DISTANCE FUNCTION, LINEAR QUASI-CONNECTIONS AND CHERN CHARACTER

Nicolae TELEMAN

Institut des Hautes Études Scientifiques

35, route de Chartres

91440 – Bures-sur-Yvette (France)

June 2004

IHES/M/04/27

[1]

Distance Function, Linear quasi-Connections and Chern Character.

Nicolae Teleman

28 May 2004

1 Abstract.

In Sect.4 we show how the Chern character of the tangent bundle of a smooth manifold may be extracted from the geodesic distance function by means of cyclic homology. In Sect.5 we introduce the notion of linear quasi-connection in vector bundles, notion which generalizes the notion of linear connection. We show next that the algebraic procedure for constructing the Chern character, discussed in Sect.4, applies also in the case of coarse linear connections.

Looking in retrospect, the constructions we present here represent the non commutative counterpart of a geometric construction of the Chern character, see Teleman N.[Tn] and Teleman C.[Tc].

The arguments discussed here may be formulated within the language of groupoids.

In a subsequent paper we are going to improve some of the considerations presented here and extend their field of application to more singular situations.

2 Introduction.

This paper is motivated by the intent to make intrinsic the main constructions of global analysis, as much as possible. The purpose of this is not only

*Dipartimento di Scienze Matematiche, Universita Politecnica delle Marche, 60131-Ancona, Italia, e-mail: teleman@dipmat.univpm.it

to find elegant formulas, but also to allow further generalizations, mostly in the case of singular spaces.

In Sect.4 of this note we present a procedure which allows one to extract, from the geodesic distance function, by means of cyclic homology, the Chern character of the tangent bundle of a smooth manifold. In Sect.5 we introduce the notion of linear quasi-connection in vector bundles, notion which generalizes the notion of linear connection. We show next that the algebraic procedure for constructing the Chern character, discussed in Sect.4, applies also to the case of coarse linear connections.

Looking in retrospect, the constructions we present here represent non commutative counterparts of a geometric construction of the Chern character due to Teleman N.[Tn] and Teleman C.[Tc].

The arguments discussed here may be formulated within the language of groupoids.

This is a preliminary version of a subsequent paper in which we propose ourselves to improve some of the considerations presented here, to extend their field of application to more singular situations and to pursue the main objective expressed above.

This research was done with partial support from the project Nr. 2002013241.003 of the Ministero dell'Istruzione dell'Universita e della Ricerca, Italia.

The author thanks the Institut des Hautes Etudes Scientifiques for hospitality.

3 Preliminaries.

3.1 Riemannian geometry preliminaries.

Let g be a smooth Riemannian metric on the manifold M of dimension n.

Let $g_{i,j}(x) =< \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} >$ be the components of the metric tensor g with respect to the local coordinate system (x) and let $g^{ij}(x)$ be the components of the inverse matrix. Customary notation is used for the components of the curvature tensor $R_{rklh} = g_{rs} R^s_{klh}$, where $R^s_{klh} = \frac{\partial \Gamma^s_{kh}}{\partial x^l} - \frac{\partial \Gamma^s_{kl}}{\partial x^h} - \sum_{i=1}^{i=n} (\Gamma^i_{kh} \Gamma^s_{il} - \Gamma^i_{kl} \Gamma^s_{ih})$.

The curvature tensor satisfies the identities, see e.g. [S]

Proposition 3.1 *-i)* $R_{rklh} = -R_{rkhl}, \quad R_{rklh} = -R_{krlh}$
-ii) $R_{rklh} = R_{lhrk}$
-iii) $R_{rklh} + R_{rlhk} + R_{rhkl} = 0.$

The following identity is a corollary of this proposition; it will be used below.

