Lusternik-Schnirelman theory and dynamics, II

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Abstract. We show how methods of the homotopy theory can be used in dynamics to study topology of the chain recurrent set. More specifically, we introduce new homotopy invariants $\mathrm{cat}^1(X, \xi)$ and $\mathrm{cat}^1_s(X, \xi)$ depending on a finite polyhedron X and a real cohomology class $\xi \in H^1(X; \mathbf{R})$ which are modifications of the invariants introduced in [5], [6]. We prove that under certain conditions $\mathrm{cat}^1_s(X, \xi)$ provides a lower bound for the Lusternik - Schnirelman category of the chain recurrent set R_ξ of the flow. The approach of the present paper compared with the one of [5, 6] applies to a wider class of flows; in particular it allows to avoid certain difficult to check assumptions which appear in [6].

Dedicated to the memory of L.V. Keldysh

1. Introduction

The notion of a Lyapunov function is well-known in dynamics. Existence of Lyapunov functions was established by C. Conley in [1], [2]. Conley's theorem states that for any smooth flow on a closed smooth manifold M there exists a smooth function $L : M \to \mathbf{R}$ which has the chain recurrent set R of the flow as the set of its critical points and such that L decreases along the orbits lying in $M - R$. Applying this this theorem one obtains that the Lusternik - Schnirelman category of the chain recurrent set R of the flow is bounded below by $\mathrm{cat}(M)$; a slightly stronger statement of this type is given by Theorem 6 below.

A generalization of the Lusternik – Schnirelman theory was constructed in [5], [6] where new homotopy invariants $\mathrm{cat}(X, \xi)$ and $\mathrm{Cat(X}, \xi)$ were introduced. Here X denotes a finite polyhedron X and $\xi \in H^1(X; \mathbf{R})$ a real cohomology class. The numbers $\mathrm{cat}(X, \xi)$ and $\mathrm{Cat(X}, \xi)$ turn into the usual Lusternik - Schnirelman category $\mathrm{cat}(X)$ in the case when $\xi = 0$. The main results of [5], [6] deal with smooth flows admitting a Lyapunov closed 1-form ω lying in a prescribed cohomology class ξ and estimate the number of the fixed points of the flow from below by the numbers $\mathrm{cat}(X, \xi)$ and $\mathrm{Cat(X}, \xi)$. One of the crucial assumptions appearing in [5] is the absence of homoclinic cycles. In [6] one imposes a slightly stronger assumption of the same spirit:

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the set of fixed points must be isolated in the chain recurrent set R of the flow.

The recent papers [8], [9] give necessary and sufficient conditions for the existence of Lyapunov 1-forms of a given flow lying in a prescribed cohomology class $\xi \in H^1(M; \mathbf{R})$. In the smooth case [9] these conditions involve the Schwartzman's asymptotic cycles of the flow which are elements of the vector space $H_1(M; \mathbf{R})$, see section 3.

The results of [5] have been generalized in two directions. Dirk Schütz [16] introduced a symmetric version $\mathrm{cat}_s(X, \xi)$ of the invariant $\mathrm{cat}(X, \xi)$ of [5]. Janko Latschev in a recent preprint [10] generalized the result of [5] about the existence of homoclinic cycles in dynamical systems. Latschev's theorem involves the invariant $\mathrm{cat}(X, \xi)$ and predicts existence of a chain of trajectories resembling a homoclinic cycle.

In this paper we introduce and study the new invariants which we denote $\mathrm{cat}^1(X, \xi)$ and $\mathrm{cat}^1_s(X, \xi)$. They are modifications of the invariants $\mathrm{cat}(X, \xi)$ and $\mathrm{Cat}(\mathrm{X}, \xi)$. Our main result Theorem 5 gives a lower bound for the Lusternik – Schnirelman category of the chain recurrent set R_ξ of a smooth flow on M (where $\xi \in H^1(M; \mathbf{R})$ is a real cohomology class) in terms of $\mathrm{cat}^1_s(M, \xi)$. Since $\mathrm{cat}^1_s(M, \xi) \leq \mathrm{Cat}(\mathrm{M}, \xi)$ this result may seem potentially weaker than the theorem of [6]. However in this paper we consider a wider class of flows and we do not require the existence of a Lyapunov 1-form as an apriori condition. Another important advantage of the present approach is that it allows to avoid assumption $(*)$, see page 108 of [6] which appears in the main Theorems 6.5 and 6.6 of [6]. This condition is difficult to verify in concrete examples.

As an illustration of Theorem 5 we shall state here a simple corollary of our main Theorem 5:

Theorem 1. *Let Φ be a smooth flow on a closed smooth manifold such that the chain recurrent set R of Φ is a union of finitely many circles and isolated points. Let $\xi \in H^1(M; \mathbf{R})$ be a cohomology class such that $\langle \xi, z \rangle \leq 0$ for the homology class $z \in H_1(M)$ of any periodic orbit. Then*

$$p_0 + p_1 + 2p_2 \geq \mathrm{cat}^1_s(M, \xi), \tag{1}$$

where p_0 denotes the number of fixed points of the flow, p_1 denotes the number of periodic orbits which are null-homotopic in M, and p_2 denotes the number of periodic orbits which are homotopically nontrivial but their homology classes $z \in H_1(M)$ satisfy $\langle \xi, z \rangle = 0$.

Our proof of Theorem 5 uses an auxiliary Theorem 9 which guarantees the existence of flow-convex neighborhoods. In our view, this latter result is of independent interest.

2. Notions of category of a space with respect to a cohomology class

Let X be a finite polyhedron and let $\xi \in H^1(X;\mathbf{R})$ be a cohomology class with real coefficients. Let ω be a closed 1-form on X representing ξ; here we refer to the formalism of closed 1-forms on topological spaces suggested in [5]. See also [7].

Definition 1. *A subset $A \subset X$ is N-movable with respect to ω (where $N \in \mathbf{Z}$ is an integer) if there exists a continuous homotopy $h_t : A \to X$, $t \in [0,1]$, such that $h_0 : A \to X$ is the inclusion and for any point $x \in A$ one has*

$$\int_{h_1(x)}^{x} \omega > N, \tag{2}$$

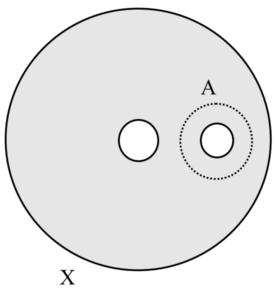
where the integral is calculated along the path $t \mapsto h_{1-t}(x) \in X$, $t \in [0,1]$.

