Self-similar Fractals in Arithmetic

Arash RASTEGAR

Institut des Hautes Études Scientifiques

35, route de Chartres

91440 – Bures-sur-Yvette (France)

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Abstract

We define a notion of self-similarity on algebraic varieties by considering algebraic endomorphisms as "similarity" maps. Self-similar objects are called fractals, for which we present several examples and define a notion of dimension in many different contexts. We also present a strong version of Roth's theorem for algebraic points on a variety approximated by fractal elements. Fractals provide a framework in which one can unite several important conjectures in Diophantine geometry.

Introduction

By self-similar fractals, we mean objects which are union of pieces similar to the whole object. However, in this paper we assume that a fractal is a finite union of its "almost disjoint" similar images. By "almost disjoint" sets, we mean sets with at most finite intersection. Each of these similarities is given by means of a "simple" map from the ambient space of the fractal into itself which is different from the identity map. In geometric context, if the ambient space is a vector space, a "simple" map could mean a linear or affine map. There are several interesting examples of geometric fractals in the literature. Serprinski carpet, Koch snowflake, and Cantor set are among the typical examples of affine fractals. In algebraic context, if the ambient space is a ring, a "simple" map could mean a ring endomorphism or a translation of such an endomorphism by an element of the ring. Finitely generated ideals are examples of fractals in this context. There are more complicated self-similar subsets of a ring which do not inherit any algebraic structure. In arithmetic context, if the ambient space is an arithmetic variety, a "simple" map could mean an endomorphism which respects the arithmetic structure. Rational points on a projective space could be thought of as a huge self-similar subset of the projective space.

The first important question about fractals is how to define their dimension. One should introduce a well-behaved notion of dimension which is compatible with the geometric intuition we have from the integer-dimensional objects or compatible with different algebraic notions of dimension. We use arithmetic height-functions to introduce such a notion of dimension for fractals in several arithmetic ambient spaces. In fact, this notion of fractal-dimension turns out to be related to the growth of the number of points of bounded height. This way, we recover several classical results in this direction. One can perform these calculations in many different contexts.

One can think of Diophantine approximation of algebraic points by a fractal whose elements are algebraic over $\mathbb{Q}$. Self-similarity of fractals imply a strong version of Roth's theorem in this case. One shall note that, fractals are not necessarily dense in the ambient space with respect to complex topology. Therefore, such approximation theorems are only interesting if we are approximating a limiting point with respect to some Riemannian metric.

In arithmetic geometry, there are a few quite well-known objects which can also be considered as fractals. For example, let A denote an abelian variety over a number-field. The set of rational points $A(\mathbb{Q})$ and the set of torsion points A^{tor} on A can be thought of as fractals with respect to self-covering maps of A. Fractals provide a common framework in which similar theorems about these two types of objects could be united in a single context. Indeed, similarity between Manin-Mumford conjecture on torsion points on a abelian variety which was proved by Raynaud [Ra], and Lang's conjecture on finitely generated subgroups of rational points on an abelian variety which was proved by Faltings [Fa] lead us to the following general conjecture about arithmetic fractals which covers a version of generalized Lang's conjecture:

Conjecture 0.1 *Let X be an irreducible variety defined over a finitely generated field K and let $F \subset X(\bar{K})$ denote a fractal on X. Then, for any reduced subscheme Z of X defined over K the Zariski closure of $Z(\bar{K}) \cap F$ is union of finitely many points and finitely many components B_i such that $B_i(\bar{K}) \cap F$ is a fractal in B_i for each i.*

Detailed evidences for the above conjecture are presented in the final chapter. Motivated by Andre-Oort's conjecture, we also present a more geometric version in which geometric correspondences are considered as self-similarities. This version covers a special case of Andre-Oort conjecture for l-Hecke orbit of a special point, and a parallel version in the function field case [Br].

1 Fractals in $\mathbb{Z}$

The idea of considering fractal subsets of $\mathbb{Z}$ is due to O. Naghshineh who proposed the following problem for "International Mathematics Olympiad" held in Scotland in July 2002.

