The Hopf algebra of rooted trees in Epstein-Glaser Renormalization

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THE HOPF ALGEBRA OF ROOTED TREES IN EPSTEIN-GLASER RENORMALIZATION

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ABSTRACT. We show how the Hopf algebra of rooted trees, in a somewhat modified presentation, encodes the combinatorics of Epstein-Glaser renormalization and position space renormalization in general. In particular we prove that the Epstein-Glaser time-ordered products can be obtained from the Hopf algebra by suitable Feynman rules, mapping trees to operator-valued distributions, and by twisting the antipode with a renormalization character, which formally solves the Bogoliubov recursion and provides local counterterms due to the Hochschild 1-closedness of the grafting operator B_+.

INTRODUCTION

The Epstein-Glaser framework [5, 15] and its modern variants [14, 2, 13] provide a mathematically rigorous approach to perturbation theory and renormalization in position space. Epstein and Glaser constructed, for a scalar field theory say, a sequence of operator-valued distributions T_n on M^n respectively, where M denotes the space-time, which replace the ill-defined time-ordered products from the standard approach to perturbation theory. The result is a perturbation theory which is a priori finite in each order - no removal of short-distance singularities is needed since all expressions are well-defined from the very beginning. Indeed, the appropriate notion of renormalization in the Epstein-Glaser framework is *extension of distributions onto diagonals* - since the objects of interest T_n are rather easily and a priori defined outside the diagonals. Finite renormalizations correspond to different ways of extending distributions onto diagonals. Moreover, in this approach the S-matrix is *local* by construction.

On the other hand the combinatorics of momentum space renormalization have been most efficiently described [4, 11] in terms of the Hopf algebra and associated Lie algebra of Feynman graphs. Renormalization and in particular the Bogoliubov recursion boils down then to twisting the antipode S of that Hopf algebra by a renormalization character into some target ring of Laurent or formal power series. This is possible due to a coproduct which disentangles 1PI graphs into divergent 1PI subgraphs. There is a universal object behind all Hopf algebras of this kind: the Hopf algebra of rooted trees [9, 3] which encodes nested subdivergences in terms of a tree and their recursive removal in terms of its coproduct and the associated antipode. We will show how the Hopf algebra of rooted trees works in the realm of Epstein-Glaser renormalization in almost complete analogy to other renormalization programs like BPHZ. In fact it is even easier to understand its role in Epstein-Glaser renormalization since no regularization is required and overlapping divergences are non-existing.

This paper is organized as follows: In the first section we give a short review of the Epstein-Glaser construction of time-ordered products, emphasizing the point of view of diagonals [2]. The second section introduces the powerful notion of a Hochschild 1-cocycle on a connected graded bialgebra, giving rise to two equivalent presentations of the Hopf algebra of rooted trees. A new convolution-like product is introduced which in cooperation

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with the antipode allows to recursively generate all terms needed for an Epstein-Glaser time-ordered product, as will be proved using explicit renormalized "Feynman rules" in the final theorem which we already state in a short version:

THEOREM 0. (Main result) *There is a an algebra homomorphism* $\phi : \mathcal{H}^{\bullet,*} \to F$ *such that the* n*-th Epstein-Glaser time-ordered product* T_n *is given by*

$$T_n = \sum_{t \in \mathcal{T}_n} \phi(S_R \odot id)(t).$$

where $\mathcal{H}^{\bullet,*}$ is a Hopf-algebra of rooted trees, F something like the tensor algebra of distributions on M, $\mathcal{T}_n$ the set of all binary trees with n leaves, S_R the twisted antipode of $\mathcal{H}^{\bullet,*}$ and $\odot$ a modified "convolution product" in $\mathcal{H}^{\bullet,*}$.

1. SOME BACKGROUND ON EPSTEIN-GLASER RENORMALIZATION

For simplicity, we restrict ourselves to a massive scalar field theory with interaction Lagrangian

$$\mathcal{L} = \frac{\lambda}{k!}\phi^k, \tag{1}$$

on flat Minkowski space-time $M := \mathbb{R}^{1,3}$. Generalizations to Quantum Electrodynamics and globally hyperbolic space-times have been worked out in [15] and [2], respectively, which though does not affect the combinatorics we are primarily interested in.