Intuitively, an N-movable subset can be continuously deformed inside X such that each of its points is winding around yielding a large quantity (2) measured in terms of the form ω. We are interested in situations when the closed 1-form ω is fixed and the number N is large.

A one-point subset $A \subset X$ is N-movable for any N, assuming that $\xi \neq 0$ and X is connected. Indeed, as $\xi \neq 0$, there exists a closed loop γ such that $\langle \xi, [\gamma] \rangle < 0$. A homotopy of A satisfying the requirements of Definition 1 can be obtained by first moving A to a point on γ and then traversing the loop γ many times.

The above example can be slightly generalized. The same argument shows that (if $\xi \neq 0$ and the space X is connected) any subset $A \subset X$ such that the inclusion $A \to X$ is null-homotopic is N-movable for any N with respect to any closed 1-form ω representing the class ξ.

In general there are topological obstructions to movability. As an example consider the planar domain $X \subset \mathbf{C}$ described by the inequalities $1 \leq |z| \leq 7$, $|z-4| \geq 1$. The standard angular form $\omega = d\theta$ is a closed 1-form on X. The circle $A \subset X$ given by $|z-4| = 2$ is not N-movable if $N \geq 1$.



Assume now that $\xi = 0$ and X is connected. Then any closed 1-form ω representing ξ is the differential of a continuous function, $\omega = df$ where

$f : X \to \mathbf{R}$. Let N be an integer greater than the total variation of f, i.e. $N > \max f - \min f$. Then inequality (2) cannot be satisfied. We conclude that *in the case* $\xi = 0$ *the empty set is the only* N*-movable with respect to* ω *subset* $A \subset X$ *assuming that the integer* N *is sufficiently large.*

Lemma 2. *If a closed subset* $A \subset X$ *is* N*-movable with respect to* ω *then there exists an open neighborhood* $B \subset X$ *of* A *which is* N*-movable with respect to* ω.

Proof. Since X is a finite polyhedron and hence an ENR[1], one may embed X into an Euclidean space $\mathbf{R}^k$ such that there is a retraction $r : U \to X$ of a small neighborhood $U \subset \mathbf{R}^k$ of X in $\mathbf{R}^k$. Let $H : A \times [0,1] \to X$ be a homotopy as in the Definition 1. By the Tietze extension theorem, H can be extended to a continuous map $G : X \times [0,1] \to \mathbf{R}^k$. The set $G^{-1}(U)$ is an open neighborhood of $A \times [0,1]$ and so it contains a neighborhood of the form $B' \times [0,1]$ where B' is an open neighborhood of A in X. The map $H' = r \circ G : B' \times [0,1] \to X$ is a homotopy extending the initial homotopy of A. Inequality (2) is satisfied for all $x \in A$. Hence it is also satisfied for all x in a small neighborhood $B \subset B'$ of A. □

The following definition of the invariant $\mathrm{cat}(X, \xi)$ is equivalent to the original definition given in [5], see comments below.

Definition 2. *Fix a closed 1-form* ω *in class* ξ. *The number* $\mathrm{cat}(X, \xi)$ *is the minimal integer* k *such that for any integer* N *there exists a closed subset* $A \subset X$ *which is* N*-movable with respect to* ω *and such that* $\mathrm{cat}_X(X - A) \leq k$.

The symbol $\mathrm{cat}_X(X - A)$ denotes the Lusternik - Schnirelman category of $X - A$ in X, i.e. the minimal integer k such that $X - A$ can be covered by k open subsets of X with each of these sets being null-homotopic in X.

If ω' is another closed 1-form in the cohomology class ξ then (as X is compact) there is a constant $C > 0$ such that

$$\left| \int_\gamma \omega - \int_\gamma \omega' \right| < C$$

for any path γ in X. Hence the number $\mathrm{cat}(X, \xi)$ given by Definition 2 is independent of the choice of ω and depends only on the class ξ.

If $\xi = 0$ any N-movable subset must be empty (for ω fixed and for sufficiently large N) and hence for $\xi = 0$ the number $\mathrm{cat}(X, \xi)$ coincides with the classical Lusternik – Schnirelman category $\mathrm{cat}(X)$.

In [5] the number $\mathrm{cat}(X, \xi)$ was defined as the smallest integer k such that for any $N > 0$ there exists an open cover $X = U \cup U_1 \cup U_2 \cup \cdots \cup U_k$ where U is N-movable and each inclusion $U_j \to X$ is null-homotopic, where $1 \leq j \leq k$. Setting $A = X - (\cup_{j=1}^k U_j)$ gives a closed N-movable subset with $\mathrm{cat}_X(X - A) \leq k$. Conversely, given a closed, N-movable subset $A \subset X$ such that $\mathrm{cat}_X(X - A) \leq k$ one finds an open cover $U_1 \cup U_2 \cup \cdots \cup U_k = X - A$

[1] see [3]

such that each $U_j \to X$ is null-homotopic. Completing this cover by an open neighborhood of A given by Lemma 2 one obtains an open cover with the properties required by the definition of [5].

The number $\mathrm{cat}(X, \xi)$ depends only on the homotopy type of the pair (X, ξ), see [5]. One of the important features of $\mathrm{cat}(X, \xi)$ is the cohomological lower bound established in [5]. This material is described also in [7].

The following definition gives a new invariant which we denote $\mathrm{cat}^1(X, \xi)$.

Definition 3. *Fix a closed 1-form ω in the class ξ. The number $\mathrm{cat}^1(X, \xi)$ is the minimal integer k such that there exists a closed subset $A \subset X$ with $\mathrm{cat}_X(X - A) \leq k$ and such that the set A is N-movable with respect to ω for any integer N.*

The distinction between Definitions 2 and 3 is clear: in Definition 2 the subset A may depend on N and in Definition 3 the set A is the same for all integers N.

Definition 3 can be equivalently expressed using the language of open covers of X:

Definition 4. *The number $\mathrm{cat}^1(X, \xi)$ is the minimal integer k with the property that there exist open subsets $U_1, \dots, U_k \subset X$ such that each inclusion $U_j \to X$ is null-homotopic and such that for any $N > 0$ there exists an open N-movable subset $U \subset X$ containing the closed set $A = X - (\cup_{j=1}^k U_j)$.*

The equivalence between Definitions 3 and 4 follows from Lemma 2.