Problem 1.1 *Let F be an infinite subset of $\mathbb{Z}$ such that $F = \bigcup_{i=1}^{n} a_i.F + b_i$ for integers a_i and b_i where $a_i.F + b_i$ and $a_j.F + b_j$ are disjoint for $i \neq j$ and $|a_i| > 1$ for each i. Prove that*

$$\sum_{i=1}^{n} \frac{1}{|a_i|} \geq 1.$$

In [Na], he explains his ideas about fractals in $\mathbb{Z}$ and suggests how to define their dimension and how to prove this notion is independent of the choice of self-similarity maps. His suggestions are carried out by H. Mahdavifar. In this section, we present their results and ideas.

Definition 1.2 *Let $\phi_i : \mathbb{Z} \to \mathbb{Z}$ for $i = 1$ to n denote linear maps of the form $\phi_i(x) = a_i.x + b_i$ where a_i and b_i are integers with $|a_i| > 1$. A subset $F \subseteq \mathbb{Z}$ is called a fractal with respect to ϕ_i if F is disjoint union of its images under the linear map ϕ_i. In this case, we write $F = \sqcup_i \phi_i(F)$ and define dimension of F to be the real number s such that*

$$\sum_{i=1}^{n} |a_i|^{-s} = 1.$$

The basic example for fractals in $\mathbb{Z}$ is the set of integers which miss a number of digits in their decimal expansion. This definition of dimension is motivated by the notion of box dimension for fractals on real vector spaces, which coincides with Hausdorff dimension [Falc]. The challenge is to prove that, this notion of dimension is independent of all the choices made, and depends only on the fractal itself as a subset of $\mathbb{Z}$. Also, smaller fractals should have smaller dimension. Having this proven, it is easy to solve the above IMO problem. Note that $\mathbb{Z}$ is a fractal of dimension one. A fractal $F \subseteq \mathbb{Z}$ is of dimension ≤ 1 which solves the problem.

Lemma 1.3 *Let $F \subseteq \mathbb{Z}$ satisfy $F \subseteq \cup_i \phi_i(F)$ where ϕ_i are as above. If s is a real number such that $\sum_i |a_i|^{-s} < 1$ then the number of elements of F in the ball $B(x)$ is bounded above by cx^s for some constant c and for large x.*

Proof: Let $F_i = \phi_i(F)$, and let $N(x)$ and $N_i(x)$ denote the number of elements of F and F_i in the ball $B(x)$, respectively. We have

$$N(x) \leq \sum_i N_i(x)$$

and since for $f \in F_i$ and $\phi_i^{-1}(f) \in F$ we have $|\phi_i^{-1}(f)| \leq (|f| + |b_i|)/|a_i|$ we can write

$$N_i(x) \leq N(\frac{x + |b_i|}{|a_i|})$$

If we let $t = Max_i\{|b_i|/|a_i|\}$ then we get the following estimate

$$N(x) \leq \sum_i N(\frac{x}{|a_i|} + t)$$

We define a function $h : [1, \infty] \to \mathbb{R}$ by $h(x) = x^{-s}N(x)$ and we shall show that h is a bounded function. The above estimate will have the form

$$h(x) \leq \sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s h(\frac{x}{|a_i|} + t)$$

There exists a constant M such that for $x > M$ we have $(x/|a_i|) + t < x$ for all i and

$$\sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s < 1$$

Now, assume $|a_1| \leq ... \leq |a_n|$ and define $x_0 = |a_n|(M-t)$ and $x_j = |a_1|(x_{j-1} - t)$ for $j \geq 1$. Then x_j is an unbounded decreasing sequence. The function h is bounded on $[M, x_0]$ and we inductively show that it has the same bound on $[x_j, x_{j+1}]$: for if $x \in [x_j, x_{j+1}]$ then $(x/|a_i|) + t \in [(x_j/|a_i|) + t, x - j + 1/|a_i|) + t] \subset [M, x_j]$ and if by induction hypothesis we have $h(x/|a_i|) + t) \leq c$ for all i then

$$h(x) \leq \sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s h(\frac{x}{|a_i|} + t) < c \sum_i (\frac{1}{|a_i|} + \frac{t}{x})^s < c$$