1.1. Motivation.

As a starting point for the Epstein-Glaser construction [5] we consider the symbolic Dyson series for the S-matrix

$$S = Te^{i\int \mathcal{L}(x)dx}$$

derived from the Schwinger differential equation of motion by transforming it into an iterated integral equation and applying the time-ordering operator T to each summand

$$\frac{i^n}{n!}\int T(\mathcal{L}(x_1)\ldots\mathcal{L}(x_n))dx_1\ldots dx_n$$

which has the benefit that we are integrating now over M^n instead of over an n-simplex$\times\mathbb{R}^{3n}$. T is usually defined by

$$T(A(x_1)B(x_2)) := \Theta(x_1^0 - x_2^0)A(x_1)B(x_2) + \Theta(x_2^0 - x_1^0)B(x_2)A(x_1) \tag{2}$$

where Θ denotes the Heaviside step function, and analogously for a product of more than two factors.
Now S and $\mathcal{L}$ are obviously to be operator-valued *distributions*, for which (2) does not make sense since distributions can not just be multiplied by noncontinuous functions like Θ. It does make sense though outside the thick diagonal $D_n = \{x \in M^n : x_i = x_j$ for some $i \neq j\}$ where the Θs are continuous.
In fact it is exactly this ill-defined notion of time-ordering due to the rather unclear nature of the $\mathcal{L}$ (and so the powers of the field ϕ) somewhere between functions and distributions that leads to short-distance singularities and makes renormalization necessary. Epstein and Glaser proposed a way of constructing well-defined time ordered products T_n, one for each power n of the coupling constant, that satisfy a set of suitable conditions explained below, the most prominent being that of *locality* or *micro-causality*. The resulting formal power series S is a priori finite in each order, and renormalization corresponds then to stepwise *extension of distributions* from $M^n - D_n$ to M^n.

The notion of *locality*, crucial to the following construction of time-ordered products, can be motivated as follows: Suppose $x = (x_1, \ldots, x_n) \in M^n$, $y = (y_1, \ldots, y_m) \in M^m$ and each x_i is not in the past lightcone of any of the y_j. We denote this situation $x_i \succsim y_j \; \forall i, j$.

Then our time ordered product T_{n+m} is supposed to satisfy (in the sense of operator-valued distributions)

(3) $$T_{n+m}(x_1,\dots,x_n,y_1,\dots,y_m) = T_n(x_1,\dots,x_n)T_m(y_1,\dots,y_m)$$

because we think of x to happen after (or at least not before) y. If both $x_i \succsim y_j$ and $y_j \succsim x_i$, $\forall i,j$, so if all pairs (x_i,y_j) are space-like, we get $[T_n,T_m]=0$.

1.2. **Construction of time-ordered products.** In this subsection we give a short review of the mathematical core of Epstein-Glaser renormalization in its modern variant [14, 2, 13] which emphasizes the point of view of nested diagonals. For the proofs, the reader is referred to [2].

The semi-Riemannian metric on M provides a relation $\succsim$ on M as follows: $x \succsim y$ iff x is not in the past causal shadow of y, that is $x \notin y + c^<$ where $c^< := \{z \in M : (z)^2 \leq 0,\, z^0 \leq 0\}$ is the closed past lightcone.

Now, for $n \in \mathbb{N}$ let $N := \{1,\dots,n\}$ and $I \subset N$, $I \neq \emptyset, N$. The set

$$C_I := \{(x_1,\dots,x_n) \in M^n : x_i \succsim x_j \,\forall i \in I,\, j \in N-I\}$$

is obviously a translation invariant open subset of M^n.

LEMMA 1 (Geometric lemma).

$$\bigcup_{\emptyset \subsetneq I \subsetneq N} C_I = M^n - \Delta_n$$

where $\Delta_n = \{x \in M^n : x_1 = \ldots = x_n\}$ *is the thin diagonal.*

The proof is an easy induction on n. The geometric lemma tells us that causality (3) determines the time-ordered product T_n everywhere outside the thin diagonal Δ_n, once the T_k for $k < n$ are known on whole M^k, respectively. It is important to understand that the geometric lemma does not really consitute a specific feature of Minkowski space-time. Indeed, the lemma holds if one replaces $\succsim$ by any relation such that $x \succsim y$ or $y \succsim x$ whenever $y \neq x$, and such that $\succsim$ is "weakly transitive" in the sense that $x \succsim y$ and $\neg(z \succsim y)$ implies $x \succsim z$.

DEFINITION 2. *Let* $n \in \mathbb{N}$. *A* causal partition of unity *is a smooth partition of unity* $\{p_{I,N-I}\}_{\emptyset \subsetneq I \subsetneq N}$ *of* $M^n - \Delta_n$, *subordinate to* $\{C_I\}_{\emptyset \subsetneq I \subsetneq N}$.

For simplicity, we will sometimes drop the curly brackets in the subscript, for example $p_{1,2}$ means $p_{\{1\},\{2\}}$.

