Weak Bézout inequality for D-modules

Dima GRIGORIEV

Institut des Hautes Études Scientifiques

35, route de Chartres

91440 – Bures-sur-Yvette (France)

Juin 2004

IHES/M/04/16

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Dima Grigoriev
IRMAR, Université de Rennes
Beaulieu, 35042, Rennes, France
dima@math.univ-rennes1.fr
http://name.math.univ-rennes1.fr/dimitri.grigoriev

Abstract

Let $\{w_{i,j}\}_{1\leq i\leq n,1\leq j\leq s} \subset L_m = F(X_1,\dots,X_m)[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_m}]$ be linear partial differential operators of orders with respect to $\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_m}$ at most d. We prove an upper bound

$$n(4m^2 d\min\{n,s\})^{4^{m-t-1}(2(m-t))}$$

on the leading coefficient of the Hilbert-Kolchin polynomial of the left L_m-module $\langle\{w_{1,j},\dots,w_{n,j}\}_{1\leq j\leq s}\rangle \subset L_m^n$ having the differential type t (also being equal to the degree of the Hilbert-Kolchin polynomial). The main technical tool is the complexity bound on solving systems of linear equations over *algebras of fractions* of the form

$$L_m(F[X_1,\dots,X_m,\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_k}])^{-1}.$$

Introduction

Denote the derivatives $D_i = \frac{\partial}{\partial X_i}, 1\leq i\leq m$ and by $A_m = F[X_1,\dots,X_m,D_1,\dots,D_m]$ the *Weyl algebra* [2] over an infinite field F. It is well-known that A_m is defined by the following relations:

$$X_iX_j = X_jX_i, D_iD_j = D_jD_i, X_iD_i = D_iX_i - 1, X_iD_j = D_jX_i, \quad i\neq j \tag{1}$$

For a family $\{w_{i,j}\}_{1\leq i\leq n,1\leq j\leq s} \subset L_m$ of elements of the *algebra of linear partial differential operators* one can consider a system

$$\sum_{1\leq i\leq n} w_{i,j}u_i = 0, \quad 1\leq j\leq s \tag{2}$$

of linear partial differential equations in the unknowns $u_1,\dots,u_n$. In particular, if the F-linear space of solutions of (2) has a finite dimension l then the quotient of the free L_m-module L_m^n over the left L_m-module $L = \langle\{w_{1,j},\dots,w_{n,j}\}_{1\leq j\leq s}\rangle \subset L_m^n$ has also the dimension l over the field $F(X_1,\dots,X_m)$ [6]. Denote by t the *differential type* of L [6], then $0\leq t\leq m$ (observe that the case treated in the previous sentence, corresponds to $t=0$).

We consider the filtration on the algebra L_m defined on the monomials by $ord(cD_1^{i_1}\cdots D_m^{i_m}) = i_1+\cdots+i_m$ where a coefficient $c\in F(X_1,\dots,X_m)$. With respect to this filtration the module L posesses the Hilbert-Kolchin polynomial [6]

$$\frac{l}{t!}z^t + l_{t-1}z^{t-1} + \cdots + l_0$$

of the degree t (which coincides with the differential type of L). The leading coefficient l is called the *typical differential dimension* [6]. In the treated above particular (holonomic) case $t = 0$ the dimension of F-linear space of solutions of (2) equals to l.

In case of a module over the ring of polynomials the leading coefficient of its Hilbert polynomial equals the degree of the module, and the classical Bézout inequality provides for the leading coefficient an upper bound being the product of the degrees of generators of the module.

In the present paper we prove (see Section 4) the following inequality which could be viewed as a weak analogue of the Bézout inequality for differential modules.

Corollary 0.1 *Let $ord(w_{i,j}) \leq d, \quad 1 \leq i \leq n, 1 \leq j \leq s$. Then the leading coefficient of the Hilbert-Kolchin polynomial*

$$l \leq n(4m^2 d \min\{n, s\})^{4^{m-t-1}(2(m-t))}$$

Actually, one could slightly improve this estimate while making it more tedious. We note that the latter estimate becomes better with a smaller value of $m - t$. In fact, for small values $m - t \leq 2$ much stronger estimates are known. In the case $m - t = 0$ the bound $l \leq n$ is evident. In the case $m - t = 1$ the bound $l \leq \max_{1 \leq i \leq s}\{ord(w_{i,1}\} + \cdots + \max_{1 \leq i \leq s}\{ord(w_{i,n}\}$ was proved [6] (moreover, the latter bound holds in the more general situation of *non-linear* partial differential equations, whereas in the situation under consideration in the present paper of *linear* partial differential equations a stronger *Jacobi conjecture* was established , see e.g. [7]). In the case $m - t = 2, \quad n = 1$ the bound $l \leq ord(w_1)ord(w_2)$ was proved for the left ideal $\langle w_1, w_2, \ldots \rangle \subset L_m$ where $ord(w_1) \geq ord(w_2) \geq \ldots$ [7] which could be viewed as a direct analogue of the Bézout inequality. In the case $m = 3, \quad t = 0, \quad n = 1$ a counter-example of a left ideal $\langle w_1, w_2, w_3 \rangle \subset L_3$ is also produced in [7] which shows that the expected upper bound $ord(w_1)ord(w_2)ord(w_3)$ on l appears to be wrong. It would be interesting to clarify how sharp is the estimate in Corollary 0.1 for large values of $m - t$.