It remains to notice that h is also bounded on $[1, M]$. □

Lemma 1.4 *Let $F \subseteq \mathbb{Z}$ satisfy $F \supseteq \sqcup_i \phi_i(F)$ where ϕ_i are as above. If r is a real number such that $\sum_i Norm(a_i)^{-r} > 1$ then the number of elements of F in the ball $B(x)$ is bounded below by cx^s for some constant c and for large x.*

Proof: We use the notation in the proof of the previous lemma. Since for $f \in F_i$ and $\phi_i^{-1}(f) \in F$ we have $|\phi_i^{-1}(f)| \geq (|f| - |b_i|)/|a_i|$ and we get

$$N_i(x) \geq N(\frac{x - |b_i|}{|a_i|}) \geq N(\frac{x}{|a_i|} - t)$$

where $t = Max_i\{|b_i|)/|a_i|\}$. Now, it remains to show that $h : [1, \infty] \to \mathbb{R}$ defined by $h(x) = x^{-r}N(x)$ is bounded below, which can be proved along the same line as the previous lemma. □

Theorem 1.5 *Let $F_1 \subseteq F_2 \subseteq \mathbb{Z}$ be fractals. Then the notion of fractal dimension is well-defined and $dim(F_1) \leq dim(F_2)$.*

Proof: Suppose $F = \sqcup_i \phi_i(F) = \sqcup_j \psi_j(F)$ where ϕ_i and ψ_i are linear functions $\phi_i(x) = a_i.x + b_i$ and $\psi_j(x) = c_j.x + d_j$. Assume $\sum_i |a_i|^{-\alpha} = 1$ and $\sum_i |c_j|^{-\beta} = 1$. We must show that $\alpha = \beta$. Suppose $\alpha < \beta$. Insert real numbers $\alpha < s < r < \beta$. Since $F \subset \cup_i \phi_i(F)$ and $\sum_i |a_i|^{-s} < 1$, we get $N(x) \leq cx^s$ for large x and since $F \supseteq \sqcup_i \psi_j(F)$ and $\sum_i |c_j|^{-r} > 1$, we get $N(x) \geq cx^r$ for large x which is a contradiction. Thus $\alpha = \beta$.

Now, for fractals $F_1 \subseteq F_2$ suppose that $F_1 = \sqcup_i \phi_i(F)$ and $F_2 = \sqcup_j \psi_j(F)$ where ϕ_i and ψ_i functions as above, and let $\sum_i |a_i|^{-\alpha} = 1$ and $\sum_i |c_j|^{-\beta} = 1$. We must show that $\alpha \leq \beta$. Suppose $\alpha > \beta$ and insert real numbers $\alpha > r > s > \beta$. Then one can get a contradiction as above. □

Naghshineh and Mahdavifar also suggest that the same calculations work for $\mathbb{Z}[i]$ if we use norm of a complex number instead of absolute value for a real number. The same arguments indicates that, the notion of dimension of a fractal is linked to asymptotic behavior of the number of points of bounded norm.

2 Fractals in algebraic geometry

Let K be a number field and O_K its ring of integers. We take polynomial maps $\phi_i : O_K \to O_K$ as self-similarities. Any element in O_K has a finite norm which suggests to define a fractal in O_K and its dimension in the following manner:

Algebraic Definition 2.1 *Let O_K be the ring of integers in a number-field and let $\phi_i : O_K \to O_K$ for $i = 1$ to n denote polynomial maps of degree n_i with coefficients in O_K where a_i is the leading coefficient. Assume $Norm(a_i) > 1$ in case ϕ_i is linear. A subset $F \subseteq O_K$ is called a fractal with respect to ϕ_i if F is almost disjoint union of its images under ϕ_i. In this case, we write $F = \sqcup_i' \phi_i(F)$ and define the dimension of F to be the real number s for which*

$$\sum_{i=1}^{n} Norm(a_i)^{-\frac{s}{n_i}} = 1.$$

Arguments of section one hold almost line by line if we replace absolute value of an integer with norm of an algebraic integer in O_K and use the fact that the set of numbers in O_K of finite norm is finite. Therefore, we can prove

Proposition 2.2 *The notion of dimension for fractals in O_K is well-defined and well-behaved with respect to inclusion of fractals, i.e. dimension of a fractal is independent of the choice of self-similarities and compatible with inclusions of fractals.*