Let $\mathcal{D}(M) = C_0^\infty(M)$ the space of test functions on M. In principle a time-ordered product is a sequence $(T_n)_{n\in\mathbb{N}}$ of operator-valued distributions $T_n : \mathcal{D}(M)^{\otimes n} \to End(D)$, D a suitable dense subspace of a Hilbert space, such that we get the S-matrix as

$$S(g) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} T_n(g^{\otimes n}).$$

As the construction of time-ordered products makes extensive use of the Wick expansion, which relates Green's functions of a ϕ^k-theory to those of the ϕ^l theories for all positive $l < k$, we actually need one time-ordered product per Wick-monomial. In fact, Epstein-Glaser solves all possible interacting theories of type (1) "at the same time". Moreover it might be generally desirable to have a time-ordered product

$$T(W_1 \ldots W_n)$$

of any given set $\{W_1 \ldots W_n\}$ of Wick monomials at hand, not just the one which fits to the interaction Lagrangian. Unfortunately notational inconveniences arise from the additional dependence on Wick-monomials. We will adopt the following notation [13]: Consider elements of the polynomial algebra $\mathbb{R}[\phi]$ as *formal* Wick polynomials such that, in particular, $\mathcal{L}/\lambda \in \mathbb{R}[\phi]$ and let $\mathcal{W}(M) := \mathcal{D}(M) \otimes \mathbb{R}[\phi]$, where we abusively denote elements $f = \sum g_k \otimes \phi^k$ of $\mathcal{W}(M)$ as $f = \sum g_k \phi^k$ in order to avoid too many tensor products later on. Then a time-ordered product will be a sequence (T_n) of multilinear maps from Wick polynomials and test functions $\mathcal{W}(M)$ to densely defined operators $End(D)$. However the reader should keep in mind that time-ordered products are essentially operator-valued *distributions*, the other aspects being of technical nature.

We are now going to define the objects of interest:

DEFINITION 3. *A sequence (T_n) of $\mathbb{R}$-linear maps $T_n : \mathcal{W}(M)^{\otimes n} \to End(D)$ is called an (Epstein-Glaser) time-ordered product if*

(i) $T_1(g\phi^k) =: \phi^k(g) :$ *where* $: \phi^k(g) :$ *means the Wick monomial* $: \phi^k :$ *smeared with the test function* g,

(ii) T_n *is symmetric*

$$T_n(f_1 \otimes \ldots \otimes f_n) = T_n(f_{\pi(1)} \otimes \ldots \otimes f_{\pi(n)})$$

when π is a permutation of $N := \{1, \ldots, n\}$. This allows for the notation

$$T(N) = T_n(f_1 \otimes \ldots \otimes f_n)$$

when the f_n are clear from the context,

(iii) T_n *splits causally: Let $I \subset N$, $I \neq \emptyset, N$. Then*

$$T(N) = T(I)T(N - I) \tag{4}$$

for all test functions with support in $C_I \subset M^n$,

(iv) T *is translation invariant*

$$U(a,1)T(f_1, \ldots, f_n)U(a,1)^{-1} = T(\tau_a f_1, \ldots, \tau_a f_n)$$

where $U(\cdot, 1)$ means the representation of the inhomogeneous part of the Poincaré group in D, and $\tau_a f(x) = f(x - a)$ is translation by a.

(v) *The Wick expansion relates time-ordered products corresponding to different Wick-powers*

$$T(g_1\phi^{k_1}, \ldots, g_n\phi^{k_n}) = \sum_{i_1,\ldots,i_n=0}^{k_1,\ldots,k_n} \frac{\langle 0| T(g_1\phi^{k_1-i_1}, \ldots, g_n\phi^{k_n-i_n}) |0\rangle}{i_1! \ldots i_n!} : \phi^{i_1} \ldots \phi^{i_n} : \tag{5}$$

with $\langle 0| \cdot |0\rangle$ the vacuum expectation value.

Note that by theorem 0 in [5] the summands in the right hand side of (5) as products of translation invariant *numerical* distributions and Wick monomials are well-defined. Once a time-ordered product $T = (T_n)$ is given, the S-matrix for the $\mathcal{L} = \frac{\lambda}{k!}\phi^k$-theory is obtained as the formal power series

$$S(g) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} T_n((g\phi^k)^{\otimes n}),$$

possibly taking the adiabatic limit $g \to \lambda$ later on, which is a highly nontrivial task we shall not be concerned about in the present work.

THEOREM 4. *Time-ordered products exist.*

A constructive proof is given in [2] and of course, in a somewhat different notation, in the original paper [5]. The idea is as follows: Provided all (T_m) for $m < n$ are constructed. Then the geometric Lemma 1 tells that T_n are determined on $M^n - \Delta_n$ by causality (iii):

$$T_n = T(I)T(N - I) \text{ as a distribution on } C_I.$$

So one shows that

$$T(I)T(N-I) = T(J)T(N-J)$$

on the overlap $C_I \cap C_J$ in order to patch them together using a causal partitions of unity $\{p_{I,N-I}\}$

$$\dot{T}_n := \sum_{\emptyset \subsetneq I \subsetneq N} p_{I,N-I} T(I)T(N-I) \tag{6}$$

