

**Approximation by analytic operator functions.  
Factorizations and very badly approximable functions**

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# APPROXIMATION BY ANALYTIC OPERATOR FUNCTIONS. FACTORIZATIONS AND VERY BADLY APPROXIMABLE FUNCTIONS

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ABSTRACT. This is a continuation of our earlier paper [PT3]. We consider here operator-valued functions (or infinite matrix functions) on the unit circle  $\mathbb{T}$  and study the problem of approximation by bounded analytic operator functions. We discuss thematic and canonical factorizations of operator functions and study badly approximable and very badly approximable operator functions.

We obtain algebraic and geometric characterizations of badly approximable and very badly approximable operator functions. Note that there is an important difference between the case of finite matrix functions and the case of operator functions. Our criteria for a function to be very badly approximable in the case of finite matrix functions also guarantee that the zero function is the only superoptimal approximant. However in the case of operator functions this is not true.

## 1. Introduction

Our previous paper [PT3] was devoted to a characterization of very badly approximable matrix functions. In this paper we consider the case of operator-valued functions or, which is equivalent, infinite matrix functions.

**Background (best approximation and badly approximable functions).** The classical problem of analytic approximation is for a given bounded function  $\varphi$  on the unit circle  $\mathbb{T}$  is to find a function  $f$  in the Hardy class  $H^\infty$  such that

$$\|\varphi - f\|_\infty = \text{dist}_{L^\infty}(\varphi, H^\infty) = \inf_{h \in H^\infty} \|\varphi - h\|_\infty.$$

Such a best approximant  $f$  always exist (a compactness argument) and as was proved by S. Khavinson [Kh] it is unique if  $\varphi$  is continuous.

A function  $\varphi \in L^\infty$  is called *badly approximable* if

$$\|\varphi\|_\infty = \text{dist}_{L^\infty}(\varphi, H^\infty),$$

There is an elegant characterization of the set of continuous badly approximable functions: *a nonzero continuous function  $\varphi$  on  $\mathbb{T}$  is badly approximable if and only*

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if it has constant modulus and its winding number  $\text{wind } \varphi$  with respect to the origin is negative (see [AAK], [Po]).

It is well known that this criterion can be extended to a broader class of functions  $\varphi$ , for which winding number is not defined. To do this, we need the notion of Hankel and Toeplitz operators. The *Toeplitz operator*  $T_\varphi : H^2 \rightarrow H^2$  and the *Hankel operator*  $H_\varphi : H^2 \rightarrow H_-^2 \stackrel{\text{def}}{=} L^2 \ominus H^2$  are defined by

$$T_\varphi f = \mathbb{P}_+ \varphi f, \quad H_\varphi f = \mathbb{P}_- \varphi f, \quad (1.1)$$

where  $\mathbb{P}_-$  and  $\mathbb{P}_+$  are the orthogonal projections onto the subspaces  $H^2$  and  $H_-^2 \stackrel{\text{def}}{=} L^2 \ominus H^2$  of  $L^2$ .

It is well known (see e.g., [D] or [Pe2]) that if  $\varphi \in C(\mathbb{T})$  and  $\varphi$  does not vanish on  $\mathbb{T}$ , then the Toeplitz operator  $T_\varphi$  on the Hardy class  $H^2$  is Fredholm and  $\text{ind } T_\varphi = -\text{wind } \varphi$  (recall that for a Fredholm operator  $A$ , its *index* is defined as  $\text{ind } A = \dim \text{Ker } A - \dim \text{Ker } A^*$ ). The above characterization of badly approximable functions can be easily generalized in the following way: if  $\varphi$  is a function in  $L^\infty$  such that the essential norm  $\|H_\varphi\|_e$  of the Hankel operator  $H_\varphi$  (i.e., the *distance from  $H_\varphi$  to the set of compact operators*) is less than its norm, then  $\varphi$  is badly approximable if and only if  $\varphi$  has constant modulus almost everywhere on  $\mathbb{T}$ ,  $T_\varphi$  is Fredholm, and  $\text{ind } T_\varphi > 0$  (see e.g., [Pe2], Ch. 7, §5).

Recall also that

$$\|H_\varphi\| = \text{dist}_{L^\infty}(\varphi, H^\infty) \quad \text{and} \quad \|H_\varphi\|_e = \text{dist}_{L^\infty}(\varphi, H^\infty + C)$$

(see, e.g., [Pe2]).

Let us proceed now to the case of matrix functions. We can consider the same problem of finding a best analytic approximant for a given bounded function  $\Phi$  with values in the space  $\mathbb{M}_{m,n}$  of  $m \times n$  matrices: for  $\Phi \in L^\infty(\mathbb{M}_{m,n})$  find a bounded analytic  $\mathbb{M}_{m,n}$ -valued function  $F$  such that

$$\|\Phi - F\|_{L^\infty} = \text{dist}_{L^\infty(\mathbb{M}_{m,n})} \left( \Phi, H^\infty(\mathbb{M}_{m,n}) \right).$$

Here

$$\|\Phi\|_{L^\infty} \stackrel{\text{def}}{=} \text{ess sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta)\|_{\mathbb{M}_{m,n}},$$

$\mathbb{M}_{m,n}$  is equipped with the standard operator norm, and  $H^\infty(\mathbb{M}_{m,n})$  is the space of bounded analytic functions with values in  $\mathbb{M}_{m,n}$ .

Again, it can be shown easily that a best approximant always exists. However, the situation with uniqueness is quite different from the scalar case. Indeed, suppose that  $m = n = 2$  and  $u$  is a scalar continuous badly approximable *unimodular function* (i.e.,  $|u(\zeta)| = 1$  almost everywhere on  $\mathbb{T}$ ). Consider the matrix function

$\Phi = \begin{pmatrix} u & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ . It is easy to see that for any scalar function  $f$  in the unit ball of  $H^\infty$ , the matrix function  $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & f \end{pmatrix}$  is a best approximation of  $\Phi$ .

While it is possible to describe badly approximable matrix- and operator-valued functions, and we give such descriptions in this paper (the case of finite matrix functions was treated in our earlier paper [PT3]), this is not our main goal. It turns out that in the matrix case it is more natural to consider superoptimal approximations and *very badly approximable functions*.

**Superoptimal approximations and very badly approximable matrix functions.** Recall that for a matrix (or a bounded linear operator on Hilbert space)  $A$  the *singular values*  $s_j(A)$ ,  $j \geq 0$ , are defined by

$$s_j(A) = \inf\{\|A - K\| : \text{rank } K \leq j\}.$$

Clearly,  $s_0(A) = \|A\|$ .

**Definition.** Given a matrix function  $\Phi \in L^\infty(\mathbb{M}_{m,n})$  we define inductively the sets  $\Omega_j$ ,  $0 \leq j \leq \min\{m, n\} - 1$ , by

$$\Omega_0 = \{F \in H^\infty(\mathbb{M}_{m,n}) : F \text{ minimizes } t_0 \stackrel{\text{def}}{=} \text{ess sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta) - F(\zeta)\|\};$$

$$\Omega_j = \{F \in \Omega_{j-1} : F \text{ minimizes } t_j \stackrel{\text{def}}{=} \text{ess sup}_{\zeta \in \mathbb{T}} s_j(\Phi(\zeta) - F(\zeta))\}, \quad j > 0.$$

Functions in  $\bigcap_{k \geq 0} \Omega_k = \Omega_{\min\{m, n\} - 1}$  are called *superoptimal approximants* of  $\Phi$  by bounded analytic matrix functions. The numbers  $t_j = t_j(\Phi)$  are called the *superoptimal singular values* of  $\Phi$ . Note that the functions in  $\Omega_0$  are just the best approximants by bounded analytic matrix functions.

As in the case of scalar functions, a bounded  $m \times n$  matrix function  $\Phi$  is called *badly approximable* if

$$\|\Phi\|_{L^\infty} = \inf\{\|\Phi - F\|_{L^\infty} : F \in H^\infty(\mathbb{M}_{m,n})\}.$$

We say that a matrix function  $\Phi \in L^\infty(\mathbb{M}_{m,n})$  is called *very badly approximable* if the zero function  $\mathbf{0}$  is a superoptimal approximant of  $\Phi$ .

The notion of superoptimal approximation can be extended to the case of operator-valued functions. If  $\mathcal{H}$  and  $\mathcal{K}$  are Hilbert spaces, we denote by  $\mathcal{B}(\mathcal{H}, \mathcal{K})$  the space of bounded linear operators from  $\mathcal{H}$  to  $\mathcal{K}$ ,

$$\mathcal{B}(\mathcal{H}) \stackrel{\text{def}}{=} \mathcal{B}(\mathcal{H}, \mathcal{H}).$$

We can identify an infinite-dimensional separable Hilbert space with  $\ell^2$  and identify operators on  $\ell^2$  with infinite matrices. Suppose that  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$ , i.e.,  $\Phi$  is a weakly measurable bounded function that takes values in  $\mathcal{B}(\ell^2)$ , we can define

the sequence  $\{\Omega_j\}_{j \geq 0}$  in the same way as for finite matrix functions. However, in the case of operator-valued functions we have to consider the infinite sequence of the sets  $\Omega_j$ . For  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$ , we say that a function  $F$  in  $H^\infty(\mathcal{B}(\ell^2))$  is a *superoptimal approximant* of  $\Phi$  by bounded analytic operator functions.