Once one gets interesting fractals via polynomial maps, it is tempting to try the more general setting of algebraic geometry. There are fairly general frameworks in which we have an understanding of asymptotic behavior of points of bounded norm. For example, start from a linear semi-simple algebraic group G and a rational representation $\rho : G \to GL(W_{\mathbb{Q}})$ defined over $\mathbb{Q}$. Let $w_0 \in W_{\mathbb{Q}}$ be a point whose orbit $V = w_0\rho(G)$ is Zariski closed. Then the stabilizer $H \subset G$ of w_0 is reductive and V is isomorphic to $H \setminus G$. By a theorem of Borel-Harish-Chandra $V(\mathbb{Z})$ breaks up to finitely many $G(\mathbb{Z})$ orbits [Bo-HC]. Thus the points of $V(\mathbb{Z})$ are parametrized by cosets of $G(\mathbb{Z})$. Fix an orbit $w_0G(\mathbb{Z})$ with w_0 in $G(\mathbb{Z})$. Then the stabilizer of w_0 is $H(\mathbb{Z}) = H \cap G(\mathbb{Z})$. Duke-Rudnick-Sarnak [D-R-S] putting some extra technical assumptions, have determined the asymptotic behavior of

$$N(V(\mathbb{Z}), x) = \sharp\{\gamma \in H(\mathbb{Z}) \setminus G(\mathbb{Z}) : ||w_0\gamma|| \leq x\}.$$

They prove that there are constants $a \geq 0, b > 0$ and $c > 0$ such that

$$N(V(\mathbb{Z}), x) \sim cx^a(logx)^b.$$

The additive structure of G allows one to define self-similar subsets of $V(\mathbb{Z})$ and study their asymptotic behavior using the idea of fractal dimension. For example, one can define self-similarities to be maps $\phi : V(\mathbb{Z}) \to V(\mathbb{Z})$ of the form

$$\phi(\omega_0\gamma) = \omega_0([n]\gamma + g_0)$$

where $[n]$ denotes multiplication by n in $G(\mathbb{Z})$ and g_0 is an element in $G(\mathbb{Z})$. These similarity maps are expansive if $n > 1$ and lead to a notion of dimension for fractals in $V(\mathbb{Z})$. Note that, the whole set $V(\mathbb{Z})$ could not be a fractal, since the asymptotic behavior of its points is not polynomial. This leads us to a very general definition.

Algebro-Geometric Definition 2.3 *Let X be an algebraic variety and let f_i for $i = 1$ to n denote endomorphisms of X of degrees > 1. A subset $F \subseteq X$ is called a fractal with respect to f_i if F is almost disjoint union of its images under the endomorphisms f_i, i.e. $F = \sqcup_i f_i(F)$.*

There are several interesting examples of fractals in algebraic geometry. Let A denote an abelian variety over $\mathbb{C}$ and let $[n]$ denote multiplication by n endomorphism of A. Then the set of torsion elements A^{tor} in A is a fractal of dimension zero with respect to $[n]$ for every integer $n > 1$. Any finitely generated subgroup $\Gamma \subset A$ is a fractal with respect to a few translations of $[n]$ for large enough n.

There also exist fractals for which, the endomorphisms of the ambient space are not induced from a group structure. For examples, let $\mathbb{P}^n(\mathbb{C})$ denote the complex projective space of dimension n and $f_i = (\phi_1, ..., \phi_n)$ be an endomorphism which consists of n homogeneous polynomials of degree m_i. Then one could construct fractals with respect to these functions. The whole $\mathbb{P}^n(\mathbb{Q})$ is self-similar but not a fractal, since it is disjoint union of infinitely many copies of itself. The subset

$$\{(2^i; 2^j) \in \mathbb{P}^1(\mathbb{Q}) | i, j \in \mathbb{N} \cup \{0\}\}$$

is a fractal with respect to $f_1(x_1; x_2) = (x_1^2; x_2^2)$ and $f_2(x_1; x_2) = (2x_1^2; x_2^2)$.