which is a well-defined distribution on $M^n - \Delta_n$. As usual, $\dot{T}_n$ is independent on the choice of the partition of unity. It remains to extend it to a distribution on M^n. Using the Wick expansion (v), theorem 0 in [5] and translation invariance, this boils down to an extension problem of a numerical distribution $\dot{t}_n$ from $M^{(n-1)d} - \{0\}$ to $M^{(n-1)d}$. Having quantified the behaviour of a numerical distribution at the origin by the *Steinmann scaling degree* (see [2] for details), a generalization of the degree of homogeneity, one can show that there is a unique extension t_n of $\dot{t}_n$ to M^n provided the scaling degree $sd(\dot{t}_n)$ of $\dot{t}_n$ is smaller or equal the dimension $(n-1)d$. Otherwise, if it is bigger but finite, there is a finite dimensional space of extensions obtained as follows: Let $\phi \in \mathcal{D}(M^{(n-1)d})$. The distribution

$$t_n : \phi \mapsto \dot{t}_n \left(\phi - \sum_\alpha \omega_\alpha \partial^\alpha \phi(0) \right) \tag{7}$$

where the sum goes over all $(n-1)d$-multiindices α such that $|\alpha| \leq sd(\dot{t}_n) - (n-1)d$ and the $\omega_\alpha \in \mathcal{D}(M^{(n-1)d})$ such that $\partial^\beta \omega_\alpha = \delta_{\alpha,\beta}$, has then scaling degree $sd(t_n) \leq (n-1)d$ and is hence uniquely extendible. There is an ambiguity due to the ω_α however, and it is exactly this ambiguity which corresponds to the freedom of finite renormalizations. We call the linear operator w on test functions

$$w : \phi \mapsto \sum_\alpha \omega_\alpha \partial^\alpha \phi(0)$$

Taylor subtraction operator and, motivated by the fact that

$$t_n = (id - w^*)\dot{t}_n$$

holds on the level of numerical distributions, we write *by abuse of notation* the extension of $\dot{T}_n$ to the diagonal by

$$T_n = (id - W^*_{1\ldots n})\dot{T}_n \tag{8}$$

although there is no linear operator W^* on operator valued distributions fullfilling this duty. Our abuse of notation is justified though because we are only concerned about the combinatorics with respect to n in the following, and the Wick expansion leaves n obviously unchanged. So we understand W^* as the symbolic "operator" which unpacks the operator valued distributions into Wick monomials and numerical distributions, Taylor subtracts the test function for that numerical distribution and produces then a "counterterm" such that $(id - W^*)$ maps a distribution on $M^n - \Delta_n$ to an extension on M^n while the possible ambiguity (depending on the scaling degrees) is fixed by a choice of the ω_α. The subscript in $W^*_{1\ldots n}$ indicating to which coordinates it applies will be useful later on.

The constructive proof of Theorem 4 immediately implies the following

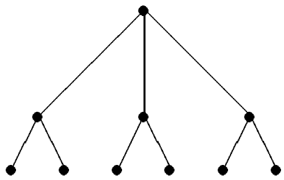
COROLLARY 5. *All time-ordered products are uniquely (up to the ω_α, more precisely up to the finite set of constants $t(\omega_\alpha)$) characterized by equations (6) and (8).*

Feynman graphs enter the game when one applies Wick's theorem. It might be instructive to have a look at the examples in [13]. We also note that the usual notions of renormalizable theories, critical dimension etc. can be traced back to the behaviour of the scaling degrees as n and the space-time dimension vary.

2. THE HOPF ALGEBRA OF ROOTED TREES IN EPSTEIN-GLASER RENORMALIZATION

The combinatorics of renormalization in position space can be most easily described in terms of rooted trees. Given some pairwise disjoint space-time points, we consider them as leaves of a tree

Whenever some of those points come together on a diagonal, we connect the corresponding vertices to a new vertex such that subdivergences (subdiagonals) correspond to subtrees



So a tree represents the (partially ordered) nested subdiagonals which are relevant to renormalization. It is now possible to construct a suitable coproduct on the free algebra generated by these trees such that the Bogoliubov recursion is essentially solved by the antipode of the resulting Hopf algebra on trees, as will be made precise in subsection 2.2. This remarkable property and the fact that local counterterms result [12] are equivalent to the closedness of a certain operator on the Hopf algebra with respect to Hochschild cohomology.