Badly approximable and very badly approximable infinite matrix functions can be defined in the same way as in the case of finite matrix functions.

Note that if  $\Phi$  is a matrix function of size  $m \times \infty$  or  $\infty \times n$ , we can add to  $\Phi$  infinitely many zero rows or zero columns and reduce the problem to the case of matrix functions of size  $\infty \times \infty$ .

**The summary of earlier results.** First of all, let us mention that superoptimal approximation is more natural in the case of matrix or operator functions because it is unique under mild natural assumptions on the function. It was shown in [PY1] that if  $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$  (i.e., all entries of  $\Phi$  belong to  $H^\infty + C$ ), then  $\Phi$  has a unique superoptimal approximation  $F$  by bounded analytic matrix functions. Moreover, it was shown in [PY1] that

$$s_j(\Phi(\zeta) - F(\zeta)) = t_j(\Phi) \quad \text{for almost all } \zeta \in \mathbb{T}. \quad (1.2)$$

Later this result was extended in [T], see also [Pe1], [PT1] to operator-valued functions  $\Phi$  for which the Hankel operator  $H_\Phi$  is compact.

The proof given in [PY1] was based on certain special factorizations (thematic factorizations, see §4 of this paper for definitions). The approach in [T] was more geometric and based on the notion of superoptimal weights.

The problem to describe the very badly approximable functions was posed in [PY1]. It follows from (1.2) that if  $\Phi$  is a very badly approximable function in  $(H^\infty + C)(\mathbb{M}_{m,n})$ , then the singular values  $s_j(\Phi(\zeta))$  are constant for almost all  $\zeta \in \mathbb{T}$ . Moreover, it was shown in [PY1] that if in addition to this  $m \leq n$  and  $s_{m-1}(\Phi(\zeta)) \neq 0$  almost everywhere, then the Toeplitz operator  $T_{z\Phi} : H^2(\mathbb{C}^n) \rightarrow H^2(\mathbb{C}^m)$  has dense range (if  $\Phi$  is a scalar function, the last condition is equivalent to the fact that  $\text{ind } T_\Phi > 0$ ). Note that the Toeplitz and the Hankel operators whose symbols are matrix functions can be defined in the same way as in the scalar case (see (1.1)). Obviously, this necessary condition is equivalent to the condition  $\text{Ker } T_{\bar{z}\Phi^*} = \{\mathbf{0}\}$ . In fact, the proof of necessity given in [PY1] allows one to obtain a more general result: if  $\Phi$  is an arbitrary very badly approximable function in  $(H^\infty + C)(\mathbb{M}_{m,n})$  and  $f \in \text{Ker } T_{\bar{z}\Phi^*}$ , then  $\Phi^* f = \mathbf{0}$ .

On the other hand, in [PY1] an example of a continuous  $2 \times 2$  function  $\Phi$  was given such that  $s_0(\Phi(\zeta)) = 1$ ,  $s_1(\Phi(\zeta)) = \alpha < 1$ ,  $\zeta \in \mathbb{T}$ ,  $T_{z\Phi}$  is invertible but  $\Phi$  is not even badly approximable.

The very badly approximable matrix functions of class  $(H^\infty + C)(\mathbb{M}_{m,n})$  were characterized in [PY1] algebraically, in terms of so-called thematic factorizations.

Later in [PT2] the above results of [PY1] were generalized to the broader context of matrix functions  $\Phi$  such that the essential norm  $\|H_\Phi\|_e$  of the Hankel operator

$H_\Phi$  is less than the smallest nonzero superoptimal singular value of  $\Phi$ . We call such matrix functions  $\Phi$  *admissible*. In particular, if  $\Phi$  is an admissible very badly approximable  $m \times n$  matrix function, then the functions  $s_j(\Phi(z))$  are constant almost everywhere on  $\mathbb{T}$  and

$$\text{Ker } T_{\bar{z}\Phi^*} = \{f \in H^2(\mathbb{C}^n) : \Phi^* f = \mathbf{0}\}.$$

In [AP] another algebraic characterization of the set of very badly approximable admissible matrix functions was given in terms of canonical factorizations (see §5 for the definition).

We refer the reader to the book [Pe2], which contains all the above information and results on superoptimal approximation and very badly approximable functions.

In [PT3] we obtained a new criterion for an admissible matrix function to be very badly approximable. In contrast with earlier criteria in terms of certain special factorizations, it is more geometric and it is easier to use it to verify whether a given matrix function is very badly approximable. This criterion is given in terms of families of subspaces spanned by Schmidt vectors of matrices  $\Phi(\zeta)$ ,  $\zeta \in \mathbb{T}$ .

Recall that if  $A$  is an  $m \times n$  matrix and  $s$  is a singular value of  $A$ , a nonzero vector  $x \in \mathbb{C}^n$  is called a *Schmidt vector* corresponding to  $s$  if  $A^*Ax = s^2x$ .

Given a matrix function  $\Phi$  in  $L^\infty(\mathbb{M}_{m,n})$  and  $\sigma > 0$ , we considered the subspace  $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$  of  $\mathbb{C}^n$  spanned by the Schmidt vectors of  $\Phi(\zeta)$  that correspond to the singular values of  $\Phi(\zeta)$  that are greater than or equal to  $\sigma$ . The subspaces  $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$  are defined for almost all  $\zeta \in \mathbb{T}$ . It was shown in [PT3] that if  $\Phi$  is an admissible very badly approximable matrix functions, then for each  $\sigma > 0$ , the family of subspaces  $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$ ,  $\zeta \in \mathbb{T}$ , is *analytic*, i.e., there exist functions  $g_1, \dots, g_k$  in  $H^2(\mathbb{C}^n)$  such that

$$\mathfrak{S}_\Phi^{(\sigma)}(\zeta) = \text{span}\{g_1(\zeta), \dots, g_k(\zeta)\} \quad \text{for almost all } \zeta \in \mathbb{T}. \quad (1.3)$$

The same analyticity condition must also be imposed on the transposed function  $\Phi^t$ . However, it was shown in [PT3] that the analyticity conditions on  $\Phi$  and  $\Phi^t$  together with the earlier necessary conditions quoted above do not guarantee that  $\Phi$  is very badly approximable.

However, it turned out that the above condition can be slightly modified to get a necessary and sufficient condition. The main result of [PT3] is the following theorem.

**Theorem.** *Let  $\Phi$  be an admissible matrix function. Then  $\Phi$  is very badly approximable if and only if for each  $\sigma > 0$  equality (1.3) holds for functions  $g_1, \dots, g_k$  in  $\text{Ker } T_\Phi$ . Moreover, this condition implies that  $\Phi$  is very badly approximable even without the assumption that  $\Phi$  is admissible.*

Note that this condition in the case of a scalar function  $\varphi$  means that  $\varphi$  has constant modulus and  $\text{Ker } T_\varphi \neq \{\mathbf{0}\}$ , i.e., our criterion is a natural generalization of the scalar results discussed above.

The uniqueness problem for superoptimal approximation of operator functions (infinite matrix functions) was studied in [T], [Pe1], and [PT1]. It was shown there that if the Hankel operator  $H_\Phi$  is compact, then  $\Phi$  has a unique superoptimal approximant by bounded analytic operator functions. In [Pe1] and [PT1] uniqueness was obtained with the help of partial thematic factorizations (see §4 of this paper). We also refer the reader to the monograph [Pe2] for the above results on superoptimal approximation of operator functions.

**The purpose and outline of the paper.** In this paper we study very badly approximable operator functions. We consider the class of *admissible operator functions*. As in the case of finite matrix functions, an operator function  $\Phi$  is called admissible if the essential norm  $\|H_\Phi\|_e$  of the Hankel operator  $H_\Phi$  is less than each nonzero superoptimal singular value of  $\Phi$ .

In §4 we consider partial thematic factorizations of admissible operator functions (without the assumption of the compactness of  $H_\Phi$  as it was done in [Pe1] and [PT1]). In §5 we consider partial canonical factorizations of operator functions.

The main result of the paper is a criterion of very bad approximability (Theorem 6.1) presented in §6. It essentially says that the theorem stated above also holds in the case of operator function.

