty}(P) = 0$.

Proof. Since the equivalence of (i) and (ii) is an immediate consequence of Proposition 6.7 we only need to prove the equivalence of (ii) and (iii).

Assume first that $\Pi_{\theta,\theta'}(P)$ is a smoothing operator. Then it defines a compact operator of $L^2(M,\mathcal{E})$ and so its range, that is the eigenspace $\ker(\Pi_{\theta,\theta'}(P)-1)$, has finite dimension. Then Proposition 4.7 implies that $\operatorname{Sp} P\cap\Lambda_{\theta,\theta'}$ is finite and $\Pi_{\theta,\theta',\infty}(P)=0$.

Conversely, suppose that $\operatorname{Sp} P\cap\Lambda_{\theta,\theta'}$ is finite and $\Pi_{\theta,\theta',\infty}(P)=0$. Then by Theorem 4.6 we have $\Pi_{\theta,\theta'}(T)=\sum_{\lambda\in\operatorname{Sp}T\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(T)$. Since each projector $\Pi_\lambda(P)$, $\lambda\in\operatorname{Sp}P$, is smoothing it follows that $\Pi_{\theta,\theta'}(P)$ is smoothing too. □

On the other hand, concerning the partial inverse we have:

Proposition 6.8. *The partial inverse of P is a ΨDO of order $-m$.*

Proof. By Proposition 5.4 we have $PP^{-1}=P^{-1}P=1-\Pi_0(P)$. Since $\Pi_0(P)$ is a smoothing operator this means that P^{-1} is a parametrix for P. Thus P^{-1} differs from any other parametrix by a smoothing operator only. However, P is elliptic and so admits a parametrix which is a ΨDO of order $-m$. Therefore, P^{-1} coincides up to a smoothing operator with a ΨDO of order $-m$, hence is itself a ΨDO of order $-m$. □

Finally, assume that the ray $L_\theta=\{\arg\lambda=\theta\}$ is a spectral cut for both P and its principal symbol. Then for $\Re s<0$ we can define the power P_θ^s as in (5.13). In fact, it can be shown that P_θ^s is a ΨDO of order $-ms$ (*cf.* [Se], [Sh]). Therefore, using (5.14) and Proposition 5.7 we see that, at the level of ΨDO's, for $k=1,2,\ldots$ we have

$$P^kP_\theta^{s-k}=P^k(P^k)^{-1}P_\theta^s=(1-\Pi_0(P))P_\theta^s=P_\theta^s. \tag{6.8}$$

Therefore, for any $s\in\mathbb{C}$ we can define P_θ^s as the ΨDO given by

$$P_\theta^s=P^kP_\theta^{s-k}, \tag{6.9}$$

where k is any integer $>\Re s$, the value of which is irrelevant. This defines a 1-parameter group of ΨDO's such that $\operatorname{ord}P_\theta^s=ms$ for any $s\in\mathbb{C}$. Moreover, for $s=0$ we get

$$P_\theta^0=PP_\theta^{-1}=PP^{-1}=1-\Pi_0(P). \tag{6.10}$$

Thus for any integer $k=1,2,\ldots$ we have

$$P_\theta^k=P^kP_\theta^0=P^k(1-\Pi_0(P))=(1-\Pi_0(P))P^k, \tag{6.11}$$

i.e. P_θ^k vanishes on $E_0(P)$ and is equal to P^k on $E_{\mathbb{C}\setminus 0}(P)$. In particular, the operators P_θ^k and P^k coincide up to a smoothing operator and if $E_0(P)=\ker P^N$ then $P_\theta^k=P^k$ for $k\geq N$ (see also [Sh, p. 88]).

7. Example 2: Elliptic ΨDO on a spectral triple

The results of the previous section for an elliptic ΨDO on a compact manifold actually hold in many other contexts of ΨDO-algebras. To illustrate this fact we shall show in this section that these results even reach analogues in the framework of Connes' noncommutative geometry [Co] when we look at elliptic ΨDO's on spectral triples.