We mention also that on p.154 [9] a (better than in Corollary 0.1) exponential bound on l (when $t = 0$) was established in case of a *homogeneous toric* ideal of the Weyl algebra.

The main technical tool in the proof of Corollary 0.1 is the complexity bound on solving linear systems over *algebras of fractions* of L_m. Let $K \subset \{1, \ldots, m\}$ be a certain subset. Denote by $A_m^{(K)} = F[X_1, \ldots, X_m, \{D_k\}_{k \in K}] \subset A_m$ the corresponding subalgebra of A_m. We consider the algebra of fractions $Q_m^{(K)} = A_m(A_m^{(K)})^{-1}$. For an element $a \in A_m$ we denote the Bernstein filtration [2] $deg(a)$ defining it on monomials $X_1^{j_1} \cdots X_m^{j_m} D_1^{i_1} \cdots D_m^{i_m}$ by $j_1 + \cdots + j_m + i_1 + \cdots + i_m$. Then for an element $ab^{-1} \in Q_m^{(K)}, \quad a \in A_m, b \in A_m^{(K)}$ we write that the degree $deg(ab^{-1}) \leq \max\{deg(a), deg(b)\}$.

In Section 1 below we study the properties of $Q_m^{(K)}$ and the complexity bounds on manipulating in $Q_m^{(K)}$. In Section 2 we establish complexity bounds on *quasi-inverse* matrices over the algebra $Q_m^{(K)}$. Finally, in Section 3 we consider the problem of solving a system of linear equations over the algebra $Q_m^{(K)}$:

$$\sum_{1 \leq i \leq p} a_{j,i} V_i = a_j, \quad 1 \leq j \leq q \tag{3}$$

where the coefficients $a_{j,i}, a_j \in A_m, \quad deg(a_{j,i}), deg(a_j) \leq d$. We prove the following theorem.

Theorem 0.2 *If (3) is solvable over* $Q_m^{(K)}$ *then (3) has a solution with*

$$deg(v_i) \leq (16m^4d^2(\min\{p,q\})^2)^{4^{m-|K|}}$$

Assume now that the ground field F is represented in an effective way, say as a finitely generated extension either of $\mathbb{Q}$ or of a finite field (see e.g. [4]). Then one can define the bit-size M of the coefficients in F of the input $\{a_{j,i}, a_j\}$.

Corollary 0.3 *One can test the solvability of (3) and if it is solvable then yield some its solution in time polynomial in*

$$M,\ \ q,\ \ p^m,\ \ (md\min\{p,q\})^{4^{m-|K|}m}$$

Theorem 0.2 and Corollary 0.3 generalize the results from [5] established for the algebra $Q_m^{(\emptyset)} = L_m$ of linear differential operators to the algebras of fractions $Q_m^{(K)}$. In [5] it is noticed that due to the example of [8] the bounds in Theorem 0.2 and Corollary 0.3 are close to sharp.

The problem in question generalizes the one of solving linear systems over the algebra of polynomials which was studied in [10] where the similar complexity bounds were proved. Unfortunately, one cannot extend directly the method from [10] (which arises to G.Hermann) to the (*non-commutative*) algebra $Q_m^{(K)}$ because the method involves the determinants. Nevertheless, we exploit the general approach of [10].

We mention also that certain algorithmical problems in the algebra of linear partial differential operators were posed in [3].

1 Algebra of fractions of differential operators

Let a matrix $B = (b_{i,j}),\quad 1 \leq i \leq p-1, 1 \leq j \leq p$ have its entries $b_{i,j} \in A_m^{(K)}$ and $deg(b_{i,j}) \leq d$. The following lemma was proved in [5].

Lemma 1.1 *There exists a vector* $0 \neq c = (c_1, \dots, c_p) \in (A_m^{(K)})^p$ *such that* $Bc = 0$ *and moreover,* $deg(c) \leq 2(m+|K|)(p-1)d = N$.

Proof. Consider an F-linear space $U \subset (A_m^{(K)})^p$ consisting of all the vectors $c = (c_1, \dots, c_p)$ such that $deg(c) \leq N$. Then $\dim U = p\binom{N+m+|K|}{m+|K|}$. For any vector $c \in U$ we have $deg(Bc) \leq N+d$, i.e. $Bc \in W$ where the F-linear space W consists of all the vectors $w = (w_1, \dots, w_{p-1}) \in (A_m^{(K)})^{p-1}$ for which $deg(w) \leq N+d$, thereby $\dim(W) = (p-1)\binom{N+d+m+|K|}{m+|K|}$.

