# Spectral Asymmetry, Zeta Functions and the Noncommutative Residue

**Raphaël PONGE**

Institut des Hautes Études Scientifiques

35, route de Chartres

91440 – Bures-sur-Yvette (France)

Juin 2004

IHES/M/04/02

# SPECTRAL ASYMMETRY, ZETA FUNCTIONS, AND THE NONCOMMUTATIVE RESIDUE

RAPHAËL PONGE

Abstract. In this paper, motivated by an approach developed by Wodzicki, we look at the spectral asymmetry of elliptic ΨDO's in terms of theirs zeta functions. First, using asymmetry formulas of Wodzicki we study the spectral asymmetry of odd elliptic ΨDO's and of geometric Dirac operators. In particular, we show that the eta function of a selfadjoint elliptic odd ΨDO is regular at every integer point when the dimension and the order have opposite parities (this generalizes a well known result of Branson-Gilkey for Dirac operators), and we relate the spectral asymmetry of a Dirac operator on a Clifford bundle to the Riemmanian geometric data, which yields a new spectral interpretation of the Einstein action from gravity. We also obtain a large class of examples of elliptic ΨDO's for which the regular values at the origin of the (local) zeta functions can easily be seen to be independent of the spectral cut. On the other hand, we simplify the proofs of two well-known and difficult results of Wodzicki: (i) The independence with respect to the spectral cut of the regular value at the origin of the zeta function of an elliptic ΨDO; (ii) The vanishing of the noncommutative residue of a zero'th order ΨDO projector. These results were proved by Wodzicki using a quite difficult and involved characterization of local invariants of spectral asymmetry, which we can bypass here. Finally, in an appendix we give a new proof of the aforementioned asymmetry formulas of Wodzicki.



## 1. Introduction

This paper is devoted to studying the spectral asymmetry of elliptic pseudodifferential operators (ΨDO's) in terms of their zeta functions. Its outline is as follows.

1.1. *Spectral asymmetry of elliptic ΨDO's.* Let $M^n$ be a compact Riemannian manifold and let $\mathcal{E}$ be a Hermitian bundle over $M$. In order to study the spectral asymmetry of an elliptic selfadjoint ΨDO $P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ Atiyah-Patodi-Singer [APS1] defined its eta invariant as the regular value at the origin of the eta function $\eta(P;s) = \operatorname{Trace} P|P|^{-(s+1)}$. Unlike the residue at the origin of the zeta function, the residue at $s=0$ of $\eta(P;s)$ needs not vanish locally (e.g. [Gi1]). Therefore, Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) had to rely on global and $K$-theoretic arguments for proving the regularity at the origin of the eta function, so that the eta invariant $\eta(P) := \eta(P;0)$ is always well defined.

Subsequently, Wodzicki ([Wo1], [Wo2]) took a different point of view. He looked at the spectral asymmetry of a (possibly nonselfadjoint) elliptic ΨDO $P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ of order $m>0$ in terms of the difference $\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)$ of two zeta functions coming from different spectral cuts $L_\theta = \{\arg\lambda = \theta\}$ and $L_{\theta'} = \{\arg\lambda = \theta'\}$ with $0 \leq \theta < \theta' < 2\pi$. He then proved:

**Theorem 1.1** (Wodzicki ([Wo1], [Wo2, 1.24])). *1) The function $\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)$ is regular at every integer $k \in \mathbb{Z}$, and there we have*

$$\lim_{s\to k}(\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = \int_M \rho^{(k)}_{\theta,\theta'}(P;x), \tag{1.1}$$

*where $\rho^{(k)}_{\theta,\theta'}$ is a density computable locally in terms of the parametric symbol of $(P-\lambda)^{-1}$.*

---

2000 *Mathematics Subject Classification.* Primary 58J50, 58J42, 46L80; Secondary 58J40.

*Key words and phrases.* Spectral asymmetry, zeta and eta functions, noncommutative residue, $K$-theory, pseudo-differential operators.

*2) The regular value $\zeta_\theta(P;0)$ is independent of the spectral cut $L_\theta$.*

In fact, the proof of the second part in [Wo2] is very difficult since it relies on a quite involved and highly non-trivial characterization of local invariants of spectral asymmetry [Wo2, 1.11] (see in particular the flowcharts 5.8 and 6.1 in that paper).

Notice that the vanishing of $\zeta_\uparrow(P;0) - \zeta_\downarrow(P;0)$ generalizes the theorem of Atiyah-Patodi-Singer and Gilkey. Indeed, if $P$ is selfadjoint and $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ denote the zeta functions respectively associated to spectral cuts in the upper halfplane $\Im\lambda > 0$ and lower halfplane $\Im\lambda < 0$ then, as first observed by Shubin [Sh, p. 114] (see also [Wo1, p. 116] and Appendix A, we have

$$\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s) = (1-e^{-i\pi s})\zeta_\uparrow(P;s) - (1-e^{-i\pi s})\eta(P;s). \tag{1.2}$$

Since $\zeta_\uparrow(s)$ and $\zeta_\downarrow(s)$ are regular at the origin we get

$$\zeta_\uparrow(P;0) - \zeta_\downarrow(P;0) = -i\pi \operatorname{res}_{s=0} \eta(P;s). \tag{1.3}$$

Therefore, in the selfadjoint case, the vanishing of $\zeta_\uparrow(P;0) - \zeta_\downarrow(P;0)$ is equivalent to the regularity at the origin of the eta function.

Shortly later, Wodzicki ([Wo3], [Wo4]) showed that the spectral asymmetry of elliptic ΨDO's was encoded by ΨDO projectors as follows. First, consider the operator given by

$$\Pi_{\theta,\theta'}(P) = \frac{1}{2i\pi}\int_{\Gamma_{\theta,\theta',r}} \lambda^{-1}P(P-\lambda)^{-1}d\lambda, \tag{1.4}$$

$$\Gamma_{\theta,\theta,r} = \{\rho e^{i\theta}; \infty > \rho \geq r\} \cup \{re^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \leq \rho < \infty\},$$

where $r$ is small enough so that no non-zero eigenvalue of $T$ lies in the disc $|\lambda| \leq r$. As it turns out $\Pi_{\theta,\theta'}(P)$ is a projector on $L^2(M,\mathcal{E})$ and either is a zero'th order ΨDO or is a smoothing operator (*cf.* [Wo4]; see also [Po1]). Moreover, Wodzicki proved that for any $s \in \mathbb{C}$ we have

$$P_\theta^s - P_{\theta'}^s = (1-e^{2i\pi s})\Pi_{\theta,\theta'}(P)P_\theta^s. \tag{1.5}$$

From this Wodzicki obtained the following refinement of (1.1).

**Theorem 1.2** (Wodzicki ([Wo3], [Wo4])). *We have the have equality of meromorphic functions,*

$$\zeta_\theta(P;s) - \zeta_{\theta'}(P;s) = (1-e^{-2i\pi s}) \operatorname{Trace} \Pi_{\theta,\theta'}(P)P_\theta^{-s}, \quad s \in \mathbb{C}. \tag{1.6}$$

*In particular, $\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)$ is regular at every integer $k \in \mathbb{Z}$ and there we have*

$$\operatorname{ord} P. \lim_{s\to k}(\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = 2i\pi \operatorname{Res} \Pi_{\theta,\theta'}(P)P^{-k}, \tag{1.7}$$

*where* Res *denote the noncommutative residue trace of Wodzicki (*[Wo2], [Wo5]*) and Guillemin* [Gu1] *(see also Section 2).*

The formulas (1.5)–(1.7) were announced in the commentary [Wo3] and can also be found in Wodzicki's thesis [Wo4] (see also Kassel's report [Ka]). They are important because, unlike (1.1), they are operator theoretic facts: on the one hand, the projector $\Pi_{\theta,\theta'}(P)$ has a clear spectral representation in [Po1] (see also Appendix A) and, on the other hand, the noncommutative residue reaches an operator theoretic analogue in the framework of Connes' noncommutative geometry (*cf.* [CM]). Therefore, those formulas hold *mutatis standis* in many other settings of ΨDO algebras as those arising in presence of foliations, manifolds with singularities, hypoelliptic calculus, groupoids, and of the quantum group $SU_q(2)$ (see [CP] and [Co3] for the last example). In fact, using the apparatus of spectral triples (see [CM]) they should ultimately hold in the framework of Connes' noncommutative geometry.

1.2. *Spectral asymmetry of odd elliptic ΨDO's and of Dirac operators.* The elliptic ΨDO's that are odd form an important and interesting class to look at. Recall that according to [KV] a ΨDO of integer order is called odd when its symbol satisfies the transmission property (*cf.* Section 3). Thus the odd ΨDO's form an algebra containing all the differential operators and their parametrices. Moreover, when the dimension of the manifold is odd the noncommutative residues of these ΨDO's all vanish locally (see [KV]). In this context we obtain:

**Theorem 1.3.** *Assume that $P$ is odd and of integer order $m \geq 1$.*

*1) If $\dim M$ is odd and $\operatorname{ord} P$ even then $\zeta_\theta(P;s)$ is regular at every integer point and its value there is independent of the spectral cut $L_\theta$.*

*2) Assume that $\dim M$ is even, $\operatorname{ord} P$ is odd and that the principal symbol of $P$ has all its eigenvalue in the open cone $\{\theta < \arg \lambda < \theta'\} \cup \{\theta + \pi < \arg \lambda < \theta' + \pi\}$. Then:*

*a) For any integer $k$ we have*

$$\operatorname{ord} P. \lim_{s \to k} (\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = i\pi \operatorname{Res} P^{-k}. \tag{1.8}$$

*b) At every integer at which they are not singular the functions $\zeta_\theta(P;s)$ and $\zeta_{\theta'}(P;s)$ take the same regular value. In particular $\zeta_\theta(P;0)$ and $\zeta_{\theta'}(P;0)$ coincide.*

Let us now assume that $P$ is selfadjoint. Then the condition in the second part of Theorem 1.3 on the location of the eigenvalues of $p_m$ is always fulfilled if we take $0 < \theta < \pi < \theta' < 2\pi$. Thus Theorem 3.1 implies that if $\dim M$ and $\operatorname{ord} P$ have opposite parities then there are many integer points at which the zeta functions $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ are not asymmetric. In particular, the regular values $\zeta_\uparrow(P;0)$ and $\zeta_\downarrow(P;0)$ coincide.

On the other hand, Theorem 1.3 also allows us to single some points at which the asymmetry of zeta functions does occur. For instance, we have:

**Proposition 1.4.** *If $\dim M$ is even and $P$ is a selfadjoint elliptic odd ΨDO of order 1, then we always have $\lim_{s \to n} \frac{1}{i}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)) > 0$.*

Now, by a well known result of Branson-Gilkey [BG] in even dimension the eta function of a geometric Dirac operator is an entire function. The following shows that this statement is actually a consequence of a more general result.

**Theorem 1.5.** *Assume that $P$ is a selfadjoint elliptic odd ΨDO.*

*1) If $\dim M$ and $\operatorname{ord} P$ have opposite parities then $\eta(P;s)$ is regular at every integer point.*

*2) If $P$ has order 1 and $\dim M$ is even then $\eta(P;s)$ is an entire function.*

In particular, Theorem 1.5 allows us to simplify the index formula of Bruüning-Seeley [BS] for a first order elliptic operator on a manifold with conical singularities when the dimension of the manifold is odd (*cf.* [Po3]).

Next, the case of a geometric Dirac operator in even dimension is of further interest. To see this let us assume that $\dim M$ is even and that $\mathcal{E}$ is a $\mathbb{Z}_2$-graded Clifford module over $M$. Then any compatible Clifford connection on $\mathcal{E}$ yields a Dirac operator $D\!\!\!/_{\mathcal{E}} : C^\infty(M, \mathcal{E}) \to C^\infty(M, \mathcal{E})$ (see [BGV]). Recall that this covers many settings, e.g. Dirac operator on spin Riemannian manifolds, Gauss-Bonnet and signature operators on an oriented Riemannian manifolds, and $\overline{\partial} + \overline{\partial}^*$-operator on a complex Kaelher manifold. Then we can relate the spectral asymmetry of $D\!\!\!/_{\mathcal{E}}$ to the Riemannian geometry of $M$ and $\mathcal{E}$ as follows.

**Theorem 1.6.** *1) The function $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}};s) - \zeta_\downarrow(D\!\!\!/_{\mathcal{E}};s)$ is entire.*

*2) At every every integer that is not an even integer between 2 and $n$ the zeta functions $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}};s)$ and $\zeta_\downarrow(D\!\!\!/_{\mathcal{E}};s)$ are non-singular and have the same regular value.*

*3) For $k=2,4,\ldots,n$ we have*

$$\lim_{s\to k}(\zeta_\uparrow(D\!\!\!/_{\mathcal{E}};s)-\zeta_\downarrow(D\!\!\!/_{\mathcal{E}};s)) = i\pi\int_M \mu_k(R^M,F^{\mathcal{E}/\$})(x)\sqrt{g(x)}d^n x, \tag{1.9}$$

*where $\mu_k(R^M,F^{\mathcal{E}/\$})(x)$ is a universal polynomial in the curvatures $R^M$ and $F^{\mathcal{E}/\$}$ and their covariant derivatives. In particular, we have*

$$\lim_{s\to n}(\zeta_\uparrow(D\!\!\!/_M;s)-\zeta_\downarrow(D\!\!\!/_M;s)) = 2i\pi(4\pi)^{-n/2}\Gamma(\frac{n}{2})^{-1}\operatorname{rk}\mathcal{E}.\operatorname{vol}M, \tag{1.10}$$

$$\lim_{s\to n-2}(\zeta_\uparrow(D\!\!\!/_M;s)-\zeta_\downarrow(D\!\!\!/_M;s)) = -i\pi c_n\operatorname{rk}\mathcal{E}.\int_M r_M(x)\sqrt{g(x)}d^n x, \tag{1.11}$$

*where $c_n=\frac{1}{12}(n-2)(4\pi)^{-n/2}\Gamma(\frac{n}{2})^{-1}$.*

In (1.11) the integral $\int_M r_M(x)\sqrt{g(x)}dx$ is the Einstein action of the metric $g$, which represents the contribution of gravity forces to the action functional in General Relativity. Thus Formula (3.22) gives a new *spectral* interpretation of the Einstein action, which in the context of Dirac operators on Clifford modules coincides with the previous one, due to Connes [Co2], in terms of $\operatorname{Res} D\!\!\!/_{\mathcal{E}}^{2-n}$.

On the other hand, from Theorem 3.6 we get:

**Proposition 1.7.** *Assume* $\dim M$ *even and* $\int_M r_M\sqrt{g(x)}dx\neq 0$. *Then for any Clifford module $\mathcal{E}$ and any Clifford connection $\nabla^{\mathcal{E}}$ as above we have* $\lim_{s\to n-2}(\zeta_\uparrow(D\!\!\!/_{\mathcal{E}};s)-\zeta_\downarrow(D\!\!\!/_{\mathcal{E}};s))\neq 0$.

We also infer that it follows from their proofs that the previous results, Theorem 1.3 through Proposition 1.7, hold locally, that is at the level of the local zeta functions the integrals of which give the global zeta functions. Therefore, we get a large class of elliptic ΨDO's for which the regular value of the local zeta functions are independent of the choice of the spectral cut. Notice that, as for the regularity at the origin of the local eta function, this result is not true in general (e.g. [Gi1], [Wo1, pp. 130-131]).

1.3. *Spectral asymmetry and the noncommutative residue of ΨDO projectors.* Another important aim of this paper is to simplify the proofs of two well known results of Wodzicki, namely, the 2nd part of Theorem 1.1 and the theorem below.

**Theorem 1.8** (Wodzicki [Wo2, 7.12]). *The noncommutative residue of any zero'th order ΨDO projector is always zero.*

In [Wo2] both results are obtained as consequences of the characterization of local invariants of spectral asymmetry proved earlier in that paper, which as alluded to before is quite difficult and involved. Note also that it follows from Theorem 1.2 that Theorem 1.8 implies the 2nd part of Theorem 1.1. Therefore we only need to prove Theorem 1.8.

Our proof of Theorem 1.8 bypasses Wodzicki's characterization of local invariants of spectral asymmetry and is actually quite close to the proof of Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) of the regularity at the origin of the eta function of a selfadjoint elliptic ΨDO. In fact, we show that once understood in $K$-theoretic terms the issue of proving the vanishing of the noncommutative residue of a ΨDO projector of order 0 becomes the same as that of proving the regularity at the origin of the eta function of any selfadjoint elliptic ΨDO of positive order on $M$. An important ingredient is the homotopy lemma below.

**Lemma 1.9.** *Let $\Pi_0$ and $\Pi_1$ be zero'th order ΨDO projectors and assume that the principal symbol of $\Pi_0$ can be connected to that of $\Pi_1$ by means of a $C^1$-path of homogeneous symbols $(\pi_t)_{0\leq t\leq 1}$ such that $\pi_t$ is a projector for every $t\in[0,1]$. Then $\Pi_0$ and $\Pi_1$ have the same noncommutative residue.*

This lemma enables us to interpret the noncommutative residue of a zero'th order ΨDO projector as a morphism from the $K$-group $K^0(S^*M)$ to $\mathbb{C}$, where $S^*M \overset{\mathrm{pr}}{\to} M$ denotes the cosphere bundle of $M$. To do so we combine this with a description of $K^0(S^*M)$ in terms of projective pairs $(\pi, \mathcal{E})$ where $\mathcal{E}$ is a smooth vector bundle over $M$ and $\pi \in C^\infty(S^*M, \operatorname{End} \operatorname{pr}^*\mathcal{E})$ is a projector. This allows us to get:

**Proposition 1.10.** *1) There exists a unique morphism $\rho_R : K^0(S^*M)/pr^*K^0(M) \to \mathbb{C}$ such that for any zero'th order $\Psi DO$ projector $\Pi$ on $M$ with principal symbol $\pi_0$ we have*

$$\operatorname{Res}\Pi = \rho_R[\operatorname{im}\pi_0]. \tag{1.12}$$

*2) There exists a unique morphism $\rho_{APS}$ from $K^0(S^*M)/pr^*K^0(M)$ to $\mathbb{C}$ such that for any elliptic selfadjoint $\Psi DO$ $P$ on $M$ of order $m > 0$ and with principal symbol $p_m$ we have*

$$\operatorname{ord}P.\operatorname{res}_{s=0}\eta(P;s) = \rho_{APS}[\operatorname{im}\Pi_+(p_m)]. \tag{1.13}$$

*3) We have $\rho_{APS} = 2\rho_R$.*

The second statement is a well-known result of Atiyah-Patodi-Singer [APS2], but an interesting fact here is that we get the second and third statements in the same time as the first one (see Section 4).

Now, we can show that the morphism $\rho_R$ is zero in the same way as did Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) for proving the vanishing of $\rho_{APS}$. Therefore, without any appeal to the characterization of local invariants of spectral asymmetry, we recover Theorem 1.8 and the 2nd part of Theorem 1.1.

Finally, we can also prove the 2nd part of Theorem 1.1 and Theorem 1.8 using an approach similar to that of Wodzicki in [Wo1] and [Wo2, Sect. 7] and which again bypasses Wodzicki's characterization of local invariants of spectral asymmetry (see Remark 4.9). However, one advantage of our previous approach is that it holds for more general traces, since the only properties of the noncommutative residue that we need in the proof of Lemma 1.9 and the first part of Proposition 1.10 are the facts that it is a continuous linear trace on $\Psi^0(M, \mathcal{E})$ and that it vanishes on $1 + \Psi^{-\infty}(M, \mathcal{E})$. For instance, the approach should hold for the exotic traces of Melrose-Nistor [MN] in the framework of Melrose's $b$-calculus on manifolds with boundary [Me].