Clearly, one has $\mathrm{cat}(X, \xi) \leq \mathrm{cat}^1(X, \xi)$. We do not know any explicit example when these two numbers are indeed distinct.

In the paper [6] invariant $\mathrm{Cat}(\mathrm{X}, \xi)$ was introduced. Recall that $\mathrm{Cat}(\mathrm{X}, \xi)$ is the minimal integer k such that for any closed 1-form ω representing ξ there is an open cover

$$X = F \cup F_1 \cup F_2 \cup \dots \cup F_k$$

with the following properties:

(a) each inclusion $F_j \to X$ is null-homotopic, where $j = 1, \dots, k$;

(b) there exists a homotopy $h_t : F \to X$ where $t \in (-\infty, \infty)$, such that

$$\lim_{t \to \pm\infty} \int_x^{h_t(x)} \omega = \pm\infty$$

uniformly in $x \in F$; see [6] for more details.

One observes that $\mathrm{cat}^1(X, \xi) \leq \mathrm{Cat}(\mathrm{X}, \xi)$. Summarizing, we obtain that the new invariant $\mathrm{cat}^1(X, \xi)$ satisfies

$$\mathrm{cat}(X, \xi) \leq \mathrm{cat}^1(X, \xi) \leq \mathrm{Cat}(\mathrm{X}, \xi). \tag{3}$$

Lemma 3. *The number $\mathrm{cat}^1(X, \xi)$ is a homotopy invariant of the pair (X, ξ).*

Lemma 3 is a direct consequence of Lemma 4 stated below.

Let (X, ξ) and (Y, η) be two pairs consisting of a finite polyhedron X, respectively Y and a one-dimensional cohomology classes $\xi \in H^1(X; \mathbf{R})$ and $\eta \in H^1(Y; \mathbf{R})$. One says that (X, ξ) *dominates* (Y, η) if there exist continuous maps $f : X \to Y$, $g : Y \to X$ with $f \circ g \simeq 1_Y$ and $\xi = f^*(\eta)$.

Lemma 4. *If* (X, ξ) *dominates* (Y, η) *then* $\operatorname{cat}^1(X, \xi) \geq \operatorname{cat}^1(Y, \eta)$.

Proof. Let ω_Y be a closed 1-form on Y lying in class η. Then $\omega_X = f^*(\omega_Y)$ is a closed 1-form in class ξ.

Let $r_t : Y \to Y$ be a homotopy with $r_0 = 1_Y$ and $r_1 = f \circ g$. Since Y is compact, there is a constant $C > 0$ such that

$$\left| \int_{\gamma_y} \omega_Y \right| < C$$

for any point $y \in Y$ where $\gamma_y : [0, 1] \to Y$ denotes the curve $\gamma_y(t) = r_t(y)$.

Let $A \subset X$ be a closed subset with $\operatorname{cat}_X(X - A) \leq k$ and such that A is N-movable with respect to ω_X for any N. Let $h_t^N : A \to X$, where $t \in [0, 1]$, be a homotopy such that

$$\int_{h_1^N(x)}^{x} \omega_X > N$$

for all $x \in A$.

$B = g^{-1}(A)$ is a closed subset of Y. We show that for any $N > 0$ the set B is N-movable with respect to ω_Y. Define the homotopy $H_t^N : B \to Y$, where $t \in [0, 1]$ by

$$H_t^N(y) = \begin{cases} r_{2t}(y) & \text{for } 0 \leq t \leq 1/2, \\ \\ f(h_{2t-1}^{N'}(g(y))) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

where N' is any integer satisfying $N' > N + C$. Then clearly for any $y \in B$ one has

$$\int_{H_1^N(y)}^{y} \omega_Y > N$$

We are left to show that $\operatorname{cat}_Y(Y - B) \leq k$. Let $U_1, \dots, U_k \subset X$ be open subsets covering $X - A$ and such that $U_j \to X$ is null-homotopic. Set $V_j = g^{-1}(U_j) \subset Y$. The sets $V_1, \dots, V_k$ cover $Y - B = g^{-1}(X - A)$. The inclusion $V_j \to Y$ is homotopic to the composition $V_j \xrightarrow{g} U_j \xrightarrow{\subset} X \xrightarrow{f} Y$ in which the middle map is null-homotopic. This shows that $V_j \to Y$ is homotopically trivial and hence $\operatorname{cat}_Y(Y - B) \leq k$. □

Note that $\operatorname{cat}^1(X, \xi) = \operatorname{cat}(X)$ in the case $\xi = 0$. Indeed, if $\xi = 0$ the subset $A \subset X$ in Definition 3 must be empty.

Another simple remark: if X is connected and $\xi \neq 0$ then

$$\mathrm{cat}^1(X,\xi) < \mathrm{cat}(X). \tag{4}$$

To show this one observes that for any open categorical covering[2] $X = F_1 \cup F_2 \cup \dots \cup F_r$ the closed subset $A = X - \cup_{i=2}^r F_r \subset F_1$ is null-homotopic in X and hence it is N-movable for any N (by the remark after Definition 1). This proves (4).

It may happen that $\mathrm{cat}^1(X,\xi) \neq \mathrm{cat}^1(X,-\xi)$. Indeed, Example 3.4 in [5] provides a situation when $\mathrm{cat}(X,\xi) = 0$ and $\mathrm{cat}(X,-\xi) > 0$. The space X (as a whole) is N-movable for any N with respect to any closed 1-form representing ξ, hence $\mathrm{cat}^1(X,\xi) = 0$. However, $\mathrm{cat}(X,-\xi) > 0$ implies $\mathrm{cat}^1(X,-\xi) > 0$ by (3).

A cohomological lower bound for $\mathrm{cat}^1(X,\xi)$ can be obtained by combining the left inequality (3) and Theorem 6.1 of [5].

A modification of the invariant $\mathrm{cat}(X,\xi)$ was introduced by Dirk Schütz in [16]. It was denoted $\mathrm{cat}_s(X,\xi)$ where s stands for "symmetric": the invariant $\mathrm{cat}_s(X,\xi)$ satisfies

$$\mathrm{cat}_s(X,\xi) = \mathrm{cat}_s(X,-\xi).$$

In this paper we will use a symmetric version of our invariant $\mathrm{cat}^1(X,\xi)$:

Definition 5. *Fix a closed 1-form ω in class ξ. The number $\mathrm{cat}^1_s(X,\xi)$ is the minimal integer k such that there exists a closed subset $A \subset X$ with $\mathrm{cat}_X(X-A) \leq k$ and such that the set A is N-movable with respect to both ω and $-\omega$ for any integer N.*

Clearly one has

$$\mathrm{cat}^1_s(X,\xi) = \mathrm{cat}^1_s(X,-\xi)$$

and

$$\max\{\mathrm{cat}^1(X,\xi), \mathrm{cat}^1(X,-\xi)\} \leq \mathrm{cat}^1_s(X,\xi) \leq \mathrm{Cat}(\mathrm{X},\xi).$$

The number $\mathrm{cat}^1_s(X,\xi)$ is a homotopy invariant of the pair (X,ξ); this fact follows from the arguments of the proof of Lemma 4.