However, it turns out that there is an important distinction between the case of finite matrix functions and the case of infinite matrix functions. In the case of finite matrix functions if  $\Phi$  satisfies the hypotheses of the above theorem, then the zero function is the only superoptimal approximant of  $\Phi$ . We show in this paper that in the case of infinite matrix functions this is not true: *under the hypotheses of the above theorem  $\Phi$  must be very badly approximable, but it can have infinitely many superoptimal approximants.*

Note also that in the case of infinite matrix functions some proofs are considerably more complicated than the proofs of the corresponding results for finite matrix functions (e.g., the proofs of Theorems 3.1 and 5.1 given below).

In §2 we define inner, outer, and co-outer operator functions and prove a theorem about inner-outer factorizations of co-outer operator functions.

In §3 we define balanced operator functions and prove that a inner and co-outer function with finitely many columns has a balanced completion.

## 2. Inner and outer operator functions

In this section we define inner, outer, and co-outer operator functions and we prove that the inner factor in the inner-outer factorization of a co-outer function with finitely many columns must also be co-outer.

Let  $\mathcal{H}$  and  $\mathcal{K}$  be separable Hilbert spaces. We denote by  $H_s^2(\mathcal{B}(\mathcal{H}, \mathcal{K}))$  the space of analytic operator functions  $F$  that take values in the space of bounded linear

operators from  $\mathcal{H}$  to  $\mathcal{K}$  and satisfy the following condition

$$F(z)x \in H^2(\mathcal{K}) \quad \text{for every } x \in \mathcal{H}.$$

A function  $F$  in  $H_s^2(\mathcal{B}(\mathcal{H}, \mathcal{K}))$  is called *inner* if  $F(\zeta)$  is an isometric operator (i.e.,  $F(\zeta)^*F(\zeta) = I$ ) for almost all  $\zeta \in \mathbb{T}$ . A function  $F$  in  $H_s^2(\mathcal{B}(\mathcal{H}, \mathcal{K}))$  is called *outer* if the set

$$\{Fq : q \text{ is a polynomial in } H^2(\mathcal{H})\}$$

is dense in  $H^2(\mathcal{K})$ .

It is well known (see e.g., [N]) that each function  $F$  in  $H_s^2(\mathcal{B}(\mathcal{H}, \mathcal{K}))$  admits an inner-outer factorization, i.e., there exist an inner operator function  $\Theta$  and an outer operator function  $G$  such that  $F = \Theta G$ .

As we have mentioned in the introduction, we are going to identify operator functions with infinite matrix functions. We say that an infinite matrix function  $F$  is *co-outer* if the transposed function  $F^t$  is outer.

**Theorem 2.1.** *Let  $F$  be a co-outer operator function in  $H_s^2(\mathcal{B}(\mathbb{C}^d, \ell^2))$ ,  $d < \infty$ . Suppose that*

$$F = \Theta G,$$

where  $\Theta$  is an inner operator function and  $G$  is an outer operator function. Then  $\Theta$  is co-outer.

**Proof.** Suppose that  $\Theta \in H^\infty(\mathcal{B}(\mathbb{C}^k, \ell^2))$  and  $G \in H^2(\mathbb{M}_{d,k})$ . Since  $G$  is outer, it follows that  $k \leq d$ . Suppose that

$$\Theta^t = \mathcal{O}Q,$$

where  $\mathcal{O}$  is an inner matrix function and  $Q$  is an outer operator function. Since  $\Theta$  is inner, it is easy to see that  $\mathcal{O}$  has size  $k \times k$ . We have

$$\Theta = Q^t \mathcal{O}^t.$$

Then

$$F = Q^t \mathcal{O}^t G,$$

and so by the hypotheses of the theorem,

$$F^t = G^t \mathcal{O}Q$$

is an outer function. It follows that  $G^t$  must be outer, and so  $k = d$ . Clearly,  $G^t \mathcal{O} H^\infty(\mathbb{M}_{d,d})$  must be dense in  $H^2(\mathbb{M}_{d,d})$ . However, the determinants of all matrix functions in  $G^t \mathcal{O} H^\infty(\mathbb{M}_{d,d})$  must be divisible by  $\det \mathcal{O}$  which is a scalar inner function. Thus  $\det \mathcal{O}$  is constant, and so  $\mathcal{O}^* = \mathcal{O}^{-1} \in H^\infty(\mathbb{M}_{d,d})$  which implies that  $\mathcal{O}$  is constant, and so  $\Theta^t$  is co-outer. ■

### 3. Balanced matrix functions

In this section we introduce the notion of balanced unitary-valued functions and prove the existence of balanced completions for inner and co-outer functions that have finitely many columns.

**Definition.** A *balanced infinite matrix function* is a unitary-valued matrix function of the form  $(\Upsilon \quad \overline{\Theta})$ , where  $\Upsilon$  and  $\Theta$  inner and co-outer matrix functions.

If  $\Upsilon$  has  $r$  columns, we say that the function  $(\Upsilon \quad \overline{\Theta})$  is *r-balanced*. 1-balanced functions are also called *thematic matrix functions*.

We are going to prove that an inner matrix function with finitely many columns can be completed to a balanced matrix function.

Let  $r$  be a positive integer and let  $\Upsilon$  be an inner matrix function in  $H^\infty(\mathbb{C}^r, \ell^2)$ . Consider the subspace  $\mathcal{L} \stackrel{\text{def}}{=} \text{Ker } T_{\Upsilon^t}$  of  $H^2(\ell^2)$ . Clearly, it is invariant under multiplication by  $z$ , and so there exists an inner matrix function  $\Theta$  such that  $\mathcal{L} = \Theta H^2(\mathcal{K})$ , where  $\mathcal{K} = \ell^2$  or  $\mathcal{K} = \mathbb{C}^m$  for some positive  $m$ . The proof of the following theorem in the special case  $r = 1$  can be found in [Pe2], Ch. 14, §18. In the general case the proof is algebraically more complicated. Note that a close result was obtained in [C], see also [H], Lect. IX.

**Theorem 3.1.** *Let  $\Upsilon$  and  $\Theta$  be as above. Then  $\Theta$  is co-outer and the matrix function  $(\Upsilon \quad \overline{\Theta})$  is unitary-valued.*

Before proceeding to the proof, we introduce a notion. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+1\ 1} & a_{r+1\ 2} & \cdots & a_{r+1\ r} \end{pmatrix}$$

be an  $(r + 1) \times r$  matrix. For  $1 \leq j \leq r + 1$ , we put

$$\alpha_j = (-1)^j \det \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{j-1\ 1} & \cdots & a_{j-1\ r} \\ a_{j+1\ 1} & \cdots & a_{j+1\ r} \\ \vdots & \ddots & \vdots \\ a_{r+1\ 1} & \cdots & a_{r+1\ r} \end{pmatrix}.$$

In other words, we multiply  $(-1)^j$  by the minor obtained from  $A$  by deleting the  $j$ th row. The vector  $A_{\text{ass}} \stackrel{\text{def}}{=} \{\alpha_j\}_{1 \leq j \leq r+1}$  is called the *vector associated with  $A$* .

**Proof.** The proof of the fact that  $\Theta$  is co-outer is exactly the same as in the case  $r = 1$ , see [Pe2], Ch. 14, Lemma 18.3. Let us show that  $(\Upsilon \ \Theta)$  is unitary-valued. The fact that  $(\Upsilon \ \Theta)$  takes isometric values almost everywhere on  $\mathbb{T}$  follows immediately from the definition of  $\Theta$ . To prove that it is unitary-valued, it suffices to show that  $\dim \text{Ker } \Theta^t(\zeta) \leq r$  for almost all  $\zeta \in \mathbb{T}$ .

Let

$$\Upsilon = \begin{pmatrix} v_{01} & v_{02} & \cdots & v_{0r} \\ v_{11} & v_{12} & \cdots & v_{1r} \\ v_{21} & v_{22} & \cdots & v_{2r} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$

Clearly, the matrix function  $\Upsilon$  has rank  $r$  almost everywhere on  $\mathbb{T}$ . Without loss of generality we may assume that

$$\det \begin{pmatrix} v_{01} & v_{02} & \cdots & v_{0r} \\ v_{11} & v_{12} & \cdots & v_{1r} \\ v_{21} & v_{22} & \cdots & v_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r-1 \ 1} & v_{r-1 \ 2} & \cdots & v_{r-1 \ r} \end{pmatrix} \neq \mathbf{0}. \quad (3.1)$$

Consider the bounded analytic matrix function  $G$  defined in the following way:

$$G = \begin{pmatrix} \alpha_0^{[0]} & \alpha_0^{[1]} & \alpha_0^{[2]} & \alpha_0^{[3]} & \cdots \\ \alpha_1^{[0]} & \alpha_1^{[1]} & \alpha_1^{[2]} & \alpha_1^{[3]} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \alpha_{r-1}^{[0]} & \alpha_{r-1}^{[1]} & \alpha_{r-1}^{[2]} & \alpha_{r-1}^{[3]} & \cdots \\ \alpha_r^{[0]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \alpha_r^{[0]} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \alpha_r^{[0]} & \mathbf{0} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where for  $k \geq 0$ , the  $H^\infty$  functions  $\alpha_m^{[k]}$ ,  $0 \leq m \leq r$ , are the components of the vector function  $A_{\text{ass}}^{[k]}$  associated with the matrix function  $A^{[k]}$  defined by

$$A^{[k]} = \begin{pmatrix} v_{01} & v_{02} & \cdots & v_{0r} \\ v_{11} & v_{12} & \cdots & v_{1r} \\ v_{21} & v_{22} & \cdots & v_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r-1\ 1} & v_{r-1\ 2} & \cdots & v_{r-1\ r} \\ v_{r+k\ 1} & v_{r+k\ 2} & \cdots & v_{r+k\ r} \end{pmatrix}.$$

Note that  $\alpha_r^{[0]}$  is nothing but the determinant on the left-hand side of (3.1).

