EXPLICIT MUMFORD ISOMORPHISM FOR HYPERELLIPTIC CURVES

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Abstract. We give an explicit version of the Mumford isomorphism on the moduli stack of hyperelliptic curves of any given genus.

A well-known application of the Grothendieck-Riemann-Roch theorem, see for instance Mumford's fundamental paper [Mu1], shows that on the moduli stack $\mathcal{M}_g$ of smooth, proper curves of genus g we have an isomorphism of line bundles, unique up to a sign,

$$\mu : (\det Rp_* \omega_{\mathcal{C}_g/\mathcal{M}_g})^{\otimes 13} \xrightarrow{\sim} \det Rp_*(\omega_{\mathcal{C}_g/\mathcal{M}_g}^{\otimes 2}) \, ,$$

where $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ denotes the relative dualising sheaf of the universal curve $p : \mathcal{C}_g \to \mathcal{M}_g$, and where $\det Rp_*$ denotes the determinant of cohomology (*cf.* [De]) along p. As the Mumford isomorphism is very fundamental, it seems natural to ask whether one can write down an explicit version of it, or in other words, whether one can give an expression for the trivialising element (unique up to sign) of

$$(\det Rp_* \omega_{\mathcal{C}_g/\mathcal{M}_g})^{\otimes 13} \otimes_{O_{\mathcal{M}_g}} \left(\det Rp_*(\omega_{\mathcal{C}_g/\mathcal{M}_g}^{\otimes 2}) \right)^{\otimes -1}$$

corresponding to μ. Using an identification of $\det Rp_*(\omega_{\mathcal{C}_g/\mathcal{M}_g}^{\otimes 2})$ with the canonical line bundle on $\mathcal{M}_g$, an answer can be given, up to a multiplicative constant, over the complex numbers by using the so-called Polyakov measure from string theory (*cf.* [BM], [Bo]). In the present note we give a precise answer over the integers, restricting however to the substack $\mathcal{I}_g$ of hyperelliptic curves of genus g, and working only with a certain power of the Mumford isomorphism. We note that our result can be used to prove a weak version of a classical identity, due to Thomae [Th], concerning Jacobian Nullwerte and Thetanullwerte associated to hyperelliptic Riemann surfaces (*cf.* Theorems 6.1 and 6.2 below). This identity is a generalisation to higher dimensions of the famous Jacobi derivative formula.

1. Statement of the results

Using the general formalism of the Deligne bracket (which we review in Section 2 below) we rewrite the Mumford isomorphism in the form

$$\text{(M)} \qquad \mu : (\det Rp_* \omega_{\mathcal{C}_g/\mathcal{M}_g})^{\otimes 12} \xrightarrow{\sim} \langle \omega_{\mathcal{C}_g/\mathcal{M}_g}, \omega_{\mathcal{C}_g/\mathcal{M}_g} \rangle \, .$$

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What we will use is that over the stack $\mathcal{I}_g$ a power of the right hand side can be identified with a line bundle involving the Weierstrass divisor W on the universal hyperelliptic curve over $\mathcal{I}_g$ (*cf.* Definition 2.6 below).

Theorem 1.1. *Let $g \geq 2$ be an integer. Let W be the Weierstrass divisor on the universal hyperelliptic curve over $\mathcal{I}_g$, and let ω be its relative dualising sheaf. We have a canonical isomorphism*

$$\nu : \langle \omega, \omega \rangle^{\otimes 4g^2+6g+2} \xrightarrow{\sim} \langle O_{\mathcal{U}_g}(W), O_{\mathcal{U}_g}(W) \otimes_{O_{\mathcal{U}_g}} \omega \rangle^{\otimes -4g(g-1)}$$

of line bundles on $\mathcal{I}_g$. Combining with (M) we obtain an isomorphism

$$(\det Rp_*\omega)^{\otimes 12(4g^2+6g+2)} \xrightarrow{\sim} \langle O_{\mathcal{U}_g}(W), O_{\mathcal{U}_g}(W) \otimes_{O_{\mathcal{U}_g}} \omega \rangle^{\otimes -4g(g-1)} ,$$

unique up to a sign.

Now the point is that, by the general properties of the Deligne bracket, the line bundle $\langle O_{\mathcal{U}_g}(W), O_{\mathcal{U}_g}(W) \otimes_{O_{\mathcal{U}_g}} \omega \rangle$ contains a canonical rational section Ξ given by adjunction (in fact, the section Ξ is the norm of the different of the induced finite, flat morphism $W \to \mathcal{I}_g$). On the other hand it is well-known (*cf.* also Proposition 4.1 below) that a certain power of the left hand side of (M) has a canonical (trivialising) section Λ over $\mathcal{I}_g$ given by the discriminant. Our explicit version of the Mumford isomorphism will be the following comparison between the two.

Theorem 1.2. *Let $g \geq 2$ be an integer. The equality of sections*

$$\left(\nu^{\otimes 8g+4} \circ \mu^{\otimes (8g+4)(4g^2+6g+2)}\right)\left(\Lambda^{\otimes 12(4g^2+6g+2)}\right) = \left(2^{-(2g+2)} \cdot \Xi\right)^{\otimes -4g(g-1)(8g+4)}$$

holds, up to a sign, in the line bundle $\langle O_{\mathcal{U}_g}(W), O_{\mathcal{U}_g}(W) \otimes_{O_{\mathcal{U}_g}} \omega \rangle^{\otimes -4g(g-1)(8g+4)}$ on $\mathcal{I}_g$.

The proof of Theorem 1.1 will be given in Section 3, generalising a line of argument developed in [BMM], Section 1, dealing with the special case of curves of genus 2. The construction of the canonical section Λ will be reviewed in Section 4. We then give the proof of Theorem 1.2 in Section 5. The application to Thomae's identity will be provided in Section 6. Finally we discuss Jacobi's derivative formula in Section 7.

2. Preliminaries

Before proving anything, we need to be clear about the objects of the stack of hyperelliptic curves of genus g. Also we will explain what we mean by the Weierstrass divisor of a hyperelliptic curve. The basic reference of all this preparatory material is [LK].

Definition 2.1. Let $g \geq 2$ be an integer, and let B be a scheme. A hyperelliptic curve of genus g over B is a smooth, proper curve $p : \mathcal{X} \to B$ of genus g admitting an involution

$\sigma \in \mathrm{Aut}_B(\mathcal{X})$ such that for every geometric point $\overline{b}$ of B, the quotient $\mathcal{X}_{\overline{b}}/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1_{k(\overline{b})}$.

For a hyperelliptic curve $p : \mathcal{X} \to B$, the involution σ is uniquely determined; this is well-known for $B = \mathrm{Spec}(k)$ with k an algebraically closed field, and the general case follows from this by the fact that $\mathrm{Aut}_B(\mathcal{X})$ is unramified over B, *cf.* [DM], Theorem 1.11.

Definition 2.2. We call σ the hyperelliptic involution of $\mathcal{X}$.

Proposition 2.3. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$ with hyperelliptic involution σ.*
(i) The quotient map $h : \mathcal{X} \to \mathcal{X}/\langle\sigma\rangle$ is a finite, faithfully flat B-morphism of degree 2 onto a smooth, proper B-curve of genus 0.
(ii) After an étale surjective base change $B' \to B$, the quotient map h becomes a finite, faithfully flat B'-morphism $h : \mathcal{X}_{B'} \to \mathbb{P}^1_{B'}$ of degree 2.
Let $\omega_{\mathcal{X}/B}$ be the relative dualising sheaf of $p : \mathcal{X} \to B$.
(iii) The image of the canonical morphism $\pi : \mathcal{X} \to \mathbb{P}(p_*\omega_{\mathcal{X}/B})$ is a smooth, proper B-curve of genus 0. Its formation commutes with base change.
(iv) There exists a closed embedding $j : \mathcal{X}/\langle\sigma\rangle \hookrightarrow \mathbb{P}(p_*\omega_{\mathcal{X}/B})$ such that $\pi = j \circ h$. After a faithfully flat base change $B' \to B$, the embedding j is isomorphic to the Veronese embedding $\mathbb{P}^1_{B'} \hookrightarrow \mathbb{P}^{g-1}_{B'}$.

Proof. See [LK], Theorem 5.5, Lemmas 5.6 and 5.7, and Remark 5.11. □

Definition 2.4. Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$ with hyperelliptic involution σ. The Weierstrass subscheme of $p : \mathcal{X} \to B$ is the fixed point subscheme of $\mathcal{X}$ under the action of $\langle\sigma\rangle$, *i.e.*, the closed subscheme defined locally on an affine open subscheme $U = \mathrm{Spec}(R)$ by the ideal generated by the set $\{r - \sigma(r) | r \in R\}$.

