Some Rationality Properties of Observable Groups and Related Questions

Thang NGUYEN QUOC and Bac DAO PHUONG

Institut des Hautes Études Scientifiques

35, route de Chartres

91440 – Bures-sur-Yvette (France)

Décembre 2004

IHES/M/04/62

SOME RATIONALITY PROPERTIES OF OBSERVABLE GROUPS AND RELATED QUESTIONS

NGUYÊÑ QUÔĆ THĂŃG AND ĐÀO PHƯƠNG BĂĆ

Abstract. We investigate in this paper some rationality questions related with observable, epimorphic, and Grosshans subgroups of linear algebraic groups over non-algebraically closed fields.

1. Introduction

Let G be a linear algebraic group defined over an algebraically closed field k. Then G acts naturally on its regular function ring $k[G]$ by right translation $(r_g.f)(x) = f(x.g)$, for all $x, g \in G, f \in k[G]$. For H a closed k-subgroup of G, we put $H' = k[G]^H := \{f \in k[G] \ : r_h.f = f$, for all $h \in H\}$. Then $k[G]^H$ is the k-subalgebra of H-invariant functions of $k[G]$. By convention, we identify the algebraic groups considered with their points in a fixed algebraically closed field. For a k-subalgebra R of $k[G]$, we put $R' = \{g \in G : r_g.f = f$, for all $f \in R\}$. Then for any closed subgroup $H \subset G$ we have

$$H \subseteq H'' \subseteq G.$$

With a motivation from representation theory, Bialynicki-Birula, Hochschild and Mostow (see [1], p. 134) introduced the concept of "observable subgroup". A closed subgroup H of G is called an *observable subgroup* of G if any finite dimensional rational representation of H can be extended to a finite dimensional rational representation on the whole group G (or, equivalently, if every finite dimensional rational H-module is a H-submodule of a finite dimensional rational G-module). In loc. cit. some equivalent conditions for a subgroup to be observable were given. Then Grosshans ([6], [7] and reference therein) has added several other conditions. It turned out later that for closed subgroups the property of being observable for a subgroup H is equivalent to the equality $H = H''$. Up to now there are known several equivalent conditions for a subgroup to be observable, which are more or less simple to verify and they are gathered in Theorem 1 below. On the opposite side, a closed subgroup $H \subseteq G$ may satisfy the equality $H'' = G$. If it is so, H is called an *epimorphic subgroup* of G. In fact, under an equivalent condition, this notion was first

2000 *Mathematics Subject Classification.* 14L24, 14L30, 20G15.

Partially supported by Fund. Res. Prog., the Abdus Salam I.C.T.P. and I.H.E.S..

introduced and studied by Bien and Borel in [2] [3] (see also [7] for recent treatment), which in turn, is based on similar notion for Lie algebras, given by Bergman (unpublished). There were given several equivalent conditions for a closed subgroup to be epimorphic (see Theorem 11 below).

In the connection with the solution of the 14th Hilbert problem, the following well-known problem is of great interest. Assume that X is an affine variety, G is a reductive group acting upon X morphically, H is a closed subgroup of G and consider the G-action on the regular function ring $k[X]$ by left translation: $(l_g.f)(x) = f(g^{-1}.x)$. It is natural to ask when $k[X]^H$ is a finitely generated k-algebra.

For a closed subgroup $H \subset G$, we have $k[X]^H = k[X]^{H''}$ (see [6], [7]). On the other hand, it is well-known (see e. g. [6], [7]) that H'' is the smallest observable subgroup of G containing H. So the problem is reduced to the case when H is an observable subgroup. To solve this problem, Grosshans ([6], [7]) introduced the codimension 2 condition for observable subgroups, and the subgroups satisfying this condition are called subsequently *Grosshans subgroups* of G (see Section 4). In this paper, we continue the study initiated in [1]. Namely, we are interested in some questions of rationality related to observable, epimorphic and Grosshans subgroups. The first rationality results regarding observable (resp. epimorphic) subgroups were already given in [1], and then in [7], [10] (resp. [2], [3] and [10]), where some arithmetical applications to ergodic actions were also given. We give some other new results related to rationality properties of observable, epimorphic and Grosshans subgroups (which were stated initially for algebraically closed fields). Some arithmetic and geometric applications will be considered in another paper under preparation. Throughout, we consider only linear algebraic groups defined over some field k, which are called also shortly as *k-groups.* For basic theory of linear algebraic groups over non-algebraically closed field we refer to [4], and for a k-group G, the notion of a rational k-module V for G is as in [6], [7].