Recall that in noncommutative geometry a (noncommutative) manifold can be represented by a spectral triple $(\mathcal{A},\mathcal{H},D)$, where:

- $\mathcal{A}$ is an involutive algebra represented in the Hilbert space $\mathcal{H}$;

- D is an unbounded selfadjoint operator on $\mathcal{H}$ which has a compact resolvent and almost commutes with $\mathcal{A}$, i.e. $[D,a]$ is bounded for any $a\in\mathcal{A}$.

In addition we assume that $\mathcal{A}$ is *smooth* in the sense that $\mathcal{A}$ is contained in $\cap_{k\geq0}\delta^k$, where δ denotes the derivation $\delta(B)=[|D|,B]$ of $\mathcal{L}(H)$. As an example we have the spectral triple

$(C^\infty(M), L^2(M,\mathcal{E}), D)$ associated to the L^2-closure $D : L^2_1(M,\mathcal{E}) \to L^2(M,\mathcal{E})$ of a first order selfadjoint elliptic ΨDO.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ Connes-Moscovici [CM, Appendix B] (see also [Hi]) introduced a class of ΨDO's defined as follows.

First, for $s \geq 0$ we define the Hilbert space $\mathcal{H}_s$ as $\operatorname{dom}|D|^s$ equipped with the Hilbertian norm,

$$\|\xi\|_{\mathcal{H}_s} = (\|\xi\|^2_{\mathcal{H}} + \||D|^s\xi\|^2_{\mathcal{H}})^{\frac{1}{2}}, \quad \xi \in \operatorname{dom}|D|^s. \tag{7.1}$$

For $s < 0$ we let $\mathcal{H}_s$ be the dual of $\mathcal{H}_{-s}$, so that we get a scale of Hilbert spaces,

$$\mathcal{H}_s \subset \mathcal{H}_{s'}, \quad s > s'. \tag{7.2}$$

Notice that the above inclusion is compact since D has a compact resolvent. We also define the Fréchet spaces,

$$\mathcal{H}_\infty = \cap_{s\in\mathbb{R}}\mathcal{H}_s \qquad \text{and} \qquad \mathcal{H}_{-\infty} = \cup_{s\in\mathbb{R}}\mathcal{H}_s. \tag{7.3}$$

Lemma 7.1. *The space $\mathcal{H}_\infty$ is dense in $\mathcal{H}_s$ for every $s \in \mathbb{R}$, hence is dense in $\mathcal{H}_{-\infty}$.*

Proof. Let $e^{-t|D|}$, $t \geq 0$, be the heat semigroup generated by $|D|$ on $\mathcal{H}$. Since the operator $e^{-t|D|}$ can be obtained by standard functional calculus of $|D|$ on $\mathcal{H}$, we see that that for any $s \in \mathbb{R}$ the operator $|D|^s e^{-t|D|}$ is bounded for $t > 0$. Therefore, for $t > 0$ the operator $e^{-t|D|}$ maps continuously to $\mathcal{H}_s$ for every $s > 0$, hence maps continuously to $\mathcal{H}_\infty$. Since for any $\xi \in \mathcal{H}$ we have $e^{-t|D|}\xi \to \xi$ as $t \to 0^+$ it follows that $\mathcal{H}_\infty$ is dense in $\mathcal{H}$.

In fact, using the heat semigroup generated by $|D|$ on $\mathcal{H}_s$, $s \in \mathbb{R}$, and arguing as above shows that $\mathcal{H}_\infty$ is dense on each space $\mathcal{H}_s$, hence is dense on $\mathcal{H}_{-\infty}$. □

Definition 7.2. *1) Op^m, $m \in \mathbb{R}$, is the the space of continuous endomorphisms of $\mathcal{H}_\infty$ that extend to a continuous linear mapping $\mathcal{H}_{s+m} \to \mathcal{H}_s$ for every $s \in \mathbb{R}$.*

2) $\operatorname{Op}^{-\infty} := \cap_{m\in\mathbb{R}} \operatorname{Op}^m$ is the space of smoothing operators.

Remark 7.3. Since for $s > s'$ the embedding $\mathcal{H}_s \subset \mathcal{H}_{s'}$ is compact we see that when $m < 0$ any $P \in \operatorname{Op}^m$ extends to a compact operator of $\mathcal{H}_s$ for every $s \in \mathbb{R}$. In particular a smoothing operator is compact on each Hilbert space $\mathcal{H}_s$.