Proposition 2.5. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. The following properties hold:*
(i) The Weierstrass subscheme of $p : \mathcal{X} \to B$ is the subscheme associated to an effective relative Cartier divisor on $\mathcal{X}$.
(ii) The Weierstrass subscheme of $p : \mathcal{X} \to B$ is finite and flat over B of degree $2g + 2$. Its formation commutes with base change.
(iii) The Weierstrass subscheme of $p : \mathcal{X} \to B$ is étale over a point $b \in B$ if and only if the residue characteristic of b is not 2.

Definition 2.6. We call Weierstrass divisor of $p : \mathcal{X} \to B$ the relative Cartier divisor associated by Proposition 2.5 to the Weierstrass subscheme of $p : \mathcal{X} \to B$. We will denote the Weierstrass divisor by $W_{\mathcal{X}/B}$.

Example 2.7. Consider the proper, flat genus 2 curve $p : \mathcal{X} \to B = \mathrm{Spec}(\mathbb{Z}[1/5])$ given by the affine equation $y^2 + x^3 y = x$. One may check that it has good reduction everywhere, hence $p : \mathcal{X} \to B$ is a hyperelliptic curve. Over the ring $R' = R[\zeta_5, \sqrt[5]{2}]$ it acquires six σ-invariant sections $W_0, \ldots, W_5$ with W_0 given by $x = 0$ and with W_k given by $x = -\zeta_5^k \sqrt[5]{4}$ for $k = 1, \ldots, 5$. The Weierstrass subscheme of $\mathcal{X}'/R'$ is supported on the images of $W_0, \ldots, W_5$. It is easy to check that $W_0, \ldots, W_5$ do not meet over points of residue characteristic $\neq 2$, which verifies that indeed the Weierstrass subscheme is étale over such points. Over a prime of characteristic 2, all σ-invariant sections meet in one point W_0 given in coordinates by $x = y = 0$. The degree 2 quotient map $h : \mathcal{X}_{\mathbb{F}_2} \to \mathcal{X}_{\mathbb{F}_2}/\langle \sigma \rangle \cong \mathbb{P}^1_{\mathbb{F}_2}$ is ramified only in this point W_0.

Remark 2.8. In general, if B is the spectrum of a field of characteristic 2, then the quotient map $h : \mathcal{X} \to \mathcal{X}/\langle \sigma \rangle$ of a hyperelliptic curve by its hyperelliptic involution ramifies in at most $g + 1$ distinct points, *cf.* [LK], Remark on p. 104.

To conclude this preliminary section we review the basic properties of the Deligne bracket, *cf.* [De]. The Deligne bracket is a rule that assigns, for every proper, flat, locally complete intersection curve $p : \mathcal{X} \to B$ over a scheme B, to every pair L, M of line bundles on $\mathcal{X}$ a line bundle $\langle L, M \rangle$ on B. It satisfies the following properties:

(i) *(Bilinearity)* For line bundles L_1, L_2, M_1, M_2 on $\mathcal{X}$ we have canonical isomorphisms
$\langle L_1 \otimes L_2, M \rangle \xrightarrow{\sim} \langle L_1, M \rangle \otimes \langle L_2, M \rangle$ and $\langle L, M_1 \otimes M_2 \rangle \xrightarrow{\sim} \langle L, M_1 \rangle \otimes \langle L, M_2 \rangle$.

(ii) *(Symmetry)* For line bundles L, M on $\mathcal{X}$ we have a canonical isomorphism
$\langle L, M \rangle \xrightarrow{\sim} \langle M, L \rangle$.

(iii) *(Base change)* The formation of the Deligne bracket commutes with base change, *i.e.*, each cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{u'} & \mathcal{X} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{u} & B \end{array}$$

gives rise to a canonical isomorphism $u^* \langle L, M \rangle \xrightarrow{\sim} \langle u'^* L, u'^* M \rangle$.

(iv) *(Pullback along a section)* For $P : B \to \mathcal{X}$ a section of p we have a canonical isomorphism $P^* L \xrightarrow{\sim} \langle O_{\mathcal{X}}(P), L \rangle$.

(v) *(Adjunction formula)* For the relative dualising sheaf $\omega_{\mathcal{X}/B}$ of p and any section $P : B \to \mathcal{X}$ of p we have a canonical adjunction isomorphism $\langle P, P \rangle^{\otimes -1} \xrightarrow{\sim} \langle P, \omega_{\mathcal{X}/B} \rangle$.

(vi) *(Riemann-Roch)* Let L be a line bundle on $\mathcal{X}$ and let $\omega_{\mathcal{X}/B}$ be the relative dualising sheaf of p. Then we have a canonical isomorphism

$$(\det Rp_* L)^{\otimes 2} \xrightarrow{\sim} \langle L, L \otimes \omega_{\mathcal{X}/B}^{-1} \rangle \otimes (\det Rp_* \omega_{\mathcal{X}/B})^{\otimes 2}$$

of line bundles on B, with $\det Rp_*$ denoting the determinant of cohomology along p.

3. Proof of Theorem 1.1

Theorem 1.1 is a direct consequence of the following result.

Theorem 3.1. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. Then we have a canonical isomorphism of line bundles*

$$\nu : \langle \omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B} \rangle^{\otimes 4g^2+6g+2} \xrightarrow{\sim} \langle O_{\mathcal{X}}(W_{\mathcal{X}/B}), O_{\mathcal{X}}(W_{\mathcal{X}/B}) \otimes_{O_{\mathcal{X}}} \omega_{\mathcal{X}/B} \rangle^{\otimes -4g(g-1)},$$

whose formation commutes with base change.

For the proof we need a lemma. At this point we follow the argument in [BMM], Section 1 very closely.

Lemma 3.2. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$ and let σ be the hyperelliptic involution of $\mathcal{X}$. For any σ-invariant section $P : B \to \mathcal{X}$ of p we have a unique isomorphism*

$$\omega_{\mathcal{X}/B} \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes_{O_{\mathcal{X}}} p^* \langle P, P \rangle^{\otimes -(2g-1)}$$

that induces, by pulling back along P, the adjunction isomorphism $\langle P, \omega_{\mathcal{X}/B} \rangle \xrightarrow{\sim} \langle P, P \rangle^{\otimes -1}$. The formation of this isomorphism commutes with base change.

Proof. First of all, let P be *any* section of p. Let $h : \mathcal{X} \to \mathcal{X}/\langle \sigma \rangle$ be the canonical map. We recall from Proposition 2.3(i) that $\mathcal{X}/\langle \sigma \rangle$ is a smooth, proper B-curve of genus 0. Let $q : \mathcal{X}/\langle \sigma \rangle \to B$ be its structure morphism. By composing P with h we obtain a section Q of q, and as a result we can write $\mathcal{X}/\langle \sigma \rangle \cong \mathbb{P}(V)$ for some locally free sheaf V of rank 2 on B (*cf.* [LK], Proposition 3.3). On the other hand, consider the canonical morphism $\pi : \mathcal{X} \to \mathbb{P}(p_*\omega)$. This gives us a natural isomorphism $\omega \cong \pi^*(O_{\mathbb{P}(p_*\omega)}(1))$. Let $j : \mathcal{X}/\langle \sigma \rangle \hookrightarrow \mathbb{P}(p_*\omega)$ be the closed embedding given by Proposition 2.3(iv). Passing to a faithfully flat cover, and then using a faithfully flat descent argument, we obtain a natural isomorphism $j^*(O_{\mathbb{P}(p_*\omega)}(1)) \cong O_{\mathbb{P}(V)}(g-1)$. By [EGA], II.4.2.7 there exists a unique line bundle L on B such that $O_{\mathbb{P}(V)}(g-1) \cong O_{\mathbb{P}(V)}((g-1) \cdot Q) \otimes q^*L$. By pulling back along h, we find a natural isomorphism $\omega \xrightarrow{\sim} O_{\mathcal{X}}((g-1) \cdot (P + \sigma(P))) \otimes p^*L$. In the special case where P is σ-invariant, this can be written as a natural isomorphism $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^*L$. Pulling back along P we find that $L \cong \langle \omega, P \rangle \otimes \langle P, P \rangle^{\otimes -(2g-2)}$ and with the adjunction isomorphism $\langle P, P \rangle \cong \langle -P, \omega \rangle$ then finally $L \cong \langle P, P \rangle^{\otimes -(2g-1)}$. It is now clear that we have an isomorphism $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^* \langle P, P \rangle^{\otimes -(2g-1)}$ that induces by pulling back along P an isomorphism $\langle P, \omega_{\mathcal{X}/B} \rangle \xrightarrow{\sim} \langle P, P \rangle^{\otimes -1}$. Possibly after multiplying with a unique global section of O_B^*, we can establish that the latter isomorphism be the canonical adjunction isomorphism. The commutativity with base change is clear from the general base change properties of the relative dualising sheaf and of the Deligne bracket. □

Corollary 3.3. *Let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. For any two σ-invariant sections P, Q of p we have a canonical isomorphism*

$$\langle \omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B} \rangle \xrightarrow{\sim} \langle P, Q \rangle^{\otimes -4g(g-1)}$$

of line bundles on B, and its formation commutes with base change.