It is obvious that $\mathrm{cat}^1_s(X,\xi) = \mathrm{cat}(X)$ for $\xi = 0$.

The importance of the new invariant $\mathrm{cat}^1_s(X,\xi)$ follows from the Main Theorem 5 of this note. This theorem suggests a new relation between the topology of the chain recurrent set of a flow and the global homotopy invariants of the manifold.

3. Asymptotic cycles of flows

In this section we recall the notion of asymptotic cycle of a flow introduced by S. Schwartzman [14].

[2]This means that each inclusion $F_j \to X$ is null-homotopic

Let V be a smooth vector field on a smooth manifold M and let $\Phi : M \times \mathbf{R} \to M$ be the flow generated by V. Consider a Borel measure μ on M which is invariant under Φ. These data determine a real homology class

$$\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M;\mathbf{R})$$

called *the asymptotic cycle of the flow* Φ *corresponding to the measure* μ. It was introduced by S. Schwartzman [14]. The class $\mathcal{A}_\mu$ is defined as follows. For a de Rham cohomology class $\xi \in H^1(M;\mathbf{R})$ the evaluation $\langle \xi, \mathcal{A}_\mu \rangle \in \mathbf{R}$ is given by the integral

$$\langle \xi, \mathcal{A}_\mu \rangle = \int_M \iota_V(\omega) d\mu, \tag{5}$$

where ω is an arbitrary closed 1-form in the class ξ. Note that $\langle \xi, \mathcal{A}_\mu \rangle$ is well-defined, i.e. it depends only on the cohomology class ξ of ω, see [14], page 277. It is clear that the RHS of (5) is a linear function of $\xi \in H^1(M;\mathbf{R})$. Hence there exists a unique real homology class $\mathcal{A}_\mu \in H_1(M;\mathbf{R})$ which satisfies (5) for all $\xi \in H^1(M;\mathbf{R})$.

4. Chain-recurrent set $R_\xi \subset R$

Let R denote the chain recurrent set of the flow Φ.

In this section we recall the definition of the chain recurrent set R_ξ of a flow $M \times \mathbf{R} \to M$ which was introduced in [8]. The set R_ξ depends on the flow and on a real cohomology class $\xi \in H^1(M;\mathbf{R})$.

First recall the definition of R. Fix a Riemannian metric on M and denote by d the corresponding distance function. Given any $\delta > 0$, $T > 1$, a (δ, T)-*chain from* $x \in M$ *to* $y \in M$ is a finite sequence $x_0 = x, x_1, \dots, x_N = y$ of points in M and numbers $t_1, \dots, t_N \in \mathbf{R}$ such that $t_i \geq T$ and $d(x_{i-1} \cdot t_i, x_i) < \delta$ for all $1 \leq i \leq N$. Here we use the notation $\Phi(x,t) = x \cdot t$. The *chain recurrent set* $R = R(\Phi)$ of the flow Φ is defined as the set of all points $x \in M$ such that for any $\delta > 0$ and $T > 1$ there exists a (δ, T)-chain starting and ending at x.

Given a cohomology class $\xi \in H^1(M;\mathbf{R})$ there is a natural covering space $p_\xi : \tilde{M}_\xi \to M$ associated with ξ. A closed loop $\gamma : [0,1] \to M$ lifts to a closed loop in $\tilde{M}_\xi$ if and only if the value of the cohomology class ξ on the homology class $[\gamma] \in H_1(M;\mathbf{Z})$ vanishes, $\langle \xi, [\gamma] \rangle = 0$. See [18].

The flow Φ lifts uniquely to a flow $\tilde{\Phi}$ on the covering $\tilde{M}_\xi$. Consider the chain recurrent set $R(\tilde{\Phi}) \subset \tilde{M}_\xi$ of the lifted flow and denote by $R_\xi = p_\xi(R(\tilde{\Phi})) \subset M$ its projection onto M. It is clear that the set $R_\xi \subset R$ is closed and flow-invariant.

A different definition of the set R_ξ which does not use the covering space $\tilde{M}_\xi$ can be found in [8], Definition 5.

5. The main result

Theorem 5. *Consider a smooth flow $\Phi : M \times \mathbf{R} \to M$ on a closed smooth manifold M. Let $\xi \in H^1(M; \mathbf{R})$ be a cohomology class such that the following conditions are satisfied:*

(1) The chain recurrent set R_ξ is isolated in the full chain recurrent set R of Φ.

(2) The restriction

$$\xi|_{R_\xi} \in \check{H}^1(R_\xi; \mathbf{R})$$

(viewed as a Čech cohomology class) vanishes.

(3) For any Φ-invariant, positive Borel measure μ on M with $\mu(R) > \mu(R_\xi)$, Schwartzman's asymptotic cycle $\mathcal{A}_\mu = \mathcal{A}_\mu(\Phi) \in H_1(M; \mathbf{R})$ satisfies

$$\langle \xi, \mathcal{A}_\mu \rangle < 0.$$

Then one has the following inequality

$$\sum_{i=1}^{r} \mathrm{cat}_M(R_\xi^i) \geq \mathrm{cat}_s^1(M, \xi), \tag{6}$$

where $R_\xi^1, \dots, R_\xi^r$ denote the connected components[3] *of the chain recurrent set R_ξ.*

A proof of Theorem 5 is given in section 7.

Let us compare Theorem 5 with Theorem 6.6 of [6]. Theorem 5 allows arbitrary sets R_ξ whereas Theorem 6.6 of [6] assumes that R_ξ consists of finitely many isolated points. Another important advantage of Theorem 5 is that it does not require condition $(*)$, see [6], page 108; this condition is difficult to check in concrete examples. On the other hand, since $\mathrm{cat}_s^1(X, \xi) \leq \mathrm{Cat}(\mathrm{X}, \xi)$, the estimate of Theorem 6.6 of [6] is potentially slightly sharper.