It is an elementary exercise in linear algebra to verify that  $\Upsilon^t G = \mathbf{0}$ . It follows that  $G$  admits a factorization  $G = \Theta Q$ , where  $Q$  is an  $H^\infty$  matrix function. Hence, to verify that  $\dim \text{Ker } \Theta^t(\zeta) \leq r$ , it suffices to show that  $\dim \text{Ker } G^t(\zeta) \leq r$ . Recall that by (3.1),  $\alpha_r^{[0]}(\zeta) \neq 0$  for almost all  $\zeta \in \mathbb{T}$ . Assume that  $\zeta \in \mathbb{T}$  and  $\alpha_r^{[0]}(\zeta) \neq 0$ . Suppose that the vector  $x = \{x_j\}_{j \geq 0}$  belongs to  $\text{Ker } G^t(\zeta)$ . If we look at the  $r$ th coordinate of the vector  $G^t(\zeta)x$ , we observe that  $\alpha_r^{[0]}(\zeta)x_r$  is uniquely determined by  $x_0, x_1, \dots, x_{r-1}$ . Since  $\alpha_r^{[0]}(\zeta) \neq 0$ , it follows that  $x_r$  is uniquely determined by  $x_0, x_1, \dots, x_{r-1}$ . If we look now at the next component of the vector  $G^t(\zeta)x$ , we observe that  $x_{r+1}$  is uniquely determined by  $x_0, x_1, \dots, x_r$ , etc. This completes the proof. ■

#### 4. Partial thematic factorizations

In the case when  $\Phi$  is an operator function such that the Hankel operator  $H_\Phi$  is compact and  $F \in \Omega_d$ , partial thematic factorizations of  $\Phi - F$  were constructed in [Pe1]. In this section we consider the more general case when  $\Phi$  is an admissible operator function.

Suppose that  $\Phi$  is function in  $L^\infty(\mathcal{B}(\ell^2))$  such that

$$\|H_\Phi\|_e < \|H_\Phi\| \tag{4.1}$$

and  $F \in H^\infty(\mathcal{B}(\ell^2))$  is a best approximant of  $\Phi$ . Then  $H_\Phi$  has a maximizing vector  $f$ , the function  $g = \|H_\Phi\|^{-1} \overline{\zeta} \overline{H_\Phi} f$  is a maximizing vector of  $H_{\Phi^t}$ . The functions  $f$  and  $g$  admit factorizations

$$f = \vartheta_1 h v, \quad g = \vartheta_2 h w,$$

where  $h$  is a scalar outer function,  $\vartheta_1$  and  $\vartheta_2$  are scalar inner functions, and  $v$  and  $w$  are inner and co-outer column functions.

By Theorem 3.1, the column functions  $v$  and  $w$  have thematic (1-balanced) completions:

$$V = \begin{pmatrix} v & \bar{\Theta} \end{pmatrix} \quad \text{and} \quad W^t = \begin{pmatrix} w & \bar{\Xi} \end{pmatrix}$$

The function  $\Phi - F$  admits the following factorization:

$$\Phi - F = W^* \begin{pmatrix} t_0 u & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} V^*,$$

where  $u = \bar{z}\bar{v}_1\bar{v}_2\bar{h}/h$  and  $\|\Psi\|_{L^\infty} \leq t_0 = t_0(\Phi) = \|H_\Phi\|$  (see [Pe2], Ch. 14, §18). Moreover, under the assumption (4.1),  $T_u$  is Fredholm and  $\text{ind } T_u > 0$ .

Such factorizations are called *partial thematic factorizations of order 1*.

As in the case of finite matrix functions (see [PT2] or [Pe2], Ch. 14, §4) the following crucial inequality holds:

$$\|H_\Psi\|_e \leq \|H_\Psi\|_e. \quad (4.2)$$

Another important result that can be established in the same way as in the case of finite matrix functions is that under the assumption (4.1) the operator functions  $\Theta$  and  $\xi$  are left-invertible in  $H^\infty$  (see [PT2] or [Pe2], Ch. 14, §4).

A function  $\Phi$  satisfying (4.1) is badly approximable if and only if it admits a partial thematic factorization of order 1:

$$\Phi = W^* \begin{pmatrix} t_0 u & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} V^*$$

(the part “if” holds even without the assumption (4.1)). Moreover,  $\Phi$  is very badly approximable if and only if  $\Psi$  is very badly approximable.

If  $\Phi$  is admissible and  $H_\Psi \neq 0$ , due to inequality (4.2) we can apply the same procedure to  $\Psi$ . If  $F \in \Omega_1$ , then  $\Phi - F$  admits a thematic factorization of order 2, i.e.,

$$\Phi - F = W^* \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & W_1^* \end{pmatrix} \begin{pmatrix} t_0 u_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & t_1 u_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Lambda \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V_1^* \end{pmatrix} V^*,$$

where  $V, V_1, W^t, W_1^t$  are thematic operator functions,  $u_0$  and  $u - 1$  are scalar very badly approximable functions such that  $\|H_{u_j}\|_e < 1$ , and  $\|\Psi\|_{L^\infty} \leq t_1$ .

If  $\Phi$  is admissible, we can continue this process and obtain partial thematic factorization of an arbitrary order.

In particular an admissible operator function  $\Phi$  is very badly approximable if and only is for each positive integer  $r$  it admits a partial thematic factorizations of order  $r$ .

## 5. Partial canonical factorizations

As in the case of finite matrix functions (see [PT3]), to obtain to obtain a geometric characterization of very badly approximable operator functions, it is more important to deal with canonical factorizations rather than with thematic factorizations.

**Theorem 5.1.** *Let  $\Phi$  be a matrix function in  $L^\infty(\mathcal{B}(\ell^2))$  such that that  $\|H_\Phi\|_e < \|H_\Phi\|$  and let  $r$  be the multiplicity of the superoptimal singular value  $t_0(\Phi)$ . Suppose that  $\mathcal{M}$  is the minimal shift invariant subspace of  $H^2(\ell^2)$  that contains all maximizing vectors of  $H_\Phi$ . Then*

$$\mathcal{M} = \Upsilon H^2(\mathbb{C}^d),$$

where  $\Upsilon$  is an inner and co-outer function of size  $\infty \times r$ .

**Proof.** Since  $\mathcal{M}$  is shift invariant, it has the form

$$\mathcal{M} = \Upsilon H^2(\mathcal{K}),$$

where  $\mathcal{K}$  is a separable Hilbert space and  $\Upsilon$  is an inner operator function. Since  $\|H_\Phi\|_e < \|H_\Phi\|$ , it is easy to see that the space of maximizing vectors of  $H_\Phi$  is finite-dimensional, and so  $\dim \mathcal{K} < \infty$ . Put  $d = \dim \mathcal{K}$  and  $\mathcal{K} = \mathbb{C}^d$ .

Let us show that  $d \geq r$ . In [PY1] (see also Lemma 1.2 of [PY2]) in the case of finite matrix functions of class  $H^\infty + C$  a finite sequence

$$f_1^{(0)}, \dots, f_{k_0}^{(0)}, f_1^{(1)}, \dots, f_{k_1}^{(1)}, \dots, f_1^{(r-1)}, \dots, f_{k_{r-1}}^{(r-1)}$$

of maximizing vectors of  $H_\Phi$  was constructed. It is easy to verify that it has the following property:

$$\max_{\zeta \in \mathbb{D}} \dim \text{span} \left\{ f_k^{(j)}(\zeta) : 0 \leq j \leq r-1, 1 \leq k \leq k_j \right\} = r. \quad (5.1)$$

This construction was generalized in [PT1] to the case of finite matrix functions  $\Phi$  satisfying the condition  $\|H_\Phi\|_e < \|H_\Phi\|$  and in [PT2] to the case of infinite matrix functions  $\Phi$  such that  $H_\Phi$  is compact (see also Chap. 14 of [Pe2]). It can easily be verified that exactly the same construction also works in the case of infinite matrix functions  $\Phi$  satisfying the condition  $\|H_\Phi\|_e < \|H_\Phi\|$  and (5.1) holds. It follows immediately from (5.1) that  $d \geq r$ .