Proof. By Lemma 3.2, we have canonical isomorphisms $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)P) \otimes p^* \langle P, P \rangle^{\otimes -(2g-1)}$ and $\omega \xrightarrow{\sim} O_{\mathcal{X}}((2g-2)Q) \otimes p^* \langle Q, Q \rangle^{\otimes -(2g-1)}$. It follows that $O_{\mathcal{X}}((2g-2)(P-Q))$ comes from the base, and hence $\langle (2g-2)(P-Q), P-Q \rangle$ is canonically trivial on B. Expanding, this gives a canonical isomorphism $\langle P, P \rangle^{\otimes 2g-2} \otimes \langle Q, Q \rangle^{\otimes 2g-2} \xrightarrow{\sim} \langle P, Q \rangle^{\otimes 2(2g-2)}$ of line bundles on B. Expanding next the right hand member of the canonical isomorphism

$$\langle \omega, \omega \rangle \xrightarrow{\sim} \langle O_{\mathcal{X}}((2g-2)P) \otimes p^* \langle P, P \rangle^{\otimes -(2g-1)}, O_{\mathcal{X}}((2g-2)Q) \otimes p^* \langle Q, Q \rangle^{\otimes -(2g-1)} \rangle$$

gives then the result. The commutativity with base change is clear. □

Proof of Theorem 3.1. By [LK], Theorem 7.3, after a faithfully flat base change the Weierstrass subscheme $W_{\mathcal{X}/B}$ is supported on $2g+2$ sections $W_1, \ldots, W_{2g+2}$ of p. By the adjunction formula for the Deligne bracket we obtain then a canonical isomorphism $\langle O_{\mathcal{X}}(W_{\mathcal{X}/B}), O_{\mathcal{X}}(W_{\mathcal{X}/B}) \otimes_{O_{\mathcal{X}}} \omega_{\mathcal{X}/B} \rangle \xrightarrow{\sim} \bigotimes_{k \neq l} \langle W_k, W_l \rangle$. On the other hand, by Corollary 3.3 we have $\langle W_k, W_l \rangle^{\otimes -4g(g-1)} \xrightarrow{\sim} \langle \omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B} \rangle$ for each $k \neq l$, and combining gives the required statement on the faithfully flat cover. The general case follows from this by faithfully flat descent. □

4. The discriminant

This section is concerned with the left hand side of (M). In particular we review the construction of a canonical (trivialising) section of the $(8g+4)$-th tensor power of $\det Rp_* \omega$ on the base of a hyperelliptic curve. It follows from the theory of duality that we have a canonical identification $\det Rp_* \omega = \det p_* \omega$.

Proposition 4.1. *Let B be an irreducible regular scheme of generic characteristic $\neq 2$ and let $p : \mathcal{X} \to B$ be a hyperelliptic curve of genus $g \geq 2$. Then the line bundle $(\det p_* \omega_{\mathcal{X}/B})^{\otimes 8g+4}$ has a canonical trivialising global section $\Lambda_{\mathcal{X}/B}$.*

For the proof we need three lemmas. At this point we follow [Ka], Section 6 almost verbatim. For a discrete valuation ring R with residue field k we denote the canonical quotient map $R \to k$, as usual, by $x \mapsto \bar{x}$.

Lemma 4.2. *Let* $B = \mathrm{Spec}(R)$ *with* R *a discrete valuation ring with quotient field* K *and residue field* k*. Assume that* $\mathrm{char}(K) \neq 2$*. Let* $p : \mathcal{X} \to B$ *be a hyperelliptic curve of genus* $g \geq 2$*. After a finite étale surjective base change with a discrete valuation ring* R' *dominating* R*, there exists an open affine subscheme* $U \cong \mathrm{Spec}(E)$ *of* $\mathcal{X}'$ *with* $E = A[y]/(y^2 + ay + b)$*, where* $A = R'[x]$ *and* $a, b \in A$*, such that* $f := a^2 - 4b \in K'[x]$ *is separable of degree* $2g + 2$ *and such that* $\deg a \leq g + 1$ *and* $\deg b \leq 2g + 2$*. For the reduced polynomials* $\overline{a}, \overline{b} \in k'[x]$ *we have* $\deg \overline{a} = g + 1$ *or* $\deg \overline{b} \geq 2g + 1$*.*

Proof. After a finite étale surjective base change with a discrete valuation ring R' dominating R, we have by Proposition 2.3 a finite faithfully flat R'-morphism $h' : \mathcal{X}' \to \mathbb{P}^1_{R'}$ of degree 2. Choose a point $\infty \in \mathbb{P}^1_{K'}$ such that $\mathcal{X}_{K'} \to \mathbb{P}^1_{K'}$ is unramified above ∞, and let x be a coordinate on $V = \mathbb{P}^1_{K'} - \infty$. We can then describe $U := h'^{-1}(V)$ as $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$ where $A = R'[x]$ and $a, b \in A$. Moreover, if we assume the degree of a to be minimal, we have $\deg a \leq g + 1$ and $\deg b \leq 2g + 2$. Next let us consider the degree of $f := a^2 - 4b$. By Proposition 2.5, the Weierstrass subscheme $W_{\mathcal{X}'/B'}$ is finite and flat over B' of degree $2g + 2$. By definition, the ideal of $W_{\mathcal{X}'/B'}$ is generated by $y - \sigma(y) = 2y + a$ on U. Note that $(2y + a)^2 = a^2 - 4b = f$, which defines the norm under h' of $W_{\mathcal{X}'/B'}$ in $\mathbb{P}^1_{R'}$. Since this norm is also finite and flat of degree $2g + 2$ over B', and since $W_{\mathcal{X}'/B'}$ is entirely supported in U by our choice of ∞, we obtain that $\deg(f) = 2g + 2$. Since the norm of $W_{\mathcal{X}'_{K'}/K'}$ in $\mathbb{P}^1_{K'}$ is étale over K' by Proposition 2.5, the polynomial $f \in K'[x]$ is separable. Consider finally the reduced polynomials $\overline{a}, \overline{b} \in k'[x]$. Regarding y as an element of $k'(X_{k'})$, we have $\mathrm{div}(y) \geq -\min(\deg \overline{a}, \frac{1}{2} \deg \overline{b}) \cdot h'^*(\overline{\infty})$ by the equation for y. On the other hand it follows from Riemann-Roch that y has a pole at both points of $h'^*(\overline{\infty})$ of order strictly larger than g. This gives then the last statement of the lemma. ☐

Lemma 4.3. *Suppose we have an open affine subscheme* $U \cong \mathrm{Spec}(E)$ *on* $\mathcal{X}$ *as in Lemma 4.2. Then the differentials* $x^i dx/(2y + a)$ *for* $i = 0, \ldots, g - 1$ *are nowhere vanishing on* U *and extend to regular global sections of* $\omega_{\mathcal{X}/B}$*.*

Proof. Let F be the polynomial $y^2 + ay + b \in A[y]$, and let F_x and F_y be its derivatives with respect to x and y, respectively. It is readily verified that the morphism $\Omega_{E/R} = (Edx + Edy)/(F_x dx + F_y dy) \to E$ given by $dx \mapsto F_y, dy \mapsto -F_x$, is an isomorphism of E-modules. This gives that the differentials $x^i dx/(2y + a)$ for $i = 0, \ldots, g - 1$ are nowhere vanishing on U. For the second part of the lemma, it suffices to show that the differentials $x^i dx/(2y + a)$ for $i = 0, \ldots, g - 1$ on the generic fiber U_K extend to global sections of $\Omega^1_{\mathcal{X}_K/K}$—but this is well-known to be true. ☐

Suppose that a polynomial $f \in K[x]$ of degree d factors over an extension of K as $f = H\prod_{i=1}^{d}(x-\alpha_i)$. Then its discriminant $D(f)$ is given as $D(f) = H^{2d-2}\prod_{i\neq j}(\alpha_i-\alpha_j)$. Recall that this element lies in R if the coefficients of f lie in R.