1.4. *Appendix: Spectral asymmetry and Cayley-Hamilton decomposition.* In an appendix (Appendix A) we give a new proof of the formulas (1.4)–(1.7) of Wodzicki. The proof is based on using the Kontsevich-Vishik canonical trace (see [KV] and Section 2) and the spectral representation of the projector $\Pi_{\theta,\theta'}(P)$, which was given in [Po1] in terms of the Cayley-Hamilton decomposition of $P$. The latter involves a family $(E_\lambda(P))_{\lambda\in\operatorname{Sp}P\cup\{\infty\}}$ of characteristic subspaces and an orthogonal family $(\Pi_\lambda)_{\lambda\in\operatorname{Sp}P\cup\{\infty\}}$ of characteristic projectors. For $\lambda \in \operatorname{Sp}P$ the subspace $E_\lambda(P)$ and the projector $\Pi_\lambda(P)$ are defined as in [RN], i.e.

$$E_\lambda(P) = \cup_{k\geq 1}E_\lambda(P) \quad \text{and} \quad \Pi_\lambda(P) = \frac{1}{2i\pi}\int_{\Gamma_{(\lambda)}}(P-\mu)^{-1}d\mu, \tag{1.14}$$

where $\Gamma_{(\lambda)}$ is a small circle about $\lambda$ which isolates it from the rest of the spectrum. In particular, $\Pi_\lambda(P)$ has finite rank and projects onto $E_\lambda(P)$ (cf. [GK] and Section 2). Furthermore, as $P$ is elliptic $\Pi_\lambda(P)$ is a smoothing operator and so $E_\lambda(P)$ is contained in $C^\infty(M, \mathcal{E})$.

It is shown in [Po1] how to associate to the "infinity" of $\mathbb{C}$ a characteristic subspace $E_\infty(P)$ which is the range of a characteristic projector $\Pi_\infty(P)$ (we refer to Section 2 for the precise definitions). Then we get the Cayley-Hamilton decomposition,

$$L^2(M, \mathcal{E}) = \overline{\dot{+}_{\lambda\in\operatorname{Sp}P\cup\{\infty\}}E_\lambda(P)} \quad \text{and} \quad \sum_{\lambda\in\operatorname{Sp}P\cup\{\infty\}}\Pi_\lambda(P), \tag{1.15}$$

where $\dot{+}$ denotes the algebraic direct sum and the series converges with respect to the strong topology of $\mathcal{L}(L^2(M, \mathcal{E}))$.

In fact, it is also proved in [Po1] that we can also associate to the "infinity" of the open angular sector $\Lambda_{\theta,\theta'} = \{\theta < \arg\lambda < \theta'\}$ a characteristic subspace $E_{\theta,\theta',\infty} \subset E_\infty(P)$ which is the range of a characteristic projector $\Pi_{\theta,\theta',\infty}(P)$. Then we have

$$\Pi_{\theta,\theta'}(P) = \Pi_{\theta,\theta',\infty}(P) + \sum_{\lambda\in \operatorname{Sp} P\cap\Lambda_{\theta,\theta'}} \Pi_\lambda(P), \tag{1.16}$$

where the series also converges with respect to the strong topology of $\mathcal{L}(L^2(M,\mathcal{E}))$. In particular, $\Pi_{\theta,\theta'}(P)$ projects onto $E_{\theta,\theta',\infty}(P) \dot{+} (\overline{\dot{+}_{\lambda\in\operatorname{Sp} P\cap\Lambda_{\theta,\theta'}} E_\lambda(P)})$ and along $E_0(P) \dot{+} E_{\theta',\theta+2\pi,\infty}(P) \dot{+} (\overline{\dot{+}_{\lambda\in\operatorname{Sp} P\cap\Lambda_{\theta',\theta+2\pi}} E_\lambda(P)})$.

Now, formulas (1.15) and (1.16) reduce the proof of (1.4) to the comparison of the operators $\Pi_\lambda(P)P^s_\theta$ and $\Pi_\lambda(P)P^s_{\theta'}$, which enables us to recover Formula (1.6). Moreover, the canonical trace of Kontsevich-Vishik [KV] provides us with the unique analytic extension of the usual trace to the class of non-integer order ΨDO's and induces on integer order ΨDO's a residual trace which coincides with the Wodzicki-Guillemin noncommutative residue trace. Since Formula (1.4) yields an equality of holomorphic families of ΨDO's the passage from (1.4) to (1.6) and (1.7) becomes immediate using the Kontsevich-Vishik trace.

As explained in Appendix A a consequence of the above approach is the following.

**Proposition 1.11.** *Suppose that the principal symbol of $P$ has no eigenvalue within the angular sector $\Lambda_{\theta,\theta'} = \{\theta < \arg\lambda < \theta'\}$. Then for any $s \in \mathbb{C}$ we have*

$$\zeta_\theta(P;s) - \zeta_{\theta'}(P;s) = (1 - e^{-2i\pi s}) \sum_{\lambda\in\operatorname{Sp} P\cap\Lambda_{\theta,\theta'}} \lambda_\theta^{-s} \dim E_\lambda(P), \tag{1.17}$$

*where the sum is actually finite. Thus $\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)$ is a holomorphic function on the whole complex plane which vanishes at every integer point and, in particular, the regular values $\zeta_\theta(P;0)$ and $\zeta_{\theta'}(P;0)$ coincide.*

1.5. *Organization of the paper.* The paper is organized as follows. In Section 2 we recall the general background needed in this paper: Cayley-Hamilton decomposition and complex powers of elliptic ΨDO's, Kontsevich-Vishik canonical trace and Wodzicki-Guillemin noncommutative residue trace, zeta and eta functions of elliptic ΨDO's. In Section 3 we look at the spectral asymmetry of odd elliptic ΨDO's and geometric Dirac operators. The Section is devoted to giving new and simple proofs of the 2nd part of Theorem 1.1 and of Theorem 1.8. In Appendix A we give a proof of formulas (1.4)–(1.7) and in Appendix B we recall how the ΨDO's can be naturally endowed with a Fréchet space structure (this is needed in Section 4).

## 2. General background

Throughout all this paper we let $M$ denote a compact Riemannian manifold of dimension $n$ and let $\mathcal{E}$ be a Hermitian vector bundle over $M$ of rank $r$.

In this section we recall the basic facts concerning Cayley-Hamilton decomposition and complex powers of elliptic ΨDO's, the canonical and noncommutative residue traces for ΨDO's, and zeta and eta functions of elliptic ΨDO's.

2.1. *Elliptic ΨDO's with spectral cuts.* Let $m \in \mathbb{C}$. Then recall that that given an open $V \subset \mathbb{R}^n$ the space of symbols $S^m(V\times\mathbb{R}^n, \operatorname{End}\mathbb{C}^r)$ consists of functions $p \in C^\infty(V\times\mathbb{R}^n, \operatorname{End}\mathbb{C}^r)$ with an asymptotic expansion,

$$p \sim \sum_{j\geq 0} p_{m-j}, \qquad p_l(x,t\xi) = t^l p_l(x,\xi) \text{ for any } t>0, \tag{2.1}$$

in the sense that, for any integer $N$ and any compact $K \subset U$, there exists $C_{NK\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta (p - \sum_{j<N} p_{-j})(x,\xi)| \leq C_{NK\alpha\beta} |\xi|^{\Re m - N - |\beta|}, \qquad x \in K, \quad |\xi| \geq 1. \tag{2.2}$$

Then we let $\Psi^m(M,\mathcal{E})$ denote the space of ΨDO's of order $m$ on $M$ that act on sections of $\mathcal{E}$, i.e. continuous operators $P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ such that:

(i) The distribution kernel of $P$ is smooth off the diagonal of $M \times M$;

(ii) For any trivialization $\tau : \mathcal{E}_{|_U} \to U \times \mathbb{C}^r$ over a local chart $\kappa : U \to V \subset \mathbb{R}^n$ we can write

$$\kappa_* \tau_* (P_{|_U}) = p^{\kappa,\tau}(x,D) + R^{\kappa,\tau}, \tag{2.3}$$

with $p^{\kappa,\tau}$ in $S^m(V \times \mathbb{R}^n, \operatorname{End}\mathbb{C}^r)$ and $R^{\kappa,\tau}$ in $\Psi^{-\infty}(V, \operatorname{End}\mathbb{C}^r)$ (i.e. $R^{\kappa,\tau}$ is a smoothing operator).

Let $P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ be an elliptic ΨDO of degree $m > 0$, so that its principal symbol $p_m(x,\xi)$ is everywhere invertible. Assume also that $p_m(x,\xi)$ has a spectral cut, i.e. there exists an open cone $\Lambda \subset \mathbb{C} \setminus 0$ such that $p_m(x,\xi) - \lambda$ is everywhere invertible for any $\lambda \in \Lambda$. Then $P$ admits an asymptotic resolvent as a parametrix for $P - \lambda$ in a suitable class $\Psi^{-m}(M,\mathcal{E};\Lambda)$ of ΨDO's with parameter (see [Se], [Sh], [GS]). Using this asymptotic resolvent one can show that, on any cone $\Lambda' \subset \Lambda$ such that $\overline{\Lambda'} \setminus 0 \subset \Lambda$, there are at most finitely many eigenvalues of $P$. Moreover, for $R$ large enough there exists $C_{\Lambda' R} > 0$ such that

$$\|(P-\lambda)^{-1}\|_{\mathcal{L}(L^2(M,\mathcal{E}))} \leq C_{\Lambda' R} |\lambda|^{-1} \qquad \lambda \in \Lambda', \quad |\lambda| \geq R. \tag{2.4}$$

Thus there are infinitely many rays $L_\theta = \{\arg \lambda = \theta\}$ contained in the spectral cut $\Lambda$, that are not through an eigenvalue of $P$.

A first consequence of this is that the spectrum of $P$ is not $\mathbb{C}$, so is an discrete subset of $\mathbb{C}$ with no accumulation point and consisting of eigenvalues with finite multiplicities. When $P$ is selfadjoint we can even diagonalize it in an orthonormal basis. As shown in [Po1] even in the non-selfadjoint case we have a Cayley-Hamilton decomposition analogous to that in finite dimension, since it involves a family $(E_\lambda(P))_{\lambda \in \operatorname{Sp} P \cup \{\infty\}}$ of characteristic subspaces and an orthogonal family $(\Pi_\lambda)_{\lambda \in \operatorname{Sp} P \cup \{\infty\}}$ of characteristic projectors defined as follows.

If $\lambda \in \operatorname{Sp} P$ then the characteristic subspace $E_\lambda(P)$ and the characteristic projector $\Pi_\lambda(P)$ are defined as in [RN], i.e.

$$E_\lambda(P) = \cup_{k \geq 1} \ker(P-\lambda)^k \quad \text{and} \quad \Pi_\lambda(P) = \frac{-1}{2i\pi} \int_{\Gamma_{(\lambda)}} (P-\mu)^{-1} d\mu, \tag{2.5}$$

where $\Gamma_{(\lambda)}$ is a small circle about $\lambda$ which isolates $\lambda$ from the rest of the spectrum. In fact, general arguments show that, first, the family $(\Pi_\lambda(P))_{\lambda \in \operatorname{Sp} P}$ is an orthogonal family of finite rank projectors and, second, $\Pi_\lambda(P)$ projects onto $E_\lambda(P)$ and along $E_{\bar\lambda}(P^*)^\perp$ and there exists an integer $N$ so that $E_\lambda(P) = \ker(P-\lambda)^N$ (see [RN], [GK], [Po1]). Moreover, since $P$ is elliptic the subspace $E_\lambda(P)$ is contained in $C^\infty(M,\mathcal{E})$ and the projector $\Pi_\lambda(P)$ is a smoothing operator (*cf.* [Sh], [Po1]).

It is shown in [Po1] that we can associate to the "infinity" of $\mathbb{C}$ a characteristic subspace $E_\infty(P)$, which is the range of a characteristic projector $\Pi_\infty(P)$. More precisely, we let

$$\Pi_\infty(P) = \lim_{R\to\infty} \frac{1}{2i\pi} \int_{|\mu|=R} \mu^{-1} P (P-\mu)^{-1} d\mu, \quad E_\infty(P) = \operatorname{im} \Pi_\infty(P), \tag{2.6}$$

where the limit exists with respect to the strong topology of $\mathcal{L}(L^2(M,\mathcal{E}))$, i.e. that of the pointwise convergence (*cf.* [Po1]). Then we obtain Cayley-Hamilton decomposition below.

**Proposition 2.1** ([Po1]). *1) The family $\{\Pi_\lambda(P)\}_{\lambda \in \operatorname{Sp} P \cup \{\infty\}}$ is an orthogonal family of projectors.*

*2) We have:*

$$L^2(M,\mathcal{E}) = \overline{\dot{+}_{\lambda \in \operatorname{Sp} P \cup \{\infty\}} E_\lambda(P)} \qquad \text{and} \qquad \sum_{\lambda \in \operatorname{Sp} P \cup \{\infty\}} \Pi_\lambda(P) = 1, \tag{2.7}$$

*where $\dotplus$ denotes the algebraic direct sum and the series converges with respect to the strong topology of $\mathcal{L}(L^2(M,\mathcal{E}))$.*

As a consequence of this we can define the partial inverse of $P$ as follows. Consider the characteristic subspace

$$E_{\bar{\mathbb{C}}\backslash 0}(P) = \overline{\dotplus_{\lambda\in \overline{\operatorname{Sp} P}\backslash 0} E_\lambda(P)}. \tag{2.8}$$

In fact $E_{\bar{\mathbb{C}}\backslash 0}(P)$ coincides with the kernel of $\Pi_0(P)$, so that $E_{\bar{\mathbb{C}}\backslash 0}(P)$ is globally invariant by $P$ (i.e. $P$ maps $E_{\bar{\mathbb{C}}\backslash 0}(P)\cap L^2_m(M,\mathcal{E})$ to $E_{\bar{\mathbb{C}}\backslash 0}(P)$) and on there $P$ induces an invertible operator (see [Po1]). Therefore, we can set-up the definition below.

**Definition 2.2.** *The partial inverse of $P$ is the bounded operator on $L^2(M,\mathcal{E})$ denoted $P^{-1}$ which is zero on $E_0(P)$ and inverts $P$ on $E_{\bar{\mathbb{C}}\backslash 0}(P)$.*

The main properties of the partial inverse are summarized below.

**Proposition 2.3** ([Po1]). *1) The partial inverse $P^{-1}$ is a $\Psi DO$ of order $-m$.*

*2) Looking at $P^{-1}$ as a $\Psi DO$ we have*

$$PP^{-1} = P^{-1}P = 1 - \Pi_0(P), \tag{2.9}$$

$$(P^*)^{-1} = (P^{-1})^* \quad \text{and} \quad (P^k)^{-1} = (P^{-1})^k \quad \text{for any integer } k\geq 1. \tag{2.10}$$

*Remark* 2.4. When $P$ is invertible the subspace $E_{\bar{\mathbb{C}}\backslash 0}(P)$ is the whole space $L^2(M,\mathcal{E})$, so that the partial inverse is the actual inverse of $P$. Also, if $P$ is normal then $E_0(P) = \ker P$ and $E_{\bar{\mathbb{C}}\backslash 0}(P) = (\ker P)^\perp$, so $P^{-1}$ is the operator that vanishes on $\ker P$ and inverts $P$ on $(\ker P)^\perp$, that is we recover the usual definition of the partial inverse of a normal operator.

2.2. *Complex powers.* Thanks to the estimates (2.4) we define a bounded operator on $L^2(M,\mathcal{E})$ by letting

$$P^s_\theta = \frac{1}{2i\pi}\int_{\Gamma_\theta} \lambda^s (P-\lambda)^s d\lambda, \qquad \Re s<0, \tag{2.11}$$

$$\Gamma_\theta = \{\rho e^{i\theta}; \infty<\rho\leq r\}\cup\{re^{it}; \theta\geq t\geq \theta-2\pi\}\cup\{\rho e^{i(\theta-2\pi)}; r\leq\rho\leq\infty\}, \tag{2.12}$$

where the power $\lambda^s$ is defined by means of the continuous determination of the argument on $\mathbb{C}\setminus L_\theta$ which takes values in $(\theta,\theta-2\pi)$ and the number $r>0$ is small enough so that there is no nonzero eigenvalue of $P$ in the disc $|\lambda|<r$.

**Proposition 2.5** ([Se, Lem. 3], [Sh], [Po1]). *1) The family $(P^s_\theta)_{\Re s<0}$ is a semigroup, i.e.*

$$P_\theta^{s_1+s_2} = P_\theta^{s_1}P_\theta^{s_2}, \quad \Re s_j<0. \tag{2.13}$$

*2) The family $(P^s_\theta)_{\Re s<0}$ interpolates the partial negative integer powers of $P$, i.e.*

$$P_\theta^{-k} = P^{-k} \quad \textit{for any integer } k\geq 1, \tag{2.14}$$

*where $P^{-k}$ denotes the partial inverse of $P^k$.*

Next, let $\Omega$ be an open subset of $\mathbb{C}$. Recall that each $\Psi$DO space $\Psi^m(M,\mathcal{E})$ carries a natural Fréchet space structure (see Appendix B). Following Wodzicki ([Wo2], [Wo5]) and Kontsevich-Vishik [KV] we then say that a family $(Q_z)_{z\in\Omega}\subset\Psi^*(M,\mathcal{E})$ is holomorphic when:

(i) The order $m_z$ of $P_z$ is a holomorphic function of $z$;

(ii) For any $\varphi$ and $\psi$ in $C^\infty(M)$ the family $(\varphi Q_z\psi)_{z\in\Omega}$ is a holomorphic family of smoothing operators (i.e. they are defined by means of a holomorphic family of smooth Schwartz kernels);

(iii) For any trivialization $\tau:\mathcal{E}_{|_U}\rightarrow U\times\mathbb{C}^r$ over a local chart $\kappa: U\rightarrow V\subset\mathbb{R}^n$ we can write

$$\kappa_*\tau_*(Q_{z|_U}) = q_z^\kappa(x,D) + R_z^\kappa, \tag{2.15}$$

with families holomorphic families $(q_z^\kappa)_{z\in\Omega}\subset C^\infty(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ and $(R_z)_{z\in\Omega}$ in such way that $q_z^\kappa$ is a symbol of degree $m_z$ and the bounds (2.2) of the corresponding asymptotic expansion $p_z\sim\sum p_{z,m_z-j}$ are locally uniform with respect to $z$.

Using the asymptotic resolvent of $P$ one can prove that $(P_\theta^s)_{\Re s<0}$ is a holomorphic family of $\Psi$DO's such that $\operatorname{ord}P_\theta^s = s.\operatorname{ord}P$ (see [Se, Thm. 3], [Sh, Thm. 11.2]). Moreover, for $\Re s<0$ and $k=1,2,\ldots$ we have $P^kP_\theta^{s-k}=P_\theta^s$. Therefore, for any $s\in\mathbb{C}$, we can directly define the power $P_\theta^s$ as the $\Psi$DO given by

$$P_\theta^s = P^kP_\theta^{s-k}, \tag{2.16}$$

where $k$ is any integer $k>\Re s$, the value of which is irrelevant.