Let us verify an inequality $p\binom{N+m+|K|}{m+|K|} > (p-1)\binom{N+d+m+|K|}{m+|K|}$ whence lemma would follow immediately. Indeed,

$$\binom{N+d+m+|K|}{m+|K|} \Big/ \binom{N+m+|K|}{m+|K|} = \frac{N+d+m+|K|}{N+m+|K|} \cdots \frac{N+d+1}{N+1} \leq \Big(\frac{N+d+1}{N+1}\Big)^{m+|K|}.$$

It suffices to check the inequality $(\frac{N+d+1}{N+1})^{m+|K|} < \frac{p}{p-1}$. The latter follows in its turn from the inequality

$$(1+\frac{1}{p-1})^{1/(m+|K|)} > 1 + \Big(\frac{1}{m+|K|}\Big)\frac{1}{p-1} + \frac{1}{2}\Big(\frac{1}{m+|K|}\Big)\Big(\frac{1}{m+|K|}-1\Big)\frac{1}{(p-1)^2} >$$

$$1 + \frac{1}{2}\Big(\frac{1}{m+|K|}\Big)\frac{1}{p-1} > 1 + \frac{d}{N+1} \blacksquare$$

Notice that Lemma 1.1 implies that $A_m^{(K)}$ is an Ore domain [2], i.e. the expressions of the form $b_1 b_2^{-1}$ where $b_1, b_2 \in A_m^{(K)}$ constitute an algebra. Below we use the following notations: letters a, α (respectively, b, β) with subscripts denote the elements from A_m (respectively, from $A_m^{(K)}$). Our nearest purpose is to show that the expressions of the form ab^{-1} also constitute an algebra $Q_m^{(K)} = A_m (A_m^{(K)})^{-1}$ (see above the Introduction) and to provide complexity bounds on performing arithmetic operations in $Q_m^{(K)}$. To verify that the sum $a_1 b_1^{-1} + a_2 b_2^{-1}$ can be represented in the desired form $a_3 b_3^{-1}$ we note first the following bound on a *(left) common multiple* of a family of elements from $A_m^{(K)}$ being a consequence of Lemma 1.1.

Corollary 1.2 *For a family $b_1, \dots, b_p \in A_m^{(K)}$ of the degrees $deg(b_1), \dots, deg(b_p) \le d$ there exist $c_1, \dots, c_p \in A_m^{(K)}$ such that $b_1 c_1 = \dots = b_p c_p \ne 0$ of the degrees $deg(c_1), \dots, deg(c_p) \le 2(m + |K|)(p-1)d$.*

Evidently, the same bound holds also for a *right* common multiple of $b_1, \dots, b_p$ which equals to $c_1' b_1 = \dots = c_p' b_p$.

To complete the consideration of the sum one can find $c_1, c_2 \in A_m^{(K)}$ such that $b = b_1 c_1 = b_2 c_2$ according to Corollary 1.2, then $a_1 b_1^{-1} + a_2 b_2^{-1} = a_1 c_1 b^{-1} + a_2 c_2 b^{-1} = (a_1 c_1 + a_2 c_2) b^{-1}$.

For an element $a \in A_m$ we denote by $ord^{(K)}(a)$ the filtration degree of a with respect to the symbols $\{\frac{\partial}{\partial X_j}\}$ for $j \notin K$ and by $deg^{(K)}(a)$ the filtration degree of a with respect to the symbols $X_1, \dots, X_m, \{\frac{\partial}{\partial X_k}\}$ for $k \in K$.

Next we verify that $(A_m^{(K)})^{-1} A_m = A_m (A_m^{(K)})^{-1}$ relying on the following lemma.

Lemma 1.3 *Let $a \in A_m, \quad b \in A_m^{(K)}$ be such that $deg^{(K)}(a), deg^{(K)}(b) \le d, \quad ord^{(K)}(a) = e$. Then there exist suitable elements $\alpha \in A_m, \quad \beta \in A_m^{(K)}$ such that $b\alpha = a\beta$ (or in other terms $\alpha\beta^{-1} = b^{-1}a$) and moreover, $ord^{(K)}(\alpha) \le ord^{(K)}(a), \quad deg^{(K)}(\alpha), deg^{(K)}(\beta) \le 2(m + |K|)\binom{e+m-|K|}{e} d$.*

Proof. Write down $\alpha = \sum_I D^I \beta_I$ where indeterminates $\beta_I \in A_m^{(K)}$ and the summation ranges over all the derivatives $D^I = \prod_{j \notin K} D_j^{i_j}$ with the orders $\sum_{j \notin K} i_j \le e$. In a similar manner $a = \sum_I D^I b_I$. Then the equality $b\alpha = a\beta$ turns into a linear system in $\binom{e+m-|K|}{e}$ equations in $\binom{e+m-|K|}{e} + 1$ indeterminates $\beta, \{\beta_I\}_I$. Applying to this system Lemma 1.1 we complete the proof. ∎

Lemma 1.3 entails that the product of two elements $a_1 b_1^{-1}$ and $a_2 b_2^{-1}$ from $Q_m^{(K)}$ has again the similar form $a_3 b_3^{-1}$: indeed, let $b_1^{-1} a_2 = a_4 b_4^{-1}$ for appropriate $a_4 \in A_m, \quad b_4 \in A_m^{(K)}$, then $a_1 b_1^{-1} a_2 b_2^{-1} = a_1 a_4 (b_2 b_4)^{-1}$.