Lemma 4.4. *Suppose we have an open affine subscheme $U \cong \mathrm{Spec}(E)$ on $\mathcal{X}$ as in Lemma 4.2. Then the modified discriminant $\Delta(f) = 2^{-(4g+4)} \cdot D(f)$ is a unit of R.*

Proof. In the case that the characteristic of k is $\neq 2$, this is not hard to see: we know that $W_{\mathcal{X}_k/k}$ is étale of degree $2g+2$ by Proposition 2.5, and hence f remains separable of degree $2g+2$ in $k[x]$ under the reduction map. So let us assume from now on that the characteristic of k equals 2. If C is any ring, and if $P(T) = \sum_{i=0}^{n} u_i T^i$ and $Q(T) = \sum_{i=0}^{m} v_i T^i$ are two polynomials in $C[T]$, we denote by $R_T^{n,m}(P,Q) \in C$ the resultant of P and Q. Recall the following property of the resultant: suppose that at least one of u_n, v_m is non-zero, and that C is a field. Then $R_T^{n,m}(P,Q) = 0$ if and only if P and Q have a root in common in an extension field of C. Let F be the polynomial $y^2 + a(x)y + b(x)$ in $A[y]$ with $A = R[x]$, and let F_x and F_y be its derivatives with respect to x and y, respectively. We set $Q = R_y^{2,1}(F, F_x)$ and $P = R_y^{2,1}(F, F_y) = 4b - a^2 = -f$. Let $H \in R$ be the leading coefficient of P, and abbreviate the modified discriminant $\Delta(f)$ of f by Δ. A calculation (*cf.* [Lo], Section 1) shows that $R_x^{2g+2,4g+2}(P,Q) = (H\cdot\Delta)^2$. We should read this equation as a formal identity between certain universal polynomials in the coefficients of $a(x)$ and $b(x)$. Doing so, we may conclude that $\Delta \in R$ and that H^2 divides $R_x^{2g+2,4g+2}(P,Q)$ in R. To finish the argument, we distinguish two cases. First assume that $\overline{H} \neq 0$. Then $\deg \overline{P} = 2g+2$ and again a calculation shows that $R_x^{2g+2,4g+2}(\overline{P},\overline{Q}) = (\overline{H}\cdot\overline{\Delta})^2$. The fact that X_k is smooth implies that $R_x^{2g+2,4g+2}(\overline{P},\overline{Q})$ is non-zero, and altogether we obtain that $\overline{\Delta}$ is non-zero. Next assume that $\overline{H} = 0$. Then since $\overline{P} = \overline{a}^2$ we obtain that $\deg \overline{a} \leq g$ and hence $\deg \overline{P} \leq 2g$. By Lemma 4.2 we have then $2g+1 \leq \deg \overline{b} \leq 2g+2$. But then from $2\deg(y) = \deg(\overline{a}y + \overline{b})$ and $\deg(y) > g$ (*cf.* the proof of Lemma 4.2) it follows that in fact $\deg \overline{b} = 2g+2$ and hence $\deg \frac{d\overline{b}}{dx} = 2g$ since we are in characteristic 2. This implies that $\deg \overline{Q} = 4g$. A calculation shows that $R_x^{2g,4g}(\overline{P},\overline{Q}) = \overline{\Delta}^2$. Again by smoothness of X_k we may conclude that $R_x^{2g,4g}(\overline{P},\overline{Q})$ is non-zero. This finishes the proof. □

Example 4.5. Consider once more the curve over $R = \mathbb{Z}[1/5]$ given by the equation $y^2 + x^3y = x^5$, *cf.* Example 2.7 above. In the notation from Lemma 4.2 we have $a = x^3, b = -x$. We compute $D(a^2 - 4b) = D(x^6 + 4x) = 2^{12}5^5$ so that $\Delta(f) = 5^5$ which is indeed a unit in R.

Proof of Proposition 4.1. After a faithfully flat base change we have, by Proposition 2.3, that the canonical quotient map $h : \mathcal{X} \to \mathcal{X}/\langle\sigma\rangle$ is a B-morphism onto a $\mathbb{P}_B^1$. Then by Lemma 4.2 we may assume that the scheme $\mathcal{X}$ is covered by affine schemes $U \cong \mathrm{Spec}(E)$ with $E = A[y]/(y^2 + ay + b)$ and A a polynomial ring $R[x]$. For such an affine scheme U, consider $V := \mathrm{Spec}(A)$. In the line bundle $(\det p_*\omega_{U/V})^{\otimes 8g+4}$ we have a rational section

$$\Lambda_{U/V} := \Delta(f)^g \cdot \left(\frac{dx}{2y+a} \wedge \ldots \wedge \frac{x^{g-1}dx}{2y+a} \right)^{\otimes 8g+4} ,$$

where $\Delta(f)$ is as in Lemma 4.4. One can check that this element does not depend on any choice of affine equation $y^2 + ay + b$ for U, and moreover, these sections coincide on overlaps. Hence they build a canonical rational section $\Lambda_{\mathcal{X}/B}$ of $(\det p_*\omega_{\mathcal{X}/B})^{\otimes 8g+4}$. By Lemma 4.3 and Lemma 4.4, this $\Lambda_{\mathcal{X}/B}$ is a global trivialising section. The general case follows by faithfully flat descent. □

5. Proof of Theorem 1.2

In the present section we give a proof of Theorem 1.2. We start with two propositions. The argument in the proof of Proposition 5.1 is taken from [HR], Lemma 2.1.

Proposition 5.1. *Let $g \geq 2$ be an integer. The stack $\mathcal{I}_g$ is a smooth, closed substack of relative dimension $2g-1$ of the stack $\mathcal{M}_g$ of smooth, proper curves of genus g. We have $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_m) = \mathbb{C}^*$ and $H^0(\mathcal{I}_g, \mathbb{G}_m) = \{-1, +1\}$.*

Proof. For the first statement, see [LL], Theorem 3. In order to see that $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_\mathrm{m}) = \mathbb{C}^*$, note that we can describe $\mathcal{I}_g \otimes \mathbb{C}$ as the space of $(2g+2)$-tuples of distinct points on $\mathbb{P}^1$ modulo projective equivalence, that is, we can write $\mathcal{I}_g \otimes \mathbb{C} = ((\mathbb{C} \setminus \{0,1\})^{2g-1} - \Delta)/S_{2g+2}$ (in the orbifold sense) where Δ denotes the fat diagonal and where S_{2g+2} is the symmetric group acting by permutation on $2g+2$ points on $\mathbb{P}^1$. According to [HM], Theorem 10.6, the first homology of $(\mathbb{C} \setminus \{0,1\})^{2g-1} - \Delta$ is isomorphic to the irreducible representation of S_{2g+2} corresponding to the partition $\{2g, 2\}$ of $2g+2$; in particular it does not contain a trivial representation of S_{2g+2}. This proves that $H_1(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{Q})$ is trivial, and hence $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_\mathrm{m}) = \mathbb{C}^*$. The statement that $H^0(\mathcal{I}_g, \mathbb{G}_\mathrm{m}) = \{-1, +1\}$ follows from this since $\mathcal{I}_g \to \mathrm{Spec}(\mathbb{Z})$ is smooth and surjective. □

Proposition 5.2. *Let $g \geq 2$ be an integer. Let W be the Weierstrass divisor of the universal hyperelliptic curve over $\mathcal{I}_g$ and let ω be its relative dualising sheaf. Let Ξ be the canonical rational section of $\langle O_{\mathcal{U}_g}(W), O_{\mathcal{U}_g}(W) \otimes_{O_{\mathcal{U}_g}} \omega \rangle$ on $\mathcal{I}_g$ given by adjunction. Then $2^{-(2g+2)} \cdot \Xi$ is a global trivialising section over $\mathcal{I}_g$.*