2. Some rationality results for observable groups

First we recall well-known results over algebraically closed fields. For an algebraic group G we denote by G° the identity connected component subgroup of G.

Theorem 1. *([1], [7], Theorems 2.1 and 1.12) Let G be a linear algebraic group defined over an algebraically closed field k and let H be a closed k-subgroup of G. Then the following conditions are equivalent.*

a) $H = H''$.

b) There exists a finite dimensional rational representation $\rho : G \to GL(V)$ and a vector $v \in V$, all defined over k such that

$$H = G_v = \{g \in G : \rho(g).v = v\}.$$

c) There are finitely many functions $f \in k[G/H]$ which separate the points in G/H.

d) G/H is a quasi-affine k-variety.

e) Any finite dimensional rational k-representation $\rho : H \to GL(V)$, can be extended to a finite dimensional rational k-representation $\rho' : G \to GL(V')$, where $V \hookrightarrow V'$, i. e., every finite dimensional rational H-module is a H-submodule of a finite dimensional rational G-module.

f) There is a finite dimensional rational k-representation $\rho : G \to GL(V)$ and a vector $v \in V$ such that $H = G_v$, the isotropy group of v, and

$$G/H \cong G.v = \{\rho(g).v : g \in G\}$$

(as algebraic varieties).

g) The quotient field of the ring of $G^\circ \cap H$-invariants in $k[G^\circ]$ is equal to the field of $G^\circ \cap H$-invariants in $k(G^\circ)$.

h) If 1-dimensional rational H-module M is a H-submodule of a finite dimensional rational G-module then the H-dual module M^* of M is also a H-submodule of a finite dimensional rational G-module.

Now let k be any field. If a closed k-subgroup H of a linear algebraic k-group G satisfies the condition *b)* (resp. *e)*) in Theorem 1 where $v \in V(k)$ and the corresponding representation ρ is defined over k, then we say that H is an *isotropy k-subgroup of G* (resp. has *extension* property over k).

First we recall the following rationality results proved in [1] (Theorems 5, 8).

Theorem 2. *([1], Theorem 5) Let G be a linear algebraic k-group, H a closed k-subgroup of G, $k \subset K$ an algebraic extension of k. Then H has extension property over k if and only if it has one over K.*

Theorem 3. *([1], Theorem 8) If H is a closed k-subgroup of a linear algebraic k-group G with extension property over k, then H is an isotropy k-subgroup of G. Conversely, if k is algebraically closed and H is a isotropy k-subgroup then it has extension property over k.*

From Theorem 2 and Theorem 3, we derive the following.

Proposition 4. *Let k be an arbitrary field and let H be a closed k-subgroup of a k-group G. The following two conditions are equivalent.*

a) H is an isotropy subgroup of G over $\overline{k}$.

b) H is an isotropy subgroup of G over k, i.e., there exists a finite dimensional k-rational representation $\rho : G \to GL(V)$ and a vector $v \in V(k)$ such that $H = G_v$.

Proof. $b) \Rightarrow a)$: trivial.

$a) \Rightarrow b)$. By Theorem 1, since H is an isotropy subgroup over $\overline{k}$, H has extension property over $\overline{k}$. Therefore by Theorem 2, H has extension property over k. By Theorem 3, H is an isotropy k-subgroup of G.

□

Remark 1. In [10], another proof of Proposition 4 was given, which is based on some ideas of Grosshans [6], under the condition (which is not essential) that $k = \mathbf{Q}$ and H is connected.