Finally, to complete the description of the algebra $Q_m^{(K)}$ we need to verify that the relation $\alpha\beta^{-1} = b^{-1}a \in Q_m^{(K)}$ being defined as $b\alpha = a\beta$, induces an equivalence relation on $Q_m^{(K)}$. To this end it suffices to show that the equalities $\alpha_1 \beta_1^{-1} = b_1^{-1} a_1, \quad b_1^{-1} a_1 = \alpha_2 \beta_2^{-1}, \quad \alpha_2 \beta_2^{-1} = b_2^{-1} a_2$ imply the equality $\alpha_1 \beta_1^{-1} = b_2^{-1} a_2$. Due to Corollary 1.2 there exist $a_3, a_4 \in A_m$ such that $a_3 a_1 = a_4 a_2$, hence $a_4 b_2 \alpha_2 = a_4 a_2 \beta_2 = a_3 a_1 \beta_2 = a_3 b_1 \alpha_2$, therefore $a_4 b_2 = a_3 b_1$. Because of that $a_4 b_2 \alpha_1 = a_3 b_1 \alpha_1 = a_3 a_1 \beta_1 = a_4 a_2 \beta_1$, thus $b_2 \alpha_1 = a_2 \beta_1$ that was to be shown.

The following corollary summarizes the established above properties of the algebra $Q_m^{(K)}$.

Corollary 1.4 *In the algebra of fractions $Q_m^{(K)} = A_m (A_m^{(K)})^{-1} = (A_m^{(K)})^{-1} A_m$ two elements $a_1 b_1^{-1}, a_2 b_2^{-1} \in A_m (A_m^{(K)})^{-1}$ are equal if and only if there exists an element $\beta^{-1}\alpha \in (A_m^{(K)})^{-1} A_m$ such that $\beta a_1 = \alpha b_1, \quad \beta a_2 = \alpha b_2$.*

2 Quasi-inverse matrices over algebras of differential operators

Let us call an $p \times p$ matrix $C = (c_{i,j})$ a right (respectively, left) *quasi-inverse* to an $p \times p$ matrix $B = (b_{i,j})$ where the entries $c_{i,j}, b_{i,j} \in A_m^{(K)}$ if the matrix BC (respectively, CB) has the diagonal form with non-zero diagonal entries. The following lemma was proved in [5].

Lemma 2.1 *If an $p \times p$ matrix B over $A_m^{(K)}$ has a right quasi-inverse (we assume that $deg(B) \leq d$) then B has also a left quasi-inverse C over $A_m^{(K)}$ such that $deg(C) \leq 2(m + |K|)(p-1)d$.*

Proof. First observe that there does not exist a vector $0 \neq b \in (A_m^{(K)})^p$ for which $bB = 0$ since $A_m^{(K)}$ is a domain (see [2] and also Section 1). Consider the $p \times (p-1)$ matrix $B^{(i)}$ obtained from B by deleting its i-th column, $1 \leq i \leq p$. Due to Lemma 1.1 there exists a vector $0 \neq c^{(i)} \in (A_m^{(K)})^p$ such that $c^{(i)}B^{(i)} = 0$ and $deg(c^{(i)}) \leq 2(m+|K|)(p-1)d$. Then the $p \times p$ matrix with the rows $c^{(i)}, \quad 1 \leq i \leq p$ is a left quasi-inverse of B. ∎

We note that a matrix G over A_m (or over $Q_m^{(K)}$) has a quasi-inverse if and only if G is *non-singular*, i.e. has an inverse over the skew-field $Q_m^{(\{1,\dots,m\})} = A_m(A_m)^{-1}$. The latter is equivalent to that G has a non-zero determinant of Dieudonné [1]. The rank $r = rk(G)$ is defined as the maximal size of non-singular submatrices of G. The following lemma was proved in [5].

Lemma 2.2 *Let $G = (g_{i,j})$ be a $p_1 \times p_2$ matrix over $A_m^{(K)}$ with the rank $rk(G) = r$ and assume that the $r \times r$ submatrix G_1 of G in its left-upper corner is non-singular. Let an $r \times r$ matrix C_1 over $A_m^{(K)}$ be a left quasi-inverse to G_1. Then one can find an $(p_1 - r) \times r$ matrix C_2 over the algebra $Q_m^{(K)}$ such that*

$$\begin{pmatrix} C_1 0 \\ C_2 E \end{pmatrix} G = \left(\begin{array}{ccc|c} g_1 & & 0 & \\ & \ddots & & * \\ 0 & & g_r & \\ \hline & \mathbf{0} & & \mathbf{0} \end{array}\right)$$

where E denotes the unit matrix.