Let X denote an algebraic variety and $f : X \to X$ be an endomorphism. The set of periodic points and the set of pre-periodic points of f are zero-dimensional fractals on X.

In order to have a functorial setting, we should extend the notion of fractal to quasi-fractal by replacing endomorphisms with geometric correspondences. This way, one can pull back a fractal to a new ambient space. Let us forget our concern about defining fractal dimension and remove the condition that similar images should have at most finite intersection.

Algebro-Geometric Definition 2.4 *Let X be an algebraic variety and let $Y_i \hookrightarrow X \times X$ for $i = 1$ to n denote correspondences on X where π_1 and π_2 are projection to the first and second factor in $X \times X$ are finite and surjective. A subset $F \subseteq X$ is called a quasi-fractal with respect to Y_i if F is union of its images under the action of correspondences Y_i,*

$$F = (\cup_i \pi_{1*} \circ \pi_2^*(F))$$

The l-Hecke orbit of a point on the moduli-space of principally polarized abelian varieties is an example of a quasi-fractal with respect to the l-Hecke correspondences associated to l-isogenies.

In general, there is no global norm on the set of points in a fractal to motivate us how to define the notion of dimension. In special cases, arithmetic height functions are appropriate replacements for the norm of an algebraic integer, particularly because finiteness theorems hold in this context.

3 Fractals in arithmetic geometry

Northcott associated a heights function to points on the projective space which are defined over number fields [No]. In course of his argument for the fact that the number of periodic points defined over a given number-field of an endomorphism of a projective space are finite, he proved that the number of points of bounded height is finite. Therefore, one can study the asymptotic behavior of rational points on a fractal hosted by the projective space.

Proposition-Definition 3.1 *Let f_i for $i = 1, ..., n$ denote homogeneous endomorphisms of a projective space defined over a global field K with each homogeneous component of degree m_i. Let $F \subseteq \mathbb{P}^n(K)$ be a fractal with respect to f_i: $F = \sqcup_i' f_i(F)$. We define the dimension of F to be the real number s for which $\sum_i m_i^{-s} = 1$. Then dimension of F is well-defined and well-bahaved with respect to fractal embeddings.*

Proof: In the number-field case, we use the logarithmic height h to control the height growth of points under endomorphisms. Again we claim that if $\sum_i m_i^{-s} < 1$ and $F \subseteq \cup_i f_i(F)$ then the number of elements of F of logarithmic height less than x which we denote again by $N(x)$ is bounded above by cx^s for some constant c and large x. Let $F_i = f_i(F)$, and $N_i(x)$ denote the number of elements of F_i of logarithmic height less than x. We have

$$N(x) \leq \sum_i N_i(x)$$

and for $f \in F_i$ and $f_i^{-1}(f) \in F$ we have $h(f_i(f)) = m_i.h(f) + O(1)$. Therefore

$$N(x) \leq \sum_i N(m_i^{-1}x + t)$$

for some t. We define a function $\bar{h} : [1, \infty] \to \mathbb{R}$ by $\bar{h}(x) = x^{-s}N(x)$. The argument of Lemma 1.3 implies that $\bar{h}$ is bounded, and hence the claim follows. By a similar argument, if $F \supseteq \sqcup_i f_i(F)$ and if r is a real number such that $\sum_i m_i^{-r} > 1$ then $N(x)$ is bounded below by cx^s for some constant c and large x. One can follow the argument of theorem 1.5 to finish the proof.

In the function field case, one could follow the same argument using an appropriate height function. Let $\mathbb{F}_q(X)$ denote the function field of an absolutely irreducible projective variety X which is non-singular in codimension one, defined over a finite field $\mathbb{F}_q$ of characteristic p. One can use the logarithmic height on $\mathbb{P}^n(\mathbb{F}_q(X))$ defined by Neron [La-Ne]. Finiteness theorem holds for this height function as well.□

Let $h, R, w, r_1, r_2, d_K, \zeta_K$ denote class number, regulator, number of roots of unity, number of real and complex embeddings, absolute discriminant and the zeta function associated to the number field K. Schanuel proved that [Scha] the asymptotic behavior of points in $\mathbb{P}^n(K)$ of logarithmic height bounded by $log(x)$ is given by

$$\frac{hR}{w\zeta_K(n+1)}\left(\frac{2^{r_1}(2\pi)^{r_2}}{d_K^{1/2}}\right)^{n+1}(n+1)^{r_1+r_2-1}x^{n+1}.$$

This shows that rational points on projective space can not be regarded as a fractal of finite dimension.