2.1. **Hochschild cohomology of bialgebras.** All algebras are supposed to be over some field k, associative and unital, analogously for coalgebras. The unit (and unit map) will be denoted $\mathbb{I}$, the counit map ϵ. *All algebra homomorphisms are supposed to be unital.* A bialgebra $A = \bigoplus_{i=0}^{\infty} A_i$ is called *graded* if $A_i A_j \subset A_{i+j}$ and $\Delta(A_i) \subset \bigoplus_{j+k=i} A_j \otimes A_k$ where Δ is the coproduct. A is called *connected* if it is graded and $A_0 = k$. We call $\ker \epsilon$ the augmentation ideal of A and denote P the projection $A \to A$ onto the augmentation ideal, $P = id - \mathbb{I}\epsilon$.
Let A be a bialgebra. We think of linear maps $L : A \to A^{\otimes n}$ as n-cochains and define a coboundary map b by

$$bL := (id \otimes L) \circ \Delta + \sum_{i=1}^{n} (-1)^i \Delta_i \circ L + (-1)^{n+1} L \otimes \mathbb{I} \tag{9}$$

where Δ denotes the coproduct, $\mathbb{I}$ the unit map, and Δ_i the coproduct Δ applied to the i-th factor in $A^{\otimes n}$. It is easy to see (using essentially the coassociativity of Δ) that $b^2 = 0$, which gives rise to a cohomology theory called Hochschild cohomology.
It is also easy to see that, for A finite dimensional, the cohomology theory (9) is the dual of the usual Hochschild homology of the algebra A.

For $n = 1$, (9) reduces to, for $L : A \to A$,

$$bL = (id \otimes L) \circ \Delta - \Delta \circ L + L \otimes \mathbb{I}. \tag{10}$$

It is known [3] that the category of objects (A, C) consisting of a commutative bialgebra A and a Hochschild 1-cocycle C on A with morphisms bialgebra morphisms commuting with the cocycles has a universal (initial) object (H, B_+), with H the Hopf algebra of (non-planar) rooted trees and the operator B_+ which grafts a product of rooted trees together to a new root as described in the next subsection.
While the higher $(n > 1)$ Hochschild cohomology of H vanishes [6], the 1-coclosedness of B_+ will turn out to be crucial for what follows.

LEMMA 6. *Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a free or free commutative graded algebra such that $A_0 = k\mathbb{I}$ and $(C_i)_{i \in I}$ a finite collection of injective linear endomorphism of A such that $\bigcap_i C_i(A) = \{0\}$ and each free generator y in degree n is the image under some C_i of* *an* $x \in A_{n-1}$ *for* $n \geq 1$. *Then there is a unique connected graded bialgebra structure* (Δ, ϵ, S) *on* A *such that the* C_i *are Hochschild 1-coclosed with respect to* Δ. *In particular,* A *is a Hopf algebra in a unique way.*

Proof. By (10) we have

$$\Delta \circ C_i = (id \otimes C_i) \circ \Delta + C_i \otimes \mathbb{I}. \tag{11}$$

$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$ by convention, so Δ is known on A_0. Now let y be a generator in A_{n+1}. By assumption there is a unique $x \in A_n$ such that $y = C_i x$. Assume Δ is known on x, then by (11) it is known on y. So we can uniquely extend Δ to an algebra homomorphism on A_{n+1}. From (11) it also follows inductively that Δ respects the grading in all orders:

$$\Delta(A_n) \subset \bigoplus_{k=0}^{n} A_k \otimes A_{n-k}.$$

For the coassociativity $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ we note that

$$\begin{aligned}
(\Delta \otimes id)\Delta C_i &= (\Delta \otimes id)((id \otimes C_i)\Delta + C_i \otimes \mathbb{I}) \\
&= (\Delta \otimes C_i)\Delta + \Delta C_i \otimes \mathbb{I} \\
&= (\Delta \otimes C_i)\Delta + (id \otimes C_i \otimes id)(\Delta \otimes \mathbb{I}) + C_i \otimes \mathbb{I} \otimes \mathbb{I} \\
&= (id \otimes id \otimes C_i)(\Delta \otimes id)\Delta + (id \otimes C_i \otimes id)(\Delta \otimes \mathbb{I}) + C_i \otimes \mathbb{I} \otimes \mathbb{I}.
\end{aligned}$$

On the other hand,

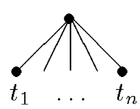
$$\begin{aligned}
(id \otimes \Delta)\Delta C_i &= (id \otimes \Delta)((id \otimes C_i)\Delta + C_i \otimes \mathbb{I}) \\
&= (id \otimes \Delta C_i)\Delta + C_i \otimes \mathbb{I} \otimes \mathbb{I} \\
&= id \otimes ((id \otimes C_i)\Delta + C_i \otimes \mathbb{I})\Delta + C_i \otimes \mathbb{I} \otimes \mathbb{I} \\
&= (id \otimes id \otimes C_i)(id \otimes \Delta)\Delta + (id \otimes C_i \otimes id)(\Delta \otimes \mathbb{I}) + C_i \otimes \mathbb{I} \otimes \mathbb{I}
\end{aligned}$$

which proves coassociativity by induction on the grading. Now setting $\epsilon(\mathbb{I}) = \mathbb{I}$ and $\epsilon = 0$ elsewhere finishes the proof. Note that any connected graded bialgebra is a Hopf algebra in a unique way.