Proof. The matrix C_2 is determined uniquely by the requirement that in the product of matrices in the right-hand side the left-lower corner is zero. Then the right-lower corner is zero as well by the definition of the rank.∎

We proceed to solving system (3). Denote $r = rk(a_{j,i})$. After renumerating the rows and columns one can suppose the $r \times r$ submatrix in the left-upper corner of $(a_{j,i})$ to be non-singular. Applying Lemma 2.1 to $r \times r$ submatrix $(a_{j,i}), \quad 1 \leq i, j \leq r$ one gets a matrix C_1, subsequently applying Lemma 2.2 one gets a matrix C_2. If the vector $(C_2 \ \ E)(a_1, \dots, a_q)$ does not vanish then system (3) has no solutions. Otherwise, if $(C_2 \ \ E)(a_1, \dots, a_q) = 0$ then system (3) is equivalent to a linear system over $Q_m^{(K)}$ of the following form (see Lemma 2.2):

$$g_j V_j + \sum_{r+1 \leq i \leq p} g_{j,i} V_i = f_j, \quad 1 \leq j \leq r \tag{4}$$

where $g_j, g_{j,i}, f_j \in A_m$. Lemma 2.1 implies that $deg(g_j), deg(g_{j,i}), deg(f_j) \leq (4m(r-1)+1)d$. Fix for the time being a certain i, $\quad r+1 \leq i \leq p$. Applying Lemma 1.1 to the $r \times (r+1)$ submatrix which consists of the first r columns and of the i-th column of the matrix in the left-hand side of (4), we obtain $h_1^{(i)}, \dots, h_r^{(i)}, h^{(i)} \in A_m$ such that

$$g_j h_j^{(i)} + g_{j,i} h^{(i)} = 0, \quad 1 \leq j \leq r \tag{5}$$

Moreover, $deg(h_j^{(i)}), deg(h^{(i)}) \leq 4mr(4m(r-1)+1)d \leq (16m^2r^2-1)d$.

3 Complexity of solving a linear system over an algebra of fractions of differential operators

In the present section we design an algorithm to solve a linear system (4) over $Q_m^{(K)}$.

Fix for the time being a certain $\gamma \notin K$. An arbitrary element $h \in A_m$ can be written as

$$h = \sum_{0 \leq s \leq t} D_\gamma^s h_s = \sum_{S=\{s_\delta\}_{\delta \notin K}} (\prod_{\delta \notin K} D_\delta^{s_\delta}) h_S \tag{6}$$

where $h_s \in A_m^{(\{1,\dots,m\}\setminus\gamma)}$, $\quad h_S \in A_m^{(K)}$. Denote the leading coefficient $lc_\gamma(h) = h_t \neq 0$. We say that h is *normalized with respect to* D_γ when $lc_\gamma(h) \in A_m^{(K)}$. The following lemma plays the role of the normalization for the algebra $Q_m^{(K)}$ (cf. Lemma 2.3 [11] or Lemma 4 [5]).

Lemma 3.1 *For any finite family $H = \{h\} \subset A_m$ there exists a non-singular F-linear transformation of the $2(m-|K|)$-dimensional F-linear subspace of A_m with the basis $\{X_\delta, \frac{\partial}{\partial X_\delta}\}_{\delta \notin K}$ under which the vector $\{\frac{\partial}{\partial X_\delta}\}_{\delta \notin K}$ is transformed as follows:*

$$\{\frac{\partial}{\partial X_\delta}\}_{\delta \notin K} \to \Omega \{\frac{\partial}{\partial X_\delta}\}_{\delta \notin K}$$

where the $(m-|K|) \times (m-|K|)$ matrix $\Omega = (\omega_{\delta_1,\delta})$, $\quad \omega_{\delta_1,\delta} \in F$, and the vector

$$\{X_\delta\}_{\delta \notin K} \to (\Omega^T)^{-1} \{X_\delta\}_{\delta \notin K}$$

such that any transformed (under the transformation continued to A_m) element $\overline{h} \in A_m$ for $h \in H$ is normalized with respect to D_γ. Moreover, $deg_{D_\gamma}(\overline{h}) = ord^{(K)}(\overline{h})$.

Proof. One can verify that this linear transformation keeps the relations (1), therefore, one can consider A_m as a Weyl algebra with respect to the variables $\{X_k\}_{k \in K} \cup (\Omega^T)^{-1}\{X_\delta\}_{\delta \notin K}$ and the corresponding differential operators $\{\frac{\partial}{\partial X_k}\} \cup \Omega\{\frac{\partial}{\partial X_\delta}\}_{\delta \notin K}$ (cf. also [5]).