Next we state some corollaries of Theorem 5.

In the special case $\xi = 0$ one has $R_\xi = R$ and the assumptions (1), (2), (3) of Theorem 5 are automatically satisfied. Hence we obtain the following result which presumably is well known but for which we failed to find a reference:

Theorem 6. *Consider a smooth flow on a closed smooth manifold M. Let R be the chain recurrent set of the flow. Then*

$$\sum_{i=1}^{r} \mathrm{cat}_M(R^i) \geq \mathrm{cat}(M). \tag{7}$$

Here $R^1, \dots, R^r$ denote the connected components of R.

[3] The number of connected components r of R_ξ can be infinite. In this case inequality (5) is trivially satisfied as its LHS is infinite.

In the special case where the chain recurrent set R consists of finitely many points (the fixed points of the flow) Theorem 6 says that *the number of fixed points is at least the Lusternik - Schnirelman category of* M. This is one of the fundamental results of the classical Lusternik – Schnirelman theory, see [4].

Consider another special case of Theorem 5 when R_ξ consists of finitely many points (they are the fixed points of the flow). Condition (2) of Theorem 5 is automatically satisfied under these assumptions. We obtain:

Theorem 7. *Consider a smooth flow on a closed smooth manifold M. Let $\xi \in H^1(M;\mathbf{R})$ be a cohomology class such that the chain recurrent set R_ξ consists of finitely many points which are isolated in the full chain recurrent set R of the flow. Assume additionally that condition (3) of Theorem 5 is satisfied. Then the flow has at least $\operatorname{cat}^1_s(M,\xi)$ fixed points.*

Here is a reformulation of Theorem 7:

Theorem 8. *Let $\xi \in H^1(M;\mathbf{R})$ be a cohomology class. Consider a flow Φ on M such that the chain recurrent set R_ξ consists of less than $\operatorname{cat}^1_s(M,\xi)$ fixed points and such that condition (3) of Theorem 5 is satisfied. Then at least one of the fixed points of the flow is not isolated in the full chain recurrent set R.*

Theorem 1 stated in the introduction follows from Theorem 5 in the special case when the chain recurrent set R consists of finitely many circles and finitely many isolated points. Condition (3) of Theorem 5 is then equivalent to the requirement that $\langle \xi, z\rangle \le 0$ for the homology class $z \in H_1(M)$ of any periodic orbit. Under the assumptions of Theorem 1 the set R_ξ is the union of the fixed points and of the periodic orbits satisfying $\langle \xi, z\rangle = 0$. If C is a periodic orbit then the number $\operatorname{cat}_M(C)$ equals 1 or 2 depending on whether C is or is not null-homotopic in M. These consideration show that Theorem 1 follows from Theorem 5.

As a side remark we point out that under the conditions of Theorem 1,

$$p_0 + p_1 + 2p_2 + 2q_2 \ \ge\ \operatorname{cat}(M) \tag{8}$$

as follows from Theorem 6; here q_2 denotes the number of periodic orbits satisfying $\langle \xi, z\rangle \neq 0$.

6. Existence of the flow-convex neighborhoods

In this section we study some auxiliary problem which will be used in the proof of Theorem 5 and is of independent interest.

Consider a smooth vector field V on a closed smooth manifold M. Let $\Phi : M \times \mathbf{R} \to M$ be the flow of V. We will write $\Phi(x,t)$ as $x \cdot t$, where $x \in M$ and $t \in \mathbf{R}$. The symbol R denotes the chain recurrent set of Φ.

Theorem 9. *Let Z be a connected component of R which is isolated in R, i.e. such that there exists a neighborhood $Z \subset U$ with $U \cap R = Z$. Let*

$W \subset M$ be a neighborhood of Z. Then there exists an open neighborhood B of Z, contained in W, with the following two properties:

(A) *For any $x \in M$, the open set*

$$J_x = \{t; x \cdot t \in B\} \subset \mathbf{R}$$

is convex (i.e. it is either empty or an interval);

(B) *Let A be the set of points $x \in M$ such that the interval J_x is nonempty and bounded below. Then the function*

$$A \to \mathbf{R}, \quad x \mapsto \inf J_x,$$

is continuous.

A neighborhood B of Z having properties (A) and (B) is called *convex* with respect to the flow Φ.

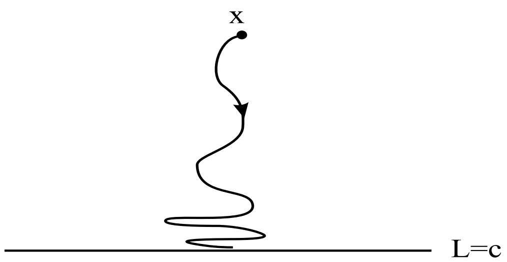
Theorem 9 could be compared to Lemma B.1 in Appendix B to [5].

By Conley's theorem [1, 2], there exists a smooth Lyapunov function $L : M \to \mathbf{R}$ for the flow; see also [9], Proposition 2, where the proof of the smooth version of Conley's theorem is given. The Lyapunov function L satisfies: $V(L) < 0$ on the complement $M - R$ of R and the differential dL vanishes on R.

Fix a point $x \in M - R$ and consider its orbit $x(t) = x \cdot t$. The function $t \mapsto L(x \cdot t)$ is strictly decreasing. Hence, as t tends to $+\infty$ the limit

$$\ell(x) = \lim_{t\to\infty} L(x \cdot t) \tag{9}$$

exists and is finite. If $x \in R$ then the function $t \mapsto L(x \cdot t)$ is constant.



For any $x \in M$ the number $\ell(x)$ is a critical value of L. Indeed, the ω-limit set $\omega(x)$ is contained in the level set $L^{-1}(\ell(x))$ and, on the other hand, $\omega(x)$ is a part of the chain recurrent set R which is the set of critical points of L.

Lemma 10. *The function $\ell : M \to \mathbf{R}$ is upper semi-continuous: if $x_n \to x \in M$ then*

$$\ell(x) \geq \limsup \ell(x_n). \tag{10}$$

Proof. Given any $\epsilon > 0$ there exists $T > 0$ such that $0 < L(x \cdot T) - \ell(x) < \epsilon$. Since L is continuous and the map $x \mapsto x \cdot T$ is continuous, there exists a neighborhood $U \subset M$ of x such that $|L(y \cdot T) - L(x \cdot T)| < \epsilon$ for any $y \in U$. Hence for all $y \in U$ one has

$$\ell(y) \leq L(y \cdot T) < L(x \cdot T) + \epsilon < \ell(x) + 2\epsilon.$$

Hence, $\ell(y) \leq \ell(x)$. □

The function $L : M \to \mathbf{R}$ restricted to Z is constant. Indeed, $L(Z) \subset \mathbf{R}$ must be connected (as the image of a connected set) and it has measure zero by Sard's theorem. Hence it is a single point. Denote $c = L(Z)$.