2.2. **The Hopf algebra of rooted trees, relation to previous work.** In this section we collect well known results [3, 4, 9, 12] on Hopf algebra methods in momentum space renormalization which will turn out to be applicable to Epstein-Glaser renormalization after some simplifications and slight modifications. Let $\mathcal{H}$ be the free commutative algebra on rooted trees. We think of a rooted tree as a one-dimensional connected simply-connected simplicial complex with a distinguished vertex (the root), subject to the following equivalence relation: Two trees are equivalent iff the numbers of their vertices having fertility f and sitting in a given distance d from (i.e. the number of edges between them and) their roots match for all f and d. The choice of the root in the simply-connected simplicial complex induces an orientation of the edges: we draw the root on top and let the rest of the tree hang down. Vertices with no outgoing edges are called *leaves*, the remaining vertices are called *internal vertices*. The commutative product in $\mathcal{H}$ will be visualized as the disjoint union of trees. Monomials in $\mathcal{H}$ (disjoint unions of trees) will be called forests. We demand that the linear operator B_+ on $\mathcal{H}$, defined by

$$B_+(\mathbb{I}) = \bullet$$

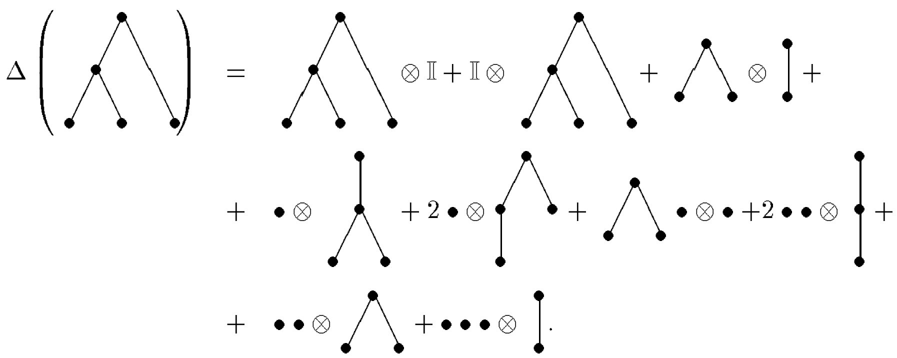
$B_+(t_1 \dots t_n) =$



is a Hochschild 1-cocycle, which makes $\mathcal{H}$ a Hopf algebra by virtue of Lemma 6 (a suitable grading is, for instance, the *weight grading*: the number of vertices). It is easy to see that the coproduct can be described as follows

$$\Delta(t) = \mathbb{I} \otimes t + t \otimes \mathbb{I} + \sum_{adm.c} P_c(t) \otimes R_c(t) \tag{12}$$

where the sum goes over all *admissible cuts* of the tree t. By a cut of t we mean a nonempty subset of the edges of t that are to be removed. The product of subtrees which "fall down" upon removal of those edges is called the *pruned part* and denoted $P_c(t)$, the part which remains connected with the root $R_c(t)$. Now a cut $c(t)$ is admissible, if for each leaf l of t it contains at most one edge on the path from l to the root. For instance,



$\mathcal{H}$ is obviously not cocommutative.

Let V be a unital ring with multiplication m_V. Given ring homomorphisms $\phi, \psi : \mathcal{H} \to V$, one can define their convolution product $\phi \star \psi : \mathcal{H} \to V$, $x \mapsto m_V(\phi \otimes \psi)\Delta x$, which is a ring homomorphism again. In particular, the antipode S is the inverse of $Id : \mathcal{H} \to \mathcal{H}$ with respect to this convolution product. Let Q be the linear endomorphism of $\mathcal{H} \otimes \mathcal{H}$ such that $Q(\mathbb{I} \otimes \mathbb{I}) = -\mathbb{I} \otimes \mathbb{I}$ and $Q = id \otimes P$ otherwise. So (up to the sign) Q is the projection onto $\mathcal{H} \otimes \ker \epsilon \oplus k\mathbb{I} \otimes k\mathbb{I}$. The shorthand notation $\phi \star_Q \psi := m_V(\phi \otimes \psi)Q\Delta$ will be useful. Now in any Hopf-algebra approach [9, 12, 3, 4] to perturbative quantum field theory, renormalization boils down to twisting the antipode which, (in any graded Hopf algebra) satisfies the recursive equation

$$S = -m(S \otimes id)Q\Delta = -S \star_Q id,$$

by a homomorphism $\phi : \mathcal{H} \to V$, called "Feynman rules", for example into a ring V of Laurent series (dimensional regularization) or formal power series (BPHZ), and a "renormalization scheme" $R : V \to V$ which delivers the counterterm. Explicitly this is