Proposition 3.2 *-i)*

$$R_{i\alpha j\beta} - R_{i\beta j\alpha} = R_{ij\alpha\beta}$$

-ii)

$$\sum_{\alpha\beta}(R_{ij\alpha\beta} - R_{i\beta j\alpha})dx^\alpha \wedge dx^\beta = \frac{3}{2}\sum_{\alpha\beta} R_{ij\alpha\beta}dx^\alpha \wedge dx^\beta.$$

Proof. -i) The identity

$$R_{i\alpha j\beta} + R_{ij\beta\alpha} + R_{i\beta\alpha j} = 0$$

given by Proposition 2.1 -iii) may be rewritten

$$R_{i\alpha j\beta} - R_{ij\alpha\beta} - R_{i\beta j\alpha} = 0,$$

which proves the desired relation.

-ii) We have

$$\sum_{\alpha\beta}(R_{ij\alpha\beta} - R_{i\beta j\alpha})dx^\alpha \wedge dx^\beta =$$

(skew-symmetry of the wedge product)

$$\sum_{\alpha\beta} R_{ij\alpha\beta}dx^\alpha \wedge dx^\beta - \frac{1}{2}\sum_{\alpha\beta}(R_{i\beta j\alpha} - R_{i\alpha j\beta})dx^\alpha \wedge dx^\beta =$$

(identity -i) above)

$$\sum_{\alpha\beta} R_{ij\alpha\beta}dx^\alpha \wedge dx^\beta + \frac{1}{2}\sum_{\alpha\beta} R_{ij\alpha\beta}dx^\alpha \wedge dx^\beta = \frac{3}{2}\sum_{\alpha\beta} R_{ij\alpha\beta}dx^\alpha \wedge dx^\beta.$$

Let $r : M \times M \longrightarrow [0,\infty)$ be the induced geodesic distance function. The function r^2 is smooth on a neighborhood of the diagonal, see e.g. [dR]. The following result is due to H. Donnelly [D]; see also A. Connes and H. Moscovici [CM].

Proposition 3.3 *Let $x_0 \in M$ and let $(x^1, x^2, ..., x^n)$ be a system of normal coordinates at the point x_0, corresponding to an ortho-normal tangent frame at x_0. Then*

-i)

$$g_{rl}(x) = \delta_{rl} - \frac{1}{3} R_{rklh}(0) x^k x^h + O(x^3)$$

-i')

$$g^{rl}(x) = \delta^{rl} + O(x^2)$$

-ii) the first terms of the Taylor expansion of the function r^2 about the origin of the normal coordinates are given by the formula

$$r^2(x,y) = \sum_{i=1}^{n} (x^i - y^i)^2 - \frac{1}{3} R_{rklh}(0) x^k x^h (y^r - x^r)(y^l - x^l) + O((x,y)^5).$$

3.2 Recall of cyclic homology.

Given a locally convex associative algebra $\mathcal{A}$, the space of k-chains $B_k(\mathcal{A})$ of its bar complex $B_*(\mathcal{A})$ is, by definition, a topological completion (usually, projective completion) of the algebraic tensor product $\otimes^{k+1}\mathcal{A}$. The boundary operator is customarily denoted by b' and it is defined by the formula

$$b'(f_0 \otimes f_1 \otimes .... f_{k-1} \otimes f_k = \sum_{r=0}^{r=k-1} (-1)^r f_0 \otimes f_1 \otimes .... \otimes (f_r . f_{r+1}) \otimes ... \otimes f_{k-1} \otimes f_k;$$

the terms of this sum are also called boundary faces.

The graded cyclic permutation $T : B_k(\mathcal{A}) \longrightarrow B_k(\mathcal{A})$ is defined on generators by

$$T(f_0 \otimes f_1 \otimes .... f_{k-1} \otimes f_k) = (-1)^k f_1 \otimes .... f_{k-1} \otimes f_k \otimes f_0.$$

The cyclic complex of the algebra $\mathcal{A}$ is by definition the subcomplex

$$C^{\lambda}_*(\mathcal{A}) = Ker(1 - T).$$

By definition, its homology, $H^{\lambda}_*(\mathcal{A})$, is the cyclic homology of the algebra $\mathcal{A}$, see A. Connes [C.1], [C.2] and J. L. Loday [L].