We put

$$H'_k = k[G]^{H(k)} = \{f \in k[G] : r_h.f = f, \forall h \in H(k)\},$$

and

$$(H'_k)' = \{g \in G : r_g.f = f, \forall f \in H'_k\}.$$

Then $k[G]^{H(k)}$ and $k[G]^H := \{f \in k[G] \ : r_h.f = f, \forall h \in H\}$ are k-subalgebras of $k[G]$. In general we have the following diagram

$$\begin{array}{ccc} H'_k = k[G]^{H(k)} & \subseteq & \bar{k}[G]^{H(k)} \\ \bigcup\vert & & \bigcup\vert \\ k[G]^H & \subseteq & \bar{k}[G]^H = H' \end{array}$$

so we have

$$\begin{array}{ccc} (H'_k)' = (k[G]^{H(k)})' & \supseteq & (\bar{k}[G]^{H(k)})' \\ \bigcap\vert & & \bigcap\vert \\ (k[G]^H)' & \supseteq & (\bar{k}[G]^H)' = H''. \end{array}$$

If, moreover, $H(k)$ is Zariski dense in H then we have

$$H'_k = k[G]^H = \overline{k}[G]^H \cap k[G].$$

We say that H is *relatively observable over* k if $H = (H'_k)'$, and H is k-*observable*, if $(k[G]^H)' = H$. It is clear that if k is algebraically closed, then these notions coincide with the observability. We have the following obvious implication

$$H \textit{ is } k\textit{-observable} \Rightarrow H \textit{ is observable.}$$

Proposition 5. *Let k be a field, and let H be a closed k-subgroup of a k-group G. Then*

a) $H' = \overline{k}[G]^H = \overline{k} \otimes_k k[G]^H$;

b) H is observable if and only if H is k-observable;

c) Assume that $H(k)$ is Zariski dense in H. Then H is observable $\Leftrightarrow$ H is k-observable $\Leftrightarrow$ H is relatively observable over k.

Proof. a) We need the following lemma.

Lemma 6. *Let X be an affine scheme of finite type over k upon which a k-group H acts k-morphically, such that the (good) quotient scheme X/H exists. Then we have*

$$\overline{k}[X]^H = \overline{k} \otimes_k k[X]^H.$$

(Here by convention $X = \mathrm{Spec}(\bar{k}[X])$, and $k[X]$ gives the k-structure of $\bar{k}[X]$.)

Since X/H is defined over k, we have $\overline{k}[X/H] = \overline{k} \otimes_k k[X/H]$. Besides, the quotient morphism $\pi : X \to X/H$ is also defined over k so the comorphism π^0 sends $k[X/H]$ into $k[X]$. On the other hand, $\pi^0 : \overline{k}[X/H] \to \overline{k}[X]^H$ is an isomorphism. So $\pi^0|_{k[X/H]} : k[X/H] \to k[X] \cap \overline{k}[X]^H = k[X]^H$ is a monomorphism. Because of the k-linearity of π^0, we have

$$\begin{aligned} \overline{k}[X]^H &= \pi^0(\overline{k}[X/H]) \\ &= \pi^0(\overline{k} \otimes k[X/H]) \\ &= \overline{k} \otimes \pi^0(k[X/H]) \\ &\subseteq \overline{k} \otimes (k[X]^H) \\ &\subseteq \overline{k}[X]^H. \end{aligned}$$

So the above equalities imply that $\pi^0(k[X/H]) = k[X]^H$. Since π^0 is an isomorphism, we have $\overline{k}[X]^H = \overline{k} \otimes_k k[X]^H$. The lemma is proved.
So $a)$ follows by taking $X = G$.
$b)$ It suffices to show that if H is observable then it is also k-observable. But this follows directly from $a)$.
$c)$ By part $b)$ we need only show that

$$H \text{ is relatively observable over } k \Leftrightarrow H \text{ is } k\text{-observable.}$$

$(\Rightarrow)$: Since $H(k)$ is Zariski-dense in H, so we have $(f \in \bar{k}[G]^{H(k)} \Leftrightarrow f \in \bar{k}[G]^H)$. Therefore $H = (k[G]^{H(k)})' \supseteq (\bar{k}[G]^{H(k)})' = (\bar{k}[G]^H)' \supseteq H$, i.e., H is observable, hence also k-observable , by $b)$.
$(\Leftarrow)$: If H is k-observable, then we have

$$H = (k[G]^H)' \supseteq (k[G]^{H(k)})' \supseteq H,$$

so H is relatively observable over k. □

Proposition 7. *Let H be a k-subgroup of a k-group G. The following are equivalent:*

a) There exist finitely many functions in $\overline{k}[G/H]$ which separate the points in G/H.

b) There exist finitely many functions in $k[G/H]$ which separate the points in G/H.