$$S_R^\phi := -Rm_V(S_R^\phi \otimes \phi)Q\Delta = -R(S_R^\phi \star_Q \phi) \tag{13}$$

which corresponds to Bogoliubov's $\bar{R}$ operation. While ϕ means application of unrenormalized Feynman rules, the renormalized expression is then given by

$$S_R^\phi \star \phi. \tag{14}$$

For details the reader is referred to [3]. In Epstein-Glaser renormalization, essentially the same happens, but in an easier way because no regularization is required. So the target ring V is most suitably chosen to be something like the tensor algebra of distributions on M, ϕ will then map a given "subdivergence situation" encoded in a rooted tree to the corresponding distribution in V. The meaning of ϕ is much easier to understand however if we give a somewhat different presentation of the Hopf algebra and define a different convolution product.

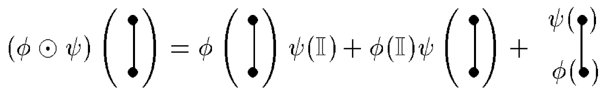
2.3. **The cut product and the Bogoliubov recursion.** We enlarge the Hopf algebra $\mathcal{H}$ to $\mathcal{H}^{\bullet,*}$ by allowing for two types of vertices: $\bullet$ and $*$. This yields two Hochschild 1-cocycles $B_{+\bullet}$ and B_{+*} depending on which type the newly adjoined root has. It is easy to see that the coproduct Δ which we endow $\mathcal{H}^{\bullet,*}$ with using $B_{+\bullet}$, B_{+*} and Lemma 6 has the same form (12) as in $\mathcal{H}$. Now let R be the algebra endomorphism of $\mathcal{H}^{\bullet,*}$ which changes the type of the root to $*$, whatever it was before. Once again we remark that all our algebra endomorphisms are supposed to be unital, so we won't specify their values at $\mathbb{I}$ explicitly.
Our aim is now to construct a new product $\odot$ called *cut product* of linear endomorphisms of $\mathcal{H}^{\bullet,*}$. The usual convolution product

$$(\phi,\psi)\mapsto\phi\star\psi=m(\phi\otimes\psi)\Delta$$

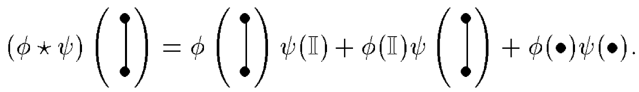
in $End_k(\mathcal{H})$ or $End_k(\mathcal{H}^{\bullet,*})$ has the disadvantage that, applied several times with the projection P onto the augmentation ideal, it gets rid of the structure of trees. For example, for any tree t there is an $n\in\mathbb{N}$ such that

$$P^{\star n}(t)=(P\star\ldots\star P)(t)=\text{polynomial in }\bullet$$

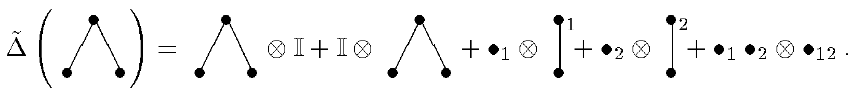
Our new product $(\phi\odot\psi)(t)$ is supposed to apply ϕ to $P_c(t)$ and ψ to $R_c(t)$ as well, but reassemble the tree afterwards instead of taking the disjoint union of pruned and root parts using m. For instance,



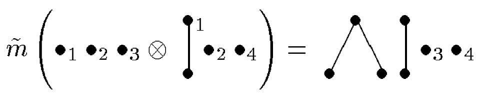
which should be compared to



This is however only possible for a rather small class of ϕ and ψ which do not change the trees too much. For example, ϕ is supposed to map trees to trees while ψ is not allowed to kill the vertices where something has been cut. We leave it to the reader to find the most general notion of those maps, because the only ones we need here are B_+ and id, P, R, where all this is possible in a rather trivial way.
Let $\tilde{\mathcal{H}}^{\bullet,*}$ be the Hopf algebra of trees as in $\mathcal{H}^{\bullet,*}$ with an additional decoration of the vertices by subsets of $\mathbb{N}$. There is an obvious forgetful projection $\pi:\tilde{\mathcal{H}}^{\bullet,*}\to\mathcal{H}^{\bullet,*}$ and an inclusion $j:\mathcal{H}^{\bullet,*}\to\tilde{\mathcal{H}}^{\bullet,*}$ decorating all vertices by the empty set. We lift one of the maps $\phi=B_+,id,P,R:\mathcal{H}^{\bullet,*}\to\mathcal{H}^{\bullet,*}$ to a map $\tilde{\phi}:\tilde{\mathcal{H}}^{\bullet,*}\to\tilde{\mathcal{H}}^{\bullet,*}$ by the prescription that newly created vertices are to be decorated by the empty set while the decorations of the old vertices is to be preserved.
We consider the coproduct $\tilde{\Delta}:\tilde{\mathcal{H}}^{\bullet,*}\to\tilde{\mathcal{H}}^{\bullet,*}\otimes\tilde{\mathcal{H}}^{\bullet,*}$ which does the same as Δ in $\mathcal{H}^{\bullet,*}$ but decorates roots in P_c and vertices in R_c separated by a cut by the same integers. Vertices not affected by cuts keep their decoration. For example,