Let $Z \subset W$ be an open neighborhood. As Z is supposed to be isolated in R, we may assume (without loss of generality) that $\overline{W} \cap R = Z$.

Fix $\epsilon > 0$ and denote by $A_+(\epsilon)$ the set of all points $x \in M$ with the properties: (a) $L(x) = c + \epsilon$; (b) $\omega(x) \subset Z$. Note that (b) implies that $\ell(x) = c$.

Lemma 11. *For any sufficiently small $\epsilon > 0$ the set $A_+(\epsilon)$ is contained in the neighborhood W.*

Proof. Assuming the contrary, there exists a convergent sequence $x_n \in M - W$ such that $L(x_n) \to c$ and $L(x_n) > c$, $\ell(x_n) = c$ and $\omega(x_n) \subset Z$, where $\omega(x)$ denotes the ω-limit set of the trajectory $x \cdot t$. If $x_0 = \lim x_n \in M$ is the limit of x_n then (using (10)) one has $\ell(x_0) \geq c$. On the other hand, $L(x_0) = c$ and hence $\ell(x_0) = c$. We obtain that x_0 belongs to the set $L^{-1}(c) \cap (R - Z)$. In other words, it lies in a connected component of R distinct from Z.

Fix a Riemannian metric on M. Denote by $d > 0$ the distance between Z and $R - Z$. Denote by $K > 0$ a constant such that the norm of the vector V is less than K at every point of M. Such K exists since M is compact.

Fix some $\delta > 0$ such that $\delta < d/2$. The function $V(L) : M \to \mathbf{R}$ restricted to the complement of the δ-neighborhood of R is negative and moreover can be estimated from above $V(L) \leq -\eta$ for some positive $\eta = \eta_\delta > 0$. Now, we may find a large n such that

$$L(x_n) - c \; < \; \frac{\eta(d - 2\delta)}{K}$$

and x_n lies in the δ-neighborhood of $R - Z$. The trajectory $x_n \cdot t$ approaches Z for large t. The length of the trajectory is given by

$$\int_{t_1}^{t_2} |\dot{x}(t)| dt = \int_{t_1}^{t_2} |V(x(t)| dt \; \leq \; K|t_2 - t_1|.$$

One concludes that the time τ_n the trajectory $x_n \cdot t$ for $t > 0$ spends in the complement of the δ-neighborhood of R can be estimated by

$$\tau_n \; \geq \; \frac{d - 2\delta}{K}.$$

Therefore, for large $t > 0$ one has

$$L(x_n \cdot t) - L(x_n) = \int_0^t V(L)(x_n \cdot t)dt \leq -\eta \cdot \tau_n \leq -\frac{\eta(d-2\delta)}{K}$$

These inequalities show (since $L(x_n)$ tends to c) that for large t one has $L(x_n \cdot t) < c$ contradicting the assumption $\ell(x_n) = c$. □

Remark 12. Define $A_-(\epsilon)$ as the set of all points $x \in M$ with $L(x) = c - \epsilon$ and $\alpha(x) \subset Z$ where $\alpha(x)$ denotes the α-limit set of the trajectory $x \cdot t$. Applying the above arguments to the time reversed flow one obtains that for any neighborhood W of Z the set $A_-(\epsilon)$ is contained in W for any sufficiently small $\epsilon > 0$.

The arguments of the proof could be summarized as follows: the point x_n is very close to a component Z' of R lying in $L^{-1}(c)$ and distinct from Z. The trajectory starting at x_n cannot approach Z since while it passes the distance separating Z and Z' it descends with respect to L so that the point $x_n \cdot t$ slips below the level c.

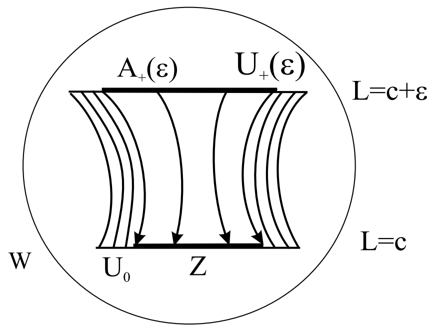
Lemma 13. *For any sufficiently small $\epsilon > 0$ the set $A_+(\epsilon)$ is closed.*

Proof. Let W be a neighborhood of Z with $\overline{W} \cap R = Z$ and let $\epsilon > 0$ be such that $A_+(\epsilon) \subset W$. Assume that a sequence $x_n \in A_+(\epsilon)$ converges to a point x_0. Then, for any $t > 0$ the point $x_n \cdot t$ lies in W. Hence we obtain that for any $t > 0$ the point $x_0 \cdot t$ lies in $\overline{W}$. It follows that $\omega(x_0) \subset \overline{W} \cap R = Z$. Hence, x_0 is an element of $A_+(\epsilon)$. □

Lemma 14. *Let W be an open neighborhood of Z such that $\overline{W} \cap R = Z$. Let $\epsilon > 0$ be such that $A_+(\epsilon')$ is contained in W for any $0 < \epsilon' \leq \epsilon$. Then there exists an open neighborhood $U_+(\epsilon)$ of the set $A_+(\epsilon)$ in the level set $L^{-1}(c+\epsilon)$ and an open neighborhood U_0 of Z in $L^{-1}(c)$ with the following properties:*

1. *If $x \in U_+(\epsilon) - A_+(\epsilon)$ then for some $\tau_x > 0$ the point $x \cdot \tau_x$ lies in $U_0 - Z$.*
2. *The mapping $x \mapsto x \cdot \tau_x$ is a homeomorphism of $U_+(\epsilon) - A_+(\epsilon) \to U_0 - Z$.*
3. *For any $0 \leq t \leq \tau_x$ the point $x \cdot t$ lies in W.*
4. *The number τ_x depends continuously on $x \in U_+(\epsilon) - A_+(\epsilon)$ and it tends to $+\infty$ as $x \in U_+(\epsilon) - A_+(\epsilon)$ approaches $A_+(\epsilon)$.*