Proof. By Proposition 2.5(iii), the section Ξ is a global and trivialising section over $\mathcal{I}_g \otimes \mathrm{Spec}(\mathbb{Z}[1/2])$, and hence by the second statement in Proposition 5.1 there is an integral power 2^α of 2 such that $2^\alpha \cdot \Xi$ trivialises our line bundle over the whole of $\mathcal{I}_g$. We claim that $\alpha = -(2g+2)$. In order to prove the claim, it suffices, by the first statement of Proposition 5.1, to place ourselves in the situation of a hyperelliptic curve $p : \mathcal{X} \to B$ of genus g with $B = \mathrm{Spec}(R)$ for a discrete valuation ring R whose fraction field K has characteristic $\neq 2$, and to prove the corresponding statement for that case. Now by Proposition 2.3(ii) and by [LK], Theorem 7.3, after making a faithfully flat cover we have that the Weierstrass subscheme $W_{\mathcal{X}/B}$ is supported on $2g+2$ sections $W_1, \ldots, W_{2g+2}$ and that the image of the canonical map $h : \mathcal{X} \to \mathcal{X}/\langle \sigma \rangle$ is a $\mathbb{P}^1_R$. Assume that the discrete valuation v of R is normalised such that $v(K^*) = \mathbb{Z}$. Then the valuation $v(\Xi_{\mathcal{X}/B})$ of $\Xi_{\mathcal{X}/B}$ at the closed point b of B is given by the sum $\sum_{k \neq l} (W_k, W_l)$ of the local intersection multiplicities (W_k, W_l) above b of pairs of sections W_k. Suppose that W_k is given by a polynomial $x - \alpha_k$, and write α_k as a shorthand for the corresponding section of $\mathbb{P}^1_R$. By the projection formula (*cf.* [Li], Theorem 9.2.12) we have for the local intersection multiplicities that $4(W_k, W_l) = (2W_k, 2W_l) = (h^* \alpha_k, h^* \alpha_l) = 2(\alpha_k, \alpha_l)$ for each $k \neq l$ hence $(W_k, W_l) = \frac{1}{2}(\alpha_k, \alpha_l)$ for each $k \neq l$. Now the local intersection multiplicity (α_k, α_l) above b on $\mathbb{P}^1_R$ is calculated to be $v(\alpha_k - \alpha_l)$. This gives that $v(\Xi) = \sum_{k \neq l} (W_k, W_l) = \frac{1}{2} \sum_{k \neq l} v(\alpha_k - \alpha_l)$. By Lemma 4.4 we have $\sum_{k \neq l} v(\alpha_k - \alpha_l) = (4g+4) v(2)$ hence the valuation of $2^{-(2g+2)} \cdot \Xi$ vanishes at b, which is what we wanted. The general case follows from this by faithfully flat descent. □

Proof of Theorem 1.2. Consider the isomorphism, unique up to a sign,

$$(\det p_* \omega)^{\otimes 12(4g^2+6g+2)} \xrightarrow{\sim} \langle O_{\mathcal{U}_g}(W), O_{\mathcal{U}_g}(W) \otimes_{O_{\mathcal{U}_g}} \omega \rangle^{\otimes -4g(g-1)}$$

given by Theorem 1.1. By Proposition 4.1 we know that $\Lambda^{\otimes 12(4g^2+6g+2)}$ is a trivialising section of the $(8g+4)$-th power of the left hand side. It should be mapped to a trivialising section of the $(8g+4)$-th power of the right hand side, hence by Proposition 5.2 its image is $(2^{-(2g+2)} \cdot \Xi)^{\otimes -4g(g-1)(8g+4)}$, up to an element of $H^0(\mathcal{I}_g, \mathbb{G}_{\mathrm{m}})$. But by Proposition 5.1 we have $H^0(\mathcal{I}_g, \mathbb{G}_{\mathrm{m}}) = \{-1, +1\}$, and thus we obtain the theorem. □

6. Thomae's identity

As an application of our results, we give in this section a proof of a weak version of a classical identity due to Thomae involving Jacobian Nullwerte and Thetanullwerte associated to hyperelliptic Riemann surfaces. The idea is that over the complex numbers, both sides of the second isomorphism of Theorem 1.1 are equipped with certain canonical hermitian metrics, and that the norm of the isomorphism with respect to these metrics is known. Comparing then the lengths of the canonical trivialising sections on both sides given by Theorem 1.2 we obtain the result. The definition of the canonical hermitian metrics is given by Arakelov theory, of which we shall recall what we need later on. We start however by formulating Thomae's identity and the weak version of it that we are able to prove.

Let $g \geq 1$ be an integer and let $\mathcal{H}_g$ be the Siegel upper half-space of degree g consisting of the symmetric complex $g \times g$-matrices with positive definite imaginary part. For vectors

$\eta', \eta'' \in \frac{1}{2}\mathbb{Z}^g$ we have on $\mathbb{C}^g \times \mathcal{H}_g$ the theta function with characteristic $\eta = [\begin{smallmatrix}\eta' \\ \eta''\end{smallmatrix}]$ given by

$$\vartheta[\eta](z;\tau) := \sum_{n\in\mathbb{Z}^g} \exp(\pi i^t(n+\eta')\tau(n+\eta') + 2\pi i^t(n+\eta')(z+\eta'')) .$$

For an analytic theta characteristic η, the corresponding theta function $\vartheta[\eta](z;\tau)$ is either odd or even as a function of z. We call the analytic theta characteristic η odd if the corresponding theta function $\vartheta[\eta](z;\tau)$ is odd, and even if the corresponding theta function $\vartheta[\eta](z;\tau)$ is even. Now let $\eta_1,\ldots,\eta_g$ be g odd characteristics in dimension g. We define the Jacobian Nullwert $J(\eta_1,\ldots,\eta_g)$ in $\eta_1,\ldots,\eta_g$ to be the jacobian

$$J(\eta_1,\ldots,\eta_g)(\tau) = \frac{\partial(\vartheta[\eta_1],\ldots,\vartheta[\eta_g])}{\partial(z_1,\ldots,z_g)}(0;\tau) .$$

These functions are modular forms on $\mathcal{H}_g$. Thomae's identity expresses such Jacobian Nullwerte as a product of Thetanullwerte, for τ coming from a hyperelliptic Riemann surface.

For any subset S of $\{1,2,\ldots,2g+1\}$ we define a theta characteristic η_S as in [Mu2], Chapter IIIa: let

$$\begin{aligned} \eta_{2k-1} &= \begin{bmatrix} {}^t(0,\ldots,0,\frac{1}{2},0,\ldots,0) \\ {}^t(\frac{1}{2},\ldots,\frac{1}{2},0,0,\ldots,0) \end{bmatrix}, \quad 1 \le k \le g+1 , \\ \eta_{2k} &= \begin{bmatrix} {}^t(0,\ldots,0,\frac{1}{2},0,\ldots,0) \\ {}^t(\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2},0,\ldots,0) \end{bmatrix}, \quad 1 \le k \le g , \end{aligned}$$

where each time the non-zero entry in the top row occurs in the k-th position. Then we put $\eta_S := \sum_{k\in S}\eta_k$ where the sum is taken modulo 1. We extend this definition to subsets S of $\{1,2,\ldots,2g+2\}$ by putting $\eta_S := \sum_{\substack{k\in S \\ k\neq 2g+2}} \eta_k$. To each subset $\{i_1,\ldots,i_g\}$ of $\{1,2,\ldots,2g+2\}$ of cardinality g we asociate a set (traditionally called a *fundamental system*) $\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}$ of characteristics as follows: let U denote the set $\{1,3,\ldots,2g+1\}$ and denote by $\circ$ the symmetric difference. Then for $k=1,\ldots,g$ we put $\eta_k := \eta_{T\circ U}$ with $T = \{i_1,\ldots,\hat{i}_k,\ldots,i_g\}$, and for $k = g+1,\ldots,2g+2$ we put $\eta_k := \eta_{T\circ U}$ with $T = \{i_1,\ldots,i_g\} \cup \{i_k\}$. The multiset of $\binom{2g+2}{g}$ systems of characteristics that we obtain in this way is denoted by $\mathcal{F}(g)$. The result of Thomae is now as follows.

Theorem 6.1. *(Thomae* [Th]*) Let $g \ge 2$ be an integer. Let $\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}$ be a fundamental system of characteristics in dimension g contained in $\mathcal{F}(g)$, and let $\tau \in \mathcal{H}_g$ be a period matrix associated to a hyperelliptic Riemann surface of genus g. Then the formula*

$$J(\eta_1,\ldots,\eta_g)(\tau) = \pm\pi^g \prod_{k=g+1}^{2g+2} \vartheta[\eta_k](0;\tau)$$

holds.

We give a proof of the following slightly weaker result.

Theorem 6.2. *Let $g \geq 2$ be an integer. Let $\mathcal{T}(g)$ be the set of subsets of $\{1,2,\ldots,2g+1\}$ of cardinality $g+1$, let $U = \{1,3,\ldots,2g+1\}$ and let $m = \binom{2g+2}{g}$. Let $\tau \in \mathcal{H}_g$ be a period matrix associated to a hyperelliptic Riemann surface of genus g. Then the formula*

$$\prod_{\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}\in\mathcal{F}(g)} J(\eta_1,\ldots,\eta_g)(\tau) = \pm\pi^{gm} \prod_{T\in\mathcal{T}(g)} \vartheta[\eta_{T\circ U}](0;\tau)^{2g+2}$$

holds.

If we take the product over all $\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\} \in \mathcal{F}(g)$ in the formula from Theorem 6.1 we obtain the formula from Theorem 6.2. We note that the right hand side of the formula from Theorem 6.2 is related to the modular discriminant $\varphi_g(\tau)$ from [Lo], Section 3, which is defined to be the function

$$\varphi_g(\tau) := \prod_{T\in\mathcal{T}(g)} \vartheta[\eta_{T\circ U}](0;\tau)^8$$

on $\mathcal{H}_g$. It is a modular form on $\Gamma_g(2) := \{\gamma \in \mathrm{Sp}(2g,\mathbb{Z}) | \gamma \equiv I_{2g} \bmod 2\}$ of weight $4r$ where $r = \binom{2g+1}{g+1}$, generalising the usual Jacobi discriminant modular form in dimension 1.