Let us now show that  $d \leq r$ . Let  $F$  be a function in  $\Omega_r$ . Consider a partial canonical factorization of  $\Phi - F$ . It has the form

$$\Phi - F = \mathfrak{W} \begin{pmatrix} t_0 \mathfrak{U} & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} \mathfrak{V},$$

where  $\mathfrak{W}$  and  $\mathfrak{V}$  are infinite unitary-valued functions,  $\mathfrak{U}$  is an  $r \times r$  unitary-valued function, and  $\|\Psi\|_{L^\infty} = t_r < t_0$ . It follows that the subspace spanned by the maximizing vectors of  $(\Phi - F)(\zeta)$  has dimension  $r$  for almost all  $\zeta \in \mathbb{T}$ .

For every function  $f \in \mathcal{M}$  the vector  $f(\zeta)$  is a maximizing vector of  $\Phi(\zeta)$  for almost all  $\zeta \in \mathbb{T}$  (see Lemma 15.2 in Ch. 14 of [Pe2], note that in [Pe2] the result is stated for finite matrix functions, but the proof given there works for infinite matrix functions too). It is easy to see now that if  $d > r$ , then the subspace spanned by the maximizing vectors of  $(\Phi - F)(\zeta)$  has dimension at least  $d$ .

It remains to show that  $\Upsilon$  is co-outer. Without loss of generality we may assume that  $\|\Phi\|_\infty = \|H_\Phi\| = 1$ . Consider the subspace of  $H^2(\ell^2)$  spanned by the maximizing vectors of  $H_\Phi$ . It must be finite-dimensional. Let  $f_1, \dots, f_s$  be a basis of this subspace and let  $F$  be the matrix function whose columns are  $f_1, \dots, f_s$ . Consider the inner-outer factorization of  $F^t$ :

$$F^t = \mathcal{O}G,$$

where  $\mathcal{O}$  is an inner matrix function of size  $s \times k$ ,  $k \geq s$ , and  $G$  is an outer matrix function of size  $k \times \infty$ . Then

$$F = G^t \mathcal{O}^t,$$

and so

$$G^t = G^t \mathcal{O}^t \overline{\mathcal{O}} = F \overline{\mathcal{O}}. \quad (5.2)$$

Since the functions  $f_j$  are maximizing vectors of  $H_\Phi$ , it follows that for almost all  $\zeta \in \mathbb{T}$ , the vectors  $f_j(\zeta)$  are maximizing vectors of  $\Phi(\zeta)$  and  $\Phi f_j \in H_-^2(\ell^2)$  (see [Pe2], Theorem 2.3 of Ch. 2). Thus for almost all  $\zeta \in \mathbb{T}$ , the restriction of  $\Phi(\zeta)$  to  $\text{Range } F(\zeta)$  is an isometry and  $\Phi F \in H_-^2(\mathcal{B}(\mathbb{C}^s, \ell^2))$ . It follows that

$$\Phi G^t = \Phi F \overline{\mathcal{O}} \in H_-^2(\mathcal{B}(\mathbb{C}^k, \ell^2)).$$

Suppose now that  $g$  is a column of  $G^t$ . Then  $\Phi g \in H_-^2(\ell^2)$ . Since

$$g(\zeta) \in \text{Range } G^t(\zeta) \subset \text{Range } F(\zeta) \quad \text{for almost all } \zeta \in \mathbb{T},$$

it follows that

$$\|\Phi(\zeta)g(\zeta)\|_{\ell^2} = \|g(\zeta)\|_{\ell^2} \quad \text{almost everywhere on } \mathbb{T}.$$

Thus

$$\|H_\Phi g\| = \|\mathbb{P}_- \Phi g\| = \|\Phi g\| = \|g\|,$$

and so all columns of  $G^t$  are maximizing vectors of  $H_\Phi$ . Since the columns of  $F$  form a basis in the space of maximizing vectors, it follows from (5.2) that  $\mathcal{O}$  is a constant isometric matrix.

Clearly,  $\mathcal{M}$  is the minimal invariant subspace of multiplication by  $z$  on  $H^2(\ell^2)$  that contains the columns of  $F$ . Consider the subspace minimal invariant subspace  $\mathcal{M}_1$  that contains the columns of  $G^t$ . Since  $F = G^t \mathcal{O}^t$  and  $\mathcal{O}$  is a constant matrix, it follows that  $\mathcal{M} \subset \mathcal{M}_1$ . On the other hand, the columns of  $G^t$  are maximizing vectors of  $H_\Phi$ , and so  $\mathcal{M}_1 \subset \mathcal{M}$ . Thus  $\mathcal{M}_1 = \mathcal{M}$ .

Now it is easy to see that  $\Upsilon$  is inner factor of the inner-outer factorization of  $G^t$ . It follows now from Theorem 2.1 that  $\Upsilon$  is co-outer. ■

Consider now the matrix function  $\Phi^t$ . Let  $\mathcal{N}$  be the shift-invariant subspace of  $H^2(\ell^2)$  spanned by the maximizing vectors of  $H_{\Phi^t}$ . Then by Theorem 5.1,  $\mathcal{N}$  has the form  $\Omega H^2(\mathbb{C}^r)$ , where  $\Omega$  is an inner and co-outer matrix function. By Theorem 3.1, there exist inner and co-outer matrix functions  $\Theta$  and  $\Xi$  such that

$$\mathcal{V} \stackrel{\text{def}}{=} \begin{pmatrix} \Upsilon & \bar{\Theta} \end{pmatrix} \quad \text{and} \quad \mathcal{W}^t \stackrel{\text{def}}{=} \begin{pmatrix} \Omega & \bar{\Xi} \end{pmatrix} \quad (5.3)$$

are unitary-valued matrix functions.

The proof of the following result is exactly the same as the proof of Theorem 15.3 of Ch. 14 of [Pe2] for finite matrix functions (see also [AP]).

**Theorem 5.2.** *Let  $\Phi$  be a function in  $L^\infty(\mathcal{B}(\ell^2))$  such that  $\|H_\Phi\|_e < t_0 = \|H_\Phi\|$ . Let  $r$  be the number of superoptimal singular values of  $\Phi$  equal to  $t_0$ . Suppose that  $F$  is a best approximation of  $\Phi$  by analytic matrix functions. Then  $\Phi - F$  admits a factorization of the form*

$$\Phi - F = \mathcal{W}^* \begin{pmatrix} t_0 U & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} \mathcal{V}^*, \quad (5.4)$$

where  $\mathcal{V}$  and  $\mathcal{W}$  are given by (5.3),  $U$  is an  $r \times r$  unitary-valued very badly approximable matrix function such that  $\|H_U\|_e < 1$ , and  $\Psi$  is a matrix-function in  $L^\infty(\mathcal{B}(\ell^2))$  such that  $\|\Psi\|_{L^\infty} \leq t_0$  and  $\|H_\Psi\| = t_r(\Phi) < \|H_\Phi\|$ . Moreover,  $U$  is uniquely determined by the choice of  $\Upsilon$  and  $\Omega$  and does not depend on the choice of  $F$ .

As in the case of finite matrix functions, under the hypotheses of Theorem 5.2 the following inequality holds

$$\|H_\Psi\|_e \leq \|H_\Phi\|_e. \quad (5.5)$$

it can be deduced from (4.2) in exactly the same way as in [AP] (see also Theorem 15.12 of Ch. 15 of [Pe2]).

Moreover, under the hypotheses of Theorem 5.2, the operator functions  $\Theta$  and  $\Xi$  in (5.3) are left-invertible in  $H^\infty$ . Again, this can be deduced from the same results for partial thematic factorizations (see §4) in the same way it was done in the case of finite matrix functions in [AP] (see also [Pe2], Ch. 14, §5). This left-invertibility property of  $\Theta$  and  $\Xi$  is important in the main result of the next section.

The following theorem can be considered as a converse of Theorem 5.2. These two theorems together give a characterization of the badly approximable matrix functions  $\Phi$  satisfying the condition  $\|H_\Phi\|_e < t_0 = \|H_\Phi\|$ . Note however that we do not need this condition to prove that functions that admit a factorization of the form (5.4). Moreover, in the following theorem we can also relax the assumptions on  $U$  imposed in Theorem 5.2.