Proof. If $L(y) = c + \epsilon$ and y is close enough to the set $A_+(\epsilon)$ then for $t > 0$ the orbit $y \cdot t$ does not leave W before it leaves the set $L \geq c$. Indeed, if this claim is false then there exists a sequence of points $y_n \in M$ and a sequence of numbers $t_n > 0$ such that $L(y_n) = c + \epsilon$, $y_n \to x \in A_+(\epsilon)$, $L(y_n \cdot t_n) \geq c$, and $y_n \cdot t_n \notin W$. Fix a Riemannian metric on M. Let $K > 0$ be such that the length of the vector $V(x)$ is less or equal than K at every point $x \in M$. Choose a small neighborhood G of Z such that its closure is contained in W. Let $d > 0$ be the distance between $\overline{G}$ and $M - W$. The function $V(L)$



satisfies in $W - G$ the inequality $V(L) \leq -\eta$ for some $\eta > 0$. Now, let $G' \subset G$ be a neighborhood of Z such that

$$L(G') \subset (-\infty, c + \eta \cdot \frac{d}{K}).$$

As x belongs to $A_+(\epsilon)$ there exists $t' > 0$ with $x \cdot t' \in G'$. Then for n large enough $y_n \cdot t' \in G'$. Arguing as in the proof of Lemma 11, one obtains that the point $y_n \cdot t$ cannot reach $M - W$ before it leaves the domain $L \geq c$.

Similarly we observe: if $L(y) = c$ where $y \in W - Z$ is sufficiently close to Z then the trajectory $y \cdot t$ for some $t < 0$ reaches the level $L = c + \epsilon$ without leaving W. Indeed, if this statement is false one finds a sequence $y_n \in W - Z$, $L(y_n) = c$ such that $y_n \to x \in Z$ and for some $t_n < 0$, $y_n \cdot t_n \in \partial W$, $L(y_n \cdot t_n) \leq c + \epsilon$. We may assume that the sequence $y_n \cdot t_n$ converges to a point $z \in \partial W$. If $\ell(z) > c$ the forward trajectory $z \cdot t$ where $t > 0$ crosses the level surface $L = c$ at a point which is not in Z and the trajectory $y_n \cdot t$ must cross this level at a nearby point as $n \to \infty$, a contradiction. Hence we have to consider only the case $\ell(z) = c$ which leaves two possibilities: either $\omega(z)$ is contained in a connected component of R distinct from Z which also lies on the level surface $L = c$, or $\omega(z) \subset Z$. The first possibility leads to a contradiction, arguing as in the proof of Lemma 11. The second possibility means that z belongs to $A_+(\epsilon')$ where $0 < \epsilon' \leq \epsilon$. This is a contradiction since we assume that $A_+(\epsilon')$ is contained in the open set W.

Let U_0 be the union of the set Z and the set of all points $y \in W \cap L^{-1}(c)$ such that the trajectory $y \cdot t$ for $t < 0$ reaches the set $W \cap L^{-1}(c+\epsilon)$ without leaving W.

Let $U_+(\epsilon)$ be the union of $A_+(\epsilon)$ and the set of all $y \in W \cap L^{-1}(c+\epsilon)$ such that the trajectory $y \cdot t$ for $t > 0$ reaches the level surface $L^{-1}(c)$ without leaving W.

U_0 is an open neighborhood of Z in $L^{-1}(c)$ and $U_+(\epsilon)$ is an open neighborhood of $A_+(\epsilon)$ in $L^{-1}(c + \epsilon)$ as was shown above. It is then easy to see that the statements of Lemma 14 hold for U_0 and $U_+(\epsilon)$. Let us show, for example, that if $x_n \in U_+(\epsilon) - A_+(\epsilon)$ and $x_n \to x \in A_+(\epsilon)$ then τ_{x_n} tends to

$+\infty$. If not one may pass to a subsequence such that the sequence τ_{x_n} has a finite limit τ. Then $L(x\cdot\tau) = \lim L(x_n \cdot \tau_{x_n}) = c$. Thus the trajectory $x \cdot t$ arrives in finite time at Z which is impossible since Z is flow-invariant and $x \notin Z$. □

Here is an analog of Lemma 14:

Lemma 15. *Let W be an open neighborhood of Z such that $\overline{W} \cap R = Z$. Let $\epsilon > 0$ be such that $A_-(\epsilon')$ is contained in W for any $0 < \epsilon' \leq \epsilon$. Then there exists an open neighborhood $U_-(\epsilon)$ of the set $A_-(\epsilon)$ in the level set $L^{-1}(c-\epsilon)$ and an open neighborhood U_0 of Z in $L^{-1}(c)$ with the following properties:*

1. *If $x \in U_-(\epsilon) - A_-(\epsilon)$ then for some $T_x < 0$ the point $x \cdot T_x$ lies in $U_0 - Z$.*
2. *The mapping $x \mapsto x \cdot T_x$ is a homeomorphism*
$$U_-(\epsilon) - A_-(\epsilon) \to U_0 - Z.$$
3. *For any $T_x < t < 0$ the point $x \cdot t$ lies in W.*
4. *The number T_x depends continuously on $x \in U_-(\epsilon) - A_-(\epsilon)$ and it tends to $-\infty$ as $x \in U_-(\epsilon) - A_-(\epsilon)$ approaches $A_-(\epsilon)$.*

Proof. It is similar to the proof of Lemma 14. □

Proof of Theorem 9. Let $W \subset M$ be an open neighborhood of Z. We may assume without loss of generality that $\overline{W} \cap R = Z$. By Lemma 11 and Remark 12 we may find $\epsilon > 0$ such that for any $0 < \epsilon' \leq \epsilon$ the sets $A_+(\epsilon')$ and $A_-(\epsilon')$ are contained in W.

Let C denote the set of all points $x \in W \cap L^{-1}(c-\epsilon, c+\epsilon)$ such that there exist numbers $t_x < 0 < \tau_x$ with

$$L(x \cdot t_x) = c + \epsilon, \quad L(x \cdot \tau_x) = c - \epsilon \tag{11}$$

and $x \cdot t$ is contained in W for any $t \in (t_x, \tau_x)$. Observe that if $x \in C$ then $x \cdot (t_x, \tau_x)$ is contained in C.