In order to prove Theorem 6.2, we first reformulate it in a way which is more convenient for us. Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ and fix an ordering $W_1,\ldots,W_{2g+2}$ of its Weierstrass points, *i.e.* the ramification points of a hyperelliptic map $X \to \mathbb{P}^1$. By [Mu2], Chapter IIIa, §5 this induces a canonical symplectic basis of the homology of X. Moreover we have an equation $y^2 = f(x)$ for X with f monic of degree $2g+1$ and putting W_{2g+2} at infinity. Denote by $\mu_1,\ldots,\mu_g$ the holomorphic differentials on X given in coordinates by $\mu_1 = dx/2y,\ldots,\mu_g = x^{g-1}dx/2y$ and denote by $(\mu|\mu')$ the period matrix of $\mu_1,\ldots,\mu_g$ on the chosen canonical symplectic basis of homology. The matrix μ is invertible and we put $\tau := \mu^{-1}\mu'$. We obtain from this a jacobian $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ and a Riemann vector κ on it corresponding to infinity, *cf.* [Mu2], Chapter IIIa, §5. Consider then the Abel-Jacobi map $u : \mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ given by $\sum m_k P_k \mapsto \sum m_k \int_\infty^{P_k} {}^t(\nu_1,\ldots,\nu_g) + \kappa$, where $\{\nu_1,\ldots,\nu_g\} := \{\mu_1,\ldots,\mu_g\} \cdot {}^t\mu^{-1}$. This map induces an identification of the set of classes of semi-canonical divisors on X (*i.e.*, divisors D with $2D$ linearly equivalent to a canonical divisor) with the set of classes mod $\mathbb{Z}^g \times \mathbb{Z}^g$ of characteristics, the class $[D]$ of D corresponding to the class of $\eta = [{\eta' \atop \eta''}]$ where $u([D]) = [\eta' + \tau \cdot \eta'']$. As follows from [Mu2], Proposition 6.2, for any subset $T = \{i_1,\ldots,i_{g-1}\}$ of $\{1,2,\ldots,2g+2\}$ of cardinality $g-1$ the semi-canonical divisor $W_{i_1} + \cdots + W_{i_{g-1}}$ corresponds to the characteristic $\eta_{T\circ U}$ and likewise, for any subset $T = \{i_1,\ldots,i_{g+1}\}$ of $\{1,2,\ldots,2g+2\}$ of cardinality $g+1$ the semi-canonical divisor $W_{i_1} + \cdots + W_{i_g} - W_{i_{g+1}}$ corresponds to the characteristic $\eta_{T\circ U}$ (recall that for any Weierstrass point W we have $2W \sim E$, with E a divisor from the hyperelliptic pencil on X, and that $(g-1)E$ is a canonical divisor). Now, recall from [Fa] and [Gu] that we have natural real-valued functions $\|\vartheta\|$ on $\mathrm{Pic}_{g-1}(X)$ and $\|J\|$ on $\mathrm{Sym}^g(X)$. It is not hard to verify that $\|\vartheta\|(W_{i_1}+\cdots+W_{i_{g-1}}) = (\det \mathrm{Im}\tau)^{1/4}|\vartheta[\eta](0;\tau)|$ with η corresponding to $W_{i_1}+\cdots+W_{i_{g-1}}$ and that $\|J\|(W_{i_1},\ldots,W_{i_g}) = (\det \mathrm{Im}\tau)^{(g+2)/4}|J(\eta_1,\ldots,\eta_g)(\tau)|$ with η_k corresponding to $W_{i_1}+\cdots+\widehat{W_{i_k}}+\cdots+W_{i_g}$ for $k=1,\ldots,g$. We add to this an invariant $\|\varphi_g\|(X)$ of X by putting $\|\varphi_g\|(X) := (\det \mathrm{Im}\tau)^{2r}\cdot|\varphi_g(\tau)|$. These definitions do not depend on the choice of symplectic basis for homology at the outset. Using the fact that $W_{i_1}+\cdots+W_{i_g}-W_{i_{g+1}} \sim W_{i'_1}+\cdots+W_{i'_g}-W_{i'_{g+1}}$ if and only if $\{i_1,\ldots,i_{g+1}\} = \{i'_1,\ldots,i'_{g+1}\}$ or $\{i_1,\ldots,i_{g+1}\}\cup\{i'_1,\ldots,i'_{g+1}\} = \{1,\ldots,2g+2\}$ (*cf.* [Mu1], Chapter IIIa, Proposition 6.1) we can write $\|\varphi_g\|(X) = \prod_{\{i_1,\ldots,i_{g+1}\}} \|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W_{i_{g+1}})^4$, with the product running over the set of subsets of $\{1,2,\ldots,2g+2\}$ of cardinality $g+1$. We will prove

Theorem 6.3. *Let X be a hyperelliptic Riemann surface of genus $g\geq 2$ with Weierstrass points $W_1,\ldots,W_{2g+2}$. Let $m=\binom{2g+2}{g}$. Then we have*

$$\prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1},\ldots,W_{i_g})^4 = \pi^{4gm}\|\varphi_g\|(X)^{g+1}\,,$$

where the product runs over the subsets of $\{1,\ldots,2g+2\}$ of cardinality g.

¿From this follows Theorem 6.2: divide left and right of the equality in Theorem 6.3 by the appropriate power of $\det \mathrm{Im}\tau$ to obtain

$$\prod_{\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}\in\mathcal{F}(g)} |J(\eta_1,\ldots,\eta_g)(\tau)|^4 = \pi^{4gm}|\varphi_g(\tau)|^{g+1}\,,$$

where τ is a period matrix associated to X. Taking 4th roots and applying the maximum principle we find

$$\prod_{\{\eta_1,\ldots,\eta_g,\eta_{g+1},\ldots,\eta_{2g+2}\}\in\mathcal{F}(g)} J(\eta_1,\ldots,\eta_g)(\tau) = \varepsilon\pi^{gm}\prod_{T\in\mathcal{T}(g)}\vartheta[\eta_{T\circ U}](0;\tau)^{2g+2}$$

for all $\tau\in\mathcal{H}_g$ associated to a hyperelliptic Riemann surface, where ε is a complex constant of modulus 1. We find the right value ±1 of ε by considering Fourier expansions on the left and the right as in [Ig1], pp. 86–88.

Theorem 6.3 will be an immediate consequence of Theorems 6.5 and 6.6 below, both of which concern the canonical Arakelov-Green function [Ar] [Fa] of X evaluated at pairs of Weierstrass points. For the moment let X be an arbitrary compact and connected Riemann surface of genus $g>0$, and let μ_X be the fundamental (1,1)-form on X as defined in [Ar] and [Fa]. Let $S(X)$ be the invariant defined by

$$\log S(X) := -\int_X \log\|\vartheta\|(gP-Q)\cdot\mu_X(P)\,,$$

where Q can be any point of X. It can be checked that the integral is well-defined and does not depend on the choice of Q. In [dJ] we proved the following explicit formula for the Arakelov-Green function $G : X \times X \to \mathbb{R}_{\geq 0}$ of X.

Theorem 6.4. *Let $\mathcal{W}$ be the classical divisor of Weierstrass points on X. For P, Q points on X, with P not a Weierstrass point, we have*

$$G(P,Q)^g = S(X)^{1/g^2} \cdot \frac{\|\vartheta\|(gP-Q)}{\prod_{W \in \mathcal{W}} \|\vartheta\|(gP-W)^{1/g^3}} \,.$$

Here the product runs over the Weierstrass points of X, counted with their weights. The formula is also valid if P is a Weierstrass point, provided that we take the leading coefficients of a power series expansion about P in both numerator and denominator.

In the same paper, we also defined a second natural Arakelov invariant $T(X)$ of X. For P on X, not a Weierstrass point, and z a local coordinate about P, we first put

$$\|F_z\|(P) := \lim_{Q \to P} \frac{\|\vartheta\|(gP-Q)}{|z(P)-z(Q)|^g} \,.$$

Next we let $W_z(\omega)(P)$ be the Wronskian at P in z of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of the differentials $H^0(X, \Omega_X^1)$ provided with the standard hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \overline{\eta}$. Then we put

$$T(X) := \|F_z\|(P)^{-(g+1)} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP-W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2 \,,$$

where again the product runs over the Weierstrass points of X, counted with their weights. It can be checked that this definition does not depend on the choice of P, nor on the choice of local coordinate z about P. A more intrinsic definition is possible, see [dJ], but we will not need this.

