The Structure of the Ladder Insertion-Elimination Lie Algebra

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THE STRUCTURE OF THE LADDER INSERTION-ELIMINATION LIE ALGEBRA

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ABSTRACT. We continue our investigation into the insertion-elimination Lie algebra $\mathcal{L}_L$ of Feynman graphs in the ladder case, emphasizing the structure of this Lie algebra relevant for future applications in the study of Dyson–Schwinger equations. We work out the relation to the classical infinite dimensional Lie algebra $\mathfrak{gl}_+(\infty)$ and we determine the cohomology of $\mathcal{L}_L$.

1. Introduction

In the last few years perturbative QFT has been shown to have a rich algebraic structure [9] leading to relations with apparently unrelated sectors of mathematics like non-commutative geometry and Riemann-Hilbert like problems [4, 5]. Such extraordinary relations can be resumed, to some extent, by the existence of a commutative, non co-commutative Hopf algebra $\mathcal{H}$ defined on the set of Feynman diagrams.

We will continue the investigation started in [15] where we discussed first relations of perturbative QFT with the representation theory of Lie algebras. In that paper we introduced the ladder Insertion-Elimination Lie algebra $\mathcal{L}_L$ and we discussed relations of this Lie algebra with some more classical (infinite dimensonal) Lie algebras.

In what follows we describe in greater detail the structure of this Insertion-Elimination Lie algebra. The plan of the paper is as follows: we will start the present paper in section two by some motivation for the relevance of the ladder insertion elimination Lie algebra $\mathcal{L}_L$ for full QFT. In particular we stress

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the relation of $\mathcal{L}_L$ with the quantum equations of motion or Dyson-Schwinger equations (DSEs).
In section three we recollect some basic fact about the Lie algebra $\mathcal{L}_L$ taken from [15].
Section four and section five are the core of this paper: in section four we give a structure theorem that stresses the relation of the Lie algebra $\mathcal{L}_L$ with the classical infinite dimensional Lie algebra $\mathfrak{gl}_+(\infty)$.
Finally, in section five we collect some basic result about the cohomology of the Lie algebra $\mathcal{L}_L$.

2. The significance of $Z_{n,m}$

The Lie algebra $\mathcal{L}_L$ on generators $Z_{n,m}$ is an insertion elimination Lie algebra [15] gained from these operations applied to a cocommutative and commutative Hopf algebra H_{comm} built on generators (ladders) t_n, $n \geq 0$, $\Delta(t_n) = \sum_{j=0}^{n} t_j \otimes t_{n-j}$, on which it acts as a derivation

$$Z_{i,j}(t_n) = \Theta(n-j)t_{n-j+i},$$

where $\Theta(n-j)$ is defined as $\Theta(n-j) = 1$ for $n-j \geq 0$, and 0 otherwise.
This seems to give just a glimmer of the full insertion elimination Lie algebra of [6], which acts as a derivation on the full Hopf algebra of Feynman graphs in a renormalizable quantum field theory.
Nevertheless, a full understanding of $Z_{n,m}$ goes a long way in understanding the full insertion elimination Lie algebra [13], using the fact that $\mathcal{L}_L$ acts on elements in the full Hopf algebra which are homogenous in the appropriate grading resulting from the Hochschild cohomology of that very Hopf algebra [10].
There are two powerful reasons for that: i) quantum field theory sums over all skeleton graphs in a symmetric fashion, ii) non-linear Dyson–Schwinger equations (DSEs) modify linear DSEs precisely by the anomalies generated by a non-vanishing β-function. The first fact ensures that we can work on homogenous elements in the Hopf algebra, the second one ensures that there are effective methods available to deal with the operadic aspects of graph insertions.
Here, DSEs are introduced combinatorially via a fixpoint equation in the Hochschild cohomology of a connected graded commutative Hopf algebra. Let us summarize the main features which emerged in recent work [11, 10, 12, 13].

Under the Feynman rules the Hochschild one-cocycles provided by the Hopf algebra of graphs map to integral operators provided by the underlying skeletons of the theory. Renormalization conditions are determined by suitable boundary conditions for the integral equations so generated. The DSEs determine the Green functions from this Hochschild cohomology of the Hopf algebra of Feynman graphs, which is itself derived from free quantum field theory and the choice of renormalizable interactions.

Indeed, following [12, 10] the identification of these one cocycles leads to a combinatorial Dyson–Schwinger equation:

$$\begin{aligned} \Gamma^{\underline{r}} &= 1 + \sum_{\substack{p \in H_L^{[1]} \\ \mathrm{res}(p) = \underline{r}}} \frac{\alpha^{|p|}}{\mathrm{Sym}(p)} B_+^p(X_p) \\ &= 1 + \sum_{\substack{\Gamma \in H_L \\ \mathrm{res}(\Gamma) = \underline{r}}} \frac{\alpha^{|\Gamma|}\Gamma}{\mathrm{Sym}(\Gamma)}, \end{aligned} \tag{1}$$

where the first sum is over a finite (or countable) set of Hopf algebra primitives p, Feynman graphs such that

$$\Delta(p) = p \otimes e + e \otimes p, \tag{2}$$

indexing the closed Hochschild one-cocycles B_+^p above, while the second sum is over all one-particle irreducible graphs contributing to the desired Green function, all weighted by their symmetry factors. Here, X_p is a polynomial in all $\Gamma^{\underline{r}}$, and the superscript $\underline{r}$ ranges over the finite set (in a renormalizable theory) of superficially divergent Green functions. It indicates the number and type of external legs reflecting the monomials in the underlying Lagrangian. We use $\mathrm{res}(p) = \underline{r}$ to indicate that the external legs of the graph p are of type $\underline{r}$. The structure of these equations allows for a proof of locality using Hochschild cohomology [10], which is also evident using a coordinate space approach [2]. These fixpoint equations are solved by an Ansatz

$$\Gamma^{\underline{r}} = 1 + \sum_{k=1}^{\infty} \alpha^k c_k^{\underline{r}}. \tag{3}$$

We grant ourselves the freedom to call such an equation a DSE or a combinatorial equation of motion for a simple reason: the DSEs of any renormalizable quantum field theory can be cast into this form. Crucially, in the above it can be shown (see [13], which we follow here) that

$$X_p = \Gamma^{\mathrm{res}(p)} X_{\mathrm{coupl}}^{|p|}, \tag{4}$$

where X_{coupl} is a connected Green functions which maps to an invariant charge under the Feynman rules. This is rather obvious: consider, as an example, the vertex function in quantum electrodynamics: a n loop primitive graph p contributing to it provides $2n+1$ internal vertices, $2n$ internal fermion propagators and n internal photon propagators. An invariant charge [8] is provided by a vertex function multiplied by the squareroot of the photon propagator and the fermion propagator. Thus the integral kernel corresponding to p is dressed by $2n$ invariant charges, and one vertex function. This is a general fact: each integral kernel corresponding to a Green function with external legs $\underline{r}$ in a renormalizable quantum field theory is dressed by a suitable power of invariant charges proportional to the grading of that kernel, and one additional apperance of $\Gamma^{\underline{r}}$ itself. This immediately shows that for a vanishing β-function the DSEs are reduced to a linear set of equations, and that the general case can be most efficiently handled by an expansion in the breaking of conformal symmetry induced by a non-vanishing β-function. Thus, a complete understanding of the linear case goes a long way in understanding the full solution. This emphasizes the crucial role which the insertion-elimination Lie algebra [6] in the ladder case [15] plays in the full theory: it defines an algebra of graphs which provide an underlying field of residues which is then extended by the contributions resulting from a non-trivial β-function: the resulting scaling anomalies extend the Hopf algebra of graphs to a non-cocommutative one, at the same time they force the appearance of new transcendental numbers and result in the appearance of non-trivial representations of the symmetric group in the operad of graph insertions [13]. Here, we study the underlying linear DSEs which would suffice for a vanishing β-function.
Indeed, we now define the linear DSE associated to the system above:

$$\Gamma^{\underline{r}}_{\mathrm{lin}} = 1 + \sum_{\substack{p \in H_L^{[1]} \\ \mathrm{res}(p) = \underline{r}}} \frac{\alpha^{|p|}}{\mathrm{Sym}(p)} B_+^p(\Gamma^{\underline{r}}_{\mathrm{lin}}). \tag{5}$$

The Hochschild closedness of B_+^p then ensures that we obtain a Hopf algebra isomorphic to the word Hopf algebra based on letters p, which we obtain as the underlying Dyson skeletons in the expansion of $\Gamma^{\underline{r}}$.