**Theorem 5.3.** *Let  $\Phi$  be an infinite matrix function of the form*

$$\Phi = \mathcal{W}^* \begin{pmatrix} \sigma U & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} \mathcal{V}^*, \quad (5.6)$$

where  $\sigma > 0$ ,  $\mathcal{V}$  and  $\mathcal{W}^t$  are  $r$ -balanced matrix functions,  $U$  is an  $r \times r$  unitary-valued matrix function such that the shift-invariant subspace of  $H^2(\mathbb{C}^r)$  spanned by the maximizing vectors of  $H_U$  coincides with  $H^2(\mathbb{C}^r)$ , and  $\|\Psi\|_\infty \leq \sigma$ . Then  $\Phi$  is badly approximable and  $t_0(\Phi) = \dots = t_{r-1}(\Phi) = \sigma$ . Moreover,  $\Phi$  is very badly approximable if and only if  $\Psi$  is very badly approximable.

The proof of Theorem 5.3 is exactly the same as the proof of Theorem 15.7 of Ch. 14 of [Pe2] for finite matrix functions (see also [AP]).

Consider now the sequence

$$t_0 = \dots = t_{r_1-1} > t_{r_1} = \dots = t_{r_2-1} > \dots > t_{r_{\ell-1}} = \dots = t_{r_\ell-1} > \dots$$

of superoptimal singular values of  $\Phi$ . Let

$$\sigma_0 > \sigma_1 > \sigma_2 > \dots$$

be the sequence of distinct superoptimal singular values of  $\Phi$ , i.e.,

$$\sigma_0 = t_0 = \dots = t_{r_1-1}, \quad \sigma_1 = t_{r_1} = \dots = t_{r_2-1}, \quad \text{etc.}$$

If  $\|H_\Phi\|_e < \sigma_1$ , we can apply Theorem 5.2 to the matrix function  $\Psi$ . Now if  $\|H_\Phi\|_e < \sigma_2$ , then by (5.5),  $\|H_\Psi\|_e < \sigma_2$ , and so we can continue this process and obtain the following result in exactly the same way as in the case of finite matrix functions in [Pe2], Ch. 14, §15.

**Theorem 5.4.** *Let  $\Phi$  be a function in  $L^\infty(\mathcal{B}(\ell^2))$  such that  $\|H_\Phi\|_e < \sigma_{d-1}$ . Let  $F$  be an arbitrary matrix function in  $\Omega_{r_d}$ . Then  $\Phi - F$  admits a factorization*

$$\Phi - F = \mathcal{W}_0^* \cdots \mathcal{W}_{d-1}^* \begin{pmatrix} \sigma_0 U_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_1 U_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \sigma_{d-1} U_{d-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \Psi \end{pmatrix} \mathcal{V}_{d-1}^* \cdots \mathcal{V}_0^*, \quad (5.7)$$

where the  $U_j$  are  $(r_{j+1} - r_j) \times (r_{j+1} - r_j)$  very badly approximable unitary-valued functions such that  $\|H_{U_j}\|_e < 1$ ,

$$\mathcal{V}_j = \begin{pmatrix} \mathbf{I}_{r_j} & \mathbf{0} \\ \mathbf{0} & \check{\mathcal{V}}_j \end{pmatrix} \quad \text{and} \quad \mathcal{W}_j = \begin{pmatrix} \mathbf{I}_{r_j} & \mathbf{0} \\ \mathbf{0} & \check{\mathcal{W}}_j \end{pmatrix}, \quad 1 \leq j \leq d-1,$$

$\check{\mathcal{V}}_j$  and  $\check{\mathcal{W}}_j^t$  are  $(r_{j+1} - r_j)$ -balanced matrix functions, and  $\Psi$  is a matrix function satisfying

$$\|\Psi\|_{L^\infty} \leq t_{r_{d-1}}, \quad \text{and} \quad \|H_\Psi\| < t_{r_{d-1}}.$$

Factorizations of the form (5.7) with the  $\sigma_j$ ,  $U_j$ ,  $V_j$ , and  $W_j$  as in Theorem 5.4 are called *partial canonical factorizations* (or *partial canonical factorizations of order  $d$* ).

Now we can state the following description of very badly approximable matrix functions.

**Theorem 5.5.** *Let  $\Phi$  be an admissible function in  $L^\infty(\mathcal{B}(\ell^2))$ . If  $\Phi$  is very badly approximable, then for each  $d$  with nonzero  $\sigma_{d-1}$  the matrix function  $\Phi$  admits a partial canonical factorization of the form (5.2).*

The proof of Theorem 5.5 is exactly the same as in the case of finite matrix functions, see [Pe2], Ch. 14, §15 (see also [AP]). Finally, we state the converse of Theorem 5.5, which is valid without the admissibility assumption.

**Theorem 5.6.** *Let  $\Phi$  be a function in  $L^\infty(\mathcal{B}(\ell^2))$  such that  $\Phi$  admits a partial canonical factorization of the form (5.7) whenever  $\sigma_{d-1} > 0$ . Then  $\Phi$  is very badly approximable and*

$$t_\varkappa(\Phi) = \begin{cases} \sigma_0, & \varkappa < r_1, \\ \sigma_j, & r_j \leq \varkappa < r_{j+1}. \end{cases}$$

**Example.** As we have mentioned in the Introduction there is an important difference between the case of finite matrix functions and the case of infinite matrix functions. In the case of finite matrix functions the hypotheses of Theorem 5.5 guarantee that the zero function the only superoptimal approximant. It turns out that in the case of infinite matrix functions this is not true. Consider the following example.

Let  $\{u_j\}_{j \geq 0}$ , be a sequence of scalar badly approximable functions such that

$$|u_j(\zeta)| = 1 \quad \text{for almost all } \zeta \in \mathbb{T} \quad \text{and} \quad \|H_{u_j}\|_e < 1$$

and let  $\{t_j\}_{j \geq 0}$  be a decreasing sequence of positive numbers such that

$$\lim_{j \rightarrow \infty} t_j > 0.$$

Consider the infinite matrix function

$$\Phi = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & t_0 u_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & t_1 u_1 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t_2 u_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Obviously, for every  $d \in \mathbb{Z}_+$ , there is constant unitary matrix  $V_d$  such that

$$\Phi = V_d^* \begin{pmatrix} t_0 u_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & t_1 u_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & t_{d-1} u_{d-1} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & t_d u_d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} V_d. \quad (5.8)$$

Clearly, the right-hand side of (5.8) is a partial canonical factorization of  $\Phi$ , and so by Theorem 5.6,  $\Phi$  is very badly approximable and  $t_j(\Phi) = t_j$

On the other hand, if  $f$  is an arbitrary scalar function in  $H^\infty$  with

$$\|f\|_\infty \leq \lim_{j \rightarrow \infty} t_j$$

and

$$F = \begin{pmatrix} f & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then

$$\begin{pmatrix} -f & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & t_0 u_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & t_1 u_1 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & t_2 u_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and since, obviously,

$$s_j((\Phi - F)(\zeta)) = t_j, \quad j \in \mathbb{Z}_+, \quad \zeta \in \mathbb{T},$$

it follows that  $F$  is a superoptimal approximant of  $\Phi$ .

To make the conclusion that an admissible infinite matrix function has a unique superoptimal approximant, we need the condition that

$$\lim_{j \rightarrow \infty} t_j(\Phi) = 0. \quad (5.9)$$

Indeed, if  $F_1$  and  $F_2$  belong to  $\mathbf{\Omega}_r$ , then we can consider partial thematic factorizations of  $\Phi - F_1$  and  $\Phi - F_2$  and see that

$$\|F_1 - F_2\|_\infty = \|(\Phi - F_1) - (\Phi - F_2)\|_\infty \leq 2t_r(\Phi).$$

In particular, if both  $F_1$  and  $F_2$  are superoptimal approximants, then by (5.9),  $F_1 = F_2$ . However, if  $\Phi$  is admissible and satisfies (5.9), then  $\lim_{j \rightarrow \infty} s_j(T_\Phi) = 0$ , and so  $H_\Phi$  is compact.

## 6. Very badly approximable functions

In this sections we obtain a necessary and sufficient condition for an admissible infinite matrix function to be very badly approximable. Let  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$ . Put

$$t_\infty(\Phi) = \lim_{j \rightarrow \infty} t_j(\Phi).$$

As in the case of finite matrix functions, for  $\sigma > t_\infty(\Phi)$ , we consider the subspace  $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$  that is the linear span of the Schmidt vectors of  $\Phi(\zeta)$  that correspond to the singular values of  $\Phi(\zeta)$  that are greater than or equal to  $\sigma$ . The subspaces  $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$  are defined for almost all  $\zeta \in \mathbb{T}$ .

**Definition.** Let  $L(\zeta)$ ,  $\zeta \in \mathbb{T}$ , be a family of subspaces of  $\ell^2$  that is defined almost everywhere on  $\mathbb{T}$ . We say that functions  $\xi_1, \dots, \xi_l$  in  $H^2(\ell^2)$  *span the family*  $L$  if  $L(\zeta) = \text{span}\{\xi_j(\zeta) : 1 \leq j \leq l\}$  for almost all  $\zeta \in \mathbb{T}$ .