**Proposition 2.6** ([Se, Thm. 3], [Sh, Thm. 11.2]). *The family $(P_\theta^s)_{s\in\mathbb{C}}$ satisfies the following.*

*1) It is a holomorphic family of $\Psi$DO's in such way that $\operatorname{ord}P_\theta^s = s.\operatorname{ord}P$ for any $s\in\mathbb{C}$.*

*2) It is a 1-parameter group of $\Psi$DO's such that for $k=1,2,\ldots$ we have*

$$P_\theta^{-k}=P^{-k},\quad P_\theta^0 = 1-\Pi_0(P),\quad P_\theta^k=(1-\Pi_0(P))P^k. \tag{2.17}$$

*In particular, the operators $P^k$ and $P_\theta^k$ coincide up to a smoothing operator and if $E_0(P)=\ker P^N$ then $P_\theta^k = P^k$ for $k\geq N$.*

2.3. *Canonical trace and noncommutative residue.* The canonical trace of Kontsevich-Vishik [KV] and the noncommutative residue trace of Wodzicki ([Wo2], [Wo5]) and Guillemin [Gu1] can be obtained as follows (see [KV], [CM], [Po2]).

First, if $Q$ is in $\Psi^{\text{int}}(M,\mathcal{E})=\cup_{\Re m<-m}\Psi^m(M,\mathcal{E})$ then the restriction of its distribution kernel to the diagonal of $M\times M$ defines a smooth density $k_Q(x,x)$ with values in $\operatorname{End}\mathcal{E}$. Hence $P$ is traceable and $\operatorname{Trace}Q=\int_M \operatorname{tr}_{\mathcal{E}} k_Q(x,x)$.

Next, as first observed by Wodzicki ([Wo3], [Wo5, 3.22]) and Kontsevich-Vishik [KV] (see also [Gu2], [CM], [Po2]) there exists a unique analytic continuation $Q\rightarrow t_Q(x)$ of the map $Q\rightarrow k_Q(x,x)$ to the class $\Psi^{\mathbb{C}\mathbb{Z}}(M,\mathcal{E})$ of $\Psi$DO's of noninteger order. Moreover, if $Q$ is in $\Psi^{\mathbb{Z}}(M,\mathcal{E})$ and $(Q_z)_{z\in\Omega}$ is a holomorphic family of $\Psi$DO's defined near $z=0$ such that $Q_0=Q$ and $\operatorname{ord}Q_z = z+\operatorname{ord}Q$, then the map $z\rightarrow t_{Q_z}(x)$ has at worst a simple pole singularity at $z=0$ in such way that in local trivializing coordinates we have

$$\operatorname{res}_{z=0} t_{Q_z}(x) = -(2\pi)^{-n}\int_{|\xi|=1} q_{-n}(x,\xi)d^{n-1}\xi, \tag{2.18}$$

where $q_{-n}(x,\xi)$ denotes the symbol of degree $-n$ of $Q$.

From this we see that $\operatorname{res}_{z=0} t_Q(x)$ depends on $Q$ only. Moreover, as pointed out in [Po2] since the property of being a density is preserved by taking residues Formula (2.18) shows that we get a well-defined $\operatorname{End}\mathcal{E}$-valued density by letting

$$c_Q(x) = (2\pi)^{-n}(\int_{|\xi|=1} q_{-n}(x,\xi)d^{n-1}\xi). \tag{2.19}$$

Thus, we recover the noncommutative residue of Wodzicki and Guillemin by letting

$$\operatorname{Res}Q = \int_M \operatorname{tr}_{\mathcal{E}} c_Q(x), \qquad Q\in\Psi^{\mathbb{Z}}(M,\mathcal{E}). \tag{2.20}$$

Now, if we define

$$\operatorname{TR}Q = \int_M \operatorname{tr}_{\mathcal{E}} c_Q(x), \qquad Q\in\Psi^{\mathbb{C}\mathbb{Z}}(M,\mathcal{E}), \tag{2.21}$$

then we obtain:

**Proposition 2.7** ([KV]). *1) The functional* TR *is the unique analytic continuation of the usual trace to* $\Psi^{\mathbb{C}\mathbb{Z}}(M,\mathcal{E})$.

*2) We have* $\mathrm{TR}[Q_1,Q_2]=0$ *whenever* $\mathrm{ord}Q_1+\mathrm{ord}Q_2\notin\mathbb{Z}$.

*3) Let* $Q\in\Psi^{\mathbb{Z}}(M,\mathcal{E})$ *and let* $(Q_z)_{z\in\Omega}$ *be a holomorphic family of* $\Psi DO$*'s defined near* $z=0$ *such that* $Q_0=Q$ *and* $\mathrm{ord}Q_z=z+\mathrm{ord}Q$. *Then near* $z=0$ *the function* $\mathrm{TR}\,Q_z$ *has at worst a simple pole singularity such that* $\mathrm{res}_{z=0}\,\mathrm{TR}\,Q_z=-\mathrm{Res}\,Q$.

From this it is easy to deduce:

**Proposition 2.8** ([Wo2], [Gu1], [Wo5]). *1)* Res *is a linear trace on the algebra* $\Psi^{\mathbb{Z}}(M,\mathcal{E})$.

*2)* Res *vanishes* $\Psi DO$*'s of integer order* $\leq -(n+1)$ *and on differential operators.*

*2) We have* $\mathrm{res}_{s=0}\,\mathrm{TR}\,QP_\theta^{-s}=m\,\mathrm{Res}Q$ *for any* $Q\in\Psi^{\mathbb{Z}}(M,\mathcal{E})$.

Note also that by a well-known result of Wodzicki ([Wo4], [Ka, Prop. 5.4]) (see also [Gu3]) if $M$ is connected and has dimension $\geq 2$ then the noncommutative residue induces the only trace on $\Psi^{\mathbb{Z}}(M,\mathcal{E})$ up to a multiplicative constant.

2.4. *Zeta and eta functions.* The canonical trace TR allows us to define the zeta function of $P$ directly as the meromorphic function on $\mathbb{C}$ given by

$$\zeta_\theta(P;s)=\mathrm{TR}\,P_\theta^{-s},\quad s\in\mathbb{C}. \tag{2.22}$$

Then from Proposition 2.7 we obtain:

**Proposition 2.9.** *1)* $\zeta_\theta(P;s)$ *is analytic outside* $\Sigma=\{\frac{n}{m},\frac{n-1}{m},\ldots,\frac{-1}{m},\frac{1}{m},\frac{2}{m},\ldots\}$.

*2) On* $\Sigma$ *the function* $\zeta_\theta(P;s)$ *has at worst simple pole singularities such that*

$$\mathrm{res}_{s=\sigma}\,\zeta_\theta(P;s)=m\,\mathrm{Res}\,P^{-\sigma}\quad \textit{for any } \sigma\in\Sigma. \tag{2.23}$$

*3) The function* $\zeta_\theta(P;s)$ *is always at* $s=0$.

Notice that the regularity at the origin follows from the 2nd part because, since $\Pi_0(P)$ is smoothing, we have $\mathrm{Res}\,P^0=\mathrm{Res}[1-\Pi_0(P)]=0$.

Finally, let us assume that $P$ is selfadjoint. Then its eta function is the meromorphic function given by

$$\eta(P;s)=\mathrm{TR}\,F|P|^{-s},\qquad s\in\mathbb{C}, \tag{2.24}$$

where $F=P|P|^{-1}$ is the sign operator of $P$. Then, as with Proposition 2.9 we get:

**Proposition 2.10.** *1)* $\eta(P;s)$ *is analytic analytic outside* $\Sigma=\{\frac{n}{m},\frac{n-1}{m},\ldots,\frac{-1}{m},\frac{1}{m},\frac{2}{m},\ldots\}$.

*2) On* $\Sigma$ *the function* $\eta(P;s)$ *has at worst simple pole singularities such that*

$$\mathrm{res}_{s=\sigma}\,\eta(P;s)=m\,\mathrm{Res}\,F|P|^{-\sigma}\quad \textit{for any } \sigma\in\Sigma. \tag{2.25}$$

However, showing the regularity at the origin of $\eta(P;s)$ is a more difficult task than for the zeta functions. Indeed, from (2.25) we get $\mathrm{res}_{s=0}\,\eta(P;s)=m\,\mathrm{Res}\,F=m\int_M \mathrm{tr}_{\mathcal{E}}\,c_F(x)$, and examples show that $c_F(x)$ need not vanish locally (e.g. [Gi1], [Wo1, pp. 130-131]). Therefore, Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) had to rely on global and $K$-theoretic arguments to prove:

**Theorem 2.11** ([APS2], [Gi2]). *The function* $\eta(P;s)$ *is always regular at* $s=0$.

This shows that the eta invariant of $P$, i.e.

$$\eta(P)=\eta(P;0), \tag{2.26}$$

is well defined for every selfadjoint elliptic operator of order $m>0$.

Since its appearance as a boundary correcting term in the index formula of Atiyah-Patodi-Singer [APS1], the eta invariant has found many applications and has been extended to various other settings. We refer to the surveys of Bismut [Bi] and Müller [Mü], and the references therein, for an overview of the main results.

## 3. Spectral asymmetry of odd elliptic ΨDO's and Dirac operators

In this section we shall look at the spectral asymmetry of odd elliptic ΨDO's and (geometric) Dirac operators. Recall that according to [KV] a ΨDO-operator $Q$ of integer order $m$ is an odd ΨDO when, in local trivializing coordinates, its symbol $q(x,\xi) \sim \sum_{j\geq 0} q_{m-j}(x,\xi)$ has the transmission property, i.e.

$$q_{m-j}(x,-\xi) = (-1)^{m-j} q_{m-j}(x,\xi), \quad j \geq 0. \tag{3.1}$$

This gives rise to a well-defined subalgebra of $\Psi^{\mathbb{Z}}(M,\mathcal{E})$, containing all the differential operators and the parametrices of elliptic odd ΨDO's. Moreover, the condition $q_{-n}(x,-\xi) = (-1)^n q_{-n}(x,\xi)$ implies that, when the dimension of $M$ is odd, the noncommutative residue of an odd ΨDO vanishes locally (i.e. the density $c_Q(x)$ given by (2.19) is zero).

From now on we let $P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ be an odd elliptic of integer order $m \geq 1$ and we assume that $L_\theta = \{\arg\lambda = \theta\}$, $0 \leq \theta < 2\pi$, and $L_{\theta'} = \{\arg\lambda = \theta\}$, $\theta < \theta' < 2\pi$, are two spectral cuts for both $P$ and its principal symbol $p_m(x,\xi)$.

**Theorem 3.1.** *1) If* $\dim M$ *is odd and* $\operatorname{ord}P$ *even then* $\zeta_\theta(P;s)$ *is regular at every integer point and its values there are independent of the spectral cut* $L_\theta$.

*2) Assume that* $\dim M$ *is even,* $\operatorname{ord}P$ *is odd and that all the eigenvalues of the principal symbol of* $P$ *lie in the open cone* $\{\theta < \arg\lambda < \theta'\} \cup \{\theta + \pi < \arg\lambda < \theta' + \pi\}$*. Then:*

*a) For any integer* $k \in \mathbb{Z}$ *we have*

$$\operatorname{ord}P. \lim_{s\to k} (\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = i\pi \operatorname{Res} P^{-k}. \tag{3.2}$$

*b) At every integer at which they are not singular the functions* $\zeta_\theta(P;s)$ *and* $\zeta_{\theta'}(P;s)$ *take the same regular value. In particular* $\zeta_\theta(P;0)$ *and* $\zeta_{\theta'}(P;0)$ *coincide.*

*Proof.* The proof relies on the computation of the symbol of $\Pi_{\theta,\theta'}(P)$ in local trivializing coordinates. First, let $p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$ and $q(x,\xi,\lambda) \sim \sum_{j\geq 0} q_{-m-j}(x,\xi,\lambda)$ be the respective symbols of $P$ and $(P-\lambda)^{-1}$. As $(p(x,\xi)-\lambda)q(x,\xi,\lambda) + \sum_{\alpha\neq 0} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi,\lambda) \sim 1$ we get

$$q_{-m}(x,\xi,\lambda) = (p_m(x,\xi)-\lambda)^{-1}, \tag{3.3}$$

$$q_{-m-j}(x,\xi,\lambda) = -(p_m(x,\xi)-\lambda)^{-1} \sum_{\substack{|\alpha|+k+l=j,\\ l\neq j}} \frac{1}{\alpha!}\partial_\xi^\alpha p_{m-k}(x,\xi) D_x^\alpha q_{-m-l}(x,\xi,\lambda), \quad j \geq 1. \tag{3.4}$$

Since the symbol $p(x,\xi)$ satisfies the transmission property (3.1), from (3.3) and (3.4) we see that $q(x,\xi,\lambda)$ satisfies the transmission property with respect to the dilation $(\xi,\lambda) \to (-\xi,(-1)^m\lambda)$, i.e.

$$q_{-m-j}(x,-\xi,(-1)^m\lambda) = (-1)^{-m-j} q_{-m-j}(x,\xi,\lambda), \quad j \geq 0. \tag{3.5}$$

On the other hand, let $s(x,\xi;\lambda) \sim \sum_{j\geq 0} s_{-j}(x,\xi;\lambda)$ be the symbol of $P(P-\lambda)^{-1}$. Then it follows from the formula (1.4) for $\Pi_{\theta,\theta'}(P)$ that $\Pi_{\theta,\theta'}(P)$ has symbol

$$\pi(x,\xi) = \frac{1}{2i\pi}\int_{\Gamma_{\theta,\theta'}} \lambda^{-1}s(x,\xi;\lambda)d\lambda = \frac{1}{2i\pi}\int_{\Gamma_{(x,\xi)}} \lambda^{-1}s(x,\xi;\lambda)d\lambda, \tag{3.6}$$

where $\Gamma_{(x,\xi)}$ is a bounded contour contained in the sector $\Lambda_{\theta,\theta'} = \{\theta < \arg\lambda < \theta'\}$ and which isolates from $\mathbb{C} \setminus \Lambda_{\theta,\theta'}$ the eigenvalues of $p_m(x,\xi)$ that lie in $\Lambda_{\theta,\theta'}$. In fact, as $P(P-\lambda)^{-1} = [(P-\lambda)+\lambda](P-\lambda)^{-1} = 1 + \lambda(P-\lambda)^{-1}$, we see that $s(x,\xi;\lambda) = 1 + \lambda q(x,\xi;\lambda)$. Therefore, the symbol of degree $-j$ of $\Pi_{\theta,\theta'}(P)$ is given by

$$\pi_{-j}(x,\xi) = \frac{1}{2i\pi}\int_{\Gamma_{(x,\xi)}} q_{-m-j}(x,\xi;\lambda)d\lambda. \tag{3.7}$$

Now, assume that $n$ is odd and $m$ is even. As alluded to above the noncommutative residue of an odd ΨDO is zero in odd dimension. Since the odd ΨDO's form an algebra it follows that for any integer $k$ the operator $P^{-k}$ is an odd ΨDO and its noncommutative residue is zero. Therefore, the zeta functions $\zeta_\theta(P;s)$ and $\zeta_{\theta'}(P;s)$ are regular at integer points.

On the other hand, since $m$ is even using (3.5) we see that $\pi_{-j}(x,-\xi)$ is equal to

$$\frac{1}{2i\pi}\int_{\Gamma_{(x,-\xi)}} q_{-m-j}(x,-\xi,\lambda)d\lambda = \frac{(-1)^{m-j}}{2i\pi}\int_{\Gamma_{(x,\xi)}} q_{-m-j}(x,\xi,\lambda)d\lambda = (-1)^{-j}\pi_{-j}(x,\xi). \tag{3.8}$$

Hence $\Pi_{\theta,\theta'}(P)$ is an odd ΨDO. Therefore, for any $k\in\mathbb{Z}$ the operator $\Pi_{\theta,\theta'}(P)P^{-k}$ is an odd ΨDO as well, and so $\operatorname{Res}\Pi_{\theta,\theta'}(P)P^{-k}=0$. Then it follows from Theorem A.2 that $\zeta_\theta(P;k)=\zeta_{\theta'}(P;k)$.

Next, suppose that $m$ is odd, $n$ is even and that all the eigenvalues of $p_m(x,\xi)$ are contained in the cone $\mathcal{C}_{\theta,\theta'}=\{\theta<\arg\lambda<\theta'\}\cup\{\theta+\pi<\arg\lambda<\theta'+\pi\}$. Then $P$ has only finitely many eigenvalues outside $\mathcal{C}_{\theta,\theta'}$. Hence $\Pi_{\theta,\theta'}(P)+\Pi_{-\theta,-\theta'}(P)=1-\sum_{\lambda\in\mathbb{C}\setminus\mathcal{C}_{\theta,\theta'}}\Pi_\lambda(P)=1 \bmod \Psi^{-\infty}(M,\mathcal{E})$. At the level of homogeneous symbols this gives

$$\frac{1}{2i\pi}\int_{\Gamma_{(x,\xi)}} q_{-m}(x,\xi,\lambda)d\lambda+\frac{1}{2i\pi}\int_{-\Gamma_{(x,\xi)}} q_{-m}(x,\xi,\lambda)d\lambda=1, \tag{3.9}$$

$$\frac{1}{2i\pi}\int_{\Gamma_{(x,\xi)}} q_{-m-j}(x,\xi,\lambda)d\lambda+\frac{1}{2i\pi}\int_{-\Gamma_{(x,\xi)}} q_{-m-j}(x,\xi,\lambda)d\lambda=0, \quad j\geq 1. \tag{3.10}$$

As $m$ is odd using (3.5) and (3.9) we see that $\pi_0(x,-\xi)-1$ is equal to

$$\frac{-1}{2i\pi}\int_{-\Gamma_{(x,\xi)}} q_{-m}(x,-\xi,\lambda)d\lambda=\frac{1}{2i\pi}\int_{\Gamma_{(x,\xi)}} q_{-m}(x,-\xi,-\lambda)d\lambda=(-1)^m\pi_0(x,\xi)=-\pi_0(x,\xi). \tag{3.11}$$

Similarly, using (3.5) and (3.10) for $j=1,2,\ldots$ we get

$$\pi_{-j}(x,-\xi)=\frac{1}{2i\pi}\int_{-\Gamma_{(x,\xi)}} q_{-m-j}(x,-\xi,-\lambda)d\lambda=(-1)^{m-j}\pi_{-j}(x,\xi)=(-1)^{j+1}\pi_{-j}(x,\xi). \tag{3.12}$$

Next, let $k\in\mathbb{Z}$ and let $p^{(k)}\sim\sum_{j\geq 0}p^{(k)}_{-km-j}$ denote the symbol of $P^{-k}$. Then the symbol $r^{(k)}_{-n}$ of degree $-n$ of $R^{(k)}=\Pi_{\theta,\theta'}(P)P^{-k}$ is given by

$$r^{(k)}_{-n}(x,-\xi)=\sum_{|\alpha|+j+l=n-km}\frac{1}{\alpha!}\partial_\xi^\alpha\pi_{-j}(x,\xi)D_x^\alpha p^{(k)}_{-km-l}(x,\xi). \tag{3.13}$$

Since $P^{-k}$ is an odd ΨDO, using (3.9) and (3.10) we obtain:

$$\begin{aligned} r^{(k)}_{-n}(x,-\xi) &= \sum_{j+l+|\alpha|=n-km}\frac{1}{\alpha!}(\partial_\xi^\alpha\pi_{-j})(x,-\xi)(D_x^\alpha p^{(k)}_{-km-l})(x,-\xi),\\ &= \sum_{l+|\alpha|=n-km}\frac{(-1)^{|\alpha|-km-l}}{\alpha!}\partial_\xi^\alpha[1-\pi_0(x,\xi)]D_x^\alpha p^{(k)}_{-km-l})(x,\xi)\\ &\quad - \sum_{j+l+|\alpha|=n-km}\frac{(-1)^{j+|\alpha|-km-l}}{\alpha!}\partial_\xi^\alpha\pi_{-j}(x,\xi)D_x^\alpha p^{(k)}_{-km-l}(x,\xi)\\ &= (-1)^n p^{(k)}_{-n}(x,\xi)-(-1)^n\sum_{|\alpha|+j+l=n-km}\frac{1}{\alpha!}(\partial_\xi^\alpha\pi_{-j})(x,\xi)(D_x^\alpha p^{(k)}_{-km-l})(x,\xi). \end{aligned} \tag{3.14}$$

Combining this with (3.13) and the fact that $n$ is even we get $r^{(k)}_{-n}(x,\xi)+r^{(k)}_{-n}(x,-\xi)=p^{(k)}_{-n}(x,\xi)$.