Proof. The assertion $b) \Rightarrow a)$ is obvious. To prove $a) \Rightarrow b)$, notice that since G/H defined over k, so

$$\overline{k}[G/H] = \overline{k} \otimes k[G/H].$$

Assume that the functions $f_1, ..., f_n \in \overline{k}[G/H]$ separate the points in G/H. We have

$$f_i = \sum_j \lambda_{ij}\varphi_{ij},$$

with $\lambda_{ij} \in \overline{k}, \varphi_{ij} \in k[G/H]$. With $xH \neq yH \in G/H$, there exists i such that $f_i(xH) \neq f_i(yH)$. So there exists j such that $\varphi_{ij}(xH) \neq \varphi_{ij}(yH)$. Hence $\{\varphi_{ij}\} \subseteq k[G/H]$ separate the points in G/H. □

Proposition 8. *Let G be a k-group, H a closed k-subgroup of G. Assume that, there exists finite dimensional k-rational representation $\rho : G \to GL(V)$, and $v \in V(k)$ such that $H = G_v$. Then there is a finite dimensional k-rational representation $\rho' : G \to GL(W)$ and $w \in W(k)$ such that $H = G_w$ and $G/H \cong_k G.w$.*

Proof. (Our original proof is lengthy and the following is based on the communications with F. Grosshans.)
By Theorem 1.12 of [7], there exists a vector space V', a representation $\rho' : G \to \mathrm{GL}(V')$, a vector $v \in V'$ such that $H = G_v$ and there is an isomorphism

$$G/H \simeq G.v.$$

Denote by $X = \overline{G.v}$ a closed subvariety of V', V'^* the dual vector space of V', and by $\{\lambda_1, \dots, \lambda_n\}$ a basis of V'^*. Thus, considered as an affine space, we have $\bar{k}[V'] = \bar{k}[\lambda_1, ..., \lambda_n]$. The morphisms

$$\varphi : G \xrightarrow{\pi} G/H \overset{p}{\hookrightarrow} X \hookrightarrow V', g \mapsto g.v$$

correspond to the comorphisms

$$\varphi^* : \bar{k}[V'] \xrightarrow{r} \bar{k}[X] \xrightarrow{p^*} \bar{k}[G/H] \overset{\pi^*}{\hookrightarrow} \bar{k}[G], \varphi^*(\lambda_i)(g) = \lambda_i(g.v),$$

where r is the restriction. We may identify $\bar{k}[G/H]$ with $\bar{k}[G]^H$, thus consider it as a subalgebra of $\bar{k}[G]$. It is clear that φ^* is G-equivariant with respect to left translation, and (by the construction)

$$\varphi^*(\bar{k}(X)) = \bar{k}(G)^H,$$

and

$$l_i := \varphi^*(\lambda_i) = p^*(r(\lambda_i)) \in \bar{k}(G)^H \cap \bar{k}[G] = \bar{k}[G]^H.$$

By Proposition 5, we may write

$$l_i = \sum_j c_{ij} \otimes \mu_{ij}, \quad c_{ij} \in \bar{k}, \mu_{ij} \in k[G]^H, \forall i, j.$$

Since G is defined over k, so the G-orbit of μ_{ij} span a finite dimensional vector subspace of $\bar{k}[G]$, which is defined over k. By adding a finite number of functions (see e.g. [4], Proposition, p. 54), we may therefore assume that the functions $\{\mu_{ij}\}$ are k-linearly independent and that the $\bar{k}$-vector space W', with the k-basis $\{\mu_{ij}\}$ is defined over k and is G-stable. Let W be the dual k-vector space of W'. Hence this gives rise to a representation $\rho : G \to \mathrm{GL}(W)$, which is defined over k. Denote by Y the affine k-variety with $\bar{k}[\mu_{ij}]$ as $\bar{k}$-algebra of functions. By considering the algebra of regular functions $\bar{k}[W]$ on the vector space W defined over k, we have the following k-homomorphisms of k-algebras

$$k[W] \to k[\mu_{ij}] \to k[G/H] \to k[G],$$

which corresponds to G-equivariant k-morphisms of k-varieties with G-action

$$G \to G/H \xrightarrow{q} Y \xrightarrow{r} W.$$

One checks that the k-morphism $q : G/H \to Y$ is dominant. Denote by $y = q(eH) \in Y$. Then Y is the closure of the G-orbit $G.y$, which is isomorphic to G/H (since it is so over $\bar{k}$), hence it is a k-isomorphism, and the representation $\rho : G \to \mathrm{GL}(W)$ is the one required. Therefore Proposition 8 is proved. □

From results proved above, we have the following theorem, which is an analog of Theorem 1 for arbitrary fields.