We rewrite (6) as

$$h = \sum_{S_0=\{s_\delta\}_{\delta \notin K}} (\prod_{\delta \notin K} D_\delta^{s_\delta}) h_{S_0} + \Sigma_1$$

where in the first sum all the terms from (6) with the maximal value of the sum $\sum_{s_\delta \in S_0} s_\delta = ord^{(K)}(h)$ are gathered. Then the leading coefficient

$$lc_\gamma(\overline{h}) = \sum_{S_0} (\prod_{\delta \notin K} \omega_{\gamma,\delta}^{s_\delta}) \overline{h}_{S_0} \in A_m^{(K)}.$$

Since the latter sum does not vanish if and only if the result of its linear transformation

$$\sum_{S_0}(\prod_{\delta\notin K}\omega_{\gamma,\delta}^{s_\delta})h_{S_0}\in A_m^{(K)}$$

with respect to Ω^T does not vanish as well, the set of the entries $\{\omega_{\gamma,\delta}\}_{\delta\notin K}$ for which $lc_\gamma(\overline{h})$ does not vanish, is open in the Zariski topology (and thereby, is non-empty taking into account that the ground field F is infinite). Hence for an open set of the entries $\{\omega_{\gamma,\delta}\}_{\delta\notin K}$ the leading coefficients $lc_\gamma(\overline{h})$ do not vanish for all $h\in H$. Therefore, $deg_{D_\gamma}(\overline{h})=ord^{(K)}(h)=ord^{(K)}(\overline{h})$ and thereby, $\overline{h}$ is normalized with respect to D_γ.■

Applying Lemma 3.1 to the family $\{h^{(i)}\}_{r+1\le i\le p}$ constructed in (5), we can assume without loss of generality that $0\ne lc_{D_\gamma}(h^{(i)})\in A_m^{(K)}, r+1\le i\le p$.

Consider a certain solution $v_i\in Q_m^{(K)},\quad 1\le i\le p$ of system (4). Fix some $r+1\le i\le p$ for the time being. One can divide (from the right) v_i by $h^{(i)}$ with the remainder in $Q_m^{(K)}$ with respect to D_γ, i.e. $v_i=h^{(i)}\phi_i+\psi_i$ for suitable $\phi_i,\psi_i\in Q_m^{(K)}$ such that $deg_{D_\gamma}(\psi_i)<deg_{D_\gamma}(h^{(i)})=t$. Let $v_i=\sum_{0\le s\le t_1}D_\gamma^s v_{i,s}$ where $v_{i,s}\in A_m^{(\{1,\dots,m\}\setminus\gamma)}$ and $v_{i,t_1}=lc_{D_\gamma}(v_i)$. Taking into account that $h^{(i)}$ is normalized with respect to D_γ, one can rewrite $lc_{D_\gamma}(h^{(i)})D_\gamma^{t_1-t}=D_\gamma^{t_1-t}lc_{D_\gamma}(h^{(i)})+\sum_{0\le s\le t_1-t-1}D_\gamma^s\eta_s$ for appropriate $\eta_s\in A_m^{(K)}$. Thus, one can put the leading term of (the quotient) ϕ_i to be $\phi_{i,t_1-t}=D_\gamma^{t_1-t}(lc_{D_\gamma}(h^{(i)}))^{-1}lc_{D_\gamma}(v_i)\in Q_m^{(K)}$. Then $deg_{D_\gamma}(v_i-h^{(i)}\phi_{i,t_1-t})<t_1$ and one can continue the process of dividing with the remainder achieving finally ϕ_i,ψ_i.

For a fixed $1\le j\le r$ we multiply each of the equalities (5) for $r+1\le i\le p$ from the right by ϕ_i and subtract it from the corresponding equality (4), as a result we get an equivalent to (4) linear system

$$g_j\psi_j+\sum_{r+1\le i\le p}g_{j,i}\psi_i=f_j,\quad 1\le j\le r \tag{7}$$

for certain $\psi_j\in Q_m^{(K)}$. Since $deg(f_j),deg(g_{j,i})\le(4m(r-1)-1)d,\quad deg_{D_\gamma}(\psi_i)<deg_{D_\gamma}(h^{(i)})\le(16m^2r^2-1)d$ (see the end of Section 2) we conclude that $deg_{D_\gamma}(\psi_j)\le N_1\le 16m^2r^2d,\quad 1\le j\le r$.

Represent $\psi_j=\sum_{0\le s\le N_1}D_\gamma^s\psi_{j,s},\quad 1\le j\le p$ for appropriate $\psi_{j,s}\in A_m^{(\{1,\dots,m\}\setminus\gamma)}(A_m^{(K)})^{-1}$.