Remark 7.4. In the sequel we endow Op^m with the weakest topology such that for any $s \in \mathbb{R}$ the natural embedding of Op^m in $\mathcal{L}(\mathcal{H}_{s+m}, \mathcal{H}_s)$ is continuous.

Definition 7.5. *Ψ^m_D, $m \in \mathbb{R}$, is the space of ΨDO's of order m and consists of continuous endomorphisms $P : \mathcal{H}_\infty \to \mathcal{H}_\infty$ such that $|D|^{-m}P$ belongs to $\cap_{k\geq 0} \operatorname{dom}\delta^k$.*

Remark 7.6. One can show that Op^0 contains $\cap_{k\geq 0} \operatorname{dom}\delta^k$, so that any ΨDO of order m belongs to Op^m (see [CM, pp. 237–239]).

Now, let $P : \mathcal{H}_\infty \to \mathcal{H}_\infty$ be a ΨDO of order $m > 0$ and suppose that P is elliptic, i.e. there exists $Q \in \Psi^{-m}_D$ such that

$$PQ = 1 - R_1 \quad \text{and} \quad QP = 1 - R_2, \tag{7.4}$$

where R_1 and R_2 are smoothing operators. Notice that P with domain $\mathcal{H}_m$ is closed on $\mathcal{H}$, since P extends to a continuous map from $\mathcal{H}_m$ to $\mathcal{H}$. In fact, as $\mathcal{H}_\infty$ is dense in $\mathcal{H}$ and Q extends to a continuous linear mapping from $\mathcal{H}$ and $\mathcal{H}_m$, the equalities (7.4) imply that the domain of the closure of P on $\mathcal{H}$ is precisely $\mathcal{H}_m$.

Let us now assume that the spectrum of P on $\mathcal{H}$ is not $\mathbb{C}$. Then as in the case of an elliptic ΨDO on a compact (commutative) manifold we have:

Lemma 7.7. *1) The spectrum of the closure of P on $\mathcal{H}_s$, $s \in \mathbb{R}$, does not depend on s, thus coincides with that of the closure of P on $\mathcal{H}$.*

2) For any $\lambda \in \mathbb{C} \setminus \operatorname{Sp} P$ the resolvent $(P-\lambda)^{-1}$ induces an element of Op^{-m} which depends analytically on λ. In particular, P has compact resolvent.

Proof. Let $\lambda \in \mathbb{C} \setminus \operatorname{Sp}_{\mathcal{H}} P$. Then $(P-\lambda)^{-1}$ is a bounded operator on $\mathcal{H}$ and maps to $\operatorname{dom} P = \mathcal{H}_m$. Moreover, by (7.4) we have $(P-\lambda)Q = 1 - \lambda Q - R_1$ and $Q(P-\lambda) = 1 - \lambda Q - R_2$. Multiplying these equalities by $(P-\lambda)^{-1}$ then gives

$$(P-\lambda)^{-1} = Q + (P-\lambda)^{-1}(\lambda Q + R_1) \quad \text{and} \quad (P-\lambda)^{-1} = Q + (\lambda Q + R_2)(P-\lambda)^{-1}. \tag{7.5}$$

Now, recall that $\lambda Q + R_1$ and $\lambda Q + R_2$ are elements of Op^{-m}, so are bounded from $\mathcal{H}_{-m}$ to $\mathcal{H}$ and from $\mathcal{H}$ to $\mathcal{H}_m$. Since $(P-\lambda)^{-1}$ is bounded from $\mathcal{H}$ to itself it follows from the equalities in (7.5) that $(P-\lambda)^{-1}$ is also bounded from $\mathcal{H}_{-m}$ to $\mathcal{H}$ and from $\mathcal{H}$ to $\mathcal{H}_m$, thus gives rise to an element of $\mathcal{L}(\mathcal{H}_s)$ for every $s \in [-m, m]$. We similarly deduce that if $(P-\lambda)^{-1}$ happens to be in $\mathcal{L}(\mathcal{H}_s)$ for some $s \in \mathbb{R}$ then it extends to an element of $\mathcal{L}(\mathcal{H}_{s-m}, \mathcal{H}_s)$ and induces an element of $\mathcal{L}(\mathcal{H}_s, \mathcal{H}_{s+m})$. Therefore, easy inductions show that $(P-\lambda)^{-1}$, on the one hand, extends to a continuous linear mapping $\mathcal{H}_s \rightarrow \mathcal{H}_{s-m}$ for every $s \leq 0$ and, on the other hand, for every $s > 0$ induces a continuous linear mapping $\mathcal{H}_s \rightarrow \mathcal{H}_{s-m}$. Hence $(P-\lambda)^{-1}$ induces an element of Op^{-m}.