Schmidt in case $K=\mathbb{Q}$ [Schm] and Thunder for general number field K [Th] generalized the estimate of Shanuel to Grassmanian varieties, and proved that

$$C(G(m,n)(K), log(x)) \sim c_{m,n,K}x^n$$

where C denotes the number of points of bounded logarithmic height and $c_{m,n,K}$ is an explicitly given constant. Also, Franke-Manin-Tschinkel provided a generalization to flag manifolds [Fr-Ma-Tsh]. Let G be a semi-simple algebraic group over K and P a parabolic subgroup and $V = P\backslash G$ the associated flag manifold. Choose an embedding of $V \subset \mathbb{P}^n$ such that the hyperplane section H is linearly equivalent to $-sK_V$ for some positive integer s, then there exists an integer $t \geq 0$ and a constant c_V such that

$$C(V(K), x)^s = c_V x(logx)^t.$$

All of these spaces are self-similar objects which have the potential to be ambient spaces for fractals, but they are too huge to be fractals themselves.

Wan proved that [Wa] the asymptotic behavior of points in $\mathbb{P}^n(K)$ of logarithmic height bounded by d is given by

$$\frac{hq^{(n+1)(1-g)}}{(q-1)\zeta_X(n+1)}q^{(n+1)d}.$$

which shows that $\mathbb{P}^n(\mathbb{F}_q(X))$ can indeed be considered as a finite dimensional fractal.

Proposition-Definition 3.2 *Let A be an abelian variety over a number-field K and let $F \subseteq A(\bar{\mathbb{Q}})$ be a fractal with respect to endomorphisms ϕ_i which are translations of multiplication maps $[n_i]$ by elements of $A(\bar{\mathbb{Q}})$. We define dimension of F to be the real number s for which $\sum_i n_i^{-s} = 1$. Then dimension of F is well-defined and well-bahaved with respect to fractal embeddings.*

Proof: We use the Neron-Tate logarithmic height $\hat{h}$ to control the growth of the heights of points under the action of endomorphisms ϕ_i. The same proof as before works except that

$$\hat{h}([n_i](f)) = (n_i)^2\hat{h}(f)$$

does not hold for translations of the form $[n_i]$. One should use the fact that for the Neron-Tate height associated to a symmetric ample bundle on A and for every $a \in A(\bar{\mathbb{Q}})$ and $n \in \mathbb{N}$, we have

$$\hat{h}([n](f)+a)+\hat{h}([n](f)-a)=2\hat{h}([n](f))+2\hat{h}(a).$$

This helps to get the right estimate. The rest of proof goes as before.□

The above notion of dimension implies that the number of points of bounded height defined over a fixed number-field has polynomial growth, which given an immediate proof for the following classical result of Neron [Ne].

Theorem 3.3 *(Neron) Let $A \subset \mathbb{P}^n$ denote an abelian variety defined over a number field K and let $r = r(A, K)$ denote the rank of the group of K-rational points in A, then there exists a constant $c_{A,K}$ such that*

$$N(A(K), x) \sim c_{A,K} x^{r/2}.$$

Analogous to abelian varieties, one also can define fractals on t-modules. By a t-module of dimension N and rank d defined over the algebraic closure $\bar{k} = \overline{\mathbb{F}_q(t)}$ we mean, fixing an additive group $(\mathbb{G}_a)^N(\bar{k})$ and an injective homomorphism Φ from the ring $\mathbb{F}_q[t]$ to the endomorphism ring of $(\mathbb{G}_a)^N$ which satisfies

$$\Phi(t) = a_0 F^0 + ... + a_d F^d$$

with a_d non-zero, where a_i are $N \times N$ matrices with coefficients in $\bar{k}$, and F is a Frobenius endomorphism on $(\mathbb{G}_a)^N$.