The solutions of the linear DSE above are graded by the order in α and by the augmentation degree

$$\Gamma^r_{\text{lin}} = 1 + \sum_{j=1}^{\infty} \alpha^j c_j = 1 + \sum_{j=1}^{\infty} d_j. \tag{6}$$

Here, c_j is the sum of all words of order α^j, where the degree $|\, w \,|$ of a word w is the sum of the degree of its letters, and the degree of a letter is the loop number of the accompanying skeleton graph. These words uniquely correspond to Feynman graphs obtained by inserting primitive graphs into each other, where insertion now happens at a single vertex or edge in accordance with that linear DSE.

On the other hand, d_j is the sum

$$d_j = \sum_{w \in H^j_{\text{aug}}} \alpha^{|w|} w, \tag{7}$$

of all words made out of j letters, and we set $|\, w \,|_{\text{aug}} = j$, the augmentation degree.

Having defined the associated linear system, the propagator-coupling dualities [3] provide the general solution once the representation theory of the symmetric group has been established, which reflects the operadic nature of graph insertions [13]. But a complete understanding of linear Dyson–Schwinger equations comes first. To this end, it is profitable to study the action of the insertion elimination Lie algebra which acts on the Hopf algebra of graphs in that case. In this paper, we start some groundwork by clarifying the structure of the insertion elimination Lie algebra which relates to the Hopf algebra structure of a linear DSE. The crucial point is always the identification of the Hochschild closed one-cocycles in the Hopf algebra of graphs B_+^p, typically parametrized by primitive elements p of the Hopf algebra of graphs [10].

We first mention that the Hopf algebra of graphs contains, as a corollary of the results in [6], a sub Hopf algebra H_w of graphs generated by the linear DSE. It is naturally based on graphs which can be regarded as words, with corresponding insertion-elimination Lie algebra L_w. It acts on the Hopf algebra H_w as

$$Z_{w_1,w_2}(w) = \begin{cases} w_1 v \ \text{ if } \ w = w_2 v \ \text{ for some } \ v \\ 0, \ \text{ if } w \text{ has not this form.} \end{cases} \tag{8}$$

The Lie bracket in L_w is then

$$\begin{aligned}[Z_{w_1,w_2}, Z_{w_3,w_4}] &= Z_{\overline{Z_{w_1,w_2}(w_3)},w_4} - Z_{w_3,\overline{Z_{w_2,w_1}(w_4)}} \\ &\quad -Z_{\overline{Z_{w_3,w_4}(w_1)},w_2} + Z_{w_1,\overline{Z_{w_4,w_3}(w_2)}} \\ &\quad -\delta^K_{w_2,w_3} Z_{w_1,w_4} + \delta^K_{w_1,w_4} Z_{w_3,w_2}. \end{aligned} \tag{9}$$

See [6] for notation.

The significance of the Lie algebra $Z_{n,m}$ comes from the fact that the map $B_+ = \sum_p \alpha^{|p|} B_+^p$ maps the linear DSE to the fundamental DSE (which also underlies the polylog [12])

$$X = 1 + B_+(X), \tag{10}$$

where B_+ is of order α. Note that B_+ is not homogenous in α (there are primitive graphs of any degree in the coupling), but it is homogenous in the augmentation degree: all terms in its defining sum enhance this degree by one. There is a natural inclusion ι_H from H_{comm} to H_w which sends $t_n \to d_n$. This induces a map

$$\iota_L : \ L \to L_w, \ Z_{n,m} \to \sum_{|w_1|_{\text{aug}}=n, |w_2|_{\text{aug}}=m} \frac{Z_{w_1,w_2}}{\#(m)}, \tag{11}$$

where $\#(m)$ is the number of words of degree m, such that $\iota_H(Z_{n,m}(t_k)) = \iota_L(Z_{n,m})(\iota(t_k))$. It is compatible with the Lie bracket:

$$[\iota_L(Z_{n_1,m_1}), \iota_L(Z_{n_2,m_2})] \left(\iota_H(t_n)\right) = \iota_H \left([Z_{n_1,m_1}, Z_{n_2,m_2}](t_n)\right). \tag{12}$$

As long as we study linear DSEs, the ladder insertion elimination Lie algebra on generators $Z_{n,m}$ suffices, where it now acts by increasing and decreasing the augmentation degree. In [13] the reader can find a discussion of the Galois theory which is missing to handle the general case. The study of such questions in QFT is a beautiful mathematical problem in its own right. It gives mathematical justification to early ideas [14] of the use of anomalous dimensions and bootstrap equations in QFT to absorb short-distance singularities. Progress along these lines following [12, 13] will be reported in future work. We now continue to treat $\mathcal{L}_L$.

3. Generalities about the Ladder Insertion-Elimination Lie algebra.

Let us recall some basic definition from [15] to which we refer for the details omitted in what follows.

Let us introduce the Lie algebra $\mathcal{L}_L$ via generators and relations:

Definition 3.1. $\mathcal{L}_L = \ span_{\mathbb{C}} < Z_{n,m} | \ n,m \in \mathbb{Z} >$, *with:*

$$\begin{aligned}\left[Z_{n,m}, Z_{l,s}\right] &= \Theta(l-m)Z_{l-m+n,s} - \Theta(s-n)Z_{l,s-n+m} \\ & \quad -\Theta(n-s)Z_{n-s+l,m} + \Theta(m-l)Z_{n,m-l+s} \\ & \quad -\delta_{m,l}Z_{n,s} + \delta_{n,s}Z_{l,m}, \end{aligned} \tag{13}$$

where:

$$\begin{cases} \Theta(l-m) & = 0 \ \ if \, l < m, \\ \Theta(l-m) & = 1 \ \ if \, l \geq m \end{cases} \tag{14}$$

and where $\delta_{n,m}$ is the usual Kronecker delta:

$$\begin{cases} \delta_{n,m} & = 1 \ \ if \, m = n, \\ \delta_{n,m} & = 0 \ \ if \, n \neq m. \end{cases} \tag{15}$$

We start with the following:

Corollary 3.2. [15] *1) $\mathcal{L}_L$ is $\mathbb{Z}$-graded Lie algebra:*

$$\mathcal{L}_L = \oplus_{i \in \mathbb{Z}} l_i$$

where each for each $Z_{n,m} \in l_i$, $deg(Z_{n,m}) = i = n - m$ and $dim_{\mathbb{C}} \ l_i = +\infty$;
2) $\mathcal{L}$ has the following decomposition:

$$\mathcal{L}_L = L^+ \oplus L^0 \oplus L^-;$$

where $L^+ = \oplus_{n>0} l_n$, $L^- = \oplus_{n<0} l_n$ and $L^0 = l_0$.