Finally we modify the Arakelov-Green function a bit by putting

$$G'(P,Q) := S(X)^{-1/g^3} \cdot G(P,Q) \,.$$

It is clear from Theorem 6.4 that this function is classical in the sense that it is obtained by carrying out only elementary operations on values of the $\|\vartheta\|$-function.

Theorem 6.5. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Let $m = \binom{2g+2}{g}$ and $n = \binom{2g}{g+1}$. Then we have*

$$\prod_{(W,W')} G'(W,W')^{n(g-1)} =$$
$$= T(X)^{-(g+2)m} \cdot \|\varphi_g\|(X)^{(g^2-1)/2} \cdot \prod_{\{i_1,\ldots,i_g\}} \|J\|(W_{i_1}, \ldots, W_{i_g})^{-(2g+4)} \,,$$

the first product running over all ordered pairs of distinct Weierstrass points of X, ignoring their weights, the second product running over the set of subsets of $\{1,\ldots,2g+2\}$ of cardinality g.

Theorem 6.6. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Then we have*

$$\prod_{(W,W')} G'(W,W')^{n(g-1)} = \pi^{-2g(g+2)m} \cdot T(X)^{-(g+2)m} \cdot \|\varphi_g\|(X)^{-\frac{3}{2}(g+1)},$$

the product running over all ordered pairs of distinct Weierstrass points of X, ignoring their weights.

We mention once more that a comparison of Theorems 6.5 and 6.6 immediately yields Theorem 6.3 and hence Theorem 6.2.

The proof of Theorem 6.5 uses the following special case of a formula due to Guàrdia (*cf.* [Gu], Corollary 2.6). It can be obtained by a limiting process from a fundamental formula of Faltings (*cf.* [Fa], p. 402), using Riemann's singularity theorem. Recall the Faltings delta-invariant $\delta(X)$ of X from [Fa], which for us will play only an auxiliary role.

Proposition 6.7. *(Guàrdia* [Gu]*) Let $W_{i_1},\ldots,W_{i_g},W$ be distinct Weierstrass points of X. Then the formula*

$$\|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W)^{g-1} = \exp(\delta(X)/8) \cdot \|J\|(W_{i_1},\ldots,W_{i_g}) \cdot \frac{\prod_{k=1}^{g} G(W_{i_k},W)^{g-1}}{\prod_{k<l} G(W_{i_k},W_{i_l})}$$

holds, where $\delta(X)$ is the Faltings delta-invariant of X.

Proof of Theorem 6.5. In [dJ] we proved the formula

$$\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X)$$

for Faltings' delta-invariant and hence by Proposition 6.7 and the definition of G' we have

$$\prod_{k=1}^{g} G'(W_{i_k},W)^{2g-2} = \frac{1}{T(X)} \cdot \frac{\|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W)^{2g-2}}{\|J\|(W_{i_1},\ldots,W_{i_g})^2} \cdot \prod_{k\neq l} G'(W_{i_k},W_{i_l})$$

for distinct Weierstrass points $W_{i_1},\ldots,W_{i_g},W$ on X. Taking the product over $W \notin \{W_{i_1},\ldots,W_{i_g}\}$ we obtain

$$\prod_{W\notin\{W_{i_1},\ldots,W_{i_g}\}} \prod_{k=1}^{g} G'(W_{i_k},W)^{2g-2}$$
$$= \frac{1}{T(X)^{g+2}} \cdot \frac{\prod_{W\notin\{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1}+\cdots+W_{i_g}-W)^{2g-2}}{\|J\|(W_{i_1},\ldots,W_{i_g})^{2g+4}} \cdot \prod_{k\neq l} G'(W_{i_k},W_{i_l})^{g+2}.$$

Taking the product over all sets of indices $\{i_1, \ldots, i_g\}$ of cardinality g we find

$$\prod_{(W,W')} G'(W,W')^{n(g-1)}$$
$$= \frac{1}{T(X)^{(g+2)m}} \cdot \prod_{\{i_1,\ldots,i_g\}} \frac{\prod_{W \notin \{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W)^{2g-2}}{\|J\|(W_{i_1}, \ldots, W_{i_g})^{2g+4}} .$$

Now it is not difficult to check that

$$\prod_{\{i_1,\ldots,i_g\}} \prod_{W \notin \{W_{i_1},\ldots,W_{i_g}\}} \|\vartheta\|(W_{i_1} + \cdots + W_{i_g} - W)^{2g-2} = \|\varphi_g\|(X)^{(g^2-1)/2} .$$

Plugging this in finishes the proof. □

For the proof of Theorem 6.6 we will invoke Theorems 1.1 and 1.2. As was said before, the theorem follows by considering the canonical hermitian metrics that we have on the occurring line bundles and a comparison of the norms of the occurring canonical sections. First however we recall a well-known relation between the algebraic discriminant and the modular discriminant of a hyperelliptic Riemann surface, *cf.* for instance [Lo], Proposition 3.2.

Proposition 6.8. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Fix an ordering $W_1, \ldots, W_{2g+2}$ of its Weierstrass points. Consider an equation $y^2 = f(x)$ for X where $f \in \mathbb{C}[x]$ is monic and separable of degree $2g+1$, putting W_{2g+2} at infinity. Write $f(x) = \prod_{k=1}^{2g+1}(x - a_k)$ and denote by $D = \prod_{k<l}(a_k - a_l)^2$ the discriminant of f. Let μ_k for $k = 1, \ldots, g$ be the holomorphic differential on X given in coordinates by $\mu_k = x^{k-1}dx/2y$ and let $(\mu|\mu')$ be the period matrix of these differentials on the canonical symplectic basis of homology determined by the chosen ordering of the Weierstrass points. Finally let $\tau = \mu^{-1}\mu'$, let $n = \binom{2g}{g+1}$ and let $r = \binom{2g+1}{g+1}$. Then we have the equality*

$$D^n = \pi^{4gr}(\det \mu)^{-4r}\varphi_g(\tau)$$

relating the discriminant D of the polynomial f to the value $\varphi_g(\tau)$ of the modular discriminant corresponding to X marked with its chosen ordering of Weierstrass points.

Sketch of the proof. A calculation shows that, for any X, the quantity $D^n(\det \mu)^{4r}\varphi_g(\tau)^{-1}$ is independent of the choice of ordering of Weierstrass points and the choice of corresponding equation $y^2 = f(x)$, hence depends only on X. As a consequence, $D^n(\det \mu)^{4r}\varphi_g(\tau)^{-1}$ descends to a well-defined holomorphic function on $\mathcal{I}_g \otimes \mathbb{C}$. We claim that it has neither zeroes nor poles: it is clear that the discriminant D is never zero, and as to the function $\varphi_g(\tau)$, we recall that any characteristic $\eta_{T \circ U}$ with T of the form $T = \{i_1, \ldots, i_{g+1}\}$ for a certain subset $\{i_1, \ldots, i_{g+1}\}$ of $\{1, \ldots, 2g+2\}$ corresponds, under the identification induced by the Abel-Jacobi map, to a semi-canonical divisor $W_{i_1} + \cdots + W_{i_g} - W_{i_{g+1}}$. By Proposition 6.1 of Chapter IIIa of [Mu1] such semi-canonical divisors are not linearly equivalent to an effective divisor, hence $\vartheta[\eta_{T\circ U}](0;\tau) \neq 0$ in this case. Now we know by Proposition 5.1 that $H^0(\mathcal{I}_g \otimes \mathbb{C}, \mathbb{G}_{\mathrm{m}}) = \mathbb{C}^*$, which means that $D^n(\det \mu)^{4r}\varphi_g(\tau)^{-1}$ is a constant. By a degeneration argument as in [Fay], pp. 48–49 we find this constant to be equal to π^{4gr}. $\square$