Theorem 3.4 *(A. Connes [C.1], [C.2])*

-i)

$$H^{\lambda}_k(C^{\infty}(M)) = \Omega^k(M)/d\Omega^{k-1}(M) \bigoplus H^{k-2}_{dR}(M) \bigoplus H^{k-4}_{dR}(M) \bigoplus .... \bigoplus H^{\epsilon}_{dR}(M),$$

where $\Omega^k(M)$ denotes the vector space of smooth k-differential forms on M, $H^*_{dR}(M)$ denotes de Rham cohomology, and ϵ = 0, or 1, depending on the parity of k.

-ii) Given the cyclic cycle $f \in C^\lambda_*(C^\infty)$, its top degree component $[f]_k$ belonging to $\Omega^k(M)/d\Omega^{k-1}(M)$, is given by the formula

$$[f(x_0, x_1, ..., x_k)]_k(x) =$$

$$\frac{1}{k!}\frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}.....\frac{\partial}{\partial x_k^{i_k}} f(x_0, x_1, ..., x_k)_{|x_0=x_1=...=x_k=x} dx_1^{i_1}\wedge dx_2^{i_2}\wedge...\wedge dx_k^{i_k} \quad mod. d\Omega^{k-1}(M).$$

4 The characteristic cyclic function Φ_k.

Introduce on $M \times M$ the double form

$$\varphi(x,y) := d_x d_y(\chi \circ r^2)(x,y) = \sum_{ij} \frac{\partial^2(\lambda \circ r^2)(x,y)}{\partial x^i \partial y^j} dx^i \otimes dy^j,$$

where χ is a cut-off smooth monotone decreasing real valued function, identically 1 on a neighborhood of 0, having support on a sufficiently small interval. We assume that the support of χ is sufficiently small so that $\lambda \circ r^2$ be well defined and smooth.

Let T_x denote the tangent space to M at x. Let $A(x,y) : T_x \to T_y$ be the linear mapping given by the formula

$$A(x,y)(\sum_i \xi^i \frac{\partial}{\partial x^i}) = \sum_{i,j,k} \xi^i \frac{\partial^2(\chi \circ r^2)(x,y)}{\partial x^i \partial y^j} g^{jk}(y) \frac{\partial}{\partial y^k}.$$

The linear mapping $A(x,y)$ is independent of the local coordinates. Let

$$A_i^k(x,y) = \sum_j \frac{\partial^2(\chi \circ r^2)(x,y)}{\partial x^i \partial y^j} g^{jk}(y)$$

denote the components of the matrix associated to A.

Taken an arbitrary chain of points $x_0, x_1, ..., x_k$ together with corresponding local coordinates about them, one defines

$$\Phi_k(x_0, x_1, ..., x_k) := Trace A(x_0, x_1)A(x_1, x_2).....A(x_{k-1}, x_k)A(x_k, x_0).$$

The explicit formula for Φ_k is

$$\Phi_k(x_0, x_1, ..., x_k) = \sum_{i_0, i_1,,, i_k, j_0, j_1,,, j_k} \frac{\partial^2 r^2(x_0, x_1)}{\partial x_0^{i_0} \partial x_1^{j_1}} g^{j_1 i_1}(x_1) \frac{\partial^2 r^2(x_1, x_2)}{\partial x_1^{i_1} \partial x_2^{j_2}} g^{j_2 i_2}(x_2)......$$

$$...\frac{\partial^2 r^2(x_{k-1}, x_k)}{\partial x_{k-1}^{i_{k-1}} \partial x_k^{j_k}} g^{j_k i_k}(x_k) \frac{\partial^2 r^2(x_k, x_0)}{\partial x_k^{i_k} \partial x_0^{j_0}} g^{j_0 i_0}(x_0).$$

The function Φ_k is a well defined smooth real valued function on a neighborhood of the diagonal in M^{k+1}.

We intend to study the function Φ_k within the context of cyclic homology.