Moreover, we have $\int_{|\xi|=1}r^{(k)}_{-n}(x,-\xi)d^{n-1}\xi=(-1)^n\int_{|\xi|=1}r^{(k)}_{-n}(x,\xi)d^{n-1}\xi=(2\pi)^{-n}c_{R^{(k)}}(x)$, where $c_{R^{(k)}}(x)$ is the residual density (2.19). Hence $2c_{R^{(k)}}(x)=c_{P^{-k}}(x)$. Therefore, using (2.20) we obtain $\operatorname{Res}\Pi_{\theta,\theta'}(P)P^{-k} = \operatorname{Res} R^{(k)} = \frac{1}{2}\operatorname{Res} P^{-k}$. Combining this with Theorem A.2 then gives the equality $\operatorname{ord}P. \lim_{s\to k}(\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = 2i\pi \operatorname{Res} P^{-k}$.

Finally, by Proposition 2.9 the functions $\zeta_\theta(P;s)$ and $\zeta_{\theta'}(P;s)$ are regular at $k \in \mathbb{Z}$ if, and only if, we have $\operatorname{Res} P^{-k} = 0$. As $\operatorname{ord}P. \lim_{s\to k}(\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)) = 2i\pi \operatorname{Res} P^{-k}$ it follows that whenever $\zeta_\theta(P;s)$ and $\zeta_{\theta'}(P;s)$ are regular at an integer their regular values there coincide. In particular, as they are regular at the origin, we have $\zeta_\theta(P;0) = \zeta_{\theta'}(P;0)$. □

*Remark* 3.2. Recall that the noncommutative residue of a differential operator is always zero. Thus, Theorem 3.1 also shows that if we further assume that $P$ is a differential operator then at every integer that is not between 1 and $\frac{n}{m}$ the functions $\zeta_\theta(P;s)$ and $\zeta_{\theta'}(P;s)$ are non-singular and have the same regular value.

Let us now assume that $P$ is selfadjoint. Then the condition in the second part of Theorem 3.1 on the location of the eigenvalues of $p_m$ is always fulfilled if we take $0 < \theta < \pi < \theta' < 2\pi$. Thus Theorem 3.1 says that if $\dim M$ and $\operatorname{ord}P$ have opposite parities, then there are many integer points at which the zeta functions $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ are not asymmetric. In particular, the regular values $\zeta_\uparrow(P;0)$ and $\zeta_\downarrow(P;0)$ coincide.

In fact, Theorem 3.1 also allows us to single some points at which the asymmetry of zeta functions does occur, for we have:

**Proposition 3.3.** *If* $\dim M$ *is even and* $P$ *is a selfadjoint elliptic odd* $\Psi$*DO of order 1, then we always have* $\lim_{s\to n}\frac{1}{i}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)) > 0$.

*Proof.* By Theorem 3.1 we have $\lim_{s\to n}\frac{1}{i}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)) = \pi \operatorname{Res} P^{-n}$. Moreover, since $P^{-n}$ has order $-n$ its symbol of degree $-n$ is its principal symbol $p_m(x,\xi)^{-n}$. Thus,

$$\operatorname{Res} P^{-n} = (2\pi)^{-n}\int_{S^*M} \operatorname{tr} p_m(x,\xi)^{-n} dx d\xi, \tag{3.15}$$

where $S^*M$ denotes the cosphere bundle of $M$ with its induced metric. As $p_m(x,\xi)$ is selfadjoint and $n$ is even we have $\operatorname{tr} p_m(x,\xi)^{-n} = \operatorname{tr}[p_m(x,\xi)^{-\frac{n}{2}*} p_m(x,\xi)^{-\frac{n}{2}}] > 0$. Hence the positivity of $\operatorname{Res} P^{-n}$ and $\lim_{s\to n}\frac{1}{i}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s))$. □

Recall that, as first observed by Shubin [Sh] (see also [Wo1, p. 116] and Appendix A), the difference $\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)$ is nicely related to the eta function $\eta(P;s)$ by means of the formula,

$$\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s) = (1 - e^{-i\pi s})\zeta_\uparrow(P;s) - (1 - e^{-i\pi s})\eta(P;s). \tag{3.16}$$

In particular, for any integer $k \in \mathbb{Z}$ we get

$$i\pi \operatorname{ord}P. \operatorname{res}_{s=k}\eta(P;s) = i\pi \operatorname{Res} P^{-k} - \operatorname{ord}P. \lim_{s\to k}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s)). \tag{3.17}$$

Now, by a well known result of Branson-Gilkey [BG] in even dimension the eta function of a geometric Dirac operator is an entire function. The following shows that this statement is actually a consequence of a more general result.

**Theorem 3.4.** *Assume that* $P$ *is a selfadjoint elliptic odd* $\Psi$*DO.*

*1) If* $\dim M$ *and* $\operatorname{ord}P$ *have opposite parities then* $\eta(P;s)$ *is regular at every integer point.*

*2) If* $P$ *has order 1 and* $\dim M$ *is even then* $\eta(P;s)$ *is an entire function.*

*Proof.* Let $k \in \mathbb{Z}$. Since $\dim M$ and $\operatorname{ord}P$ have opposite parities Theorem 3.1 tells us that the terms $i\pi \operatorname{Res} P^{-k}$ and $\operatorname{ord}P. \lim_{s\to k}(\zeta_\uparrow(P;s) - \zeta_\downarrow(P;s))$ in (3.17) either are both equal to zero (when $\dim M$ is odd and $\operatorname{ord}P$ even) or are equal to each other (when $\dim M$ is even and $\operatorname{ord}P$ is odd). In any case we deduce that $\eta(P;s)$ is regular at $s = k$.

On the other hand, when $P$ has order 1 Proposition 2.10 implies that $\eta(P;s)$ is holomorphic on $\mathbb{C}\setminus\mathbb{Z}$. Thus, when $\dim M$ is even and $P$ has order 1 the function $\eta(P;s)$ is entire. □

*Remark* 3.5. Theorem 3.4 also allows us to simplify the general index formula of Bruñing-Seeley [BS] for a first order elliptic operator on a manifold with conical singularities when the dimension of the manifold is odd, since the residues of the eta function coming into play now vanish (*cf.* [Po3]).

Finally, the case of a geometric Dirac operator in even dimension is of further interest. To see this let us assume that $\dim M$ is even and that $\mathcal{E}$ is a $\mathbb{Z}_2$-graded Clifford module over $M$, so that there is a $\mathbb{Z}_2$-grading $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ intertwined by the Clifford action of $M$. Then given a compatible Clifford connection $\nabla^{\mathcal{E}}$ we can form a Dirac operator $D\!\!\!/_{\mathcal{E}}$ acting on the sections of $\mathcal{E}$ by means of the composition,

$$D\!\!\!/_{\mathcal{E}} : C^\infty(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^\infty(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} C^\infty(M, \mathcal{E}), \tag{3.18}$$

where $c$ denotes the action of $T^*M$ on $\mathcal{E}$ by Clifford representation (see [BGV]). Then $D\!\!\!/_{\mathcal{E}}$ anticommutes with the $\mathbb{Z}_2$-grading of $\mathcal{E}$ and if $e_1, \ldots, e_n$ is a local orthonormal tangent frame then by the Lichnerowicz formula ([BGV], [LM]) we have

$$D\!\!\!/_{\mathcal{E}}^2 = (\nabla^{\mathcal{E}})^* \nabla^{\mathcal{E}} + \mathcal{F}^{\mathcal{E}/\$} + \frac{1}{4} r_M = -g^{ij}(\nabla_i^{\mathcal{E}} \nabla_j^{\mathcal{E}} - \Gamma_{ij}^k \nabla_k^{\mathcal{E}}) + \frac{1}{2} F^{\mathcal{E}/\$} c(e^i) c(e^j) + \frac{1}{4} r_M, \tag{3.19}$$

where the $\Gamma_{ij}^k$'s are the Christoffel symbols of the metric, $r_M$ is the scalar curvature of $M$ and $F^{\mathcal{E}/\$}$ is the twisted curvature of $\mathcal{E}$ as defined in [BGV, Prop. 115].

Moreover, this setting cover the following geometric examples (see [BGV]):

- The Dirac operator on a spin Riemannian manifold (with or without coefficients in a Hermitian bundle);

- The Gauss-Bonnet and signature operators on an oriented Riemannian manifold;

- The $\overline{\partial} + \overline{\partial}^*$-operator on a complex Kaelher manifold.

Now, we can relate the spectral asymmetry of $D\!\!\!/_{\mathcal{E}}$ to the Riemmanian geometry of $M$ and $\mathcal{E}$ as follows.

**Theorem 3.6.** *1) The function $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; s) - \zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; s)$ is entire.*

*2) At every every integer that is not an even integer between 2 and $n$ the zeta functions $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; s)$ and $\zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; s)$ are non-singular and have the same regular value.*

*3) For $k = 2, 4, \ldots, n$ we have*

$$\lim_{s \to k} (\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; s) - \zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; s)) = i\pi \int_M \mu_k(R^M, F^{\mathcal{E}/\$})(x) \sqrt{g(x)} d^n x, \tag{3.20}$$

*where $\mu_k(R^M, F^{\mathcal{E}/\$})(x)$ is a universal polynomial in the curvatures $R^M$ and $F^{\mathcal{E}/\$}$ and their covariant derivatives. In particular, we have*

$$\lim_{s \to n} (\zeta_\uparrow(D\!\!\!/_M; s) - \zeta_\downarrow(D\!\!\!/_M; s)) = 2i\pi (4\pi)^{-n/2} \Gamma(\frac{n}{2})^{-1} \operatorname{rk} \mathcal{E}. \operatorname{vol} M, \tag{3.21}$$

$$\lim_{s \to n-2} (\zeta_\uparrow(D\!\!\!/_M; s) - \zeta_\downarrow(D\!\!\!/_M; s)) = -i\pi c_n \operatorname{rk} \mathcal{E}. \int_M r_M(x) \sqrt{g(x)} d^n x, \tag{3.22}$$

*where $c_n = \frac{1}{12}(n-2)(4\pi)^{-n/2}\Gamma(\frac{n}{2})^{-1}$.*

*Proof.* First, since $D\!\!\!/_{\mathcal{E}}$ has order 1 Proposition 2.9 and Theorem A.2 imply that $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; s) - \zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; s)$ is an entire function. Moreover, since $n$ is even and $D\!\!\!/_{\mathcal{E}}$ is a also differential operator it follows from Theorem 3.1 and Remark 3.2 that at every integer $k$ not between 1 and $n$ the functions $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; s)$ and $\zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; s)$ are regular and we have $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; k) = \zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; k)$.

Second, by construction $D\!\!\!/_{\mathcal{E}}$ anticommutes with respect to the $\mathbb{Z}_2$-grading $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ of $\mathcal{E}$. Thus, if $k$ is odd then $D\!\!\!/_{\mathcal{E}}^{-k}$ anticommutes with the $\mathbb{Z}_2$-grading. At the level of symbols, and henceforth at the level of the residual density $c_{D\!\!\!/_{\mathcal{E}}^{-k}}(x)$, this means that they take values in endomorphisms of

$\mathcal{E}$ intertwining $\mathcal{E}^+$ and $\mathcal{E}^-$. Thus $\operatorname{tr}_{\mathcal{E}} c_{D\!\!\!/_{\mathcal{E}}^{-k}}(x)$ vanishes, and so $\operatorname{Res} P^{-k} = 0$. Then $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; s)$ and $\zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; s)$ are regular at $s = k$ and using Theorem 3.1 we get $\zeta_\uparrow(D\!\!\!/_{\mathcal{E}}; k) = \zeta_\downarrow(D\!\!\!/_{\mathcal{E}}; k)$.

Now, let us assume that $k = 2l$ where $l$ is an integer between 0 and $\frac{n}{2}$, and let us compute the density $c_{D\!\!\!/_{\mathcal{E}}^{-k}}(x)$. To this end set $\operatorname{tr}_{\mathcal{E}_x} c_{D\!\!\!/_{\mathcal{E}}^{-k}}(x) = \mu_k(x)\sqrt{g(x)}dx$, where $\sqrt{g(x)}dx$ denotes the Riemannian density and $\mu_k(x)$ is a smooth function. For instance, in normal coordinates centered at $x_0$ we have $g(0) = \mathrm{id}$, so using (2.19) we get

$$\mu_k(x_0) = (2\pi)^{-n} \int_{|\xi|=1} \operatorname{tr} q_{-n}^{(l)}(0,\xi) d^{n-1}\xi, \tag{3.23}$$

where $q_{-n}^{(l)}(x,\xi)$ denotes the symbol of degree $-n$ of $D\!\!\!/_{\mathcal{E}}^{-2l} = D\!\!\!/_{\mathcal{E}}^{-k}$ relatively to the coordinates.

Notice that thanks to the Lichnerowicz Formula (3.19) in the above coordinates the principal symbol of $D\!\!\!/_{\mathcal{E}}^2$ is $p_2(0,\xi) = \|\xi\|^2 := g^{ij}(x)\xi_i\xi_j$. Since $n$ is even at $x = 0$ this gives $q_{-n}^{(n)}(0,\xi) = |\xi|^{-n}$. Thus $\mu_n(x_0) = (2\pi)^{-n} \operatorname{rk}\mathcal{E}|S^{n-1}| = \operatorname{rk}\mathcal{E}.2\Gamma(\frac{n}{2})^{-1}(4\pi)^{-n/2}$, from which we deduce Formula (3.21).

Similarly, using an explicit computation of $q_{-n}^{(n-2)}(0,\xi)$ Kalau-Walze [KW] and Kastler [Kas] showed that $\mu_{n-2}(x_0) = -c_n r_M(x_0)$ with $c_n = \frac{1}{12}(n-2)(4\pi)^{-n/2}\Gamma(\frac{n}{2})^{-1}$, which yields Formula (3.22).

Next, as it is well-known (e.g. [Wo5, 3.23]) the densities $c_{(D\!\!\!/_{\mathcal{E}}^2)^{-l}}(x)$, $l = 1, \ldots, \frac{n}{2}$, are related to the heat kernel asymptotics $k_t(x,x) \simeq_{t\rightarrow 0^+} t^{-\frac{n}{2}} \sum_{j\geq 0} t^j a_j(D\!\!\!/_{\mathcal{E}}^2)(x)$, where $k_t(x,y)$, $t > 0$, denotes the (smooth) distribution kernel of $e^{-tD\!\!\!/_{\mathcal{E}}^2}$. Namely, we have

$$c_{(D\!\!\!/_{\mathcal{E}}^2)^{-l}}(x) = \frac{2}{(l-1)!} a_{\frac{n}{2}-l}(D\!\!\!/_{\mathcal{E}}^2)(x). \tag{3.24}$$

On the other hand, it is also well-known (see [ABP, p. 303], [Gi3, Sect. 4.1]) that the coefficients $a_j(D\!\!\!/_{\mathcal{E}}^2)(x)$ are of the form $a_j(D\!\!\!/_{\mathcal{E}}^2)(x) = \lambda_j(R^M, F^{\mathcal{E}/\$})\sqrt{g(x)}dx$, where $\lambda_j(R^M, F^{\mathcal{E}/\$})$ is a universal polynomial in the curvatures $R^M$ and $F^{\mathcal{E}/\$}$. Combining this with (3.24) we see that $\mu_{2l}(x) = \mu_{2l}(R^M, F^{\mathcal{E}/\$})(x)$, where $\mu_{2l}(R^M, F^{\mathcal{E}/\$})(x)$ is a universal polynomial in $R^M$ and $F^{\mathcal{E}/\$}$ and their covariant derivatives. Hence Formula (3.20).

Alternatively, using the same kind arguments as those used by Atiyah-Bott-Patodi [ABP] and Gilkey [Gi3] we can directly prove Formula (3.20) without making any use of the short time heat kernel asymptotics.

We proceed locally using a synchronous frame for $\mathcal{E}$ over normal coordinates centered at $x_0$, so that we then have to give a universal expression for $\mu_k(x_0) = \mu_k(0)$. To this end let $p^{(l)} \sim \sum_{j\geq 0} p_{2l-j}^{(l)}$ and $q^{(l)} \sim \sum_{j\geq 0} q_{-2l-j}^{(l)}$ denote the symbols of $D\!\!\!/_{\mathcal{E}}^{2l}$ and $D\!\!\!/_{\mathcal{E}}^{-2l}$. Notice that $\sum \frac{1}{\alpha!}\partial_\xi^\alpha p^{(l)} D_x^\alpha q^{(l)} \sim 1$ and from (3.19) we get $p_{2l}^{(l)} = \|\xi\|^l$. Thus,

$$q_{-2l}(x,\xi) = \|\xi\|^{-2l}, \tag{3.25}$$

$$\|\xi\|^{2l} q_{-2l-j}(x,\xi) = -\sum_{\substack{j_1+j_2+|\alpha|=j \\ j_2\neq 0}} \frac{1}{\alpha!}\partial_\xi^\alpha p_{2l-j_1}^{(l)}(x,\xi) D_x^\alpha q_{-2l-j_2}^{(l)}(x,\xi), \quad j \geq 1. \tag{3.26}$$

Therefore $\|\xi\|^{2l} q_{-n}^{(l)}(x,\xi)$ is a universal polynomial in the derivatives of the homogeneous components of the symbol of $D\!\!\!/_{\mathcal{E}}^{2l}$, which are themselves given by universal polynomial in the derivatives of homogeneous components of the symbol of $D\!\!\!/_{\mathcal{E}}^2$. Thus, using (3.23) we see that $\mu_k(x_0)$ is a universal polynomial in the derivatives at $x = 0$ of the coefficients of the differential operator $D\!\!\!/_{\mathcal{E}}^2$ with respect to the synchronous frame and normal coordinates centered at $x_0$.

On the other hand, by (3.19) the coefficients of $D\!\!\!/_{\mathcal{E}}^2$ depends in a uniform and polynomial way on the coefficients of the metric $g(x)$ and of the connection $\nabla^{\mathcal{E}}$ and their derivatives. However, relatively to synchronous trivializations and normal coordinates centered at $x_0$ the coefficients of the Taylor expansions at the origin of the coefficients of the metric $g$ and of the connection $\nabla^{\mathcal{E}}$ are universal polynomials in the covariant derivatives at $x_0$ of the coefficients of the curvatures $R^M$ and $F^{\mathcal{E}/\$}$ (see [ABP, Prop. 2.11, Prop. 3.7]). Going back to the manifold we then see that $\mu_k(x)$ is given by a universal polynomial in the covariant derivatives of the curvatures $R^M$ and $F^{\mathcal{E}/\$}$. Hence Formula (3.20). □

*Remark* 3.7. In [Gi3, Sect. 4.1] Gilkey made use of Invariant Theory to obtain a more precise form of the coefficients $a_j(\not{D}_{\mathcal{E}}^2)(x)$ coming into the heat kernel asymptotics for $\not{D}_{\mathcal{E}}^2$. In fact, the analysis is carried out for more general operators, i.e. Laplace type operators, and in particular by combining with (3.24) and (3.2) it allows us to recover Formula (3.22).