Proposition-Definition 3.4 *Think of polynomials $P_i \in \mathbb{F}_q[t]$ of degrees r_i for $i = 1$ to n as self-similarities of the t-module $(\mathbb{G}_a)^N$ and let $F \subseteq (\mathbb{G}_a)^N(\bar{k})$ be a fractal with respect to P_i , i.e. $F = \sqcup_i \Phi(P_i)(F)$. We define the fractal dimension of F to be the real number s such that $\sum_i (r_i d)^{-s} = 1$. Then dimension of F is well-defined and well-bahaved with respect to inclusions.*

Proof: Denis defines a canonical height $\hat{h}$ on t-modules which satisfies

$$\hat{h}[\Phi(P)(\alpha)] = q^{dr}.\hat{h}[\alpha]$$

for all $\alpha \in (\mathbb{G}_a)^N$, where P is a polynomial in $\mathbb{F}_q[t]$ and degree r [De]. This can be used to prove the result in the same lines as before.□

One can get information on the asymptotic behavior of $N(\mathbb{G}_a^N(\bar{k}), x)$ by representing $\mathbb{G}_a^N(\bar{k})$ as a fractal.

4 Diophantine approximation by fractals

Roth's theorem on Diophantine approximation of points on projective line implies a version on projective spaces, which in turn induces the corresponding statement on projective varieties defined over number-fields. Self-similarity of rational points on abelian varieties strengthens the estimates. This argument can be imitated in case of arbitrary fractals defined over a fixed number-field.

Theorem 4.1 *Fix a number-field K and $\sigma : K \hookrightarrow \mathbb{C}$ a complex embedding. Let V be a smooth projective algebraic variety defined over K and let L be an ample line-bundle on V. Denote the arithmetic height function associated to the line-bundle L by h_L. Suppose $F \subset V(K)$ is a fractal subset with respect to finitely many height-increasing self-endomorphisms $\phi_i : V \to V$ defined over K such that for all i we have*

$$h_L(\phi_i(f)) = m_i h_L(f) + 0(1)$$

where $m_i > 1$. Fix a Riemannian metric on $V_\sigma(\mathbb{C})$ and let d_σ denote the induced metric on $V_\sigma(\mathbb{C})$. Then for every $\delta > 0$ and every choice of an algebraic point $\alpha \in V(\bar{K})$ which is not a critical value of any of the ϕ_i's and all choices of a constant C, there are only finitely many fractal points $\omega \in F$ approximating α in the following manner

$$d_\sigma(\alpha, \omega) \leq Ce^{-\delta h_L(\omega)}.$$

Proof: Note that, we have assumed all points on F are defined over a fixed number-field K. Therefore, Roth's theorem implies that the above is true for some $\delta_0 > 0$ without any assumption on ϕ_i or on α.

Fix $\epsilon > 0$ such that $\epsilon < \delta_0 < m_i\epsilon$ for all i. Suppose that w_n is an infinite sequence of elements in F such that $\omega_n \to \alpha$ which satisfies the estimate

$$d_\sigma(\alpha, \omega_n) \leq Ce^{-\epsilon h_L(\omega_n)}.$$

then infinitely many of them are images of elements of F under the same ϕ_i. By going to a subsequence, one can find a sequence ω'_n in F and an algebraic point α' in $V(\bar{K})$ such that $\omega'_n \to \alpha'$ and for a fixed ϕ_i we have $\phi_i(\alpha') = \alpha$ and $\phi_i(\omega'_n) = \omega_n$ for all n. Then

$$d_\sigma(\alpha, \omega_n) \leq Ce^{-\epsilon h_L(\omega_n)} \leq C' e^{-\epsilon m_i h_L(\omega'_n)}$$

for an appropriate constant C'. On the other hand,

$$d_\sigma(\alpha', \omega'_n) \leq C'' d_\sigma(\alpha, \omega_n)$$

holds for an appropriate constant C'' and large n by injectivity of $d\phi_i^{-1}$ on the tangent space of α. This contradicts Roth's theorem because $\delta_0 < m_i\epsilon$. $\square$

Remark 4.2 *The condition that points of F are defined over some number-field is equivalent to the assumption that there are only finitely many points in F of bounded height for any given bound.*

The preceding theorem can not hold for general fractals in $V(\bar{K})$. For example, torsion points of an abelian variety are dense in complex topology, and have vanishing height. Therefore, no analogue of Roth's theorem could hold in this case.