Theorem 9. *Let G be a linear algebraic group defined over a field k and let H be a closed k-subgroup of G. Then the following are equivalent.*

a) $H = H''$, i. e., H is observable.

a′) $H = (k[G]^H)'$, i.e., H is k-observable.

b′) There exists a k-rational representation $\rho : G \longrightarrow GL(V)$ and a vector $v \in V(k)$ such that

$$H = G_v = \{g \in G : g.v = v\}.$$

c′) There are finitely many functions $f \in k[G/H]$ which separate the points in G/H.

d′) G/H is a quasiaffine variety defined over k.

e′) Any k-rational representation $\rho : H \longrightarrow GL(V)$, can be extended to a k-rational representation $\rho' : G \longrightarrow GL(V')$.

f′) There is a k-rational representation $\rho : G \longrightarrow GL(V)$ and a vector $v \in V(k)$ such that $H = G_v$ and

$$G/H \cong_k G.v = \{\rho(g)v : g \in G\}.$$

g′) The quotient field of the ring of $G^\circ \cap H$-invariants in $k[G^\circ]$ is equal to the field of $G^\circ \cap H$-invariants in $k(G^\circ)$.

If, moreover, $H(k)$ is Zariski dense in H, then the above conditions are equivalent to the relative observability of H over k.

Proof. First, by Proposition 4, with conditions labelled as in Theorem 1 we have *b*) ⇔ *b*′), by Proposition 5 we have *a*) ⇔ *a*′), and by Proposition 7 we have *c*) ⇔ *c*′). The fact that *d*) ⇔ *d*′) is trivial, and we have *e*) ⇔ *e*′) by Theorem 2, and the same proof as in [7], Theorem 1.12, shows that we have *f*) ⇔ *f*′).

To prove the equivalence of *g*′) with the other conditions, we may use the other equivalent conditions. We need the following

Lemma 10. *With above assumption, H is k-observable in G if and only if $H \cap G^\circ$ is k-observable in G°.*

For, first observe that since H and G^0 are defined over k, so is $H \cap G^0$. We have *a*) ⇔ *a*′), so H is observable in G if and only if H is k-observable in G, and $H \cap G^\circ$ is observable in G° if and only if $H \cap G^\circ$ is k-observable in G°. By [7], Corollary 1.3, H is observable in G if and only if $H \cap G^\circ$ is observable in G°. It follows that, H is k-observable in G if and only $H \cap G^\circ$ is k-observable in G°. The lemma is proved.

Now by [1], Theorem 3, $H \cap G^\circ$ is k-observable in G° if and only if *g*′) holds. □

3. Rationality properties for epimorphic subgroups

Recall after Grosshans ([7]) that epimorphic subgroups $H \subseteq G$ are those closed subgroups of G satisfying the condition $H'' = G$. We have the following characterizations of epimorphic subgroups over an algebraically closed fields.

Theorem 11. *(*[2]*, Théorème 1,* [7]*, Lemma 23.7) Let H be a closed subgroup of G, all defined over an algebraically closed field k. Then the following are equivalent.*

a) H is epimorphic, i. e., $H'' = G$.

b) $k[G/H] = k$.

c) $k[G/H]$ is finite dimensional over k.

d) If V is any rational G-module then the spaces of fixed points of G and H in V coincide.

e) If V is a rational G-module such that $V = X \oplus Y$, where X, Y are H-invariant, then X, Y are also G-invariant.

f) Morphisms of algebraic groups from G to another L are defined by their values on H.

Remark 2. The initial definition of epimorphic subgroups was given in [2], by only requiring that the condition *f*) above hold.