We consider in this section the following condition:

(C) *for each  $\sigma > t_\infty(\Phi)$ , the family of subspaces  $\mathfrak{S}_\Phi^{(\sigma)}$  is analytic and spanned by finitely many functions in  $\text{Ker } T_\Phi$ .*

As in the case of finite matrix functions (see [PT3]), it is easy to see that condition (C) implies that the functions  $\zeta \mapsto s_j(\Phi(\zeta))$ ,  $j \in \mathbb{Z}_+$ , are constant almost everywhere on  $\mathbb{T}$ .

The following theorem is the main result of this section.

**Theorem 6.1.** *If  $\Phi$  is an admissible very badly approximable matrix function in  $L^\infty(\mathcal{B}(\ell^2))$ , then  $\Phi$  satisfies (C).*

*Conversely, if  $\Phi$  is an arbitrary function in  $L^\infty(\mathcal{B}(\ell^2))$  that satisfies (C), then  $\Phi$  is very badly approximable.*

**Remark.** As we have already mentioned, there is an important difference between the case of finite matrix functions and the case of infinite matrix functions. In the case of finite matrix functions condition (C) also implies that the zero function is the only superoptimal approximant. In the case of infinite matrix functions this is not true. Indeed, it is easy to see that the matrix function given in the example at the end of the previous section satisfies condition (C). However, it has infinitely many superoptimal approximants.

The necessity of condition (C) can be obtained from Theorem 5.5 in exactly the same way as it was done in [PT3], Theorem 4.1 in the case of finite matrix functions. On the other hand, the proof of the sufficiency of (C) given in [PT3] works only for finite matrices. It has to be slightly modified to work in the case of infinite matrix functions.

Here we present a proof based on canonical factorization. Note that the proof based on superoptimal weights that was presented in §5 of [PT3] in the case of finite matrix functions also works (with obvious modifications).

**Proof of the sufficiency of (C).** Suppose that  $\Phi$  satisfies (C). As we have already observed, the functions  $\zeta \mapsto s_j(\Phi(\zeta))$ ,  $j \in \mathbb{Z}_+$ , are constant almost everywhere on  $\mathbb{T}$ . Let

$$\sigma_0 > \sigma_1 > \sigma_2 > \cdots \quad (6.1)$$

be positive numbers (finitely many or infinitely many) such that for almost all  $\zeta \in \mathbb{T}$ , the numbers (6.1) are all nonzero distinct singular values of  $\Phi(\zeta)$ . It suffices to prove that if  $\sigma_{d-1} > 0$ , then  $\Phi$  admits a partial canonical factorization of order  $d$ . We prove it by induction on  $d$ .

Suppose first that  $d = 1$ . Let  $r = \dim \mathfrak{S}_{\Phi}^{(\sigma_0)}(\zeta)$  for almost all  $\zeta \in \mathbb{T}$ . Obviously,  $\dim \mathfrak{S}_{\Phi^t}^{(\sigma_0)}(\zeta) = r$  for almost all  $\zeta \in \mathbb{T}$ . Let us show that  $\Phi$  admits a factorization of the form (5.6) with  $r$ -balanced functions  $\mathcal{V}$  and  $\mathcal{W}^t$ . It is easy to verify that a function  $\xi \in H^2(\ell^2)$  is a maximizing vector of  $H_{\Phi}$  if and only if  $\eta \stackrel{\text{def}}{=} \bar{z} \overline{H_{\Phi}} \xi$  is a maximizing vector of  $H_{\Phi^t}$  (see [Pe2], Ch. 14, §2). Let  $\mathcal{M}$  be the minimal invariant subspace of multiplication by  $z$  on  $H^2(\ell^2)$  that contains all maximizing vectors of  $H_{\Phi}$  and let  $\mathcal{N}$  be the minimal invariant subspace of multiplication by  $z$  on  $H^2(\ell^2)$  that contains all maximizing vectors of  $H_{\Phi^t}$ .

By Theorem 5.1, there exist inner and co-outer functions  $\Upsilon$  and  $\Omega$  in  $H^\infty(\mathcal{B}(\mathbb{C}^r, \ell^2))$  such that  $\mathcal{M} = \Upsilon H^2(\mathbb{C}^r)$  and  $\mathcal{N} = \Omega H^2(\mathbb{C}^r)$ . By Theorem 3.1, there exist  $r$ -balanced matrix functions  $\mathcal{V}$  and  $\mathcal{W}^t$  of the form

$$\mathcal{V} = \begin{pmatrix} \Upsilon & \bar{\Theta} \end{pmatrix} \quad \text{and} \quad \mathcal{W}^t = \begin{pmatrix} \Omega & \bar{\Xi} \end{pmatrix}.$$

In exactly the same way as in the proof of Theorem 3.2 of [PT3] it can be shown that  $\Phi$  admits a factorization

$$\Phi = \mathcal{W}^* \begin{pmatrix} \sigma_0 U & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} \mathcal{V}^*,$$

where  $U$  is an  $r \times r$  unitary-valued matrix function. The proof of the fact that the shift-invariant subspace spanned by the maximizing vectors of  $H_U$  is  $H^2(\mathbb{C}^r)$  is the same as it was done in the proof of Theorem 4.1 of [PT3].

In exactly the same way as in the proof of Theorem 4.1 of [PT3] one can prove that  $\Psi$  satisfy condition (C). Clearly, for almost all  $\zeta \in \mathbb{T}$ ,

$$\sigma_1 > \sigma_2 > \cdots$$

are all nonzero distinct singular values of  $\Psi(\zeta)$ .

Suppose now that  $d > 1$ . By the inductive hypothesis,  $\Psi$  admits a partial canonical factorization of order  $d - 1$ . Thus  $\Phi$  admits a partial canonical factorization of order  $d$ . ■

**Remarks on uniqueness.** As we have mentioned above, unlike the case of finite matrix functions, in the infinite-dimensional case a very badly approximable function satisfying condition (C) can have infinitely many superoptimal approximants. However, in certain important cases the zero function is the only superoptimal approximant of a very badly approximable function  $\Phi$ :

- (i) if the Hankel operator  $H_\Phi$  is compact, then  $\Phi$  has a unique superoptimal approximant ([T], [Pe1], [PT1]), and so in this case (note that such functions  $\Phi$  are automatically admissible)  $\Phi$  is very badly approximable if and only if condition (C) holds and in this case the zero function is the only superoptimal approximant of  $\Phi$ ;
- (ii) if  $\text{rank } \Phi(\zeta)$  is uniformly bounded for almost all  $\zeta \in \mathbb{T}$  (this happens, for example, if  $\Phi$  has finitely many columns or rows), then the family of subspaces  $\mathfrak{S}_\Phi^{(\sigma)}$  stabilizes and we have the situation similar to the case of finite matrix functions; in this case again the zero function is the only superoptimal approximant of  $\Phi$  provided  $\Phi$  satisfies condition (C);
- (iii) if  $\Phi$  satisfies condition (C) and if for almost all  $\zeta \in \mathbb{T}$  the subspaces  $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$ ,  $\sigma > t_\infty(\Phi)$  span  $\ell^2$ , it is not hard to see that the zero function is the only superoptimal approximant of  $\Phi$ .

Let us explain (iii) in more detail.

Suppose that  $\Phi$  satisfies condition (C) and let  $\sigma_k$ ,  $k \geq 0$ , be the decreasing sequence such that for almost all  $\zeta \in \mathbb{T}$ , the  $\sigma_k$  are all nonzero distinct singular values of  $\Phi(\zeta)$  (we have already mentioned above that (C) implies that the singular values of  $\Phi(\zeta)$  are constant for almost all  $\zeta \in \mathbb{T}$ ).

It was shown in §5 of [PT3] that if  $F$  is a superoptimal approximation of  $\Phi$ , then

$$\Phi(\zeta) - F(\zeta) | \mathfrak{S}_\Phi^{(\sigma_k)}(\zeta) = \Phi(\zeta) | \mathfrak{S}_\Phi^{(\sigma_k)}(\zeta) \quad \text{for almost all } \zeta \in \mathbb{T}.$$

(this was done in [PT3] for finite matrix functions, but the same proof also works in the infinite-dimensional case). Thus if we assume that the subspaces  $\mathfrak{S}_\Phi^{(\sigma_k)}(\zeta)$  span  $\ell^2$  for almost all  $\zeta \in \mathbb{T}$ , we obtain  $\Phi - F = 0$ , and so the zero function is the only superoptimal approximant of  $\Phi$ .

## 7. Badly approximable operator functions

In [PT3] we obtained a description of badly approximable matrix functions. Now we can obtain the same result for operator function.