As alluded to in the introduction, since in (3.22) the integral $\int_M r_M(x)\sqrt{g(x)}dx$ is the Einstein action of the metric $g$, Theorem 3.6 yields a new *spectral* interpretation of this action (which in the context of Dirac operators on Clifford modules thus coincides with the previous one, due to Connes [Co2]).

On the other hand, from Theorem 3.6 we also obtain:

**Proposition 3.8.** *Assume* $\dim M$ *even and* $\int_M r_M\sqrt{g(x)}dx \neq 0$. *Then for any Clifford data* $(\mathcal{E}, \nabla^{\mathcal{E}})$ *as above we have* $\lim_{s\to n-2}(\zeta_\uparrow(\not{D}_{\mathcal{E}};s) - \zeta_\downarrow(\not{D}_{\mathcal{E}};s)) \neq 0$.

*Remark* 3.9. As it follows from their proofs, all the results of this section hold locally at the level of densities. For instance the local version of Formula (3.2) states that, under the same hypotheses, for any $k \in \mathbb{Z}$ we have

$$\lim_{s\to k}(t_{P_\theta^{-s}}(x) - t_{P_{\theta'}^{-s}}(x)) = \frac{2i\pi}{m}c_{P^{-k}}(x). \tag{3.27}$$

Therefore, we get a wide class of examples of elliptic ΨDO's for which, not only the regular values at the origin of the zeta functions are independent of the spectral cut, but the densities $\lim_{s\to 0} t_{P_\theta^{-s}}(x)$ themselves are independent of the spectral cut. Notice that, as for the vanishing at the origin of residue of the local eta function, this fact is not true in general (e.g. [Gi1], [Wo1, pp. 130–131]).

## 4. Spectral asymmetry and noncommutative residue of a ΨDO projector

In this section we shall simplify the proofs of the following results of Wodzicki.

**Theorem 4.1** ([Wo2, 1.24]). *The regular value at the origin of the zeta function of any elliptic ΨDO of positive order on M is independent of the choice of the spectral cut.*

*Remark* 4.2. The above result was first announced [Wo1], but the initial argument of Wodzicki was not quite complete (see the corrigenda in [Wo2, 1.25]).

**Theorem 4.3** ([Wo2, 7.12]). *The noncommutative residue of any zero'th order ΨDO projector on M is always zero.*

*Remark* 4.4. Even though it is not explicitly written in [Wo2, 7.12], Wodzicki really meant ΨDO projectors of order zero, and not ΨDO projectors of any order, since he assumed that the ΨDO projectors he considered had either order 0 or are smoothing.

In [Wo2, 1.24] Wodzicki obtained Theorem 4.1 as a consequence of the characterization of local invariants of spectral asymmetry that he proved in that paper. Then he obtained Theorem 4.3 by proving that it was equivalent to Theorem 4.1 (see [Wo2, 7.9–12]). Note that if $P$ is an elliptic ΨDO of positive order and $L_\theta$, $0 \leq \theta \leq 2\pi$, and $L_{\theta'}$, $\theta < \theta' < 2\pi$, are spectral cuts for both $P$ and its principal symbol $p_m$, then by Theorem A.2 we have

$$\operatorname{ord} P.(\zeta_\theta(P;0) - \zeta_{\theta'}(P;0)) = 2i\pi \operatorname{Res} \Pi_{\theta,\theta'}(P). \tag{4.1}$$

Since $\Pi_{\theta,\theta'}(P)$ either is a $\Psi$DO projector of order 0 or is smoothing we immediately see that Theorem 4.3 implies Theorem 4.1.

As alluded to in Introduction the proof of Wodzicki's characterization of local invariants of spectral asymmetry is quite difficult and very involved. We shall give here a proof of Theorem 4.1 and Theorem 4.3 which bypasses this characterization and is quite close to the proof of Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) of the regularity at the origin of the eta function of a selfadjoint elliptic $\Psi$DO. The starting point is the following homotopy lemma, which will be proven later on in this section.

**Lemma 4.5.** *Let $\Pi_0$ and $\Pi_1$ be zero'th order $\Psi$DO projectors and assume that the principal symbol of $\Pi_0$ can be connected to that of $\Pi_1$ by means of a $C^1$-path of homogeneous symbols $(\pi_t)_{0\leq t\leq 1}$ such that $\pi_t$ is a projector for every $t$. Then $\Pi_0$ and $\operatorname{Res}\Pi_1$ have same noncommutative residue.*

Let $S^*M \overset{\text{pr}}{\to} M$ denote the cosphere bundle of $M$ and recall that while $K^0(S^*M)$ is the group of formal differences of stable homotopy classes of (smooth) vector bundles over $S^*M$, the group $K_0(C^\infty(S^*M))$ consists of formal differences of (smooth) homotopy classes of idempotents in $M_\infty(C^\infty(S^*M)) = \varinjlim M_q(C^\infty(S^*M))$. In fact, by the Serre-Swan Theorem the map $e \to \operatorname{im} e$ from idempotents of $M_\infty(C^\infty(S^*M))$ to (smooth) vector bundles over $S^*M$ induces an isomorphism from $K_0(C^\infty(S^*M))$ onto $K^0(S^*M)$ (e.g. [At]).

Now, it is convenient to describe $K^0(S^*M)$ as follows. We call *projective pair* a pair $(\pi, \mathcal{E})$ where $\mathcal{E}$ is a smooth vector bundle over $M$ and $\pi \in C^\infty(S^*M, \operatorname{End}\text{pr}^*\mathcal{E})$ is a projector. The sum of projective pairs is given by taking direct sums, i.e. $(\pi_1, \mathcal{E}_1) \oplus (\pi_2, \mathcal{E}_2) = (\pi_1 \oplus \pi_2, \mathcal{E}_1 \oplus \mathcal{E}_2)$, and we will say that two projective pairs $(\pi_1, \mathcal{E}_1)$ and $(\pi_2, \mathcal{E}_2)$ are equivalent when there exist smooth vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ over $M$ such that $\mathcal{E}_1 \oplus \mathcal{F}_1$ and $\mathcal{E}_2 \oplus \mathcal{F}_2$ are isomorphic vector bundles and under the corresponding identification of $\text{pr}^*\mathcal{E}_1 \oplus \text{pr}^*\mathcal{F}_1$ and $\text{pr}^*\mathcal{E}_2 \oplus \text{pr}^*\mathcal{F}_2$ the projectors $\pi_1 \oplus 0_{\text{pr}^*\mathcal{F}_1}$ and $\pi_2 \oplus 0_{\text{pr}^*\mathcal{F}_2}$ are homotopic.

Notice that if $e_1 \in M_{q_1}(C^\infty(S^*M))$ and $e_2 \in M_{q_2}(C^\infty(S^*M))$ are idempotents which are homotopic as idempotents in $M_\infty(C^\infty(S^*M))$, then they define equivalent projective pairs $(e_1, M \times \mathbb{C}^{q_1})$ and $(e_2, M \times \mathbb{C}^{q_2})$.

Conversely, let $(\pi, \mathcal{E})$ be a projective pair. Then there exists a (smooth) vector bundle $\mathcal{F}$ such that $\mathcal{E}\oplus\mathcal{F} \simeq M\times\mathbb{C}^q$ (cf. [At]). This yields an identification of $\text{pr}^*\mathcal{E}\oplus\text{pr}^*\mathcal{F}$ with $S^*M\times\mathbb{C}^q = \text{pr}^*(M\times\mathbb{C}^q)$ under which $\pi \oplus 0_{\text{pr}^*\mathcal{F}}$ corresponds to an idempotent $e$ in $C^\infty(S^*M, \operatorname{End}\mathbb{C}^q) = M_q(C^\infty(S^*M))$. This idempotent depends on $\mathcal{F}$ and on the isomorphism $\mathcal{E} \oplus \mathcal{F} \simeq M \times \mathbb{C}^q$ only up to similarity in $M_\infty(C^\infty(S^*M))$, hence only up to homotopy. Moreover, any other element in the equivalence class of $(\pi, \mathcal{E})$ yields an idempotent which is homotopic to $e$.

The two constructions above give rise to maps between homotopy classes of idempotents of $M_\infty(C^\infty(S^*M))$ and equivalence classes of projective pairs. These maps are inverse of each other and are compatible with direct sums, so they give rise to isomorphisms between $K_0(C^\infty(S^*M))$ and the group of formal differences of equivalence classes of projective pairs. Combining this with the aforementioned Serre-Swan isomorphism $K^0(S^*M) \simeq K_0(C^\infty(S^*M))$ we obtain:

**Proposition 4.6.** *The map $(\pi, \mathcal{E}) \to \operatorname{im}\pi$ from projective pairs to smooth vector bundles over $S^*M$ induces an isomorphism from the group of formal differences of equivalence classes of projective pairs onto the $K$-group $K^0(S^*M)$.*

Now, let $(\pi, \mathcal{E})$ be a projective pair. Endowing $\mathcal{E}$ with a smooth Hermitian structure we may assume that $\pi$ is selfadjoint, since $\pi$ is homotopic to the selfadjoint projector onto its range (see [Bl, Prop. 4.6.2]). Set $f_0 = 2\pi - 1 = \pi - (1 - \pi)$ and given some $m > 0$ define

$$p_m(x, \xi) = f_0(x, \xi|\xi|^{-1})|\xi|^m, \quad (x, \xi) \in T^*M \setminus 0. \tag{4.2}$$

Then $p_m$ is a selfadjoint invertible homogeneous symbol of degree $m$. Let $P_{(\pi,\mathcal{E})} \in \Psi^m(M, \mathcal{E})$ have principal symbol $p_m(x, \xi)$ and be selfadjoint. Then $P_{(\pi,\mathcal{E})}$ is elliptic and the principal symbol of

$\Pi_+(P_{(\pi,\mathcal{E})}) = \frac{1}{2}(1+P|_{(\pi,\mathcal{E})}P|^{-1}_{(\pi,\mathcal{E})})$ is equal to $\Pi_+(p_m) = \frac{1}{2}(1+f_0) = \pi(x,\xi)$, so is homotopic to $\pi(x,\xi)$ whatever hermitian structure is put on $\mathcal{E}$ and whatever choice is made for $P_{(\pi,\mathcal{E})}$.

Let $(\pi_1,\mathcal{E}_1)$ be a projective pair in the equivalence class of $(\pi,\mathcal{E})$. Then there exist smooth vector bundles $\mathcal{F}$ and $\mathcal{F}_1$ over $M$ and a smooth isomorphism of vector bundles $\phi : \mathcal{E}\oplus\mathcal{F} \to \mathcal{E}_1\oplus\mathcal{F}_1$ such that $\phi(x)^{-1}(\pi_1(x,\xi)\oplus 0_{\text{pr}^*\mathcal{F}_{1,x}})\phi(x)$ is homotopic to $\pi(x,\xi)\oplus 0_{\text{pr}^*\mathcal{F}_x}$ as projectors in $C^\infty(S^*M, \text{End}(\text{pr}^*\mathcal{E}\oplus\text{pr}^*\mathcal{F}))$. Endowing $\mathcal{F}$ with a Hermitian structure we give $\mathcal{E}_1\oplus\mathcal{F}_1$ the Hermitian structure corresponding under $\phi$ to that on $\mathcal{E}\oplus\mathcal{F}$. Then $\phi$ becomes unitary and the principal symbols of $\Pi_+[\phi^{-1}(P_{(\pi_1,\mathcal{E}_1)}\oplus 0_{\mathcal{F}_1})\phi] = \phi^{-1}[\Pi_+(P_{(\pi_1,\mathcal{E}_1)})\oplus 0_{\mathcal{F}_1}]\phi$ and $\Pi_+(P_{(\pi,\mathcal{E})}\oplus 0_{\mathcal{F}}) = \Pi_+(P_{(\pi,\mathcal{E})})\oplus 0_{\mathcal{F}}$ are homotopic. Therefore, using Lemma 4.5 (and the invariance of the noncommutative residue by isomorphisms of vector bundles) we get

$$\text{(4.3)}\qquad \operatorname{Res}\Pi_+(P_{(\pi,\mathcal{E})}) = \operatorname{Res}\Pi_+(P_{(\pi,\mathcal{E})}\oplus 0_{\mathcal{F}}) = \operatorname{Res}\phi^{-1}[\Pi_+(P_{(\pi_1,\mathcal{E}_1)})\oplus 0_{\mathcal{F}_1}]\phi = \operatorname{Res}P_{(\pi_1,\mathcal{E}_1)},$$

i.e. $\operatorname{Res}\Pi_+(P_{(\pi,\mathcal{E})})$ depends only on the equivalence class of $(\pi,\mathcal{E})$. Thus we define a map from equivalence classes of projective pairs to $\mathbb{C}$ by letting

$$\text{(4.4)}\qquad \rho_R[\pi,\mathcal{E}] = \operatorname{Res}\Pi_+(P_{(\pi,\mathcal{E})}).$$

This map is additive since given two projective pairs $(\pi_1,\mathcal{E}_1)$ and $(\pi_2,\mathcal{E}_2)$ the ΨDO projectors $\Pi_+(P_{(\pi_1\oplus\pi_2,\mathcal{E}_1\oplus\mathcal{E}_2)})$ and $\Pi_+(P_{(\pi_1,\mathcal{E}_1)})\oplus\Pi_+(P_{(\pi_2,\mathcal{E}_2)})$ have homotopic principal symbols. Thus,

$$\text{(4.5)}\qquad \rho_R[\pi_1\oplus\pi_2,\mathcal{E}_1\oplus\mathcal{E}_2] = \operatorname{Res}[\Pi_+(P_{(\pi_1,\mathcal{E}_1)})\oplus\Pi_+(P_{(\pi_2,\mathcal{E}_2)})] = \rho_R[\pi_1,\mathcal{E}_1]+\rho_R[\pi_2,\mathcal{E}_2].$$

Therefore, in view of Proposition 4.6 this gives rise to an additive map $\rho_R : K^0(S^*M)\to\mathbb{C}$ such that for any projective pair $(\pi,\mathcal{E})$ we have

$$\text{(4.6)}\qquad \rho_R[\operatorname{im}\pi] = \operatorname{Res}\Pi_+(P_{(\pi,\mathcal{E})}).$$

On the other hand, the subgroup $\text{pr}^*K^0(M)\subset K^0(S^*M)$ corresponds to formal differences of equivalence classes of projective pairs of the form $(\text{pr}^*1_{\mathcal{E}},\mathcal{E})$. As $\Pi_+(P_{(\text{pr}^*1_{\mathcal{E}'},\mathcal{E})})$ has principal symbol $\text{pr}^*1_{\mathcal{E}}$, which is also the principal symbol of $1_{\mathcal{E}}$ on which Res vanishes, we deduce that $\rho_R$ vanishes on $\text{pr}^*K^0(M)$.

Finally, let $\Pi : C^\infty(M,\mathcal{E})\to C^\infty(M,\mathcal{E})$ be a ΨDO projector of order 0 and let $\pi_0$ be the principal symbol of Π. Then $(\pi_0,\mathcal{E})$ is a projective pair and, as the principal symbol of $\Pi_+(P_{(\pi_0,\mathcal{E})})$ is homotopic to $\pi_0$, we get

$$\text{(4.7)}\qquad \rho_R[\operatorname{im}\pi_0] = \operatorname{Res}\Pi_+(P_{(\pi_0,\mathcal{E})}) = \operatorname{Res}\Pi.$$

Moreover, if $P$ is a selfadjoint elliptic ΨDO of positive order then, as $P|P|^{-1} = 2\Pi_+(P)-1$, we get

$$\text{(4.8)}\qquad \operatorname{ord}P.\operatorname{res}_{s=0}\eta(P;s) = \operatorname{Res}P|P|^{-1} = 2\operatorname{Res}\Pi_+(P) = 2\rho_R[\operatorname{im}\Pi_+(p_m)],$$

where $p_m(x,\xi)$ denotes the principal symbol of $P$. Therefore, we obtain:

**Proposition 4.7.** *1) There exists a unique morphism $\rho_R : K^0(S^*M)/pr^*K^0(M)\to\mathbb{C}$ such that for any zero'th order ΨDO projector Π on $M$ with principal symbol $\pi_0$ we have*

$$\text{(4.9)}\qquad \operatorname{Res}\Pi = \rho_R[\operatorname{im}\pi_0].$$

*2) There exists a unique morphism $\rho_{APS}$ from $K^0(S^*M)/pr^*K^0(M)$ to $\mathbb{C}$ such that for any elliptic selfadjoint ΨDO $P$ on $M$ of order $m>0$ and with principal symbol $p_m$ we have*

$$\text{(4.10)}\qquad \operatorname{ord}P.\operatorname{res}_{s=0}\eta(P;s) = \rho_{APS}[\operatorname{im}\Pi_+(p_m)].$$

*3) We have $\rho_{APS} = 2\rho_R$.*

*Remark* 4.8. The second part above is a well-known result of Atiyah-Patodi-Singer [APS2, Sect. 4]. In fact, the above description $K^0(S^*M)/\mathrm{pr}^*K^0(M)$ in terms of projective pairs is merely a variant of the description in [APS2, Sect. 4] in terms of selfadjoint elliptic symbols. It is anyway interesting to note that we obtained the last two statements of Proposition 4.7 in the same time as we got the first one.

Now, we can prove the vanishing of the morphism $\rho_R$ in the same way as Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) did for proving the vanishing of $\rho_{APS}$. For sake of completeness we give a brief outline below.

First, possibly by passing to the oriented double cover of $M$ we can assume $M$ oriented (e.g. [APS2, p. 84]). Then we have to distinguish between the odd and even dimensional cases.

- *Odd dimensional case* ([APS2]). In this case $K^0(S^*M)/\mathrm{pr}^*K^0(M)$ is (rationally) generated by twisted signatures classes, i.e. those classes of the form $[\mathrm{im}\,\pi_{\mathcal{E},0}]$ where $\pi_{\mathcal{E},0}$ is the principal symbol of the positive projector $\Pi_+(\not{D}_{\mathcal{E}})$ associated to the signature operator $\not{D}_{\mathcal{E}}$ with coefficients in an auxiliary bundle $\mathcal{E}$ (see [APS2, Prop. 4.4]). Then it is enough to show that $\rho_R$ vanishes on these classes. In fact as $\not{D}_{\mathcal{E}}$ is a geometric Dirac operator the vanishing of $\mathrm{Res}\,\Pi_+(\not{D}_{\mathcal{E}}) = \frac{1}{2}\,\mathrm{res}_{s=0}\,\eta(\not{D}_{\mathcal{E}};s)$ can be obtained by using invariant theory ([APS2], [Gi3]) or, alternatively, by means of direct heat kernel arguments which show that $\eta(\not{D}_{\mathcal{E}};s)$ is analytic for $\Re s > -\frac{1}{2}$ (cf. [BF, Thm. 2.6]; see also [Me, Sect. 8.13]). In any case we deduce that $\rho_R$ vanishes on odd dimensional manifolds.

- *Even dimensional case* ([Gi2], [Gi3]). In this case we cannot make use of twisted signature operators to generate $K^0(S^*M)/\mathrm{pr}^*K^0(M)$. Nevertheless, Gilkey [Gi3] showed that, on the one hand, $K_0(S^*M)$ was generated by those classes $[\mathrm{im}\,\Pi_+(p_m)]$ coming from principal symbols of selfadjoint elliptic *differential* operators and, on the other hand, if $P$ is a selfadjoint elliptic differential operator on $M$ then there exists a selfadjoint elliptic differential operator $Q$ on $M\times S^1$ such that $\mathrm{Res}\,\Pi_+(P) = \mathrm{Res}\,\Pi_+(Q)$ (since $\mathrm{res}_{s=0}\,\eta(P;s) = \mathrm{res}_{s=0}\,\eta(Q;s)$). Thus the vanishing of $\rho_R$ in the even dimensional case follows from the vanishing of $\rho_R$ in the odd dimensional case.