Proof The statements follow from the definition of graded Lie algebra, i.e $\mathcal{L}_L$ is G-graded (where G is any abelian group) if $\mathcal{L}_L = \oplus_{i \in G} l_i$ and $[l_i, l_j] \subset l_{i+j}$, and from the formula (13).

□

We conclude this section with the following:

Proposition 3.3. *Each element $Z_{n,m} \in \mathcal{L}_L$ can be written in the following form:*

$$Z_{n,m} = [Z_{n,0}, Z_{0,m}] + \Theta(n-m)Z_{n-m,0} + \Theta(m-n)Z_{0,m-n} - \delta_{n-m,0}Z_{0,0}. \tag{16}$$

Proof The statement follows trivially applying formula (13) to the elements $Z_{n,0}$ and $Z_{0,m}$ in the cases $n > m$, $n < m$ and $n = m$. □

Remark 3.4. *The previous proposition is equivalent to the following (vector space) decomposition of the Lie algebra $\mathcal{L}_L$:*

$$\mathcal{L}_L = [\mathcal{D}, \mathcal{D}] \oplus \mathcal{D};$$

where we defined:

$$\mathcal{D} = \mathfrak{a}_+ \oplus \mathfrak{a}_- \oplus \mathbb{C}$$

and $\mathfrak{a}_+ = span_{\mathbb{C}}\{Z_{n,0} : n > 0\}$, $\mathfrak{a}_- = span_{\mathbb{C}}\{Z_{0,n} : n > 0\}$ *and* $\mathbb{C}$ *is the trivial Lie algebra generated by* $Z_{0,0}$.

4. Structure of the Lie algebra $\mathcal{L}_L$.

Let us start this section with two statements whose proofs are collected at the end of the this section.

Theorem 4.1. *The center of the Lie algebra $\mathcal{L}_L$ has dimension one and it is generated by the element $Z_{0,0}$.*

Theorem 4.2. *l^0 is a maximal abelian sub-algebra of $\mathcal{L}_L$.*

In what follows we will show that the Lie algebra $\mathcal{L}_L$ is not simple. Let us introduce the following:

Definition 4.3. [15]

$$\mathfrak{gl}_+(\infty) = span_{\mathbb{C}} < E_{i,j} : Z_{i,j} - Z_{i+1,j+1} |\, i,j \in \mathbb{Z}_{\geq 0} > .$$

We have:

Proposition 4.4. *1)* $[E_{i,j}, E_{r,k}] = E_{i,k}\delta_{j,r} - E_{r,j}\delta_{k,i}$*;*
2) $\mathfrak{gl}_+(\infty)$ *is an ideal in* $\mathcal{L}_L$.

Proof The proof of 1) and 2) is a simple but tedious application of the commutator formula (13).

□

We can then define the quotient Lie algebra:

$$C = \mathcal{L}_L / \mathfrak{gl}_+(\infty)$$

and consider the exact sequence:

$$0 \longrightarrow \mathfrak{gl}_+(\infty) \longrightarrow \mathcal{L}_L \xrightarrow{\pi} C \longrightarrow 0. \qquad (17)$$

We want to understand the Lie algebra C. To accomplish this goal we need the following

Proposition 4.5.

$$\mathfrak{gl}_+(\infty) = [\mathcal{L}_L, \mathcal{L}_L].$$

PROOF Let us prove the two inclusions.
$\mathfrak{gl}_+(\infty) \subset [\mathcal{L}_L, \mathcal{L}_L]$ since from the definition of $\mathfrak{gl}_+(\infty)$:

$$E_{i,j} = Z_{i,j} - Z_{i+1,j+1} = [Z_{i,0}, Z_{0,j}] - [Z_{i+1,0}, Z_{0,j+1}],$$

where the second equality follows from the formula (16).
To show the other inclusion, i.e $[\mathcal{L}_L, \mathcal{L}_L] \subset \mathfrak{gl}_+(\infty)$, it suffices to observe that for any two generators $Z_{h,p}$ and $Z_{r,q}$ of $\mathcal{L}_L$ we have that their commutator is given by the difference between two elements having same degree (see formula (13)), say $Z_{n,m}$ and $Z_{l,s}$ with $n - m = l - s$.
Under the hypothesis that $k = n - m = l - s > 0$ and that $s > m$ (the other cases are completely analogous) we can write their difference as follows:

$$\begin{aligned} Z_{n,m} - Z_{l,s} &= Z_{m+k,m} - Z_{s+k,s} = Z_{m+k,m} - Z_{m+k+1,m+1} + \\ &\quad + Z_{m+k+1,m+1} - .... - Z_{s+k-1,s-1} + Z_{s+k-1,s-1} - Z_{s+k,s}, \end{aligned}$$

which expresses the difference between $Z_{n,m}$ and $Z_{l,s}$ as finite linear combination of elements in $\mathfrak{gl}_+(\infty)$. □
In particular we can rephrase the previous proposition in the following way:

Lemma 4.6. *Two generators $Z_{n,m}$ and $Z_{l,s}$ are $\mathfrak{gl}_+(\infty)$-equivalent if and only if they have the same degree, i.e:*

$$Z_{n,m} \sim Z_{l,s} \iff deg(Z_{m,n}) = deg(Z_{l,s}).$$

PROOF If $Z_{n,m}$ and $Z_{l,s}$ are have the same degree then they are equivalent by the argument used to prove the proposition above.
Suppose now that the difference between $Z_{n,m}$ and $Z_{l,s}$ can be written as a (finite) linear combination of elements in $\mathfrak{gl}_+(\infty)$ and also that $n - m \neq l - s$ (w.l.o.g. we can assume that $n - m > 0$ and that $l - s > 0$).
Under these assumptions and from formula (16), it follows also that: $Z_{n-m,0} - Z_{l-s,0} = \sum_{finite} a_i E_{p_i,q_i}$ But this has as consequence that each of these two elements are finite linear combinations of (homogeneous) elements in $\mathfrak{gl}_+(\infty)$, so that we can write: $Z_{n-m,0} = \sum_{finite} c_i E_{r_i,k_i}$ and $Z_{l-s,0} = \sum_{finite} c_i E_{t_i,v_i}$. Rewriting the right hand side of each of those two equalities in terms of the generators $Z_{n,m}$, it is clear that such equations can not hold. □

From the proposition 4.5 it follows that C is a (maximal) commutative Lie algebra coming as quotient of $\mathcal{L}_L$.
Let us now introduce a set of (natural) generators for C. Since the set

$$< Z_{n,m}|\ n, m \in \mathbb{Z}_{\geq 0} >$$

is a basis for $\mathcal{L}_L$ and since $\pi : \mathcal{L}_L \longrightarrow C$ is a surjection, we have that:

$$< \overline{Z}_{n,m} = \pi(Z_{n,m})|\ n, m \in \mathbb{Z}_{\geq 0} >$$

is a set of generators for C. Moreover it follows from the lemma 4.6 that for $n > m$ we have $Z_{n,m} \sim Z_{n-m,0}$, for $m > n$ we have $Z_{n,m} \sim Z_{0,m-n}$ and for $n = m$ we have $Z_{n,m} \sim Z_{0,0}$. So defining $Z_n = \overline{Z}_{n,0}$, $Z_{-n} = \overline{Z}_{0,n}$ for $n > 0$ and $Z_0 = \overline{Z}_{0,0}$, we get

$$C = span_{\mathbb{C}} < Z_n | n \in \mathbb{Z} > .$$

The fact that such elements are also linearly independent (i.e they form a base for C) follows also from the lemma 4.6.
In what follows in this section we want to prove that:

Theorem 4.7. *The exact sequence (17) does not split, i.e the Lie algebra $\mathcal{L}_L$ is not the semi-direct product of the Lie algebra $\mathfrak{gl}_+(\infty)$ with the (commutative) Lie algebra C.*

Before addressing the proof of theorem 4.7 we need to introduce some preliminaries to make the paper as self-contained as possible.
The exact sequence (17) implies that the Lie algebra $\mathcal{L}_L$ is a non-abelian extension of the commutative Lie algebra C by the Lie algebra $\mathfrak{gl}_+(\infty)$ (for generalities about non-abelian extension of Lie algebras we refer to the paper [1] and references therein). If $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{e}$ are Lie algebras:

Definition 4.8. [1] *We will say that the Lie algebra $\mathfrak{e}$ is an extension of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$ if $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{e}$ fit in the following exact sequence:*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0, \tag{18}$$

Moreover we will say that two such extensions $\mathfrak{e}$ and $\mathfrak{e}'$ are equivalent if and only if $\mathfrak{e}$ and $\mathfrak{e}'$ are isomorphic as Lie algebras.

Let $Der(\mathfrak{h})$ be the Lie algebra of derivations of $\mathfrak{h}$, $\alpha', \alpha \in \mathrm{H}om_{\mathbb{C}}(\mathfrak{g}, \mathrm{D}er(\mathfrak{h}))$ and $\rho', \rho \in \mathrm{H}om_{\mathbb{C}}(\Lambda^2\mathfrak{g}, \mathfrak{h})$. On the set of the couples (α, ρ) introduced above we define the equivalence relation:

$$(\alpha, \rho) \sim (\alpha', \rho') \iff \exists\ b \in Hom_{\mathbb{C}}(\mathfrak{g}, \mathfrak{h})$$

such that:

$$\alpha'(x).\xi = \alpha(x).\xi + [b(x), \xi]_{\mathfrak{h}},$$
$$\rho'(x, y) = \rho(x, y) + \alpha(x).b(y) - \alpha(y).b(x) - b([x, y]_{\mathfrak{g}}) + [b(x), b(y)]_{\mathfrak{h}}.$$

Then we have the following:

Theorem 4.9. [1] *1) The classes of isomorphism of the extensions of the Lie algebra* $\mathfrak{g}$ *by the Lie algebra* $\mathfrak{h}$ *given by the exact sequence (18), are in one-to-one correspondence with the classes of equivalence* $[(\alpha, \rho)]$ *such that:*

$$[\alpha(x), \alpha(y)]_{Der(\mathfrak{h})}.\xi - \alpha([x, y]_{\mathfrak{g}}).\xi = [\rho(x \wedge y), \xi]_{\mathfrak{h}};$$
$$\sum_{\text{cyclic}} \Big(\alpha(x).\rho(y, z) - \rho([x, y]_{\mathfrak{g}}, z)\Big) = 0;$$

for every $x, y, z \in \mathfrak{g}$ *and* $\xi \in \mathfrak{h}$.
2) The Lie algebra structure induced on the vector space $\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{g}$ *by the datum* (α, ρ) *is given by:*

$$[(\xi_1, x_1), (\xi_2, x_2)]\mathfrak{e} = ([\xi_1, \xi_2]_{\mathfrak{h}} + \alpha(x_1).\xi_2 - \alpha(x_2).\xi_1 + \rho(x_1, x_2), [x_1, x_2]_{\mathfrak{g}}). \tag{19}$$

We apply this result to our setting, where $\mathfrak{g} = C$ and $\mathfrak{h} = \mathfrak{gl}_+(\infty)$.
The exact sequence (17) tells us that we have:

$$\mathcal{L}_L \simeq \mathfrak{gl}_+(\infty) \oplus C$$

where such splitting holds in the category of vector spaces. We first show the following:

Proposition 4.10. *The Lie algebra structure on* $\mathcal{L}_L$ *given by the bracket (13) corresponds to the couple* (α, ρ) *defined by:*

$$\alpha(Z_n).(E_{i,j}) = \Theta(n) \sum_{k \geq 0} (E_{n+k,j}\delta_{i,k} - E_{i,k}\delta_{n+k,j}) +$$
$$\Theta(-n) \sum_{k \geq 0} (E_{k,j}\delta_{k+n,i} - E_{i,k+n}\delta_{j,k})$$

for $n \neq 0$ *and*

$$\alpha(Z_0) \equiv 0;$$

while:

$$\rho(Z_n, Z_m) = 0$$

if $n, m \geq 0$ *or* $n, m \leq 0$ *and*

$$\rho(Z_n, Z_{-m}) = \sum_{k=0}^{m-1} E_{n-m+k,k}$$

if $n > m$, and:

$$\rho(Z_n, Z_{-m}) = \sum_{k=0}^{m-1} E_{k,m-n+k},$$

if $n < m$.

Proof The proof follows comparing formula (13) with formula (19). □
We now remark that:

Lemma 4.11. [1] *Given:*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0, \tag{20}$$

as in (18), any splitting $s : \mathfrak{g} \longrightarrow \mathfrak{e}$ (at the vector space level) of the previous exact sequence, induces a map $\alpha_s \in Hom_{\mathbb{C}}\Big(\mathfrak{g}, Der(\mathfrak{h})\Big)$, via the following:

$$\alpha_s(X).\xi = [s(X), \xi];$$

for each $X \in \mathfrak{g}$ and each $\xi \in \mathfrak{h}$.

Proposition 4.12. *The map $\alpha \in Hom_{\mathbb{C}}\Big(C, Der(\mathfrak{gl}_+(\infty))\Big)$, defined in proposition, 4.10 is induced by the linear map $s : C \longrightarrow \mathcal{L}_L$, where:*

$$s(Z_n) = \Theta(n)Z_{n,0} + \Theta(-n)Z_{0,n} - \delta_{n,0}Z_{0,0}. \tag{21}$$

Proof The map s defined in formula (21) is a section of the projection $\pi : \mathcal{L}_L \longrightarrow C$ defined by the exact sequence (17), i.e $s \in Hom_{\mathbb{C}}(C, \mathcal{L}_L)$ and $s \circ \pi = Id_C$. From the lemma (4.11) induces a linear map:

$$\alpha_s : C \longrightarrow Der(\mathfrak{gl}_+(\infty)),$$

defined by:

$$\alpha_s(x).\xi = [s(x), \xi]_{\mathcal{L}_L}.$$

It is now easy to check that this map is the same defined in the proposition 4.10. □

We are now almost ready to prove theorem 4.7. We only need to remark the following: from theorem 4.9 we have that a given extension (α, ρ) of the Lie algebra C by the Lie algebra $\mathfrak{gl}_+(\infty)$ will split, i.e will be equivalent to a semi-direct product of the these two Lie algebras, if and only if $(\alpha,\rho)\sim(\alpha',0)$ i.e if and only if α' is a morphism of Lie algebras. From the theorem 4.9 this is equivalent to ask for the existence of a linear map $b: C\longrightarrow\mathfrak{gl}_+(\infty)$ such that $s+b: C\longrightarrow\mathcal{L}_L$ is a morphism of Lie algebras. Note that we are working in the category of graded Lie algebras so the requirement for such map is that it has to preserve the grading. Showing that such a map does not exist we will be able to conclude that the exact sequence does not split.