Proof of Theorem 6.6. Consider the smooth, proper complex curve $p : \mathcal{X} \to B = \mathrm{Spec}(\mathbb{C})$ corresponding to X. As usual, fix an ordering $W_1, \ldots, W_{2g+2}$ of the Weierstrass points of X and let $y^2 = f(x)$ with f monic and separable of degree $2g+1$ be an equation for X. A small computation shows that we may write

$$\Lambda_{\mathcal{X}/B} = \left(2^{-(4g+4)} \cdot D\right)^g \left(\frac{dx}{y} \wedge \ldots \wedge \frac{x^{g-1}dx}{y}\right)^{\otimes 8g+4}$$

for the canonical element of $\det H^0(\mathcal{X}, \omega_{\mathcal{X}/B})$, where D is the discriminant of f. Let μ_k for $k = 1, \ldots, g$ be the holomorphic differential on X given in coordinates by $\mu_k = x^{k-1}dx/2y$ and let $(\mu|\mu')$ be the period matrix of these differentials on the canonical symplectic basis of homology determined by the chosen ordering of the Weierstrass points. Let $\tau = \mu^{-1}\mu'$, let $n = \binom{2g}{g+1}$ and let $r = \binom{2g+1}{g+1}$. Also put $\Delta_g := 2^{-(4g+4)n} \cdot \varphi_g$. We can then write, by Proposition 6.8,

$$\begin{aligned}
\Lambda^{\otimes n} &= \left(2^{-(4g+4)} \cdot D\right)^{gn} \left(\frac{dx}{y} \wedge \ldots \wedge \frac{x^{g-1}dx}{y}\right)^{\otimes (8g+4)n} \\
&= 2^{-(4g+4)gn}\pi^{4g^2r}(\det \mu)^{-4gr}\varphi_g(\tau)^g \left(\frac{dx}{y} \wedge \ldots \wedge \frac{x^{g-1}dx}{y}\right)^{\otimes (8g+4)n} \\
&= (2\pi)^{4g^2r}(\det \mu)^{-4gr}\Delta_g(\tau)^g \left(\frac{dx}{2y} \wedge \ldots \wedge \frac{x^{g-1}dx}{2y}\right)^{\otimes (8g+4)n} .
\end{aligned}$$

Recall that we have a canonical isomorphism $j : \det H^0(\mathcal{X}, \omega_{\mathcal{X}/B}) \xrightarrow{\sim} \det H^0(\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g, \Omega^g)$. Letting $z_1, \ldots, z_g$ be the standard euclidean coordinates on $\mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$ we obtain from the above calculation

$$j^{\otimes(8g+4)n}(\Lambda^{\otimes n}) = (2\pi)^{4g^2r}\Delta_g(\tau)^g(dz_1 \wedge \ldots \wedge dz_g)^{\otimes(8g+4)n} .$$

Now consider the isomorphism

$$(\det p_*\omega_{\mathcal{X}/B})^{\otimes 12(4g^2+6g+2)} \xrightarrow{\sim} \langle O_{\mathcal{X}}(W_{\mathcal{X}/B}), O_{\mathcal{X}}(W_{\mathcal{X}/B}) \otimes_{O_{\mathcal{X}}} \omega_{\mathcal{X}/B}\rangle^{\otimes -4g(g-1)}$$

that we have on B by Theorem 1.1. We endow both sides of this isomorphism with their canonical Faltings-Arakelov metrics as in [Fa], [De]. This means in particular that Λ has norm given by

$$\|\Lambda\|(B)^n = (2\pi)^{4g^2r}\|\Delta_g\|(X)^g ,$$

where $\|\Delta_g\|(X) = 2^{-(4g+4)n} \cdot \|\varphi_g\|(X)$; indeed, by definition the norm of $dz_1 \wedge \ldots \wedge dz_g$ is $\|dz_1 \wedge \ldots \wedge dz_g\| = \sqrt{\det \mathrm{Im}\tau}$. Further, the section Ξ given by adjunction has norm

$$\|\Xi\|(B) = \prod_{k \neq l} G(W_k, W_l) = S(X)^{(4g^2+6g+2)/g^3} \prod_{k \neq l} G'(W_k, W_l)\,.$$

Now following the proof of Theorem 1.1 it is clear, as in [BMM], Section 3, that the isomorphism ν is an isometry for the canonical Faltings-Arakelov metrics. Further it follows from [Mo1], [Mo2] that the norm of the Mumford isomorphism (M) is equal to $(2\pi)^{-4g}\exp(\delta(X))$, where $\delta(X)$ is the Faltings delta-invariant of X. Combining and using Theorem 1.2 we obtain

$$\left((2\pi)^{-4g}\exp(\delta(X))\right)^{(8g+4)(4g^2+6g+2)} \cdot \|\Lambda\|(B)^{12(4g^2+6g+2)} = \|2^{-(2g+2)} \cdot \Xi\|(B)^{-4g(g-1)(8g+4)}\,.$$

Using the formula $\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X)$ from [dJ] and plugging in the results for $\|\Lambda\|$ and $\|\Xi\|$ above we obtain from this the required formula. $\square$

Remark 6.9. We remark that the traditional approach to Thomae's identity as for example in [Fr], §3 is based on the heat equation for the theta function. The use of the heat equation is circumvented in our approach. In some sense, we replaced the infinitesimal variation of the moduli τ in the heat equation by a global variation, involving all prime numbers.

Remark 6.10. It is an intriguing question how Thomae's identity generalises to general $\tau \in \mathcal{H}_g$. The question seems not at all resolved, but for a discussion of what can be proved see [Ig1]. It would be interesting to know to what extent our method can be generalised to deal with other families of curves.

7. Elliptic curves

To conclude this note we show how our arguments specialise in the case of genus 1 to give the celebrated Jacobi derivative formula, *cf.* for instance [Mu2], Chapter I, §13.

Proposition 7.1. *(Jacobi's derivative formula) Let η_1 be an odd characteristic in dimension 1, and let η_2, η_3, η_4 be even ones, different modulo $\mathbb{Z} \times \mathbb{Z}$. Then the identity*

$$\vartheta[\eta_1]'(0;\tau) = \pm\pi\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)$$

holds for τ in the complex upper half plane.

Proof. We consider the universal elliptic curve $p : \mathcal{U}_1 \to \mathcal{M}_{1,1}$. Let ω be the relative dualising sheaf of p. It is well-known, see for instance [Li], Corollary 9.3.27, that the canonical homomorphism $p^*p_*\omega \to \omega$ is an isomorphism; this implies that the line bundle $\langle\omega,\omega\rangle$ on $\mathcal{M}_{1,1}$ is canonically trivial, even as a metrised line bundle. This in turn implies that $\langle\omega,\omega\rangle$ contains a canonical global trivialising section Ξ, with unit length over the complex numbers. On the other hand, consider the line bundle $(p_*\omega)^{\otimes 12}$ on $\mathcal{M}_{1,1}$. The theory developed in Section 4 applies also in this case, and in particular we have the existence of a canonical global trivialising section Λ, which we can write as

$$\Lambda = (2\pi)^{12}\Delta_1(\tau)(dz)^{\otimes 12}\,,$$

with $\Delta_1(\tau) = 2^{-8}(\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau))^8$, on the base B of an elliptic curve $\mathcal{E} \to B = \mathrm{Spec}(\mathbb{C})$ with analytification $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. Now consider the Mumford isomorphism $\mu : (p_*\omega)^{\otimes 12} \xrightarrow{\sim} \langle\omega,\omega\rangle$, which is unique up to sign. As in Proposition 5.1 we have $H^0(\mathcal{M}_{1,1},\mathbb{G}_{\mathrm{m}}) = \{-1,+1\}$ (*cf.* [Mo1] for instance). As in the proof of Theorem 1.2 we conclude that $\mu(\Lambda) = \pm\Xi$. Now $\|\Lambda\| = (2\pi)^{12}(\mathrm{Im}\tau)^6|\Delta_1(\tau)|$ and $\|\Xi\| = 1$. Considering that by [Mo1], [Mo2] the Mumford isomorphism μ itself has norm $(2\pi)^{-4}\exp(\delta)$, where δ is the delta-invariant, we find that $(2\pi)^8(\mathrm{Im}\tau)^6|\Delta_1(\tau)|\exp(\delta(X)) = 1$ for a complex elliptic curve $X = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. On the other hand, by Proposition 6.7 we have for the delta-invariant that $\exp(-\delta(X)) = (\mathrm{Im}\tau)^6|\vartheta[\eta_1]'(0;\tau)|^8$ (this can also be seen by the formula $\exp(\delta(X)/4) = T(X)$ and by working out the definition of $T(X)$ from Section 6, *cf.* [dJ], Section 2). Eliminating $\exp(\delta(X))$ from both formulas gives

$$|\vartheta[\eta_1]'(0;\tau)| = \pi|\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)|\,.$$

By the maximum principle we find an equality of holomorphic functions

$$\vartheta[\eta_1]'(0;\tau) = \varepsilon\pi\vartheta[\eta_2](0;\tau)\vartheta[\eta_3](0;\tau)\vartheta[\eta_4](0;\tau)$$

of τ in the complex upper half plane, where ε is a complex constant of modulus 1. We find the right value $\varepsilon = \pm 1$ by considering the q-expansions on the left and on the right, as in [Mu2], Chapter I, §13. □

Remark 7.2. The function $\Delta_1(\tau)$ appearing in the above proof is equal to Jacobi's discriminant function $\Delta_1(q) = q\prod_{k=1}^{\infty}(1-q^k)^{24}$ where $q = \exp(2\pi i\tau)$.

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