Next, let $s \in \mathbb{R}$. As $(P-\lambda)^{-1}$ inverts $P - \lambda$ on $\mathcal{H}$ by density this continues to be true on $\mathcal{H}_s$, so that λ is not in the spectrum of the closure of P on $\mathcal{H}_s$. Hence $\operatorname{Sp}_{\mathcal{H}_s} P \subset \operatorname{Sp}_{\mathcal{H}} P$. Conversely, if λ is not in $\operatorname{Sp}_{\mathcal{H}_s} P$ then similar arguments as those above show that the resolvent $(P-\lambda)^{-1}$ on $\mathcal{H}_s$ induces an element of Op^{-m}, in such way that λ is not in $\operatorname{Sp}_{\mathcal{H}} P$ either. Thus $\operatorname{Sp}_{\mathcal{H}_s} P = \operatorname{Sp}_{\mathcal{H}} P$ for every $s \in \mathbb{R}$, i.e. $\operatorname{Sp}_{\mathcal{H}_s} P$ does not depend on s. □

Since P has compact resolvent the results of this paper apply. In fact, as the resolvent $(P-\lambda)^{-1}$ defines an element of $\mathcal{L}(\mathbb{C} \setminus \operatorname{Sp} P, \mathrm{Op}^{-m})$ similar arguments as those of the proofs of Lemma 6.1 and Lemma 6.2 give:

Lemma 7.8. *1) For every $\lambda \in \operatorname{Sp} P$ the characteristic projector $\Pi_\lambda(P)$ is smoothing and so the characteristic subspace $E_\lambda(P)$ is contained in $\mathcal{H}_\infty$.*

2) The projector $\Pi_\infty(P)$ belongs to Op^0.

As in the commutative case this allows us to set-up the definition below.

Definition 7.9. *For any $s \in \mathbb{R} \cup \{\pm\infty\}$ we let*

$$E_\infty^{(s)}(P) = \Pi_\infty(P)(\mathcal{H}_s). \tag{7.6}$$

Now, as the Cayley-Hamilton decomposition of Theorem 3.5 holds in every Hilbert space $\mathcal{H}_s$, $s \in \mathbb{R}$, along the same lines as that of the proof of Theorem 6.4 we obtain:

Theorem 7.10. *Let $s \in \mathbb{R} \cup \{\pm\infty\}$. Then:*

$$\mathcal{H}_s = E_\infty^{(s)}(P) \dotplus (\overline{\dotplus_{\lambda \in \operatorname{Sp} P} E_\lambda(P)}) \qquad \text{and} \qquad \sum_{\lambda \in \operatorname{Sp} P \cup \{\infty\}} \Pi_\lambda(P) = 1, \tag{7.7}$$

where the closure is taken with respect to the topology of $\mathcal{H}_s$ and the series converges with respect to the strong topology of $\mathcal{L}(\mathcal{H}_s)$.

Next, let $L_\theta = \{\arg \lambda = \theta\}$ and $L_{\theta'} = \{\arg \lambda = \theta'\}$ be rays of minimal growth for the closure of P in $\mathcal{H}$ such that $\theta < \theta' < \theta + 2\pi$ and let $\Pi_{\theta,\theta'}(P)$ denote the corresponding Wodzicki projector. For technical sake we further assume that as $\lambda \in L_\theta \cup L_{\theta'}$ becomes large we have

$$\|(P-\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_m)} = \mathrm{O}(|\lambda|^{-1}). \tag{7.8}$$

Then we get partial analogues of Proposition 6.6 and Proposition 6.7 as follows.

Proposition 7.11. *1) The projector $\Pi_{\theta,\theta'}(P)$ induces an element of* Op^0.