For each $0\le s\le N_1$ we have

$$g_jD_\gamma^s=\sum_{0\le l\le N_0}D_\gamma^l g_{j,s,l}^{(1)},\quad g_{j,i}D_\gamma^s=\sum_{0\le l\le N_0}D_\gamma^l g_{j,i,s,l}^{(1)} \tag{8}$$

for appropriate $g_{j,s,l}^{(1)},g_{j,i,s,l}^{(1)}\in A_m^{(\{1,\dots,m\}\setminus\gamma)}$ where $N_0,deg(g_{j,s,l}^{(1)}),deg(g_{j,i,s,l}^{(1)})\le 16m^2r^2d$. Substituting the expressions (8) in (7) and subsequently equating the coefficients at the same powers of D_γ, we obtain the following linear system over $A_m^{(\{1,\dots,m\}\setminus\gamma)}(A_m^{(K)})^{-1}$:

$$\sum_{j,s}g_{j,s,l}^{(2)}\psi_{j,s}=g_l^{(2)} \tag{9}$$

being equivalent to system (7) and thereby, to system (3), in other words, these systems are solvable simultaneously. Moreover, $g_{j,s,l}^{(2)},g_l^{(2)}\in A_m^{(\{1,\dots,m\}\setminus\gamma)},\quad deg(g_{j,s,l}^{(2)}),deg(g_l^{(2)})\le 16m^2r^2d$, the number of the equations in system (9) does not exceed $16m^2r^2d$ and the number of the indeterminates $\psi_{j,s}$ is less than $16pm^2r^2d$.

We summarize the proved above in this section in the following lemma.

Lemma 3.2 *A linear system (3) of q equations in p indeterminates with the degrees of the coefficients $a_{j,i}, a_j$ at most d is solvable over the algebra $Q_m(K)$ if and only if the linear system (9) is solvable over the algebra $A_m^{(\{1,\dots,m\}\setminus\gamma)}(A_m^{(K)})^{-1}$. System (9) in at most $16pm^2r^2d$ indeterminates and in at most $16m^2r^2d$ equations has the coefficients from the algebra $A_m^{(\{1,\dots,m\}\setminus\gamma)}$ of the degrees less than $16m^2r^2d$ where $r \le \min\{p,q\}$ is the rank of the system (3).*

Moreover, if system (9) has a solution with the degrees not exceeding a certain λ then system (3) has a solution with the degrees not exceeding $\lambda + 16m^2r^2d$.

Thus, we have eliminated the symbol D_γ. Continuing by recursion applying Lemma 3.2 we eliminate consecutively D_δ for all $\delta \notin K$ and finally yield a linear system

$$\sum_{1\le l\le N_3} g_{s,l}^{(0)} V_l^{(0)} = g_s^{(0)}, \quad 1 \le s \le N_2 \tag{10}$$

over the skew-field $A_m^{(K)}(A_m^{(K)})^{-1}$ with the coefficients $g_{s,l}^{(0)}, g_s^{(0)} \in A_m^{(K)}$ where $N_2, deg(g_{s,l}^{(0)}), deg(g_s^{(0)}) \le N_4 = (2m)^{4^{m-|K|}}(dr)^{3^{m-|K|}}$ and the number of the indeterminates $N_3 \le pN_4$. Notice that system (10) is solvable simultaneously with system (3).

As in Section 2 one can reduce (with the help of Lemma 2.2) system (10) to the diagonal-trapezium form similar to (4) with the coefficients from the algebra $A_m^{(K)}$ having the degrees less than $2(m+|K|)N_4^2$ due to Lemma 2.1. Therefore, if system (10) has a solution in the skew-field $A_m^{(K)}(A_m^{(K)})^{-1}$ it should have a solution of the form $v_l^{(0)} = (b_l^{(1)})^{-1}b_l^{(2)} \in (A_m^{(K)})^{-1}A_m^{(K)}$ with the degrees $deg(b_l^{(1)}), deg(b_l^{(2)}) \le 2(m+|K|)N_4^2$ taking into account the achieved diagonal-trapezium form. Applying Corollary 1.2 to $v_l^{(0)}$ one can represent $v_l^{(0)} = v_l^{(3)}(v_l^{(4)})^{-1}$ for suitable $v_l^{(3)}, v_l^{(4)} \in A_m^{(K)}$ with the degrees $deg(v_l^{(3)}), deg(v_l^{(4)}) \le 4(m+|K|)^2N_4^2, \quad 1 \le l \le N_3$. Hence due to Lemma 3.2 it provides a solution of system (3) over the algebra $Q_m(K) = A_m(A_m^{(K)})^{-1}$ with the bounds on the degrees $N_5 = 4(m+|K|)N_4^2$. This completes the proof of Theorem 0.2. ∎

Finally we observe that if system (3) has a solution it has also a solution of the form $v_i = c_i b^{-1}$ for appropriate $c_i \in A_m, \quad b \in A_m^{(K)}$ with the degrees $deg(c_i), deg(b) \le (2(m+|K|)p+1)N_5, \quad 1 \le i \le q$ due to Corollary 1.2. The algorithm looks for a solution of system (3) just in this form with the indeterminate coefficients over the field F at the monomials in the symbols $X_1, \dots, X_m, D_1, \dots, D_m$ and treat (3) or equivalently, $\sum_{1\le i\le p} a_{j,i}c_i = a_j b, \quad 1 \le j \le q$ as a linear system over F in the indeterminate coefficients. This completes the proof of Corollary 0.3. ∎

4 A bound on the leading coefficient of the Hilbert-Kolchin polynomial of a linear differential module

In the sequel we use the notations from the Introduction. If the degree $0 \le t \le m$ of the Hilbert-Kolchin polynomial of the left L_m-module L equals to m then the leading coefficient l is at most n [6].