Now, combining the vanishing of $\rho_R$ and Formula (4.1) we see that, without any appeal to Wodzicki's characterization of local invariants of spectral asymmetry, we recover Theorem 4.3 and Theorem 4.1.

Finally, it remains to carry out the proof of Lemma 4.5.

*Proof of Lemma 4.5.* First, as $\Pi_0$ and $\Pi_1$ are projectors for any $\lambda$ in $\Lambda_0 := \mathbb{C}\setminus\{0,1\}$ we have

$$(\Pi_j-\lambda)^{-1} = -\lambda^{-1}(1-\Pi_j) + (1-\lambda)^{-1}\Pi_j, \quad j=0,1. \tag{4.11}$$

Therefore, if $\Gamma$ denotes a (direct) circle about $\lambda = 1$ with radius $r\in(0,1)$, then we have

$$\Pi_j = \frac{1}{2i\pi}\int_\Gamma (\Pi_j-\lambda)^{-1}d\lambda, \quad j=0,1. \tag{4.12}$$

In the sequel we say that $P\in\Psi^0(M,\mathcal{E})$ is an almost projector when its principal symbol is a projector and, given any Fréchet algebra $\mathcal{A}$, we let $\mathcal{A}_t$ denote the Fréchet algebra $C^1([0,1],\mathcal{A})$. Also, recall that $\Psi^0(M,\mathcal{E})$ carries a natural Fréchet algebra structure (see Corollary B.5 in Appendix).

*Claim.* 1) There exists a $C^1$-path $P_t\in\Psi^0(M,\mathcal{E})_t$ of almost projectors connecting $\Pi_0$ to $\Pi_1$.

2) There exist $Q_t(\lambda)$ in $\mathrm{Hol}(\Lambda_0,\Psi^0(M,\mathcal{E})_t)$ and $R_t$ and $S_t$ in $\mathrm{Hol}(\Lambda_0,\Psi^{-\infty}(M,\mathcal{E})_t)$ such that

$$(P_t-\lambda)Q_t(\lambda) = 1-R_t(\lambda) \quad \text{and} \quad Q_t(\lambda)(P_t-\lambda) = 1-S_t. \tag{4.13}$$

*Proof of the claim.* 1) Let us cover $M$ by finitely many domains $U_i$ of local charts $\kappa_i : U_i\to V_i$ over which there is a trivialization $\tau_i : \mathcal{E}_{|U_i}\to U\times\mathbb{C}^r$. Also, let $(\varphi_i)\subset C^\infty(M)$ be a partition of the unity on $M$ subordinated to the covering $(U_i)$ and for each index $i$ let $\psi_i\in C^\infty(U_i)$ be such that $\psi_i = 1$ near $\operatorname{supp}\varphi_i$. Then by Lemma B.3 in Appendix we can define a $C^1$-family of ΨDO's of order 0 by letting

$$\tilde{P}_t = \sum \varphi_i \tau_i^* \kappa_i^* [\tilde{\pi}_{i,t}(x, D)]\psi_i, \quad 0 \leq t \leq 1, \tag{4.14}$$

where $\tilde{\pi}_{i,t}(x,\xi) \in S^0(U \times V_i, \operatorname{End}\mathbb{C}^r)_t$ is a smoothed version of $(\kappa_{i*}\tau_{i*}\pi_{t,0|_{T^*U\setminus 0}})(x,\xi)$. Notice that the principal of $\tilde{P}_t$ is just $\pi_t$. As by hypothesis $\pi_t$ is a projector we see that $\tilde{P}_t$ is an almost projector for every $t$. Moreover, since $\pi_0$ and $\pi_1$ are the principal symbols of $\Pi_0$ and $\Pi_1$, the ΨDO's $\tilde{P}_0$ and $\tilde{P}_1$ have the same the principal symbols as $\Pi_0$ and $\Pi_1$, respectively. Therefore, a $C^1$-path of almost projectors joining $\Pi_0$ to $\Pi_1$ is given by

$$P_t = \tilde{P}_t + (1-t)(\Pi_0 - \tilde{P}_0) + t(\Pi_1 - \tilde{P}_1). \tag{4.15}$$

2) Since $P_t$ has principal symbol $\pi_t$, which is projector, we see that $P_t - \lambda$ is elliptic for every $\lambda \in \Lambda_0$. Let us now show that we can carry over the parametrix construction for $P_t - \lambda$ in such way to get an element of $\operatorname{Hol}(\Lambda_0, \Psi^0(M,\mathcal{E})_t)$. To do this notice that on $V_i$ each homogeneous component $q_{t,-j}^{(i)}(\lambda)$ of the symbol of a parametrix for $\kappa_{i*}\tau_{i*}(P_{t|_{U_i}}) - \lambda$ is analytic with respect to $\lambda$ and is $C^1$ with respect to $t$. Then by an easy adaptation of the proof of the Borel's Lemma for symbols (e.g. [Hö, Prop. 18.1.3]) we can construct $q_t^{(i)}(\lambda)$ in $\operatorname{Hol}(\Lambda_0, S^0(V_i, \operatorname{End}\mathbb{C}^q)_t)$ such that $q_t^{(i)}(\lambda) \sim \sum q_{t,-j}^{(i)}(\lambda)$. Therefore, we get a global parametrix on $M$ by letting

$$Q_t(\lambda) = \sum \varphi_i \tau_i^* \kappa_i^* [q_t^{(i)}(\lambda)(x, D_x)]\psi_i. \tag{4.16}$$

Moreover, by Lemma B.3 in the appendix this defines an element of $\operatorname{Hol}(\Lambda_0, \Psi^0(M,\mathcal{E})_t)$.

Finally, for $\lambda \in \Lambda_0$ and $t \in [0,1]$ let $R_t(\lambda) = 1 - Q_t(\lambda)(P_t - \lambda)$ and $S_t = 1 - (P_t - \lambda)Q_t(\lambda)$. As $\Psi^0(M,\mathcal{E})$ is a Fréchet algebra $R_t(\lambda)$ and $S_t(\lambda)$ both belong to $\operatorname{Hol}(\Lambda_0, \Psi^0(M,\mathcal{E})_t)$. Moreover, for every $t$ and every $\lambda$ the operators $R_t(\lambda)$ and $S_t(\lambda)$ are smoothing, since $Q_t(\lambda)$ is a parametrix for $P_t - \lambda$. Therefore, on any local trivializing chart their symbols have no nonzero homogeneous components, so $R_t(\lambda)$ and $S_t(\lambda)$ are in $\operatorname{Hol}(\Lambda_0, \Psi^{-\infty}(M,\mathcal{E})_t)$. □

Since the asymptotic resolvent $Q_t(\lambda)$ belongs to $\operatorname{Hol}(\Lambda_0, \Psi^0(M,\mathcal{E})_t)$ we can interpolate Formula (4.12) "modulo smoothing operators" by letting

$$Q_t = \frac{1}{2i\pi}\int_\Gamma Q_t(\lambda)d\lambda, \qquad 0 \leq t \leq 1. \tag{4.17}$$

This defines an element of $\Psi^0(M,\mathcal{E})_t$. Observe also that, for $j = 0, 1$, the equality (4.12) shows that $(\Pi_j - \lambda)^{-1}$ is in $\operatorname{Hol}(\Lambda_0, \Psi^0(M,\mathcal{E}))$. Thus setting $t = j$ in (4.13) and multiplying the two sides of the first equality there by $(\Pi_j - \lambda)^{-1}$ show that $Q_j(\lambda) = (\Pi_j - \lambda)^{-1}$ modulo $\operatorname{Hol}(\Lambda_0, \Psi^{-\infty}(M,\mathcal{E}))$. Combining this with (4.12) we get $Q_j = \Pi_j \mod \Psi^{-\infty}(M,\mathcal{E})$. Hence

$$\operatorname{Res} Q_j = \operatorname{Res}\Pi_j, \quad j = 0, 1. \tag{4.18}$$

Next, since $Q_t$ belongs to $\operatorname{Hol}(\Lambda_0, \Psi^0(M,\mathcal{E})_t)$ and (2.19) shows that Res is a continuous linear functional on the Fréchet algebra $\Psi^0(M,\mathcal{E})$ we have:

$$\frac{d}{dt}\operatorname{Res} Q_t = \operatorname{Res}\dot{Q}_t = \frac{1}{2i\pi}\int_\Gamma \operatorname{Res}\partial_t \dot{Q}_t(\lambda)d\lambda. \tag{4.19}$$

Now, differentiating with respect to $t$ the first equality in (4.13) gives

$$(P_t - \lambda)\dot{Q}_t(\lambda) = -\dot{P}_t Q_t(\lambda) - \dot{R}_t. \tag{4.20}$$

Then multiplying this equality by $Q_t(\lambda)$ and using the second equality in (4.13) we obtain

$$(1 - S_t(\lambda))\dot{Q}_t(\lambda) = -Q_t(\lambda)\dot{P}_t Q_t(\lambda) - Q_t(\lambda)\dot{R}_t, \tag{4.21}$$

$$\dot{Q}_t(\lambda) = -Q_t(\lambda)\dot{P}_t Q_t(\lambda) \qquad \operatorname{mod} \operatorname{Hol}(\Lambda_0, \Psi^{-\infty}(M,\mathcal{E})_t). \tag{4.22}$$

Similarly, repeating the same arguments with differentiation with respect to $\lambda$ we get

$$\partial_\lambda Q_\lambda(\lambda) = -Q_t(\lambda)[\partial_\lambda(P_t - \lambda)]Q_t(\lambda) = Q_t(\lambda)^2 \qquad \mod \operatorname{Hol}(\Lambda_0, \Psi^{-\infty}(M,\mathcal{E})_t). \tag{4.23}$$

Now, since Res is a continuous trace on $\Psi^0(M,\mathcal{E})$ and vanishes on $\Psi^{-\infty}(M,\mathcal{E})$ the equalities (4.22) and (4.23) show that

$$\operatorname{Res} \dot{Q}_t(\lambda) = -\operatorname{Res}[\dot{P}_t Q_t(\lambda)^2] = -\operatorname{Res}[\dot{P}_t Q_t(\lambda)^2] = \partial_\lambda \operatorname{Res}[\dot{P}_t Q_t(\lambda)]. \tag{4.24}$$

Combining this with (4.19) gives $\frac{d}{dt}\operatorname{Res} Q_t = \frac{1}{2i\pi}\int_\Gamma \partial_\lambda \operatorname{Res}[\dot{P}_t Q_t(\lambda)]d\lambda = 0$. Hence $\operatorname{Res} Q_0 = \operatorname{Res} Q_1$. Then using (4.18) we deduce that $\operatorname{Res}\Pi_0 = \operatorname{Res}\Pi_1$. □

*Remark* 4.9. It is possible to prove Theorem 4.1 and Theorem 4.3 by using an approach similar to that of Wodzicki in [Wo1] and [Wo2, Sect. 7] and which also bypasses his characterization of local invariants of spectral asymmetry. To see this consider two rays $L_\theta$ and $L_{\theta'}$ with $0 \leq \theta < \theta' < 2\pi$ and for $m > 0$ let $\operatorname{Ell}^m_{\theta,\theta'}(M)$ consist of all $m$'th order elliptic $\Psi$DO's $P$ on $M$ such that $L_\theta$ and $L_{\theta'}$ are spectral cuts for both $P$ and its principal symbol. Then Wodzicki ([Wo1, pp. 126–129, 133-135], [Wo2, pp. 148–150]) proved that there exists a unique morphism $\rho_{\theta,\theta'}$ from $K^0(S^*M)/\mathrm{pr}^*K^0(M)$ to $\mathbb{C}$ such that for any $P \in \operatorname{Ell}^m_{\theta,\theta'}(M)$ with principal symbol $p_m$ we have

$$\operatorname{ord} P.(\zeta_\theta(P;0) - \zeta_{\theta'}(P;0)) = \rho_{\theta,\theta'}[\operatorname{im}\Pi_{\theta,\theta'}(p_m)]. \tag{4.25}$$

Moreover, in [Wo2, 7.9] Wodzicki showed that for any zero'th order $\Psi$DO projector $\Pi$ with principal symbol $\pi_0$ there exists $P_\Pi \in \operatorname{Ell}^m_{\theta,\theta'}(M)$ with principal symbol $p_m$ such that:

(i) We have $[\operatorname{im}\pi_0] = [\operatorname{im}\Pi_{\theta,\theta'}(p_m)]$ in $K^0(S^*M)$;

(ii) We have $2i\pi \operatorname{Res}\Pi = \operatorname{ord} P_\Pi.(\zeta_\theta(P_\Pi;0) - \zeta_{\theta'}(P_\Pi;0))$.

This uses the analytic continuation of $\operatorname{ord} P.\zeta_\theta(P;0)$ and $\operatorname{ord} P.\zeta_{\theta'}(P;0)$ to $\operatorname{Ell}^0_{\theta,\theta'}(M)$ (see [Wo2, 7.1–8]). Combining the properties (i) and (ii) with (4.25) we can recover Lemma 4.7 and the first part of Proposition 4.7. In fact, we can also recover second and third parts of Proposition 4.7 by using (1.3) and specializing the arguments of [Wo2, 7.9] to selfadjoint elements of $\operatorname{Ell}^0_{0,\pi}(M)$. Then by proceeding as before we can show the vanishing of the morphisms $\rho_{\theta,\theta'}$ and $\rho_R$, and so recover Theorem 4.1 and Theorem 4.3.

In fact, our previous approach allows us to easily recover the existence of the morphism $\rho_{\theta,\theta'}$. Indeed, from (4.1) and Proposition (4.7) we get

$$\operatorname{ord} P.(\zeta_\theta(P;0) - \zeta_{\theta'}(P;0)) = 2i\pi \operatorname{Res}\Pi_{\theta,\theta'}(P) = 2i\pi\rho_R[\operatorname{im}\Pi_{\theta,\theta'}(p_m)], \tag{4.26}$$

since $\Pi_{\theta,\theta'}(p_m)$ is the principal symbol of $\Pi_{\theta,\theta'}(P)$. Moreover, if $(\pi,\mathcal{E})$ is a projective pair then $\pi$ is homotopic to $\Pi_{\theta,\theta'}(p_m)$, where $p_m$ is the principal symbol of any element of $\operatorname{Ell}_{\theta,\theta'}(P)$ of the form

$$P^{(\theta,\theta')}_{(\pi,\mathcal{E})} = e^{i\omega}\Pi_+(P_{(\pi,\mathcal{E})})P_{(\pi,\mathcal{E})} - e^{i\omega'}\Pi_-(P_{(\pi,\mathcal{E})})P_{(\pi,\mathcal{E})}, \quad \theta < \omega < \theta' < \omega' < \theta + 2\pi. \tag{4.27}$$

Thus $K^0(S^*M)$ is generated by the classes $[\operatorname{im}\Pi_{\theta,\theta'}(p_m)]$ coming from principal symbols of elements of $\operatorname{Ell}^m_{\theta,\theta'}(M)$. Therefore, Formula (4.25) defines a unique morphism $\rho_{\theta,\theta'}$ from $K^0(S^*M)/\mathrm{pr}^*K^0(M)$ to $\mathbb{C}$ and, in fact, thanks to (4.26) we have $\rho_{\theta,\theta'} = 2i\pi\rho_R = i\pi\rho_{APS}$.

On the other hand, as alluded to in Introduction, our first approach has the advantage to hold for more general traces, since the only properties we need to prove Lemma 4.5 and the first part of Proposition 4.7 are the facts that Res is a continuous linear trace on $\Psi^0(M,\mathcal{E})$ and that it vanishes on $1 + \Psi^{-\infty}(M,\mathcal{E})$.

## Appendix A. Spectral asymmetry and Cayley-Hamilton decomposition

In this appendix we give a new proof of the formulas (1.4)–(1.7) of Wodzicki ([Wo3], [Wo4]) by making use of the spectral representation of the projector $\Pi_{\theta,\theta'}(P)$ given in [Po1] and of the Kontsevitch-Vishik canonical trace. Here we let $P : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ be an elliptic $\Psi$DO of order $m>0$ and assume that $L_\theta=\{\arg\lambda=\theta\}$, $0\leq\theta<2\pi$, and $L_{\theta'}=\{\arg\lambda=\theta\}$, $\theta<\theta'<2\pi$, are spectral cuts for both $P$ and its principal symbol $p_m(x,\xi)$.

Let us first suppose that $P$ is selfadjoint. Then we have

$$P_\uparrow^s=\Pi_+(P)|P|^s+e^{-i\pi s}\Pi_-(P)|P|^s \quad\text{and}\quad P_\downarrow^s=\Pi_+(P)|P|^s+e^{i\pi s}\Pi_-(P)|P|^s, \tag{A.1}$$

where $\Pi_+(P)$ and $\Pi_-(P)$ denote the orthogonal projectors onto the positive and negative eigenspaces of $P$. Hence

$$P_\uparrow^s-P_\downarrow^s=(e^{-i\pi s}-e^{i\pi s})\Pi_-(P)|P|^s=(1-e^{2i\pi s})\Pi_-(P)P_\uparrow^s. \tag{A.2}$$

Therefore, in the selfadjoint case, the spectral asymmetry of $P$ is encoded by the projector $\Pi_-(P)$.

Now, let us deal with the general case. Then Formula (1.4) tells us that for any $s\in\mathbb{C}$ we have

$$P_\theta^s-P_{\theta'}^s=(1-e^{2i\pi s})\Pi_{\theta,\theta'}(P)P_\theta^s, \tag{A.3}$$

where $\Pi_{\theta,\theta'}(P)$ is the operator given by

$$\Pi_{\theta,\theta'}(T)=\frac{1}{2i\pi}\int_{\Gamma_{\theta,\theta',r}}\mu^{-1}T(T-\mu)^{-1}d\mu, \tag{A.4}$$

$$\Gamma_{\theta,\theta,r}=\{\rho e^{i\theta};\infty>\rho\geq r\}\cup\{re^{it};\theta\leq t\leq\theta'\}\cup\{\rho e^{i\theta'};r\leq\rho<\infty\},$$

with $r$ small enough so that no non-zero eigenvalue of $T$ lies in the disc $|\lambda|\leq r$. In fact, $\Pi_{\theta,\theta'}(P)$ is a $\Psi$DO projector which either has order zero or is smoothing (cf. [Wo4], [Po1]).

In the selfadjoint case the projector $\Pi_{\theta,\theta'}(P)$ coincides with $\Pi_-(P)$ when $0<\theta<\pi<\theta'<2\pi$, so that it has a clear spectral interpretation as the orthogonal projector onto the negative eigenspace of $P$. As shown in [Po1] in the non-selfadjoint case as well we have a spectral interpretation of $\Pi_{\theta,\theta'}(P)$ in terms of Cayley-Hamilton of $P$, which was briefly reviewed in Section 2. The point is that we can associate to the "infinity" of the open angular sector $\Lambda_{\theta,\theta'}=\{\theta<\arg\lambda<\theta'\}$ a characteristic subspace $E_{\theta,\theta',\infty}\subset E_\infty(P)$ which is the range of a characteristic projector $\Pi_{\theta,\theta',\infty}(P)$ by letting

$$\Pi_{\theta,\theta',\infty}(P)=\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_{\theta,\theta',R}}\mu^{-1}P(P-\mu)^{-1}d\mu \quad\text{and}\quad E_{\theta,\theta',\infty}(P)=\operatorname{im}\Pi_{\theta,\theta',\infty}(P), \tag{A.5}$$

$$\Gamma_{\theta,\theta,r}=\{\rho e^{i\theta};\infty>\rho\geq r\}\cup\{re^{it};\theta\leq t\leq\theta'\}\cup\{\rho e^{i\theta'};r\leq\rho\leq\infty\},$$

where the limit exists with respect to the strong topology of $\mathcal{L}(L^2(M,\mathcal{E}))$. Then we get the following spectral interpretation of $\Pi_{\theta,\theta'}(P)$.