Proof (Theorem 4.7) Suppose we can define a linear map $b: C\longrightarrow\mathfrak{gl}_+(\infty)$ such that $s+b: C\longrightarrow\mathcal{L}_L$ is a morphism of (graded) Lie algebras. That means that we can find elements $\sum_{i=1}^M a_{h_i}E_{h_i+1,h_i}\in\mathfrak{gl}_+(\infty)$ and $\sum_{i=1}^N b_{k_j}E_{k_j,k_j+1}\in\mathfrak{gl}_+(\infty)$ such that $b(Z_1)=\sum_{i=1}^M a_{h_i}E_{h_i+1,h_i}$, $b(Z_{-1})=\sum_{i=1}^N b_{k_j}E_{k_j,k_j+1}$ and furthermore

$$0=[(s+b)(Z_1),(s+b)Z_{-1}]=[Z_{1,0}+\sum_{i=1}^M a_{h_i}E_{h_i+1,h_i},Z_{0,1}+\sum_{j=1}^N b_{k_j}E_{k_j,k_j+1}].$$

We can calculate such a commutator by re-writing each of the terms $E_{i,j}$ in the sums in terms of the generators $Z_{n,m}$, and applying to such terms the brackets given in formula (13). The result, written in terms of the generators $E_{i,j}$, takes the form:

$$-E_{0,0}+\sum_{j=1}^N b_{k_j}(E_{k_j+1,k_j+1}-E_{k_j,k_j})+\sum_{i=1}^M a_{h_i}(1+b_{h_i})(E_{h_i+1,h_i+1}-E_{h_i,h_i})=0.$$

The right hand side us of the previuos sum can be reorganized in term of the summands $E_{j+1,j+1}-E_{j,j}$ as follows:

$$-\sum_{i\geq 0}^L\phi_j(E_{j+1,j+1}-E_{j,j}),$$

for L is the biggest between N and M and the ϕ_j's are coefficients.

So that we have that:

$$E_{0,0}=\sum_{i\geq 0}^L\phi_j(E_{j+1,j+1}-E_{j,j})=-\phi_0E_{0,0}+\sum_{j\geq 0}(\phi_{j+1}-\phi_j)E_{j,j}+\phi_LE_{L+1,L+1},$$

that clearly give us a contradiction. □

Proof of Theorems 4.1 and 4.2. We now conclude this section by giving the proofs for theorems 4.1 and 4.2.

We recall from [15] that the Lie algebra $\mathcal{L}_L$ has an obvious module:

Definition 4.13. [15]

$$\mathcal{S} = \bigoplus_{n\geq 0} \mathbb{C}t_n = \mathbb{C}[t_0, t_1, t_2, t_3.....].$$

We will assign a degree equal to k to the generator t_k for each $k \geq 0$. $\mathcal{L}_L$ acts on $\mathcal{S}$ via the following:

$$Z_{n,m}t_k = 0 \ if \ \ m > k,$$
$$Z_{n,m}t_k = t_{k-m+n} \ if \ \ m \leq k. \tag{22}$$

In what follows we will indicate by $\mathcal{Z}(\mathcal{L}_L)$ the center of the Lie algebra $\mathcal{L}_L$.
Proof (Theorem 4.1). It is obvious that $\mathbb{C}Z_{0,0} \subset \mathcal{Z}(\mathcal{L}_L)$. Let us prove the other inclusion. Let us suppose that there is some element $\alpha \in \mathcal{L}_L$, not proportional to $Z_{0,0}$ and that belongs to the center of $\mathcal{L}_L$. W.l.o.g. we assume

$$\alpha = \sum_{i=1}^{k} a_i Z_{n_i,m_i} = \sum_{i:\, n_i,m_i\neq 0} b_i Z_{n_i,m_i} + \sum_{i:\, \tilde{n}_i\neq 0} c_i Z_{\tilde{n}_i,0} + \sum_{i:\ \tilde{m}_i\neq 0} d_i Z_{0,\tilde{m}_i}, \tag{23}$$

where all the $\tilde{n}_i$'s ($\tilde{m}_i$'s) are different from 0 and $\tilde{n}_i \neq \tilde{n}_j$ ($\tilde{m}_i \neq \tilde{m}_j$) if $i \neq j$, and $(n_i, m_i) \neq (n_j, m_j)$ if $i \neq j$.
We will prove that α defined above is equal to zero by showing that the coefficients b_i, c_i and d_i are all equal to zero. We will split the proof of this assertion into two lemmas.

Lemma 4.14. *If $\alpha \in \mathcal{Z}(\mathcal{L}_L)$, α defined as above, then $b_i = d_i = 0$ for each i.*

Proof Let us consider some element $Z_{n,0} \in \mathcal{L}_L$ such that $0 < n \leq \ min_i\{m_i, \tilde{m}_i\}$. Then using formula (13), we get:

$$[Z_{n,0}, \alpha] = \sum_i b_i[Z_{n,0}, Z_{n_i,m_i}] + \sum_i d_i[Z_{n,0}, Z_{0,\tilde{m}_i}] =$$
$$\sum_i b_i(Z_{n_i+n,m_i} - Z_{n_i,m_i-n}) + \sum_i d_i(Z_{n,\tilde{m}_i} - Z_{0,\tilde{m}_i-n}).$$

Note the all the $\tilde{m}_i$'s are different (and different from 0), while in the set of the m_i's (also all different from 0) we can have repetitions.
Let us now define the set $M \doteq \{m_1, ..., m_k, \tilde{m}_1, ..., \tilde{m}_r\}$ and let us consider the disjoint union:

$$M = M_1 \cup \ldots \cup M_s,$$

where each M_i corresponds to the set of all indices in M which are equal to some given index l_i, say. We remark once more that for each i $M_i \cap \{\tilde{m}_1, \cdots, \tilde{m}_r\}$ contains at most one element since in the set $\{\tilde{m}_1, ..., \tilde{m}_r\}$ we do not have repetitions.
Now let us consider $p_1 = l_1 - n$ (≥ 0, by the condition we imposed on n), and the corresponding element $t_{p_1} \in \mathcal{S}$. Since α belongs to $\mathcal{Z}(\mathcal{L}_L)$ and since $n > 0$, we have:

$$0 = [Z_{n,0}, \alpha](t_{p_1}) = -\Big(\sum_{i:\, m_i \in M_1} b_i t_{p_1 - m_i + n + n_i} + \sum_{i:\, \tilde{m}_i \in M_1} d_i t_{p_1 - \tilde{m}_i + n} \Big). \tag{24}$$

Remark 4.15. *We observe that all the indices in M_1 are equal to l_1 and that $p_1 = l_1 - n$. Moreover the n_i's in the first sum of the right hand side in formula (24) are all different (since by assumption we have that $(n_i, m_i) \neq (n_j, m_j)$ unless $i = j$ and in our case all the m_i belong to the class M_1). Finally we notice that the last sum, if not equal to zero, contains only one term.*