Let notation be as in Section 2, and let k be an arbitrary field. Then for a k-subgroup H of a k-group G we say that H is *relatively epimorphic over k* if $(H'_k)' = G$, and that H is *$k-$epimorphic* if $(k[G]^H)' = G$. Recall that we have the following inclusions

$$\begin{array}{ccc} (H'_k)' = (k[G]^{H(k)})' & \supseteq & (\bar{k}[G]^{H(k)})' \\ \cap\vert & & \cap\vert \\ (k[G]^H)' & \supseteq & (\bar{k}[G]^H)' = H'', \end{array}$$

therefore, the following implications holds

$$H \text{ is epimorphic} \Rightarrow H \text{ is } k\text{-epimorphic},$$

$$H \text{ is } k\text{-epimorphic} \Leftarrow H \text{ is relatively epimorphic over } k.$$

In fact we have

Proposition 12. *With above notation, if H is either (a) relatively epimorphic over k or (b) k-epimorphic, then it is also epimorphic.*

Proof. We need only check that $H'' \supseteq G$. Assume that (a) holds. Let $g \in G$ be an arbitrary element, and let $f \in H'$. Then $r_h(f) = f$ for all $h \in H$. By Proposition 5, we have

$$H' = \bar{k}[G]^H = \bar{k} \otimes_k k[G]^H.$$

Therefore we may write $f = \sum_i c_i f_i, c_i \in \bar{k}, f_i \in k[G]^H$. Since $f_i \in k[G]^H \subset k[G]^{H(k)} = H'_k$ and by assumption, $g \in G = (H'_k)'$, so $r_g(f_i) = f_i$, for all i. Therefore $r_g(f) = f$, i.e., $g \in H''$, thus $G = H''$.

Now assume (b) holds. Then by Proposition 5 again, we have

$$\bar{k}[G]^H = k[G]^H \otimes \bar{k},$$

so

$$\begin{aligned} (\bar{k}[G]^H)' &= (k[G]^H \otimes \bar{k})' \\ &= (k[G]^H)' = G \end{aligned}$$

so H is also epimorphic. □

We have the following analog of Theorem 11 over an arbitrary field.

Theorem 13. *Let k be any field and let H be a closed k-subgroup of a k-group G. Then the following are equivalent.*

a') H is k-epimorphic, i.e., $(k[G]^H)' = G$.

b') $k[G/H] = k$.

c') $k[G/H]$ is finite dimensional over k.

d') For any rational G-module V defined over k, the spaces of fixed points of G and H in V coincide.

e') For any rational G-module V defined over k, if $V = X \oplus Y$, where X, Y are H-invariant, then X, Y are also G-invariant.

f') Morphisms defined over k of algebraic k-groups from G to another one are defined by their values on H.

Proof. In what follows we refer to Theorem 11 for the properties $a) - f)$. By Proposition 12 and the implications before it, we have $a) \Leftrightarrow a')$. Since G/H is defined over k, we have $b) \Leftrightarrow b')$ and $c) \Leftrightarrow c')$.

The proof of Théorème 1, the direction $i) \Rightarrow ii)$ of [2] (i.e., $f) \Rightarrow b)$ above) gives also the proof of the direction $f') \Rightarrow b')$. The direction $b') \Rightarrow c')$ is trivial. We have $c') \Leftrightarrow c) \Leftrightarrow d) \Rightarrow d')$ and the same proof of Théorème 1 of [2] shows that $d') \Rightarrow e') \Rightarrow f')$, thus we have the equivalence of statements $b'), c'), d'), e'), f')$. Since the statements $a), b), c), d), e), f)$ are equivalent and $a) \Leftrightarrow a')$, the theorem follows. □

Remark 3. It was mentioned in [10], p. 195, that Bien and Borel (unpublished) have also proved that if G is connected, then $d) \Leftrightarrow d')$.

4. Some rationality properties for Grosshans subgroups

One of the main results related with the finite generation problem (hence also with the Hilbert's 14-th problem) mentioned in the Introduction is the following result of Grosshans (Theorem 15). First we recall the following very useful result which reduces to the case of connected groups.

Theorem 14. *([7], Theorem 4.1) Let k be an algebraically closed field. For any closed subgroup H of G, if one of the following k-algebras $k[G]^H$, $k[G]^{H^\circ}$, $k[G^\circ]^{H \cap G^\circ}$, $k[G^\circ]^{H^\circ}$ is a finitely generated k-algebra, then the same holds for the other.*

Theorem 15. *([7], Theorem 4.3) For an observable subgroup H of a linear algebraic group G, all defined over an algebraically closed field k, the following are equivalent.*

$a)$ There is a finite dimensional rational representation $\varphi : G \to GL(V)$, an element $v \in V$, such that $H = G_v$ and each irreducible component of $\overline{G.v} - G.v$ has codimension ≥ 2 in $\overline{G.v}$.