Theorem 4.1 *For any smooth Riemannian metric on the smooth manifold M,*

-i) $\Phi_k, k = even$, is a cyclic cycle over the algebra $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$,

-ii) the top degree component of the cyclic homology class of Φ_k is

$$[\Phi_k]_k = -\frac{2^{\frac{k}{2}+1}}{k!}(\frac{k}{2})!(2\pi i)^{\frac{k}{2}} Ch_k(M),$$

where $Ch_k(M)$ is the k-component of the Chern character of the tangent bundle of M.

Proof.

The proof will use the Taylor formulas from Proposition 2.1.

-i) The cyclicity of the function Φ_k follows from the parity of k and the cyclicity of the trace.

It remains to check that Φ_k is a b'-cycle. For, let's evaluate the first boundary face $\Phi_k(x_0, x_0, x_1, ..., x_{k-1})$.

We need to evaluate $\frac{\partial^2 r^2(x_0,x_1)}{\partial x_0^{i_0} \partial x_1^{j_1}} g^{j_1 i_1}(x_1)$ at $x_1 = x_0$. To do this we choose a system of normal coordinates centered at the point x_0 and we apply the Taylor formula for the distance function. We have

$$\frac{\partial^2 r^2(x_0, x_1)}{\partial x_0^i \partial x_1^j}_{|x_0 = x_1 = 0} = \delta_{ij}, \quad g^{ij}(0) = \delta^{ij}.$$

This proves the

Lemma 4.2 *For sufficiently close points x, y, $A(x, y)$ is an isomorphism.*

Therefore,

$$\Phi_k(x_0, x_0, x_1, ..., x_{k-1}) = \Phi_{k-1}(x_0, x_1, ..., x_{k-1}).$$

The needed property follows from the fact that the $b'(\Phi_k)$ consists of a sum of an even number of equal terms multiplied by alternating signs.

-ii) We use Theorem 3.4.(ii) to compute the top degree component $[\Phi_k]_k \in \Omega^k(M)/d\Omega^{k-1}(M)$. The envisioned formula produces a differential form. We intend to determine this form at an arbitrary point $\mathbf{x} \in M$.

We choose a common normal coordinate system with the center at $\mathbf{x}$ for all variable points $x_0, x_1, ..., x_k$. We have to consider first order partial derivatives of the function Φ_k with respect to each of the variables $x_1^{\alpha_1}, x_2^{\alpha_2}, ..., x_k^{\alpha_k}$ and then to evaluate these derivatives at $x_0 = x_1 = ... = x_k = 0$.

Let us see how such derivatives $\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\beta}$ operate on a typical factor $A_i^k(x,y) = \sum_j \frac{\partial^2 r^2(x,y)}{\partial x^i \partial y^j} g^{jk}(y)$ of Φ_k.

If one of these partial derivatives operates on a factor $g^{jk}(y) = \delta^{jk} + O(y^2)$, then this factor is $O(y^1)$ and hence vanishes at 0. Such a factor could not be differentiated twice, or more, because it depends only on one point, y. Therefore, each such factor contributes in the formula for $[\Phi_k]_k$ by its value $g^{jk}(0) = \delta^{jk}$.

Let us analyze the effect of the derivatives $\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\beta}$ on the factors

$$\frac{\partial^2 r^2(x,y)}{\partial x^i \partial y^j} =$$

$$-2\delta_{ij} - \frac{1}{3}\frac{\partial^2}{\partial x^i \partial y^j} R_{rklh}(0) x^k x^h (y^r - x^r)(y^l - x^l) + O((x,y)^3).$$

If such a term is differentiated once, it becomes $O((x,y)^1)$ at 0 and hence it vanishes at the origin. Therefore, only those factors which are differentiated either no time or two times could give a non zero contribution into the formula for $[\Phi_k]_k$.