From now on assume that $t < m$. For each $1 \le i_0 \le n$ and any family $K = \{k_0, \dots, k_t\} \subset \{1, \dots, n\}$ of $t+1$ integers there exists an element $0 \ne (0, \dots, 0, b_{i_0}^{(0)}, 0, \dots, 0) \in L$ with a single non-zero coordinate at the i_0-th place where $b_{i_0}^{(0)} \in A_m^{(K)}(F[X_1, \dots, X_m])^{-1}$, taking into account that the differential type of L equals to t (cf. Proposition 2.4 [11]). Rewriting the latter condition as a system of linear equations

$$\sum_{1\leq j\leq s} C_j w_{i,j} = 0, \quad i \neq i_0, \quad \sum_{1\leq j\leq s} C_j w_{i_0,j} = 1$$

in the indeterminates $C_1, \dots, C_s$ over the algebra $Q_m^{(K)}$ and making use of Theorem 0.2 one can find a solution of this system in the form $c_1 = (b_{i_0})^{-1}a_{1,i_0}, \dots, c_s = (b_{i_0})^{-1}a_{s,i_0} \in Q_m^{(K)}$ for suitable $b_{i_0} \in A_m^{(K)}$, $a_{1,i_0}, \dots, a_{s,i_0} \in A_m$ with the degrees $deg(b_{i_0}), deg(a_{1,i_0}), \dots, deg(a_{s,i_0}) \leq (16m^4d^2(\min\{n,s\})^2)^{4^{m-t-1}}$. Thus, $0 \neq (0, \dots, 0, b_{i_0}, 0, \dots, 0) \in L$.

Applying Lemma 3.1 to the family $\{b_{i_0}\}_{1\leq i_0\leq n}$ we conclude that after an appropriate F-linear transformation Ω of the subspace with the basis $D_{k_0}, \dots, D_{k_t}$ and the corresponding transformation $(\Omega^T)^{-1}$ of the subspace with the basis $X_{k_0}, \dots, X_{k_t}$, one can suppose that $b_{i_0} = \alpha_e D_{k_0}^e + \beta_{e-1} D_{k_0}^{e-1} + \dots + \beta_0$ is normalized with respect to D_{k_0} where $0 \neq \alpha_e \in F[X_1, \dots, X_m]$ and $\beta_{e-1}, \dots, \beta_0 \in A_m^{(K\setminus\{k_0\})}$. The Hilbert-Kolchin polynomial does not change under the F-linear transformation Ω. Taking into account that these transformations keep the relations (1) of the Weyl algebra (see the proof of Lemma 3.1), in the applications of these transformations below we may preserve the same notations for the basis of the resulting Weyl algebra after transformations.

First we apply the described above construction to the family $K = \{1, \dots, t+1\}$ and obtain normalized elements $(0, \dots, 0, b_{i_0}^{(1)}, 0, \dots, 0) \in L$, $1 \leq i_0 \leq n$ with respect to D_1. Thereupon consecutively we take $K = \{2, \dots, t+2\}, \dots, K = \{m-t, \dots, m\}$ and obtain elements $(0, \dots, 0, b_{i_0}^{(2)}, 0, \dots, 0), \dots, (0, \dots, 0, b_{i_0}^{(m-t)}, 0, \dots, 0) \in L$, $1 \leq i_0 \leq n$ being normalized with respect to $D_2, \dots, D_{m-t}$, correspondingly.

Hence any element in the quotient $F(X_1, \dots, X_m)$-vector space L_m^n over the left L_m-module L can be reduced to the form $(\sum_I h_{1,I} D_1^{i_1} \cdots D_m^{i_m}, \dots, \sum_I h_{n,I} D_1^{i_1} \cdots D_m^{i_m})$ where the coefficients $h_{j,I} \in F(X_1, \dots, X_m)$ and $i_1, \dots, i_{m-t} \leq (16m^4d^2(\min\{n,s\})^2)^{4^{m-t-1}}$. This completes the proof of Corollary 0.1. ∎

Acknowledgements. The paper was done partially under the support of the Humboldt-Preis. The author is grateful to Michel Granger, Fritz Schwarz, Serguey Tsarev for their attention and to SCAI (Fraunhofer Institut) for the hospitality during the stay there.

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