$b)$ The k-algebra $k[G]^H$ is a finitely generated k-algebra.

If $b)$ holds, let X be an affine variety with $k[X] = k[G]^H$, and with G-action via left translations of G on G/H. There is a point $x \in X$ such that $G.x$ is open in X and $G.x \simeq G/H$ via $gH \mapsto g.x$ and each irreducible component of $X \setminus G.x$ has codimension ≥ 2 in X.

The observable subgroups which satisfy one of the equivalent conditions in Theorem 15 are called *Grosshans subgroups* (see [7]). There are some nice geometrical characterizations and examples of Grosshans subgroups presented in [7] and the reference therein.

For a field k, a k-group G and an observable k-subgroup $H \subset G$, we say that H satisfies the *codimension 2 condition over k* if H satisfies condition $a)$ above where V, φ are all defined over k and $v \in V(k)$.

We call H a *Grosshans subgroup relatively over k* (resp. *k-Grosshans subgroup*) of G if $k[G]^{H(k)}$ (resp. $k[G]^H$) is a finitely generated k-algebra.

We have a result similar to Theorem 15 for k-Grosshans subgroups.

Theorem 16. *Let k be any perfect field with infinitely many elements, G a connected k-group. Assume that H is a observable k-subgroup of G. Consider the following conditions.*

a') H satisfies codimension 2 condition over k.

b') One of the k-algebras $k[G]^H$, $k[G]^{H^\circ}$, $k[G^\circ]^{H\cap G^\circ}$, $k[G^\circ]^{H^\circ}$ is a finitely generated k-algebra.

c') H is a Grosshans subgroup relatively over k of G (i.e $k[G]^{H(k)}$ is finitely generated k-algebra).

Then, together with conditions in Theorem 15, we have the following implications

$$a) \Leftrightarrow a') \Leftrightarrow b) \Leftrightarrow b') \Rightarrow c').$$

If, moreover, $H(k)$ is Zariski dense in H, then all these conditions are equivalent.

Proof. $a) \Leftrightarrow a')$. We have trivially $a') \Rightarrow a)$. Combined with the proof of Proposition 8, the proof of Theorem 4.3 of [7] (regarding dimension computation) shows that in fact we have $a) \Rightarrow a')$, thus $a) \Leftrightarrow a')$.

$b) \Leftrightarrow b')$. Recall that we have $a) \Leftrightarrow b)$ (Theorem 15 above). By Lemma 6 we have

$$\bar{k}[G]^H \simeq \bar{k} \otimes_k k[G]^H.$$

Therefore, it is clear that $\bar{k}[G]^H$ is finitely generated $\bar{k}$-algebra if and only if so is $k[G]^H$. By the same way, $\bar{k}[G]^{H^\circ} \simeq k[G]^{H^\circ} \otimes_k \bar{k}$ is finitely generated as $\bar{k}$-algebra if and only if $k[G]^{H^\circ}$ is finitely generated k-algebra. This is also true if H° is replace by $H \cap G^\circ$ etc... Thus $b) \Leftrightarrow b')$.

$b') \Rightarrow c')$. If H is connected, then $H(k)$ is Zariski dense in H (cf e.g. [4], 18.3, or [5]), and we have

$$(*) \qquad k[G]^H = k[G]^{H(k)},$$

and the assertion is trivial. Otherwise, asume that $H \neq H^\circ$. Then we can make use of Theorem 14 above. In fact, $H^\circ(k)$ is a normal subgroup of finite index in $H(k)$, and we see that

$$k[G]^{H(k)} = (k[G]^{H^\circ(k)})^{H(k)/H^\circ(k)}$$

is finitely generated k-algebra, since from the equivalence $a) \Leftrightarrow a') \Leftrightarrow b) \Leftrightarrow b')$ and from Theorem 14 it follows that $k[G]^{H^\circ(k)}$ is finitely generated k-algebra, and $H(k)/H^\circ(k)$ is a finite group.

Assume further that $H(k)$ is Zariski dense in H, then $(*)$ holds, so the theorem is proved. □

Remark 4. It is of interest to find examples where the condition c') holds but the other conditions do not. It will, perhaps, ultimately lead to counter-examples to (generalized) 14-th Hilbert's Problem in the case of char.$k > 0$. (Various extensions of classical results in (geometric) invariant theory to the case of characteristic $p > 0$ were discussed at length in [9], Appendices.) It will be more interesting to have examples with G, H connected groups.