Let us now suppose that $M_1 \cap \{m_1, \cdots, m_k\}$ and $M_1 \cap \{\tilde{m}_1, \cdots, \tilde{m}_r\}$ are both not empty (the cases where one of those intersections or both of them are empty are completely analogous). From the previuos remark it follows that

$$0 = [Z_{n,0}, \alpha](t_{p_1}) = -\Big(\sum_{i:\, m_i \in M_1} b_i t_{l_1 - n - l_1 + n + n_i} + \sum_{i:\, \tilde{m}_i \in M_1} d_i t_{l_1 - n - l_1 + n} \Big) =$$

$$-\Big(\sum_i b_i t_{n_i} + d_1 t_0 \Big).$$

Now since all the n_i in the first sum are different we have that $d_1 = 0$ and all $b_i = 0$.
Now using the same argument we can proceed with the sets M_2,...,M_s, to show that each of the coefficients b_i and c_i are equal to 0. □
From the lemma 4.14 we conclude that if $\alpha \in \mathcal{Z}(\mathcal{L}_L)$, α defined as in equation (23), then:

$$\alpha = \sum_i c_i Z_{n_i,0}.$$

To conclude the proof of the theorem (4.1) we have to show that:

Lemma 4.16. *If $\alpha \in \mathcal{Z}(\mathcal{L}_L)$ and $\alpha = \sum_i c_i Z_{n_i,0}$, then $c_i = 0$ for each i.*

Proof We first notice that we can suppose all $n_i \neq 0$ and $n_1 < n_2 \ldots$. Let us now consider some element $Z_{0,n}$, such that $n \geq \max_i\{n_i\}$. Since we suppose $\alpha = \sum_i c_i Z_{n_i,0}$ to be in the center of $\mathcal{L}_L$, we can write:

$$0 = [\alpha, Z_{0,n}] = \sum_i c_i [Z_{n_i,0}, Z_{0,n}] = \sum_i c_i (Z_{n_i,n} - Z_{0,n-n_i}).$$

By the hypothesis on n and on the n_i's we conclude that all the c_i's are equal to zero. □

Proof (Theorem 4.2). Let us suppose that l^0 is not maximal abelian subalgebra of $\mathcal{L}_L$, i.e that there exists $\mathcal{L}_L \ni \alpha \notin l^0$, $\alpha = \sum_{i=1}^n a_i Z_{n_i,m_i}$, such that:

$$[\alpha, Z_{k,k}] = 0, \ \forall \ k > 0.$$

Without loss of generality we can suppose that in each of (n_i, m_i)'s, $n_i \neq m_i$ (if no, $\alpha = \beta + \sum_i f_i Z_{n_i,n_i}$ and $[\beta, Z_{k,k}] = [\alpha, Z_{k,k}]$).
Such an element can be written as:

$$\alpha = \sum_{i:\, m_i \neq 0,\, n_i \neq 0} b_i Z_{n_i,m_i} + \sum_{i:\, \tilde{n}_i \neq 0} c_i Z_{\tilde{n}_i,0} + \sum_{i:\, \tilde{m}_i \neq 0} d_i Z_{0,\tilde{m}_i}. \tag{25}$$

Remark 4.17. *We note that in formula (25) all the n_i's and the m_i's are different from* 0 *and also that $\tilde{n}_i \neq \tilde{n}_j$ and $\tilde{m}_i \neq \tilde{m}_j$ for each $i \neq j$.*

We will prove that such element is identically equal to zero, showing that each of the coefficients in the equation (25) is equal to zero. We will divide the proof of this statement in two lemmas.

Lemma 4.18. *Given $\alpha \in l^0$, defined as in formula (25), we have that $c_i = d_i = 0$ for all i.*

Proof Let us fix integer k, $0 < k \leq \min_i \{n_i, m_i, \tilde{n}_i, \tilde{m}_i\}$. Then we get:

$$[\alpha, Z_{k,k}] = \sum_{i:\, m_i \neq 0, n_i \neq 0} b_i [Z_{n_i,m_i}, Z_{k,k}] + \sum_{i:\, \tilde{n}_i \neq 0} c_i [Z_{\tilde{n}_i,0}, Z_{k,k}] + \sum_{i:\, \tilde{m}_i \neq 0} d_i [Z_{0,\tilde{m}_i}, Z_{k,k}] =$$

$$\sum_{i:\, \tilde{n}_i \neq 0} c_i (Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) + \sum_{i:\, \tilde{m}_i \neq 0} d_i (Z_{0,\tilde{m}_i} - Z_{k,k+\tilde{m}_i}),$$

since:

$$[Z_{n_i,m_i}, Z_{k,k}] = 0, \ \forall \ \{n_i, m_i\} \text{ such that } n_i \geq k, \ m_i \geq k,$$

$$[Z_{\tilde{n}_i,0}, Z_{k,k}] = Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0} \text{ if } 0 < k \leq \tilde{n}_i, \text{ and}$$

$$[Z_{0,\tilde{m}_i}, Z_{k,k}] = -Z_{k,\tilde{m}_i+k} + Z_{0,\tilde{m}_i} \text{ if } 0 < k \leq \tilde{m}_i.$$

Since α commutes with all the elements of the sub-algebra l^0, we have:

$$0 = \sum_{i:\,\tilde{n}_i \neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) + \sum_{i:\,\tilde{m}_i \neq 0} d_i(Z_{0,\tilde{m}_i} - Z_{k,k+\tilde{m}_i}).$$

But in the right hand side of the previous formula the first sum contains only elements of positive degree while the second sum contains only those of negative degree, thus the sum is equal to zero if and only if separately

$$\sum_{i:\,\tilde{n}_i \neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) = 0 \text{ and } \sum_{i:\,\tilde{m}_i \neq 0} d_i(Z_{0,\tilde{m}_i} - Z_{k,k+\tilde{m}_i}) = 0.$$

From this it follows that all c_i's and d_i's are equal to zero. Indeed, consider the sum containing the c_i's (the one contains the d_i's can be treated in the same way):

$$\sum_{i:\,\tilde{n}_i \neq 0} c_i(Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}) = 0.$$

Since $k \neq 0$ and since $\tilde{n}_i \neq \tilde{n}_j$ if $i \neq j$, all the elements $Z_{k+\tilde{n}_i,k} - Z_{\tilde{n}_i,0}$ are linearly independent. □

Summarizing, so far we have proved that if a given element α commutes with each of the elements in l^0, then:

$$\alpha = \sum_{i:\,n_i \neq 0, m_i \neq 0} b_i Z_{n_i,m_i}. \tag{26}$$

Lemma 4.19. *If* $\left[\alpha, l^0\right] = 0$*, with* α *defined as in (26), then all the* b_i*'s are equal to* 0*.*

Proof Let us decompose the element α in term of elements of positive and negative degree, i.e:

$$\alpha = \sum_i a_i Z_{n_i,m_i} = \sum_j \Big(\sum_{i \geq 0} b_i Z_{r_i+s_j,r_i} \Big) + \sum_j \Big(\sum_{i \geq 0} c_i Z_{p_i,p_i+t_j} \Big),$$