5 Evidence for conjecture 0.1

The concept of fractal provides an appropriate language for us to unify many different results and conjectures in arithmetic geometry. The common geometric structures appearing in the context of dynamics of endomorphisms of algebraic varieties or dynamics of geometric self-correspondences on them, motivates us to formulate two conjectures in the spirit of fractals.

Consider the following result of Raynaud on torsion points of abelian varieties lying on a subvariety [Ra].

Theorem 5.1 *(Raynaud) Let A be an abelian variety over an algebraically closed field $\bar{K}$ of characteristic zero, and Z a reduced subscheme of A. Then every irreducible component of the Zariski closure of $Z(\bar{K}) \cap A(\bar{K})_{tor}$ is a translation of an abelian subvariety of A by a torsion point.*

The following result of Faltings on subgroups of abelian varieties has a very similar feature [Falt].

Theorem 5.2 *(Faltings) Let A be an abelian variety over an algebraically closed field $\bar{K}$ of characteristic zero and Γ be a finitely generated subgroup of $A(\bar{K})$. For a reduced subscheme Z of A, every irreducible component of the Zariski closure of $Z(\bar{K}) \cap \Gamma$ is a translation of an abelian subvariety of A.*

Considering the fact that one could think of $A(\bar{K})_{tor}$ and $A(K)$ as fractals in A, the immediate formulation of the above results in terms of fractals would be the conjecture 0.1 in introduction. A consequence of conjecture 0.1 would be the generalized Lang's conjecture

Conjecture 5.3 *(Lang) Let X be an algebraic variety defined over a number-field K and let $f : X \to X$ be a surjective endomorphism defined over K. Suppose that the subvariety Y of X is not pre-periodic in the sense that the orbit $\{Y, f(Y), f^2(Y), ...\}$ is not finite, then the set of pre-periodic points in Y is not Zariski-dense in Y.*

Lang's conjecture is confirmed by Raynaud's result mentioned above in the case of abelian varieties and by results of Laurent [Lau] and Sarnak [Sa] and [Zh] in the case of multiplicative groups.

Andre-Oort conjecture on sub-varieties of Shimura varieties is motivated by conjectures of Lang and Manin-mumford which were proved by Raynaud and Faltings as mentioned above. Motivated by the Andre-Oort conjecture (look at [Ed] for an exposition of this conjecture), we also present another conjecture in the same lines for quasi-fractals in an algebraic variety X, where self-similarities are allowed to be induced by geometric self-correspondences on X instead of self-maps. This time, we drop the requirement that similar images shall be almost-disjoint.

Conjecture 5.4 *Let X be an irreducible variety defined over a finitely generated field K and let $Q \subset X(\bar{K})$ denote a quasi-fractal on X with respect to correspondences Y_i in $X \times X$ with both projection maps finite and surjective. Then, for any* *reduced subscheme Z of X defined over K the Zariski closure of $Z(\bar{K}) \cap Q$ is union of finitely many points and finitely many components B_i such that for each i the intersection $B_i(\bar{K}) \cap Q$ is a quasi-fractal in B_i with respect to correspondences induced by Y_i.*

The following special case of Andre-Oort conjecture proved by Edixhoven is relevant to the above conjecture [Ed]:

Theorem 5.5 *(Edixhoven) Let S be a Hilbert modular surface and let C be a closed irreducible curve containing infinitely many CM points corresponding to isogeneous abelian varieties, then C is of Hodge type.*

By this result, if a curve C cuts the quasi-fractal Q of the l-Hecke orbit of a CM point inside a Hilbert modular surface, then C is of Hodge type and therefore inherits l-Hecke correspondences which makes $C \cap Q$ a quasi-fractal.

This conjecture also covers a parallel version of Andre-Oort conjecture for l-Hecke orbit of a special point in the function field case [Br].

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Sharif University of Technology, e-mail: rastegar@sharif.edu
Institut des Hautes Etudes Scientifiques, e-mail: rastegar@ihes.fr