A relation with the subalgebra of invariants of a Grosshans subgroup of a reductive group acting rationally upon a finitely generated commutative algebra is given in

Theorem 17. *([7], Theorem 9.3.) Let k be an algebraically closed field. For any closed subgroup H of a reductive group G, all defined over k, the following are equivalent.*

a) $k[G]^H$ is a finitely generated k-algebra.

b) For any finitely generated, commutative k-algebra A on which G acts rationally, the algebra of invariants A^H is a finitely generated k-algebra.

We consider the following relative version of this theorem.

Theorem 18. *Let k be a perfect field with infinitely many elements, H a closed k-subgroup of a connected reductive k-group G. Consider the following conditions.*

a') $k[G]^H$ is a finitely generated k-algebra.

b') For any finitely generated, commutative k-algebra A_k on which G acts k-rationally, the algebra of invariants A_k^H is a finitely generated k-algebra.

c') For any finitely generated, commutative k-algebra A_k on which G acts k-rationally, the algebra of invariants $A_k^{H(k)}$ is a finitely generated k-algebra.

Then with notations as in Theorem 17 we have

$$a) \Leftrightarrow a') \Leftrightarrow b) \Leftrightarrow b') \Rightarrow c').$$

If, moreover, $H(k)$ is Zariski dense in H, then all conditions above are equivalent.

Proof. The proof follows the same lines as in the proof of Theorem 16 by using Theorem 17. □

Acknowledgements. We would like to thank F. Grosshans for very useful comments and suggestions on the first draft of this paper, which were incorporated in the present version, and the referee for useful suggestions. We are also indebted to A. Bialynicki-Birula, A. Borel and F. Grosshans for sending their works, which were of great help for the work over this paper. Thanks are due to Abdus Salam I.C.T.P. and I. H. E. S. for support, where the present version of the paper was written.

References

[1] A. Bialynicki-Birula, G. Hochschild and G. D. Mostow, Extensions of representations of algebraic linear group. *Amer. J. Math.*, vol. 85 (1963), 131 - 144.

[2] F. Bien et A. Borel, Sous-groupes épimorphiques de groupes linéaires algébriques, I, *C. R. Acad. Sci. Paris.* Ser. I, 315 (1992), 649 - 653.

[3] F. Bien et A. Borel, Sous-groupes épimorphiques de groupes linéaires algébriques, II, *C. R. Acad. Sci. Paris.* Ser. I, 315 (1992), 1341 - 1346.

[4] A. Borel, *Linear Algebraic Groups.* Second enlarged edition. Graduate Texts in Mathematics **126**. Springer - Verlag, Berlin - Heidelberg - New York, 1991.

[5] M. Demazure et A. Grothendieck, *Schémas en groupes*, Séminaire de Géometrie Algébrique Du Bois - Marie, Lecture Notes in Math. vv. **151 - 153**, Berlin - Heidelberg - New York, Springer, 1972.

[6] F. Grosshans, Observable groups and Hilbert's fourteenth problem, *Amer. J. Math.* v. 95 (1973), 229 - 253.

[7] F. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lecture Notes in Mathematics v.**1673**. Springer - Verlag, Berlin - Heidelberg - New York, 1997.

[8] J. E. Humphreys, *Linear Algebraic Groups.* Graduate Texts in Mathematics **21**. Springer - Verlag, Berlin - Heidelberg - New York, 1975.

[9] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, Ergebnisse der Math., Springer - Verlag, Berlin - Heidelberg - New York, 1994.

[10] B. Weiss, Finite dimensional representations and subgroup actions on homogeneous spaces, *Israel J. Math.* 106 (1998), 189 - 207.

Institute of Mathematics, 18-Hoang Quoc Viet 10307-Cau Giay, Hanoi - Vietnam and Institute des Hautes Études Scientifiques, le Bois-Marie, 35 route de Chartres, Bures-sur-Yvette, F-91440, France

E-mail address: nqthang@math.ac.vn, nqt@ihes.fr

Department of Mathematics, Hanoi University, Nguyen Trai, Hanoi - Vietnam

E-mail address: daophuongbac@yahoo.com