Remark 4.20. *We remark that in* α *elements of the same (negative or positive) degree could be present; as an example of such element (of positive degree) we can consider:*

$$\beta_j = \sum_i b_i Z_{r_i+s_j,r_i}, \textit{ for a given } j$$

or the element (of negative degree):

$$\gamma_j = \sum_i c_i Z_{p_i, p_i+t_j}, \text{ for a given } j \ .$$

From the previuos remark let us re-write α as:

$$\alpha = \sum_j \beta_j + \sum_j \gamma_j,$$

each $\beta_j \in L^+$ and each $\gamma_j \in L^-$.
Let us now consider some element $Z_{k,k} \in l^0$ and let us take the commutator of such element with α

$$[\alpha, Z_{k,k}] = \sum_j [\beta_j, Z_{k,k}] + \sum_j [\gamma_j, Z_{k,k}].$$

Since $\mathcal{L}_L$ is a graded Lie algebra and since deg $Z_{k,k} = 0$, we have that

$$\deg\, [\beta_j, Z_{k,k}] = s_j \ , \ \forall j$$

and similarly

$$\deg\, [\gamma_j, Z_{k,k}] = -t_j \ , \ \forall j.$$

Hence

$$[\alpha, Z_{k,k}] = 0 \iff [\beta_j, Z_{k,k}] = 0 \text{ and } [\gamma_j, Z_{k,k}] = 0 \ , \ \forall j.$$

We are left to prove that any homogeneous element commuting with all the elements in l^0 can not exist.
So, to fix ideas, let us now consider some element of positive degree s, say, $\beta = \sum_{i=1}^{l} a_i Z_{n_i+s, n_i}$ and let us suppose that

$$[\beta, Z_{k,k}] = 0 \ \forall \ k \geq 1. \tag{27}$$

Without loss of generality we can further assume that $0 < n_1 < n_2 < ... < n_k$ (that β fulfills the hypothesis is constrained by the assumptions given for the element α defined in formula (26), which translates for β into the condition $n_i \neq 0$). To conclude, it suffices to show that each of the a_i's of $\beta = \sum_{i=1}^{l} a_i Z_{n_i+s, n_i}$ is equal to 0. So let us consider $k = n_2$ in formula (27). Applying the formula (13) to this case, we get:

$$[\beta, Z_{k,k}] = a_1 [Z_{n_1+s, n_1}, Z_{n_2, n_2}] + \sum_{i \geq 2} a_i [Z_{n_i+s, n_i}, Z_{n_2, n_2}] =$$

$$= a_1 \Big(Z_{n_2+s, n_2} - \Theta(n_2 - n_1 - s) Z_{n_2, n_2-s} - \Theta(n_1 + s - n_2) Z_{n_1+s, n_1} + \delta_{n_1+s, n_2} Z_{n_2, n_1} \Big),$$

since $\sum_{i \geq 2} a_i [Z_{n_i+s, n_i}, Z_{n_2, n_2}] = 0$.
By the previous formula and the hypothesis for the n_i's, we conclude that

$[\beta, Z_{n_2,n_2}] = 0 \iff a_1 = 0$. Taking $k = n_3, n_4, ...,$ and using the same argument, we can conclude that each of the a_i's is equal to zero. □

5. Cohomology of the Lie algebra $\mathcal{L}_L$.

In what follows we will describe in some details the cohomology of the Lie algebra $\mathcal{L}_L$. We will start with an explicit calculation for the dimension of the 1st cohomology group (with trivial coefficients) and we will continue using the general machinery to calculate the higher cohomology groups.
Let us introduce on the Lie algebra $\mathcal{L}_L$ the derivation Y via the following:

$$Y.Z_{n,m} \equiv [Y, Z_{n,m}] = (n-m)Z_{n,m}. \tag{28}$$

Let now consider the extension of the Lie algebra $\mathcal{L}_L$ obtained by adding to $\mathcal{L}_L$ the derivation Y.

Definition 5.1.

$$\check{\mathcal{L}}_L = span_{\mathbb{C}} < Z_{n,m}, Y | n, m \in \mathbb{Z}_{\geq 0} >;$$

where the commutator $[Z_{n,m}, Z_{l,s}]$ *is given by formula (13) and* $[Y, Z_{n,m}] = (n-m)Z_{n,m}$.

We have the following:

Theorem 5.2.

$$1.\ dim_{\mathbb{C}}\ H^1(\check{\mathcal{L}}_L, \mathbb{C}) = 1;$$

$$2.\ dim_{\mathbb{C}}\ H^1(\mathcal{L}_L, \mathbb{C}) = +\infty.$$

Proof The elements of $H^1(\mathcal{L}_L, \mathbb{C})$ are in one to one correspondence with the elements $\phi \in \ \mathrm{Hom}_{\mathbb{C}}(\mathcal{L}_L, \mathbb{C})$ such that:

$$\phi([Z_{n,m}, Z_{l,s}]) = 0$$

for each $Z_{n,m}, Z_{l,s} \in \mathcal{L}_L$. As a consequence of proposition 3.3, equation (16), we have that the value of ϕ on a given element $Z_{n,m}$, depends only on the degree of such element. In fact, given:

$$Z_{n,m} = [Z_{n,0}, Z_{0,m}] + \Theta(n-m)Z_{n-m,0} + \Theta(m-n)Z_{0,m-n} - \delta_{n-m,0}Z_{0,0},$$

$$\phi(Z_{n,m}) = \Theta(n-m)\phi(Z_{n-m,0}) + \Theta(m-n)\phi(Z_{0,m-n}) - \delta_{n-m,0}\phi(Z_{0,0}).$$

From this remark, it follows that $\dim_{\mathbb{C}} H^1(\mathcal{L}_L, \mathbb{C}) = +\infty$.
To show that also 1. holds, let us observe that, since:

$$0 = \phi([Y, Z_{n,m}]) = (n-m)\phi(Z_{n,m}),$$

for each $Z_{n,m} \in \mathcal{L}_L$, then $\phi(Z_{n,m}) = 0$ for any element $Z_{n,m}$ with degree different by zero, so that we will write:

$$\phi(Z_{n,n}) = c_\phi(n) \in \mathbb{C}.$$

On the other hand, since the value of ϕ on a given element, depends only on the degree of such element, we have that:

$$\phi(Z_{n,m}) = c_\phi \delta_{n-m,0},$$

or, in other words:

$$H^1(\check{\mathcal{L}}_L, \mathbb{C}) \simeq \mathbb{C}.$$

□

To go to the higher cohomology groups we need to introduce some notation and some (classical) results about the cohomology of Lie algebras. Let us start with the following result.

Lemma 5.3. *Let $\mathfrak{gl}(n)$ the (Lie) algebra of $n \times n$ matrices (with entries in $\mathbb{C}$). Let us define the direct system of (Lie) algebras:*

$$\ldots \to \mathfrak{gl}(n-1) \to \mathfrak{gl}(n) \to \mathfrak{gl}(n+1) \to \ldots \tag{29}$$

where the arrows are given by the standard inclusions, i.e $A \in \mathfrak{gl}(n)$ is mapped into $\tilde{A} \in \mathfrak{gl}(n+1)$ such that $\tilde{a}_{i,j} = a_{i,j}$, $\forall i \leq n,\ j \leq n$ and $\tilde{a}_{i,j} = 0$ if $i > n$ or $j > n$. Then the direct limit of such a direct system is isomorphic to the Lie algebra $\mathfrak{gl}_+(\infty)$ introduced in definition 4.3.

Proof
The proof follows immediately from the definition 4.3.