2) Let $\Lambda_{\theta,\theta'}$ denote the open angular sector $\{\theta < \arg\lambda < \theta'\}$. Then the following are equivalent:

(i) The projector $\Pi_{\theta,\theta'}(P)$ is a smoothing operator;

(ii) P has at most finitely many eigenvalues in the angular sector $\Lambda_{\theta,\theta'}$ and $\Pi_{\theta,\theta',\infty}(P) = 0$.

Proof. Let us first observe that the condition (7.8) implies that for any $s \in \mathbb{R}$ as $\lambda \in L_\theta \cup L_{\theta'}$ becomes large we have

$$\|(P-\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{m+s})} = \mathrm{O}(|\lambda|^{-1}). \tag{7.9}$$

To see this notice that with $R_s = |D|^{-s}P|D|^s - P$ for $\lambda \in L_\theta \cup L_{\theta'}$ we have

$$|D|^{-s}(P-\lambda)|D|^s = P - \lambda + R_s = (P-\lambda)[1 + (P-\lambda)^{-1}R_s]. \tag{7.10}$$

Here R_s is a ΨDO of order m which does not depend on λ. Thus, using (7.8) we see that as $\lambda \in L_\theta \cup L_{\theta'}$ becomes large we have $\|[1 + (P-\lambda)^{-1}R_s]^{-1}\|_{\mathcal{L}(\mathcal{H}_m)} = \mathrm{O}(1)$. Thus,

$$\begin{aligned}(7.11)\quad \||D|^{-s}(P-\lambda)^{-1}|D|^s\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_m)} \leq \\ \|(P-\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_m)}\|[1 + (P-\lambda)^{-1}R_s]^{-1}\|_{\mathcal{L}(\mathcal{H}_m)} = \mathrm{O}(|\lambda|^{-1}).\end{aligned}$$

Since there exists a constant $C_s > 0$ independent of λ and such that

$$\|(P-\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}_s,\mathcal{H}_{m+s})} \leq C_s\||D|^{-s}(P-\lambda)^{-1}|D|^s\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_m)}, \tag{7.12}$$

the estimate (7.9) follows.

Now, thanks to (7.9) the integral in (4.8) which defines $\Pi_{\theta,\theta'}(P)$ converges in $\mathcal{L}(\mathcal{H}_s)$ for every $s \in \mathbb{R}$. Thus $\Pi_{\theta,\theta'}(P)$ induces an element of Op^0.

Finally, since any smoothing operator is compact, arguing as in the proof of Proposition 6.7 allows us to show that $\Pi_{\theta,\theta'}(P)$ is a smoothing operator if, and only if, $\mathrm{Sp}\, P \cap \Lambda_{\theta,\theta'}$ is finite and $\Pi_{\theta,\theta',\infty}(P) = 0$. ☐

Finally, as in Proposition 6.8 we have:

Proposition 7.12. *The partial inverse of P belongs to Ψ_D^{-m}.*

Proof. The argument is very much the same as that in the proof of Proposition 6.8. Let P^{-1} be the partial inverse of P. By Proposition 5.4 we have $PP^{-1} = 1 - \Pi_0(P)$ on $\mathcal{H}$ and $P^{-1}P = 1 - \Pi_0(P)$ on $\mathrm{dom}_{\mathcal{H}}\, P = \mathcal{H}_m$. Since $\Pi_0(P)$ is smoothing this shows that P^{-1} inverts P on $\mathcal{H}$ modulo $\mathrm{Op}^{-\infty}$.

On the other hand, as P is elliptic there exists $Q \in \Psi_D^{-m}$ which inverts P modulo $\mathrm{Op}^{-\infty}$ (see (7.4)). Necessarily, P^{-1} and Q coincide on $\mathcal{H}$ up to a smoothing operator. Since Ψ_D^{-m} contains $\mathrm{Op}^{-\infty}$ it follows that P^{-1} is the closure in $\mathcal{H}$ of an element of Ψ_D^{-m}. ☐

Acknowledgements. I am grateful to Maxim Braverman and Henri Moscovici for stimulating discussions. I also thank for their warm hospitality the Max Planck Institute for Mathematics (Bonn, Germany), the Institute of Mathematics of Jussieu-Chevaleret (Paris, France) and the IHES (Bures-sur-Yvette, France) where parts of this work were done.

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