**Proposition A.1** ([Po1]). *1) The family $\{\Pi_{\theta,\theta',\infty}(P)\}\cup\{\Pi_\lambda(P)\}_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}$ is an orthogonal family of projectors and with respect to the strong topology of $\mathcal{L}(L^2(M,\mathcal{E}))$ we have*

$$\Pi_{\theta,\theta'}(P)=\Pi_{\theta,\theta',\infty}(P)+\sum_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(P). \tag{A.6}$$

*2) The projector $\Pi_{\theta,\theta'}(P)$ projects onto $E_{\theta,\theta',\infty}(P)\dotplus\overline{(\dotplus_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}E_\lambda(P))}$ and along the subspace $E_0(P)\dotplus E_{\theta',\theta+2\pi,\infty}(P)\dotplus\overline{(\dotplus_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta',\theta+2\pi}}E_\lambda(P))}$.*

We shall now make use of (A.6) and of the Cayley-Hamilton decomposition (2.7) for proving Formula (A.4).

*Proof of Formula (A.4).* Let us assume $\Re s<0$. Since by the Cayley-Hamilton decomposition (2.7) we have $\sum_{\lambda\in\operatorname{Sp}\cup\{\infty\}}\Pi_\lambda(P)=1$ we have

$$P_\theta^s-P_{\theta'}^s=\sum_{\lambda\in\operatorname{Sp}P\cup\{\infty\}}\Pi_\lambda(P)(P_\theta^s-P_{\theta'}^s). \tag{A.7}$$

Similarly, thanks to (A.6) we get

$$\Pi_{\theta,\theta'}(P)P_\theta^s = \Pi_{\theta,\theta',\infty}(P)P_\theta^s + \sum_{\lambda\in \operatorname{Sp} P\cap\Lambda_{\theta,\theta'}} \Pi_\lambda(P)P_\theta^s. \tag{A.8}$$

Now, let $\lambda \in \operatorname{Sp} P\setminus 0$. We may suppose that the radius of the circle $\Gamma_{(\lambda)}$ in Formula (2.5) which defines $\Pi_\lambda(P)$ is small enough so that $\Gamma_{(\lambda)}$ isolates $\lambda$ from the origin and the rest of the spectrum of $P$ and $\Gamma_{(\lambda)}$ does not intersect the rays $L_\theta$ and $L_{\theta'}$. Then using the identity $(P-\mu)^{-1}(P-\nu)^{-1} = (\mu-\nu)^{-1}[(P-\mu)^{-1} - (P-\nu)^{-1}]$ we get:

(A.9)

$$\begin{aligned}\Pi_\lambda(P)P_\theta^s &= -\frac{1}{4\pi}\int_{\Gamma_{(\lambda)}}\int_{\Gamma_\theta} \mu^{-1}\nu_\theta^s P(P-\mu)^{-1}(P-\nu)^{-1}d\mu d\nu,\\ &= \frac{1}{4\pi}\int_{\Gamma_{(\lambda)}} \mu^{-1}P(P-\mu)^{-1}(\int_{\Gamma_\theta}\frac{\nu_\theta^s d\nu}{\nu-\mu})d\mu + \frac{1}{4\pi}\int_{\Gamma_\theta}\nu_\theta^s P(P-\nu)^{-1}(\int_{\Gamma(\lambda)}\frac{\mu^{-1}d\mu}{\mu-\nu})d\nu,\\ &= \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}}\mu_\theta^{s-1}P(P-\mu)^{-1}d\mu,\end{aligned}$$

where the powers $\nu_\theta^s$ and $\mu_\theta^{s-1}$ are defined relatively to the continuous determination of the argument on $\mathbb{C}\setminus L_\theta$ with values in $(\theta,\theta-2\pi)$, e.g. $\mu_\theta^{s-1} = e^{i(s-1)\arg_\theta \mu}|\mu|^{s-1})$. Similarly, we have

$$\Pi_\lambda(P)P_{\theta'}^s = \frac{-1}{2i\pi}\int_{\Gamma_{(\lambda)}}\mu_{\theta'}^{s-1}P(P-\mu)^{-1}d\mu, \tag{A.10}$$

where $\mu_{\theta'}^{s-1}$ is defined by means of the continuous determinations of the argument on $\mathbb{C}\setminus L_{\theta'}$ that take values in $(\theta',\theta'-2\pi)$.

Notice that since $0\leq\theta<\theta'<2\pi$ we see that if $\theta'-2\pi<\arg_\theta\mu<\theta$ then $\arg_{\theta'}\mu = \arg_\theta\mu$ and $\mu_{\theta'}^s = \mu_\theta^s$, whereas if $\theta-2\pi<\arg_\theta\mu<\theta'-2\pi$ then $\arg_{\theta'}\mu = \arg_\theta\mu + 2\pi$ and $\mu_{\theta'}^s = e^{2i\pi s}\mu_\theta^s$. Thus,

$$\Pi_\lambda(P)P_{\theta'}^s = \begin{cases}\Pi_\lambda(P)P_{\theta'}^s = \Pi_\lambda(P)P_\theta^s & \text{if } \theta'-2\pi<\arg_\theta\mu<\theta,\\ \Pi_\lambda(P)P_{\theta'}^s = e^{2i\pi s}\Pi_\lambda(P)P_\theta^s & \text{if } \theta-2\pi<\arg_\theta\mu<\theta'-2\pi.\end{cases} \tag{A.11}$$

On the other hand, we have

$$\Pi_\infty(P)P_\theta^s = \lim_{R\to\infty}\int_{|\mu|=R}\int_{\Gamma_\theta}\mu^{-1}\nu^s P(P-\mu)^{-1}(P-\nu)^{-1}d\mu d\nu. \tag{A.12}$$

In fact, when $R>r$ arguing as in (A.9) enables us to get

$$\int_{|\mu|=R}\int_{\Gamma_\theta}\mu^{-1}\nu^s P(P-\mu)^{-1}(P-\nu)^{-1}d\mu d\nu = \frac{-1}{2i\pi}\int_{|\mu|=R}\mu_{\theta'}^{s-1}P(P-\mu)^{-1}d\mu. \tag{A.13}$$

Since we have $\Re s<0$ the r.h.s. above converges to zero as $R\to\infty$. Hence $\Pi_\infty(P)P_\theta^s = 0$. In the same way we get $\Pi_\infty(P)P_{\theta'}^s = \Pi_{\theta,\theta',\infty}(P)P_\theta^s = 0$.

Moreover, using the group property of the complex powers we obtain

$$\Pi_0(P)P_\theta^s = \Pi_0(P)P_\theta^0 P_\theta^s = \Pi_0(P)(1-\Pi_0(P))P_\theta^s = 0. \tag{A.14}$$

All this shows that in the summations in (A.7) and (A.8) only the eigenvalues contained in $\Lambda_{\theta,\theta'}$ give a non-zero contribution. Therefore, using (A.11) we get

$$P_\theta^s - P_{\theta'}^s = \sum_{\lambda\in\operatorname{Sp} P\cup\Lambda_{\theta,\theta'}}(1-e^{2i\pi s})\Pi_\lambda(P)P_\theta^s = (1-e^{2i\pi s})\Pi_{\theta,\theta'}(P)P_\theta^s. \tag{A.15}$$

This proves the equality (A.4) for $\Re s<0$. Since both side of (A.4) are holomorphic families of $\Psi$DO's we get the equality on the whole complex plane by analytic continuation. $\square$

Next, since Formula (A.4) yields an equality between holomorphic families of ΨDO's, composing it with the Kontsevich-Vishik canonical trace allows us to recover Theorem 1.2 in the form below.

**Theorem A.2.** *We have the following equality of meromorphic functions*

$$\zeta_\theta(P;s) - \zeta_{\theta'}(P;s) = (1-e^{-2i\pi s})\operatorname{TR}\Pi_{\theta,\theta'}(P)P_\theta^{-s}, \quad s\in\mathbb{C}. \tag{A.16}$$

*In particular, at any integer $k\in\mathbb{Z}$ the function $\zeta_\theta(P;s)-\zeta_{\theta'}(P;s)$ is regular, and there we have*

$$\operatorname{ord}P.\lim_{s\to k}(\zeta_\theta(P;s)-\zeta_{\theta'}(P;s)) = 2i\pi\operatorname{Res}\Pi_{\theta,\theta'}(P)P^{-k}. \tag{A.17}$$

**Corollary A.3.** *Let $k$ be an integer and assume that* $\operatorname{Res}P^{-k}=0$, *so that $\zeta_\theta(P;s)$ and $\zeta_{\theta'}(P;s)$ are regular at $s=k$. Then we have:*

$$\zeta_\theta(P;k)=\zeta_{\theta'}(P;k) \Longleftrightarrow \operatorname{Res}\Pi_{\theta,\theta'}(P)P^{-k}=0. \tag{A.18}$$

Moreover, as a consequence of the spectral interpretation of $\Pi_{\theta,\theta'}(P)$ and of Theorem A.2 we will obtain:

**Proposition A.4.** *Assume that the principal symbol of $P$ has no eigenvalue within the angular sector $\Lambda_{\theta,\theta'}$. Then for any $s\in\mathbb{C}$ we have*

$$\zeta_\theta(P;s)-\zeta_{\theta'}(P;s) = (1-e^{-2i\pi s})\sum_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}\lambda_\theta^{-s}\dim E_\lambda(P), \tag{A.19}$$

*where the sum is actually finite. Thus $\zeta_\theta(P;s)-\zeta_{\theta'}(P;s)$ is an entire function vanishing at every integer point and, in particular, the regular values $\zeta_\theta(P;0)$ and $\zeta_{\theta'}(P;0)$ coincide.*

*Proof.* First, by a result in [Po1] if the principal symbol of $P$ has no eigenvalue in $\Lambda_{\theta,\theta'}$ then $\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}$ is finite and $\Pi_{\theta,\theta',\infty}(P)=0$. Therefore, using (A.6) we get

$$\Pi_{\theta,\theta'}(P)P_\theta^s = \sum_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}\Pi_\lambda(P)P_\theta^s, \quad s\in\mathbb{C}. \tag{A.20}$$

Since each operator $\Pi_\lambda(P)P_\theta^s$ is smoothing, hence trace-class, combining this with (A.4) shows that for $\Re s<0$ we have

$$\begin{aligned}\zeta_\theta(P;s)-\zeta_{\theta'}(P;s) &= (1-e^{-2i\pi s})\sum_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}\operatorname{Trace}\Pi_\lambda(P)P_\theta^s,\\ &= (1-e^{-2i\pi s})\sum_{\lambda\in\operatorname{Sp}P\cap\Lambda_{\theta,\theta'}}\int_{\Gamma_\theta}\mu^s\operatorname{Trace}[\Pi_\lambda(P)(P-\mu)^{-1}]d\mu.\end{aligned} \tag{A.21}$$

*Claim.* For any $\lambda\in\operatorname{Sp}P$ and any $\mu\in\mathbb{C}\setminus\operatorname{Sp}P$ we have

$$\operatorname{Trace}[\Pi_\lambda(P)(P-\mu)^{-1}] = (\lambda-\mu)^{-1}\dim E_\lambda(P). \tag{A.22}$$

*Proof of the claim.* Without any loss of generality we may assume $\lambda=0$. Recall that, as alluded to in Section 3, there exists an integer $N\geq 1$ such that $E_0(P)=\ker P^N$. Since $\Pi_0(P)$ commutes with $P$ it follows that $(\Pi_0(P)P)^N = P^N\Pi_0(P)=0$. Thus $\Pi_0(P)P$ is a nilpotent operator with finite rank. Thus,

$$\Pi_0(P)[\sum_{j=0}^N\mu^{-(j+1)}P^j](P-\mu) = \Pi_0(P)(P^N-1) = -\Pi_0(P). \tag{A.23}$$

Composing this equality with $(P-\mu)^{-1}$ yields $\Pi_\lambda(P)(P-\mu)^{-1} = -\sum_{j=0}^N\mu^{-(j+1)}\Pi_0(P)P^j$. Since each operator $\Pi_0(P)P^j$ is nilpotent for $j\geq 1$, we get

$$\operatorname{Trace}[\Pi_\lambda(P)(P-\mu)^{-1}] = -\sum_{j=0}^N\mu^{-(j+1)}\Pi_0(P)P^j = -\mu\operatorname{Trace}\Pi_0(P) = -\mu\dim E_0(P). \tag{A.24}$$

Hence the claim. □

Combining the claim with (A.21) shows that $\zeta_\theta(P;s) - \zeta_{\theta'}(P;s)$ is equal to
(A.25)
$$(1-e^{-2i\pi s}) \sum_{\lambda\in \operatorname{Sp} P\cap\Lambda_{\theta,\theta'}} \int_{\Gamma_\theta} \mu^s(\mu-\lambda)^{-1}\dim E_\lambda(P)d\mu = (1-e^{-2i\pi s}) \sum_{\lambda\in \operatorname{Sp} P\cap\Lambda_{\theta,\theta'}} \lambda_\theta^s \dim E_\lambda(P).$$

This proves (A.19) for $\Re s<0$. The general equality follows by analytic continuation. □

Finally, let us look at Theorem A.2 and Proposition A.4 when $P$ is selfadjoint. To this end let $F=\Pi_+(P)-\Pi_-(P)$ be the sign operator of $P$. Then using (A.1) we obtain:

$$P_\uparrow^s - F|P|^s = (1+e^{-i\pi s})\Pi_-(P)|P|^s. \tag{A.26}$$

Combining this with (A.2) and using the fact that $(1-e^{i\pi s})(e^{-i\pi s}+1)=e^{-i\pi s}-e^{i\pi s}$ we get

$$P_\uparrow^s - P_\downarrow^s = (e^{-i\pi s}-e^{i\pi s})(1+e^{-i\pi s})^{-1}(P_\uparrow^s - F|P|^s) = (1-e^{i\pi s})(P_\uparrow^s - F|P|^s). \tag{A.27}$$

As the eta function of $P$ is $\eta(P;s)=\operatorname{TR} F|P|^{-s}$ we get the following (see also [Wo1, Prop. 6]).

**Proposition A.5.** *1) We have:*

$$\zeta_\uparrow(P;s)-\zeta_\downarrow(P;s) = (1-e^{-i\pi s})\zeta_\uparrow(P;s) - (1-e^{-i\pi s})\eta(P;s), \tag{A.28}$$

$$\operatorname{ord} P.\lim_{s\to k}(\zeta_\uparrow(P;s)-\zeta_\downarrow(P;s)) = i\pi\operatorname{Res} P^{-k} - i\pi \operatorname{ord} P.\operatorname{res}_{s=k}\eta(P;s), \qquad k\in\mathbb{Z}. \tag{A.29}$$

*2) Let $k\in\mathbb{Z}$ and suppose that $\operatorname{Res} P^{-k}=0$, so that $\zeta_\uparrow(s)$ and $\zeta_\downarrow(s)$ are both regular at $s=k$. Then we have:*

$$\zeta_\uparrow(P;k)=\zeta_\downarrow(P;k) \Longleftrightarrow \eta(P;s) \text{ is regular at } s=k. \tag{A.30}$$

*Remark* A.6. The equality (A.28) was first observed by Shubin [Sh, p. 114].

*Remark* A.7. From (A.30) we see that once $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ are regular at $s=k$ the non-regularity of the eta function at $s=k$ is the only obstruction to their non-asymmetry at $s=k$. However, when $\zeta_\uparrow(P;s)$ and $\zeta_\downarrow(P;s)$ are regular at $s=k$ there are examples for which $\eta(P;s)$ is regular at some integer, although $\lim_{s\to k}(\zeta_\uparrow(P;s)-\zeta_\downarrow(P;s))$ is non-zero (*cf.* Proposition 3.3).

On the other hand, since $P$ is selfadjoint the characteristic space $E_\lambda(P)$, $\lambda\in\operatorname{Sp}P$, coincides with $\ker(P-\lambda)$ (e.g. [Po1]). Moreover, the condition on the principal symbol in Proposition A.4 means that it has to be positive and we have $\lambda_\uparrow^{-s}=e^{i\pi s}|\lambda|^s$. Therefore, in the selfadjoint case Proposition A.4 becomes:

**Proposition A.8.** *Suppose that $P$ is selfadjoint and has a positive principal symbol. Then for any $s\in\mathbb{C}$ we have*

$$\zeta_\downarrow(P;s)-\zeta_\uparrow(P;s) = 2i\sin(\pi s)\sum_{\lambda<0}|\lambda|^{-s}\dim\ker(P-\lambda). \tag{A.31}$$

## Appendix B. Fréchet algebra structure for ΨDO's.

In this second appendix we briefly recall how the ΨDO's can be endowed with a natural Fréchet space structure.

First, let $m\in\mathbb{C}$ and let $V$ be an open subset of $\mathbb{R}^n$. As the homogeneous components $p_{-j}$'s in the asymptotic expansion $p(x,\xi)\sim\sum_{j\geq 0}p_{-j}(x,\xi)$ for a symbol $p\in S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ are uniquely determined by $p$, we turn $S^m(V\times\mathbb{R}^n, \operatorname{End}\mathbb{C}^r)$ into a Fréchet space by means of the seminorms,

$$\sup_{x\in K}\sup_{\xi\in\mathbb{R}^n}|(1+|\xi|)^{-\Re m+|\beta|}\partial_x^\alpha\partial_\xi^\beta p(x,\xi)|, \qquad \sup_{x\in K}\sup_{|\xi|=1}|\partial_x^\alpha\partial_\xi^\beta p_{-j}(x,\xi)|, \tag{B.1}$$

$$\sup_{x\in K}\sup_{|\xi|\geq 1}|\xi|^{-\Re m+N+|\beta|}|\partial_x^\alpha\partial_\xi^\beta(p-\sum_{j<N}p_{-j})(x,\xi)|. \tag{B.2}$$

where $N$ ranges over integers, $K$ over compacts subset of $U$ and $\alpha$ and $\beta$ over multiorders. In this setting the quantization map $p\rightarrow p(x,D)$ is continuous from $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ to the Fréchet space $\mathcal{L}(C_c^\infty(V,\mathbb{C}^r),C^\infty(V,\mathbb{C}^r))$ and for any $p$ in $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ the distribution kernel $k_p(x,y)$ of $p(x,D)$ is represented outside the diagonal by a smooth function depending continuously on $p$ (see [Hö, Thms. 18.1.6 and 18.1.16]).

Now, we cannot immediately turn $\Psi^m(M,\mathcal{E})$ into a topological space by means of the above topology, because in (2.3) the pair $(p^{\kappa,\tau},R^{\kappa,\tau})$ is not unique. To remedy this we need:

**Lemma B.1** ([Hö, Prop. 18.1.19]). *Let $P\in\mathcal{L}(C^\infty(M,\mathcal{E}))$ have a distribution kernel which is smooth off the diagonal. Then the following are equivalent:*

*(i) $P$ is a $\Psi DO$ of order $m$;*

*(ii) For any trivialization $\tau:\mathcal{E}_{|_U}\rightarrow U\times\mathbb{C}^r$ over a local chart $\kappa:U\rightarrow V\subset\mathbb{R}^n$ and for any $\varphi$ and $\psi$ in $C_c^\infty(U)$ we have $\kappa_*\tau_*(\varphi P\psi)=\sigma^{\kappa,\tau}(\varphi P\psi)(x,D)$ with $\sigma^{\kappa,\tau}(\varphi P\psi)$ in $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$.*

In the sequel we let $\mathcal{K}^\infty(M\times M\setminus\Delta,\mathcal{E})$ denote the Fréchet space of the smooth sections of the vector bundle $\mathcal{E}\boxtimes(\mathcal{E}^*\otimes|\Lambda|(M))$ over the complement $M\times M\setminus\Delta$ of the diagonal $\Delta$ in $M\times M$.