**Theorem 7.1.** *Let  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$  and  $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$ . If  $\Phi$  is badly approximable, then*

- (i)  $\|\Phi(\zeta)\|_{\mathcal{B}(\ell^2)}$  is constant for almost all  $\zeta \in \mathbb{T}$ ;

(ii) *there exists a function  $f$  in  $\text{Ker } T_\Phi$  such that  $f(\zeta)$  is a maximizing vector of  $\Phi(\zeta)$  for almost all  $\zeta \in \mathbb{T}$ .*

*Conversely, if  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$  and satisfies (i) and (ii), then  $\Phi$  is badly approximable.*

The proof is exactly the same as in the proof of Theorem 6.1 of [PT3].

Another result of §6 of [PT3] is a characterization of the set of badly approximable functions  $\Phi$  such that  $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$  and  $\mathbf{0}$  is the only best approximant of  $\Phi$ . We can ask the same question in the case of infinite matrix functions. However, if  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$  and  $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$ , then  $\mathbf{0}$  cannot be the only best approximant. Indeed  $\Phi$  is a badly approximable function satisfying  $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$ , then by Theorem 5.3, it admits a partial canonical factorization

$$\Phi = \mathcal{W}^* \begin{pmatrix} \sigma U & \mathbf{0} \\ \mathbf{0} & \Psi \end{pmatrix} \mathcal{V}^*$$

and  $\|H_\Psi\| < \sigma$ . Then there are infinitely many functions  $Q$  in  $H^\infty(\mathcal{B}(\ell^2))$  such that  $\|\Psi - Q\|_{L^\infty} < \sigma$ . Now it is easy to verify (see Theorem 1.8 of Ch. 14 of [Pe2]) that

$$\Phi - \Xi Q \Theta^t = \mathcal{W}^* \begin{pmatrix} \sigma U & \mathbf{0} \\ \mathbf{0} & \Psi - Q \end{pmatrix} \mathcal{V}^*,$$

where  $\Theta$  and  $\Xi$  are as in (5.3). Thus  $\Phi$  has infinitely many best approximants.

However, we still can obtain a sufficient condition for a badly approximable operator function to have a unique best approximant. Clearly, such a function  $\Phi$  cannot satisfy the inequality  $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$ . It is convenient to normalize  $\Phi$  with the condition  $\|\Phi\|_{L^\infty} = 1$ .

**Theorem 7.2.** *Let  $\Phi \in L^\infty(\mathcal{B}(\ell^2))$  be a function such that  $\Phi(\zeta)$  is an isometry for almost all  $\zeta \in \mathbb{T}$  or  $\Phi^*(\zeta)$  is an isometry for almost all  $\zeta \in \mathbb{T}$ . Suppose that  $\{f(\zeta) : f \in \text{Ker } T_\Phi\}$  is a dense subset of  $\text{Ker } \Phi(\zeta)^\perp$  for almost all  $\zeta \in \mathbb{T}$ . Then  $\Phi$  is very badly approximable and the zero function is the only best approximant of  $\Phi$ .*

**Proof.** Clearly, the fact that  $\Phi$  is very badly approximable is an immediate consequence of Theorem 7.1. Let  $F$  be a best approximant of  $\Phi$  and let  $\Psi = \Phi - F$ . Take  $f \in \text{Ker } T_\Phi$ . Suppose that  $\Phi(\zeta)$  be a coisometry for almost all  $\zeta \in \mathbb{T}$ . By the assumption of the theorem  $f(\zeta) \in \text{Ker } \Phi(\zeta)^\perp$  for almost all  $\zeta \in \mathbb{T}$ . Hence,

$$\|\Phi(\zeta)f(\zeta)\| = \|f(\zeta)\| \quad \text{for almost all } \zeta \in \mathbb{T} \quad (7.1)$$

(in the case when  $\Phi(\zeta)$  is an isometry the above identity holds automatically). Since  $f \in \text{Ker } T_\Phi$ , we conclude that  $H_\Phi f = \Phi f$ , and (7.1) implies that

$$\|H_\Phi f\|_2 = \|\Phi f\|_2 = \|f\|_2.$$

Since  $F \in H^\infty$ , we have  $H_\Phi = H_\Psi$ . Consider the following chain of inequalities:

$$\|f\|_2 = \|H_\Phi f\|_2 = \|H_\Psi f\|_2 \leq \|H_\Psi f\|_2 \leq \|\Psi\|_\infty \|f\|_2 \leq \|f\|_2.$$

Therefore all inequalities in this chain are, in fact, equalities, and so

$$\Psi f = H_\Psi f = H_\Phi f = \Phi f.$$

Since the set  $\{f(\zeta) : f \in \text{Ker } T_\Phi\}$  is a dense subset of  $\text{Ker } \Phi(\zeta)^\perp$  for almost all  $\zeta \in \mathbb{T}$ , we obtain

$$\Phi(\zeta)|_{\text{Ker } \Phi(\zeta)^\perp} = \Psi(\zeta)|_{\text{Ker } \Phi(\zeta)^\perp} \quad \text{for almost all } \zeta \in \mathbb{T}. \quad (7.2)$$

If  $\Phi(\zeta)$  is an isometry for almost all  $\zeta \in \mathbb{T}$ , then  $\text{Ker } \Phi(\zeta)$  is trivial, and therefore  $\Phi = \Psi$ .

If  $\Phi(\zeta)$  is a coisometry for almost all  $\zeta \in \mathbb{T}$ , then (7.2) implies that for almost all  $\zeta \in \mathbb{T}$

$$(\Phi(\zeta)x, y) = (\Psi(\zeta)x, y), \quad x \in \text{Ker } \Phi(\zeta)^\perp, \quad y \in \ell^2.$$

Since  $\|\Psi\|_\infty \leq 1$ , it follows that  $\Phi^*(\zeta) = \Psi^*(\zeta)$  for almost all  $\zeta \in \mathbb{T}$ . ■

## REFERENCES

- [AAK] V.M. ADAMYAN, D.Z. AROV, AND M.G. KREIN, *On infinite Hankel matrices and generalized problems of Carathéodory-Fejér and F. Riesz*, Funktsional. Anal. i Prilozhen. **2:1** (1968), 1–19 (Russian); English transl.: Functional Anal. Appl. **2:1** (1968).
- [AP] R.B. ALEXEEV AND V.V. PELLER, *Badly approximable matrix functions and canonical factorizations*, Indiana Univ. Math. J. **49** (2000), 1247–1285.
- [C] M. CAMBERN, *Analytic range functions*, J. Math. Anal. Appl. **12** (1965), 413–424.
- [D] R.G. DOUGLAS, *Banach algebra techniques in operator theory*, Acad. Press, New York, 1972.
- [H] H. HELSON, *Lectures on Invariant subspaces*, Acad. Press, New York, 1964.
- [HL] H. HELSON and D. LOWDENSLAGER, *Prediction theory and Fourier series in several variables, II*, Acta Math. **106** (1961), 175–213.
- [Kh] S.YA. KHAVINSON, *On some extremal problems of the theory of analytic functions*, Uchen. Zapiski Mosk. Universiteta, Matem. **144:4** (1951), 133–143. English transl.: Amer. Math. Soc. Translations (2) **32** (1963), 139–154.
- [N] N.K. NIKOL'SKII, *Treatise on the shift operator. Spectral function theory*, Springer-Verlag, Berlin, 1986.
- [Pe1] V.V. PELLER, *Approximation by analytic operator-valued functions*. In: Harmonic Analysis and Operator Theory, (Caracas, 1994), Contemp. Math. **189** (1995), Amer. Math. Soc., Providence, RI, 431–438.
- [Pe2] V.V. PELLER, *Hankel operators and their applications*, Springer-Verlag, New York, 2003.
- [PT1] V.V. PELLER AND S.R. TREIL, *Superoptimal singular values and indices of infinite matrix functions*, Ind. Univ. Math. J. **44** (1995), 243–255.
- [PT2] V.V. PELLER AND S.R. TREIL, *Approximation by analytic matrix functions. The four block problem*, J. Funct. Anal. **148** (1997), 191–228.
- [PT3] V.V. PELLER AND S.R. TREIL, *Very badly approximable matrix functions*, to appear in Selecta Mathematica.

- [PY1] V.V. PELLER AND N.J. YOUNG, *Superoptimal analytic approximations of matrix functions*, J. Funct. Anal. **120** (1994), 300–343.
- [PY2] V.V. PELLER AND N.J. YOUNG, *Superoptimal singular values and indices of matrix functions*, Int. Eq. Op. Theory **20** (1994), 350–363.
- [Po] S. J. POREDA, *A characterization of badly approximable functions*, Trans. Amer. Math. Soc., **169** (1972), 249-256.
- [T] S.R. TREIL, *On superoptimal approximation by analytic and meromorphic matrix-valued functions*, J. Funct. Anal. **131** (1995), 386-414.
- [V] V.I. VASYUNIN, *Formula for multiplicity of contractions with finite defect indices*, Oper. Theory: Adv. Appl., Birkhäuser **4** (1989), 281-304.

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