Define the set
$$B = C \cup Z \cup \bigcup_{0<\epsilon'<\epsilon} A_\pm(\epsilon').$$
We are going to show that it satisfies the requirements of Theorem 1. This set is open. Indeed, the terms in (11) are pairwise disjoint. The set C is clearly open. Any point of $A_\pm(\epsilon')$ (where $0 < \epsilon' < \epsilon$) has a neighborhood contained entirely in B (by Lemmas 14 and 15). Similarly, any sufficiently small neighborhood G of a point $x \in Z$ is contained in B: if $y \in G$ and $L(y) > c$ then the trajectory $y \cdot t$ for $t > 0$ either approaches Z (in that case y lies in $A_+(\epsilon')$ for some $0 < \epsilon' < \epsilon$) or it hits the level surface $L = c$. In the second case the intersection point is very close to Z and hence by Lemmas 14 and 15 the trajectory continues all the way till it reaches the level $L = c - \epsilon$ without leaving W.

Given $x \in M$ consider the set $J_x = \{t; x \cdot t \in B\}$. Let us assume that $L(x) > c$. Then there exist the following possibilities:

(1) $\ell(x) > c$; then $J_x = \emptyset$.
(2) $\omega(x) \subset J_x)$; then J_x is a half infinite interval $(a, +\infty)$.
(3) $\ell(x) = c$ and $\omega(x) \cap Z = \emptyset$; then $J_x = \emptyset$.
(4) $\ell(x) < c$; then J_x is either the empty set or a finite interval (a, b).

In the case $L(x) \le c$ the arguments are similar.

The continuity of $x \mapsto \inf J_x$ follows from the Implicit Function Theorem: the number $\inf J_x = t$ is the solution of the equation $L(x \cdot t) = c + \epsilon$ and the partial derivative of $L(x \cdot t)$ with respect to t is strictly negative.

7. Proof of Theorem 5

Consider a smooth flow $\Phi : M \times \mathbf{R} \to M$, $\Phi(x, t) = x \cdot t$ and a real cohomology class $\xi \in H^1(M; \mathbf{R})$ satisfying the conditions of Theorem 5.

If the chain recurrent set R_ξ has infinitely many connected components then the RHS of inequality (6) is infinite and so this inequality is obviously satisfied. Hence without loss of generality we may assume that the number of connected components of R_ξ is finite.

Let $R_\xi = R_\xi^1 \cup R_\xi^2 \cup \dots \cup R_\xi^r$ be the connected components of R_ξ. Fix mutually disjoint open neighborhoods $W_i \supset R_\xi^i$, where $i = 1, 2, \dots, r$. We shall assume that W_i is so small that

$$\mathrm{cat}_M(W_i) = \mathrm{cat}_M(R_\xi^i), \quad i = 1, \dots, r. \tag{12}$$

Compare [4], Lemmas 19.2 and 19.6.

Applying Theorem 9 we may find an open neighborhood $B_i \subset W_i$ of R_ξ^i which is convex with respect to the flow Φ. Clearly,

$$\mathrm{cat}_M(\overline{B}_i) = \mathrm{cat}_M(R_\xi^i), \quad i = 1, \dots, r, \tag{13}$$

as a consequence of (12).

Let U_i be an open neighborhood of R_ξ^i such that the closure $\overline{U}_i$ is contained in B_i. We have the inclusions $R_\xi^i \subset U_i \subset \overline{U}_i \subset B_i \subset W_i$.

Let $A \subset M$ denote the set of points $x \in M$ such that for any $t \in \mathbf{R}$ the point $x \cdot t$ does not belong to the union $U = \cup_{j=1}^r U_j$. The set A is closed and flow-invariant.

Applying Theorem 1 of [9] we find that there exists a smooth Lyapunov 1-form ω for the pair (Φ, R_ξ) lying in the cohomology class $\xi = [\omega] \in H^1(M; \mathbf{R})$. This means that the function $\omega(V) : M \to \mathbf{R}$ is negative on $M - R_\xi$ (where V denotes the vector field on M generating the flow Φ) and $\omega(V)|_{R_\xi} = 0$. Moreover, the restriction of ω on some open neighborhood of R_ξ is the differential of a smooth function (this is equivalent to condition (1) of Theorem 5).

Let us show that for any integer N the set A is N-movable with respect to both ω and $-\omega$. Since $M - U$ is compact there exists $\epsilon > 0$ such that

$\omega(V) < -\epsilon$ on $M - U$. Then for any $x \in A$ one has

$$\int_x^{x\cdot t} \omega \; < \; -\epsilon t, \quad \text{for} \quad t > 0.$$

Hence a continuous homotopy $h_t : A \to M$, $t \in [0,1]$ defined by

$$h_t(x) = x \cdot \left(\frac{Nt}{\epsilon}\right)$$

satisfies the condition of Definition 1. This shows that A is N-movable with respect to ω. In a similar way one shows that for any integer N the set A is N-movable with respect to $-\omega$.

Now the inequality (6) follows once one shows that

$$\operatorname{cat}_M(M - A) \; \leq \; \sum_{j=1}^{r} \operatorname{cat}_M(R^j_\xi), \tag{14}$$

see Definition 3.

Let $F_i \subset M$ denote the set of points $x \in M$ such that $x \cdot t$ belongs to B_i for some $t \in \mathbf{R}$. Then $J^i_x = \{t \in \mathbf{R}; x \cdot t \in B_i\}$ is a nonempty open interval. For $x \in F_i$ define

$$\tau_i(x) = \begin{cases} 0, & \text{if} \quad 0 \in J^i_x, \\ \inf J^i_x, & \text{if} \quad J^i_x \subset (0, \infty), \\ \sup J^i_x, & \text{if} \quad J^i_x \subset (-\infty, 0). \end{cases}$$

The function $\tau_i : F_i \to \mathbf{R}$ is continuous as follows from Theorem 9.

The mapping $x \mapsto x \cdot \tau_i(x)$ is a deformation retraction $F_i \to \overline{B}_i$ and hence

$$\operatorname{cat}_M(F_i) \leq \operatorname{cat}_M(\overline{B}_i) = \operatorname{cat}_M(R^i_\xi)$$

in view of (13). Here we use Lemma 19.5 from [4]. Since $M - A$ is contained in the union $\cup_{i=1}^r F_i$ one obtains

$$\operatorname{cat}_M(M - A) \leq \sum_{i=1}^{r} \operatorname{cat}_M(F_i) \leq \sum_{i=1}^{r} \operatorname{cat}_M(R^i_\xi).$$

This proves inequality (14) and hence completes the proof of Theorem 5.

□

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