If this factor is differentiated twice, its value at 0 is

$$-\frac{1}{3}R_{rklh}(0)\frac{\partial^2}{\partial x^\alpha \partial y^\beta}\frac{\partial^2}{\partial x^i \partial y^j}\{x^k x^h (y^r - x^r)(y^l - x^l)\} =$$

$$-\frac{1}{3}R_{rklh}(0)\frac{\partial^2}{\partial x^\alpha \partial x^i}\frac{\partial^2}{\partial y^\beta \partial y^j}\{x^k x^h (y^r - x^r)(y^l - x^l)\} =$$

$$-\frac{1}{3}R_{rklh}(0)\frac{\partial^2}{\partial x^\alpha \partial x^i}\{x^k x^h\}\frac{\partial^2}{\partial y^\beta \partial y^j}\{(y^r - x^r)(y^l - x^l)\} =$$

$$-\frac{1}{3}R_{rklh}(0)(\delta_\alpha^k \delta_i^h + \delta_i^k \delta_\alpha^h)(\delta_\beta^r \delta_j^l + \delta_j^r \delta_\beta^l) =$$

$$-\frac{1}{3}(R_{\beta\alpha j i}(0) + R_{j\alpha\beta i}(0) + R_{\beta i j\alpha}(0) + R_{j i\beta\alpha}(0)) =$$

$$-\frac{2}{3}(R_{ij\alpha\beta}(0) - R_{i\beta j\alpha}(0).)$$

This factor contributes further into the formula for $[\Phi_k]_k$ by juxtaposing the external product factor $dx^\alpha \wedge dx^\beta$, i.e. it contributes through the factor

$$-\frac{2}{3}(R_{ij\alpha\beta}(0) - R_{i\beta j\alpha}(0))dx^\alpha \wedge dx^\beta = -R_{ij\alpha\beta}(0)dx^\alpha \wedge dx^\beta,$$

the last equality being given by Proposition 3.2. (ii).

We notice that if a factor

$$\frac{\partial^2 r^2(x_r, x_{r+1})}{\partial x_r^{i_r} \partial x_{r+1}^{j_{r+1}}} g^{j_{r+1}i_{r+1}}(x_{r+1})$$

of the formula for Φ_k is differentiated (twice), the consecutive factor may not be differentiated because the two factors have one variable in common. Additionally, notice that the first and the last factors do not contribute differentiated into the final formula for $[\Phi_k]_k$ because they contain the variable x_0.

We conclude that

$$[\Phi_k]_k =$$

$$\frac{-2^{\frac{k}{2}+1}}{k!} R_{i_1 i_2 \alpha_1 \alpha_2} R_{i_2 i_3 \alpha_3 \alpha_4} ......R_{i_{\frac{k}{2}} i_1 \alpha_{k-1}\alpha_k} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge ... \wedge dx^{\alpha_k} =$$

$$\frac{-2^{\frac{k}{2}+1}}{k!} Trace(R^{\frac{k}{2}}),$$

where summation is understood with respect to all repeating indices and where R is the curvature tensor seen as an endomorphism valued 2-form.

Recall that the k-component of the Chern character of the tangent bundle of M is $Ch_k(M) = 1/(\frac{k}{2})!.Trace(\frac{R}{2\pi i})^{\frac{k}{2}}$.

Therefore,

$$[\Phi_k]_k = -\frac{2^{\frac{k}{2}+1}}{k!}(\frac{k}{2})!(2\pi i)^{\frac{k}{2}} Ch_k(M).$$

5 Linear quasi-connections and Chern character.

A *linear quasi-connection* τ in the vector bundle ξ over the smooth manifold M consists of assigning to any two points $x, y \in M$, sufficiently close one to each other, an isomorphism $\tau(y, x) : \xi_x \to \xi_y$, where ξ_x is the fiber at x, such that $\tau(x, x) = identity$.

The connection τ is *smooth* provided the isomorphism $\tau(y,x)$ depends smoothly on the pair x, y.

The parallel transport defined by a linear connection in ξ along the small geodesics of an affine connection in M induce a linear quasi-connection in ξ.

Remark, however, that a linear quasi-connection does not necessarily have to satisfy the property $\tau(y,x)\tau(x,y) = id.$, and hence there exist linear quasi-connections which do not derive from linear connections.