**Definition B.2.** *The topology of $\Psi^m(M,\mathcal{E})$ is the weakest topology such that:*

*- The map $P\rightarrow k_P(x,y)_{|_{M\times M\setminus\Delta}}$ is continuous from $\Psi^m(M,\mathcal{E})$ to $\mathcal{K}^\infty(M\times M\setminus\Delta,\mathcal{E})$;*

*- For any quadruple $(\kappa,\tau,\varphi,\psi)$ as in Proposition B.1 (ii) the map $P\rightarrow\sigma^{\kappa,\tau}(\varphi P\psi)$ is continuous from $\Psi^m(M,\mathcal{E})$ to $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$.*

Let $(\varphi_i)_{1\leq i\leq N}\subset C^\infty(M)$ be a (finite) partition of the unity on $M$ which is subordinated to an open covering $(U_i)$ by domains of local charts $\kappa_i:U_i\rightarrow V_i$ under a trivialization $\tau_i:\mathcal{E}_{|_{U_i}}\rightarrow U_i\times\mathbb{C}^r$, and for each index $i$ let $\psi_i$ and $\chi_i$ in $C_c^\infty(U_i)$ be such that $\psi_i=1$ near $\operatorname{supp}\varphi_i$ and $\chi_i=1$ near $\operatorname{supp}\psi_i$.

**Lemma B.3.** *Let $\tau:\mathcal{E}_{|_U}\rightarrow U\times\mathbb{C}^r$ be a trivialization over a local chart $\kappa:U\rightarrow V$ and let $\varphi$ and $\psi$ be in $C_c^\infty(U)$. Then:*

*1) The map $p\rightarrow\varphi\tau^*\kappa^*(p(x,D))\psi$ is continuous from $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ to $\Psi^m(M,\mathcal{E})$.*

*2) There exist continuous linear maps $\pi^i$ from $S^m(V_i\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ to $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ and $r$ from $\mathcal{K}^\infty(M\times M\setminus\Delta,\mathcal{E})$ to $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ such that, for any $P\in\Psi^m(M,\mathcal{E})$, we have*

$$\sigma^{\kappa,\tau}(\varphi P\psi)=\sum_{i=1}^N\pi^i[\sigma^{\kappa_i,\tau_i}(\chi_iP\chi_i)]+r(k_{P|_{M\times M\setminus\Delta}}). \tag{B.3}$$

*Proof.* 1) First, recall that by [Hö, Thm. 18.1.7] if $\phi:V\rightarrow V'$ is a diffeomorphism from $V$ onto another open subset $V'$ of $\mathbb{R}^n$ and if $\chi$ and $\rho$ are in $C_c^\infty(V)$, then for any $p$ in $S^m(V\times\mathbb{R}^n)$ there exists a unique symbol $p^\phi_{\chi,\rho}\in S^m(V'\times\mathbb{R}^n)$ such that

$$\phi_*(\chi p(x,D)\rho)=p^\phi_{\chi,\rho}(x',D_{x'}). \tag{B.4}$$

We claim that the map $p\rightarrow p^\phi_{\chi,\rho}$ is continuous. To see this let $(p_k)_{k\geq 0}\subset S^m(V\times\mathbb{R}^n)$ be a sequence such that $p_k\rightarrow p$ in $S^m(V\times\mathbb{R}^n)$ and $p^\phi_{k,\chi,\rho}\rightarrow q$ in $S^m(V'\times\mathbb{R}^n)$. Since the quantization map $p\rightarrow p(x,D)$ is continuous from $S^m(V\times\mathbb{R}^n)$ to $\mathcal{L}(C_c^\infty(V),C^\infty(V))$ we have

$$\phi_*(\varphi p(x,D)\psi)=\lim\phi_*(\varphi p_k(x,D)\psi)=\lim p^\phi_{k,\chi,\rho}(x',D_{x'})=q(x',D_{x'}). \tag{B.5}$$

Thus $q$ and $p^{\phi}_{\varphi\psi}$ coincide by injectivity of the above quantization map. As $S^m(V\times\mathbb{R}^n)$ is a Fréchet space it then follows from the closed graph theorem that the linear map $p\rightarrow p^{\kappa}_{\varphi\psi}$ is continuous from $S^m(V\times\mathbb{R}^n)$ to $S^m(V'\times\mathbb{R}^n)$.

Now, let $\tilde{\tau}:\mathcal{E}_{|_U}\rightarrow U\times\mathbb{C}^r$ be a trivialization over a local chart $\tilde{\kappa}:\tilde{U}\rightarrow\tilde{V}$ and let $\tilde{\varphi}$ and $\tilde{\psi}$ be in $C^\infty_c(\tilde{U})$. Thus there is a transition map $T\in C^\infty(\kappa(U\cap\tilde{U}),GL(\mathbb{C}^r))$ such that

$$[(\tilde{\kappa}\otimes 1_{\mathbb{C}^r})\circ\tilde{\tau}]\circ[(\kappa\otimes 1_{\mathbb{C}^r})\circ\tau]^{-1}(x,\xi)=(\tilde{\kappa}\circ\kappa^{-1}(x),T(x)\xi),\quad (x,\xi)\in\kappa(U\cap\tilde{U})\times\mathbb{C}^r. \tag{B.6}$$

Let $p\in S^m(\tilde{V}\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$. Then we have

$$\tilde{\kappa}_*\tilde{\tau}^*[\tilde{\varphi}\tau^*\kappa^*(\varphi p(x,D)\psi)\tilde{\psi}]=(\tilde{\kappa}\circ\kappa^{-1})_*[\chi Tp(x,D)T^{-1}\eta], \tag{B.7}$$

where we have let $\chi=(\tilde{\varphi}\circ\kappa^{-1})(\varphi\circ\kappa^{-1})$ and $\eta=(\psi\circ\kappa^{-1})(\tilde{\psi}\circ\kappa^{-1})$. Therefore, thanks to the continuity of the maps (B.4) there exists a continuous linear map $\pi^{\kappa,\tau,\varphi,\psi}_{\tilde{\kappa},\tilde{\tau},\tilde{\varphi},\tilde{\psi}}$ from $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ to $S^m(\tilde{V}\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ such that

$$\sigma^{\tilde{\kappa},\tilde{\tau}}[\tilde{\varphi}\tau^*\kappa^*(\varphi p(x,D)\psi)\tilde{\psi}]=\pi^{\kappa,\tau,\varphi,\psi}_{\tilde{\kappa},\tilde{\tau},\tilde{\varphi},\tilde{\psi}}(p). \tag{B.8}$$

Since we already know that the distribution kernel of $p(x,D)$ is represented outside the diagonal by a smooth function depending continuously on $p$, this shows that the map $p\rightarrow\varphi\kappa^*(p(x,D))\psi$ is continuous from $S^m(V\times\mathbb{R}^n,\operatorname{End}\mathbb{C}^r)$ to $\Psi^m(M,\mathcal{E})$.

2) Let $P\in\Psi^m(M,\mathcal{E})$. Since $(\varphi_i)_{1\leq i\leq m}$ is a partition of the unity we have

$$\tau_*\kappa_*(\varphi P\psi)=\sum_i\tau_*\kappa_*(\varphi\varphi_iP\psi_i\psi)+\tau_*\kappa_*R, \tag{B.9}$$

where we have let $R=\sum_{i=1}^N\varphi_iP(1-\psi_i)$. Also, as $\chi_i=1$ near $\operatorname{supp}\psi_1$, by Lemma B.1 we have

$$\varphi_iP\psi_i=\varphi_i(\chi_iP\chi_i)\psi_i=\varphi_i\tau_i^*\kappa_i^*[\sigma^{\kappa_i,\tau_i}(\chi_iP\chi_i)](x,D)\psi_i. \tag{B.10}$$

Therefore, using (B.8) and letting $\pi^i=\pi^{\kappa_i,\tau_i,\varphi_i,\psi_i}_{\kappa,\tau,\varphi,\psi}$, we can write

$$\tau_*\kappa_*(\varphi\varphi_iP\psi_i\psi)=\tau_*\kappa_*[\varphi_i\tau_i^*\kappa_i^*[\sigma^{\kappa_i,\tau_i}(\chi_iP\chi_i)(x,D)]\psi_i]=\pi^i[\sigma^{\kappa_i,\tau_i}(\chi_iP\chi_i)](x,D). \tag{B.11}$$

On the other hand, we have $\kappa_*(\varphi R\psi)=r(x,D)$ with $r=\mathcal{F}_{y\rightarrow\xi}[k_{\kappa_*(\varphi R\psi)}(x,x+y)]$, because the Schwartz kernel of $\kappa_*(\varphi R\psi)$ is smooth and has compact support. This defines an element $r=r(k_{P|_{M\times M\setminus\Delta}})$ of $S^{-\infty}(V\times\mathbb{R}^n)$ depending continuously on $k_{P|_{M\times M\setminus\Delta}}$.

All this shows that $\tau_*\kappa_*(\varphi P\psi)$ is equal to $\sigma^{\kappa,\tau}(\varphi P\psi)(x,D)$ with

$$\sigma^{\kappa,\tau}(\varphi P\psi)=\sum_{i=1}^N\pi^i[\sigma^{\kappa_i,\tau_i}(\chi_iP\chi_i)+r(k_{P|_{M\times M\setminus\Delta}}). \tag{B.12}$$

Since $r$ and the $\pi^i$'s are continuous linear maps this proves the second part of the lemma. □

Now, Lemma B.3 implies that the topology of $\Psi^m(M,\mathcal{E})$, which is complete, can actually be defined by means of a countable family of seminorms. Thus $\Psi^m(M,\mathcal{E})$ is a Fréchet space. Furthermore, we have:

**Proposition B.4.** *The composition of $\Psi$DO's is continuous.*

*Proof.* We need to check that for any complex numbers $m_1$ and $m_2$ the composition of $\Psi$DO's induces a continuous bilinear map from $\Psi^{m_1}(M,\mathcal{E})\times\Psi^{m_2}(M,\mathcal{E})$ to $\Psi^{m_1+m_2}(M,\mathcal{E})$. To do this let $(P_k)_{k\geq 0}\subset\Psi^{m_1}(M,\mathcal{E})$ and $(Q_k)_{k\geq 0}\subset\Psi^{m_2}(M,\mathcal{E})$ be sequences such that $P_k\rightarrow P$, $Q_k\rightarrow Q$ and $P_kQ_k\rightarrow R$ in $\Psi^{m_1}(M,\mathcal{E})$, $\Psi^{m_2}(M,\mathcal{E})$ and $\Psi^{m_1+m_2}(M,\mathcal{E})$, respectively. Then in $\mathcal{L}(C^\infty(M,\mathcal{E}))$ we have $R=\lim P_kQ_k=PQ$. Since $\Psi^{m_1}(M,\mathcal{E})$, $\Psi^{m_2}(M,\mathcal{E})$ and $\Psi^{m_1+m_2}(M,\mathcal{E})$ are Fréchet spaces, the closed graph theorem then tells us that the bilinear map $(P,Q)\rightarrow PQ$ is continuous from $\Psi^{m_1}(M,\mathcal{E})\times\Psi^{m_2}(M,\mathcal{E})$ to $\Psi^{m_1+m_2}(M,\mathcal{E})$. Hence the result. □

Finally, as a consequence of Proposition B.4 we get:

**Corollary B.5.** *$\Psi^0(M, \mathcal{E})$ is a Fréchet algebra.*

*Acknowledgements.* I am indebted to Maxim Braverman, Sasha Gorokhovsky, Henri Moscovici, Victor Nistor and Mariusz Wodzicki for useful and stimulating discussions. I am also grateful to Sarah Reznikoff for her fruitful English language comments on a preliminary version of this paper and to Sasha Dynin who provided me with a copy of [Wo3]. I also thank for their warm hospitality the Max Planck Institute for Mathematics (Bonn, Germany) and the IHES (Bures-sur-Yvette, France) where parts of this work were done.

## References


[At] M. F. Atiyah: *K-theory.* 2nd edition, Addison-Wesley, Redwood City, CA, 1989.

[ABP] Atiyah, M.F.; Bott, R.; Patodi, V.K.: *On the heat equation and the index theorem.* Invent. Math. **19** (1973) 279–330. *Errata* in Invent. Math. **28** (1975) 277–280.

[APS1] Atiyah, M.F.; Patodi, V.K.; Singer, I.M.: *Spectral asymmetry and Riemannian geometry. I.* Math. Proc. Camb. Philos. Soc. **77** (1975) 43–69 .

[APS2] Atiyah, M.F.; Patodi, V.K.; Singer, I.M.: Spectral asymmetry and Riemannian geometry. III. Math. Proc. Camb. Philos. Soc. **79** (1976) 71–99 .

[BG] Beals, R.; Greiner, P.C.: *Calculus on Heisenberg manifolds.* Annals of Mathematics Studies, vol. 119. Princeton University Press, Princeton, NJ, 1988.

[BGV] Berline, N; Getzler, E; Vergne, M.: *Heat kernels and Dirac operators.* Springer-Verlag, Berlin, 1992.

[Bl] Blackadar, B.: *K-theory for operator algebras.* Second edition. MSRI Publications, 5. Cambridge University Press, 1998.

[Bi] Bismut, J.-M.: *Local index theory, eta invariants and holomorphic torsion: a survey.* Surveys in differential geometry, Vol. III (Cambridge, MA, 1996), 1–76, Int. Press, Boston, MA, 1998.

[BF] Bismut, J.-M.; Freed, D.S.: *The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem.* Comm. Math. Phys. **107** (1986), no. 1, 103–163.

[BG] Branson, T; Gilkey, P.B.: *Residues of the eta function for an operator of Dirac type.* J. Funct. Anal. **108** (1992), no. 1, 47–87.

[BS] Brüning, J.; Seeley, R.T.: *An index theorem for first order regular singular operators.* Amer. J. Math. **110** (1988) 659–714.

[CP] Chakraborty, P.S.; Pal, A.: *Equivariant spectral triples on the quantum $SU(2)$ group.* $K$-Theory **28** (2003) 107–126.

[Co2] Connes, A.: *Gravity coupled with matter and the foundation of non-commutative geometry.* Comm. Math. Phys. **182** (1996), no. 1, 155–176.

[Co3] Connes, A.: *Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$.* E-print `arXiv:math.QA/0209142`, 57 pages.

[CM] Connes, A.; Moscovici, H.: *The local index formula in noncommutative geometry.* Geom. Funct. Anal. **5** (1995), no. 2, 174–243.

[Gi1] Gilkey, P.B.: *The residue of the local eta function at the origin.* Math. Ann. **240** (1979), no. 2, 183–189.

[Gi2] Gilkey, P.B.: *The residue of the global $\eta$ function at the origin.* Adv. in Math. **40** (1981), no. 3, 290–307.

[Gi3] Gilkey, P.B.: *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* 2nd edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[GK] Gohberg, I.C.; Kreĭn, M.G.: *Introduction to the theory of linear nonselfadjoint operators.* Trans. of Math. Monographs 18, AMS, Providence, 1969.

[GS] Grubb, G.; Seeley, R.T.: *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems.* Invent. Math. **121** (1995), no. 3, 481–529.

[Gu1] Guillemin, V.W.: *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues.* Adv. in Math. **55** (1985), no. 2, 131–160.

[Gu2] Guillemin, V.W.: *Gauged Lagrangian distributions.* Adv. Math. **102** (1993), no. 2, 184–201.

[Gu3] Guillemin, V.W.: *Residue traces for certain algebras of Fourier integral operators.* J. Funct. Anal. **115** (1993), no. 2, 391–417.

[Hö] Hörmander, L.: *The analysis of linear partial differential operators. III. Pseudo-differential operators.* Grundlehren der Mathematischen Wissenschaften, 274. Springer-Verlag, 1994.

[KW] Kalau, W.; Walze, M.: *Gravity, non-commutative geometry and the Wodzicki residue.* J. Geom. Phys. **16** (1995), no. 4, 327–344.

[Ka] Kassel, C.: *Le résidu non commutatif (d'aprés M. Wodzicki).* Séminaire Bourbaki, Vol. 1988/89. Astérisque No. 177-178, (1989), Exp. No. 708, 199–229.

[Kas] Kastler, D.: *The Dirac operator and gravitation.* Comm. Math. Phys. **166** (1995), no. 3, 633–643.

[KV] Kontsevich, M.; Vishik, S.: *Geometry of determinants of elliptic operators. Functional analysis on the eve of the 21st century*, Vol. 1 (New Brunswick, NJ, 1993), 173–197, Progr. Math., 131, Birkhäuser, 1995.

[LM] Lawson, B.; Michelson, M.L.: *Spin Geometry.* Princeton Univ. Press, 1993.

[Me] Melrose, R.: *The Atiyah-Patodi-Singer theorem.* A.K. Peters, 1993.

[MN] Melrose, R.; Nistor, V.: *Homology of pseudodifferential operators I. Manifolds with boundary.* E-print `arXiv:funct-an/9606005`, 47 pages. To appear in Amer. J. Math..

[Mü] Müller, W.: *The eta invariant (some recent developments).* Séminaire Bourbaki, Vol. 1993/94. Astérisque No. 227 (1995), Exp. No. 787, 5, 335–364.

[Po1] Ponge, R.: *Cayley-Hamilton decomposition and spectral asymmetry.* E-print `arXiv:math.SP/0309422`, 23 pages.

[Po2] Ponge, R.: *Noncommutative residue, zeta functions and CR geometry.* E-print ArXiv, Jan. 2004.

[Po3] Ponge, R.: *On the Bruening-Seeley index formula on manifold with conical singularities.* In preparation.

[RN] Riesz, F.; Sz.-Nagy, B.: *Leçons d'analyse fonctionnelle.* Akadémiai Kiadó, Budapest, 1952.

[Se] Seeley, R.T.: *Complex powers of an elliptic operator*, *Singular integrals* (Proc. Sympos. Pure Math., Vol. X, Chicago, Ill., 1966), pp. 288–307. Amer. Math. Soc., Providence, R.I., 1967.

[Sh] Shubin, M.: *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics. Springer-Verlag, 1987.

[Wo1] Wodzicki, M.: *Spectral asymmetry and zeta functions*, Invent. Math. **66** (1982) 115–135. *Corrigenda* in [Wo2, 1.25].

[Wo2] Wodzicki, M.: *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984) 143–177.

[Wo3] Wodzicki, M.: Commentary, in *Hermann Weyl's selected papers* (in Russian), edited by V.I. Arnold and A.N. Parshin. Nauka, Moscow, 1985.

[Wo4] Wodzicki, M.: *Spectral asymmetry and noncommutative residue* (in Russian), Thesis, Steklov Institute, (former) Soviet Academy of Sciences, Moscow, 1984.

[Wo5] Wodzicki, M.: *Noncommutative residue. I. Fundamentals. K-theory, arithmetic and geometry* (Moscow, 1984–1986), 320–399, Lecture Notes in Math., 1289, Springer, 1987.

Department of Mathematics, Ohio State University, Columbus, USA.
*E-mail address*: `ponge@math.ohio-state.edu`