□

Let us next quote a result about the cohomology of the Lie algebra of the general linear group $\mathfrak{gl}(n)$ [7]:

Theorem 5.4. *1). The cohomology ring of the Lie algebra $\mathfrak{gl}(n)$ is an exterior algebra whose generators are only of odd degree:*

$$H^\bullet(\mathfrak{gl}(n)) = \Lambda[c_1, c_3, ..., c_{2n-1}];$$

2) for any given n, the (inclusion) map, defined in formula (29), lemma 5.3:

$$i : \mathfrak{gl}(n) \longrightarrow \mathfrak{gl}(n+1)$$

induce a map i^* in cohomology:

$$i^* : H^\bullet(\mathfrak{gl}(n+1)) \longrightarrow H^\bullet(\mathfrak{gl}(n)),$$

such that:

$$i^* : H^p(\mathfrak{gl}(n+1)) \longrightarrow H^p(\mathfrak{gl}(n))$$

is an isomorphism for $p < n$, and it maps to zero the generator of degree $n+1$; 3) from 1), 2) and the previous lemma 5.3, it follows that the cohomology ring of the Lie algebra $\mathfrak{gl}$ is a (non finitely generated) exterior algebra having generators only in odd degree:

$$H^\bullet(\mathfrak{gl}_+(\infty)) = \Lambda[c_1, c_3, ...].$$

Proof

We refer to the book [7] for the proof of 1) and 2). Part 3) follows from lemma (5.3) where we have identified the direct limit of the Lie algebras $\mathfrak{gl}(n)$ with the Lie algebra $\mathfrak{gl}_+(\infty)$. □

Now let us go back to the exact sequence (17). This induces the following exact sequence in cohomology ([7], [16]):

$$\begin{aligned} 0 \to H^1(C) &\xrightarrow{f^1} H^1(\mathcal{L}_L) \to H^1(\mathfrak{gl}_+(\infty))^C \xrightarrow{\delta} H^2(C) \\ &\xrightarrow{f^2} H^2(\mathcal{L}_L) \to H^2(\mathfrak{gl}_+(\infty))^C \xrightarrow{\delta} \cdots. \end{aligned} \tag{30}$$

Here, by $H^i(\mathfrak{gl}_+(\infty))^C$ we understand the (sub)-vector space of C-invariant elements in $H^i(\mathfrak{gl}_+(\infty))$, i.e the space $\{a \in H^i(\mathfrak{gl}_+(\infty)) | x.a = 0, \ \forall\, x \in C\}$. Using theorem (5.4) and the previous exact sequence we have one of our main results.

Theorem 5.5. *The cohomology groups $H^i(\mathcal{L}_L)$ are infinite dimensional. In particular the Lie algebra $\mathcal{L}_L$ has infinite many non equivalent central extensions.*

Proof The theorem follows easily analysing the following segment of the exact sequence (30):

$$\cdots \longrightarrow H^p(\mathfrak{gl}_+(\infty))^C \xrightarrow{\delta} H^{p+1}(C) \xrightarrow{f^{p+1}} H^{p+1}(\mathcal{L}_L) \longrightarrow H^{p+1}(\mathfrak{gl}_+(\infty))^C \longrightarrow \cdots \quad (31)$$

From this we see that for p odd f^{p+1} is surjective as $H^{p+1}(\mathfrak{gl}_+(\infty)) = 0$ (by Thm.5.4). On the other hand dim $H^p(\mathfrak{gl}_+(\infty)) < \infty$, so that ker(f) is finite dimensional.
We can argue in an analogous way in the case of p even; in this case we use the fact that the map f^{p+1} is injective as $H^p(\mathfrak{gl}_+(\infty)) = 0$.
The statement about central extensions follows now from the fact that those are in one-to-one correspondence with the elements of the group $H^2(\mathcal{L}_L)$. □
We want to make the statement in the previous theorem about the central extensions of the Lie algebra $\mathcal{L}_L$ more precise. In particular we have that:

Proposition 5.6. *The map f^1, defined in the exact sequence (30), is not an isomorphism.*

The proof of such statement will follow from the following:

Claim 5.7.

$$H^1(\mathfrak{gl}_+(\infty))^C \simeq \mathbb{C}.$$

Proof Let us start to observing that

$$H^1(\mathfrak{gl}_+(\infty)) \simeq \Big(\mathfrak{gl}_+(\infty)/[\mathfrak{gl}_+(\infty), \mathfrak{gl}_+(\infty)]\Big)' \simeq \mathbb{C},$$

and identifying $[\mathfrak{gl}_+(\infty), \mathfrak{gl}_+(\infty)]$ with $\mathfrak{sl}_+(\infty)$, i.e with the Lie algebra of infinite matrices of finite rank, having trace equal to zero.
In particular this implies that the only non trivial class $[\phi] \in H^1(\mathfrak{gl}_+(\infty))$ corresponds to a (closed) cochain $\phi \in C^1(\mathfrak{gl}_+(\infty))$ whose kernel is $\mathfrak{sl}_+(\infty)$.
Let us now define the action of the (abelian) Lie algebra C on $H^1(\mathfrak{gl}_+(\infty))$: for any $\phi \in C^1(\mathfrak{gl}_+(\infty))$ and $[Z] \in C \simeq \mathcal{L}_L/\mathfrak{gl}_+(\infty)$, define:

$$([Z].\phi)(\alpha) = \phi([Z+\beta, \alpha]). \quad (32)$$

where $Z \in \mathcal{L}_L$ and $\beta \in \mathfrak{gl}_+(\infty))$. On the other hand, being ϕ a cocycle, we have that:

$$\phi([Z+\beta, \alpha]) = \phi([Z, \alpha]).$$

It is a simple calculation to show that $[\mathcal{L}_L, \mathfrak{gl}_+(\infty)] \subset \mathfrak{sl}_+(\infty)$ so that, from the hypothesis on ϕ, we conclude that $\phi([Z,\alpha]) = 0$, i.e:

$$[Z].\phi = 0,$$

or that ϕ is C-invariant. □

Remark 5.8. *We want to remark the following: as in the finite dimensional case, also the Lie algebra $\mathfrak{gl}_+(\infty)$ is not simple. In fact $\mathfrak{sl}_+(\infty)$ is an ideal in $\mathfrak{gl}_+(\infty)$. At the same time we want to underline the difference between the infinite dimensional case and the finite dimensional one. In the finite dimensional case, i.e for a given $n \in \mathbb{Z}_{>0}$, we have that the quotient $\mathfrak{gl}(n)/\mathfrak{sl}(n) \simeq \mathbb{C} \simeq \mathcal{Z}(\mathfrak{gl}(n))$, where $\mathcal{Z}(\mathfrak{gl}(n))$ is the center of $\mathfrak{gl}(n)$.*
In the infinite dimensional case we still have that $\mathfrak{sl}_+(\infty) \subset \mathfrak{gl}_+(\infty)$ is an ideal and that the quotient algebra is one dimensional, but in this case such a quotient does not correspond to any ideal in $\mathfrak{gl}_+(\infty)$. In particular $\mathcal{Z}(\mathfrak{gl}_+(\infty)) = \{0\}$.

6. Conclusion and Outlook.

In this paper we gave results about the structure of the the Lie algebra $\mathcal{L}_L$. We first discussed its relevance for the structure of quantum field theory. Having motivated its study, we showed that $\mathcal{L}_L$ is the (non-abelian) extension via $\mathfrak{gl}_+(\infty)$ of a commutative Lie algebra. We also showed that this extension does not split. Furthermore, we described the cohomology of $\mathcal{L}_L$ and proved that the second cohomology group of this Lie algebra is infinite dimensional, allowing for infinitely many non-equivalent central extensions.
It should be very interesting to understand the physical meaning of the central extensions of this Lie algebra in the future, in particular their relations with those DSEs. In future work we will study more closely the representation theory of this Lie algebra with the hope to shed some light on these problems.

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