As in the previous section we associate with τ the function $\Phi_k : M^{2k+1} \rightarrow C$ by the formula

$$\Phi_k(x_0, x_1, ..., x_k) := Trace\, \tau(x_0, x_1)\tau(x_1, x_2).....\tau(x_{k-1}, x_k)\tau(x_k, x_0).$$

Proposition 5.1 *Any two smooth linear quasi-connections in a smooth vector bundle are smoothly homotopic.*

Proof. Let τ_0, τ_1 be two coarse connections in ξ. Let x, y be two arbitrary close points in M. The function A defined by $F(y,x) := \tau_1(y,x)\tau_0^{-1}(y,x)$ is a smooth section defined over a small neighborhood $\mathcal{U}$ of the diagonal in $M \times M$ with values in the bundle $\pi_2^*(Iso(\xi))$, where $\pi_2 : M \times M \rightarrow M$ is the projection on the second factor and $Iso(\xi)$ is the bundle over M whose fibers are linear isomorphisms of ξ. As the points x, y are close, $F(y,x) \in Iso(\xi_y)$ is close to the identity. By choosing a smooth affine connection in each fiber, smoothly depending on the fibers too, we may join $F(y,x)$ with the identity of $Iso(\xi_y)$ by arcs of small geodesics. This allows one to produce within each fibre, a continuous homotopy, $F_t(y,x)$ of $F(x,y)$ with the identity: $F_1(y,x) = F(y,x)$ and $F_0(y,x) = id$. The homotopy $\tau_1(y,x)$ may be smoothed, so that we may assume that $F_-(-,-)$ is already smooth.

The desidered homotopy is given by $\tau_t(y,x) = F_t(y,x)\tau_1(y,x)$. This homotopy may be used to produce a linear quasi-connection in the pull back of the bundle ξ over $M \times [0,1]$.

Theorem 5.2 *For any smooth linear quasi-connection τ in the smooth vector bundle ξ over the manifold M,*

-i) $\Phi_k, k = even$, is a cyclic cycle over the algebra $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$,

-ii) the top degree component of the cyclic homology class of Φ_k is

$$[\Phi_k]_k = -\frac{2^{\frac{k}{2}+1}}{k!}(\frac{k}{2})!(2\pi i)^{\frac{k}{2}} Ch_k(M),$$

where $Ch_k(M)$ is the k-component of the Chern character of the tangent bundle of M.

The result follows from Theorem 4.1 along with the homotopy invariance of the cyclic homology.

Remark: For linear connections, the fact that the differential form representing $[\Phi_k]_k$ is closed under the exterior derivative is a consequence of the Bianchi identity. The notion of curvature of a linear quasi-connection and the corresponding Bianchi identity have to be investigated.

References.

[C.1] Connes A.: Non-commutative differential Geometry, Publ. Math. IHES 62 (1985), pp.257 - 360

[C.2] Connes A.: Non Commutative Geometry, Academic Press, 1994

[CM] Connes A., Moscovici H.: Cyclic cohomology, the Novikov conjecture and hyperbolic groups. Topology Vol. 29, Nr. 3 pp.345-388, 1990.

[dR]de Rham G.: Varietes Differentiables, Hermann Paris, 1960.

[D] Donnelly H.: Spectrum and the Fixed Point Sets of Isometries. I, Math. Ann. 224, pp. 161-170, 1976.

[L] Loday J.-L.: Cyclic Homology, Springer Verlag, 1992.

[S] Spivak M.: A Comprehensive Introduction to Differential Geometry, Vol. II, Publish or Perish, 1979.

[Tc] Teleman C.: Sur le charactere de Chern d'un fibre complexe differentiable. Rev. Roumaine Math. Pures Applic. 12 (1967) pp.725-731.

[Tn] Teleman N.:A geometrical definition of some Andre Weil's forms which can be associated with an infinitesimal connection, St. Cerc. Math. Tom. 18, No. 5, Bucarest (1966) 753-762.

Institut des Hautes Etudes Scientifiques

Author's permanent address:

Dipartimento di Scienze Matematiche
Universita Politecnica delle Marche
60161 - Ancona, ITALIA
E-mail: teleman@dipmat.univpm.it