The decoration has the only purpose to provide "cutting" resp. "glueing" information. We define a map $\tilde{m}:\tilde{\mathcal{H}}^{\bullet,*}\otimes\tilde{\mathcal{H}}^{\bullet,*}\to\tilde{\mathcal{H}}^{\bullet,*}$ which reconstructs the preimage of $\tilde{\Delta}$ by inserting edges between vertices that have been decorated by the same integers and discards the used decoration afterwards. So $\tilde{m}=\tilde{\Delta}^{-1}$ on the image of $\tilde{\Delta}$ and otherwise, if no decorations match, $\tilde{m}$ is the free multiplication $m_{\tilde{\mathcal{H}}^{\bullet,*}}$ of $\tilde{\mathcal{H}}^{\bullet,*}$. For instance,



$\tilde{m}$ is obviously not an algebra homomorphism.

DEFINITION 7. *Let* $\phi \in \{id, P, R\}$ *and* $\psi \in \{id, P, R, B_+\}$. *Then the linear endomorphism* $\phi \odot \psi$ *of* $\mathcal{H}^{\bullet,*}$,

$$(\phi \odot \psi) = \pi \tilde{m}(\tilde{\phi} \otimes \tilde{\psi})\tilde{\Delta} j$$

is called the cut product *of* ϕ *and* ψ.

It is easy to see that if ϕ and ψ are algebra endomorphisms, so is $\phi \odot \psi$.
As a shorthand notation, we will be using

$$(\phi \odot_Q \psi) := \pi \tilde{m}(\tilde{\phi} \otimes \tilde{\psi})\tilde{Q}\tilde{\Delta} j$$

where $\tilde{Q}$ is the obvious lift of Q to $\tilde{\mathcal{H}}^{\bullet,*}$. In analogy to the approach presented in the preceding subsection, we recursively define the twisted antipode by $\tilde{S}_R(\mathbb{I}) = \mathbb{I}$ and

$$\tilde{S}_R := -\tilde{R}\tilde{m}(\tilde{S}_R \odot_Q id)\tilde{\Delta} = -R\tilde{m}(\underbrace{-R\tilde{m}(\ldots \otimes id)\tilde{Q}\tilde{\Delta}}_{\tilde{S}_R} \otimes id)\tilde{\Delta}. \tag{15}$$

Let $S_R := \pi \tilde{S}_R j$. If one is willing to ignore the fact that $j\pi \neq id$, one can view S_R as defined by

$$S_R := -R(S_R \odot_Q id)$$

which might be a helpful motivation when compared to (13). Note that these are recursive definition indeed since $\tilde{Q}\tilde{\Delta}$ reduces the number of edges and $S_R(\mathbb{I}) = \mathbb{I}$ terminates the recursion. S_R will turn out to be the Bogoliubov $\bar{R}$-operation in the Epstein-Glaser framework. Note that R is an idempotent algebra endomorphism, so in particular a Rota-Baxter operator, hence S_R and $S_R \odot id$ are algebra endomorphism as well by a general inductive argument [10].

LEMMA 8. $(S_R \odot id)B_{+\bullet} = (id - R)B_{+\bullet}(S_R \odot id)$.

Proof. We use the Hochschild closedness of $B_{+\bullet}$,

$$\Delta B_{+\bullet} = (id \otimes B_{+\bullet})\Delta + B_{+\bullet} \otimes \mathbb{I} \tag{16}$$

Now we want to lift this equation to $\tilde{\mathcal{H}}^{\bullet,*}$ in order to apply it to $(S_R \odot id)$:

$$\tilde{\Delta}\tilde{B}_+ = C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} + \tilde{B}_{+\bullet} \otimes \mathbb{I} \tag{17}$$

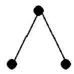
where C is a map $\tilde{\mathcal{H}}^{\bullet,*} \otimes \tilde{\mathcal{H}}^{\bullet,*} \to \tilde{\mathcal{H}}^{\bullet,*} \otimes \tilde{\mathcal{H}}^{\bullet,*}$ which decorates vertices affected by a cut by the same integer. This is the only adjustment we have to make when going from (16) to (17) because $\tilde{\Delta} j$ and $j\Delta$ differ only by decoration. This yields

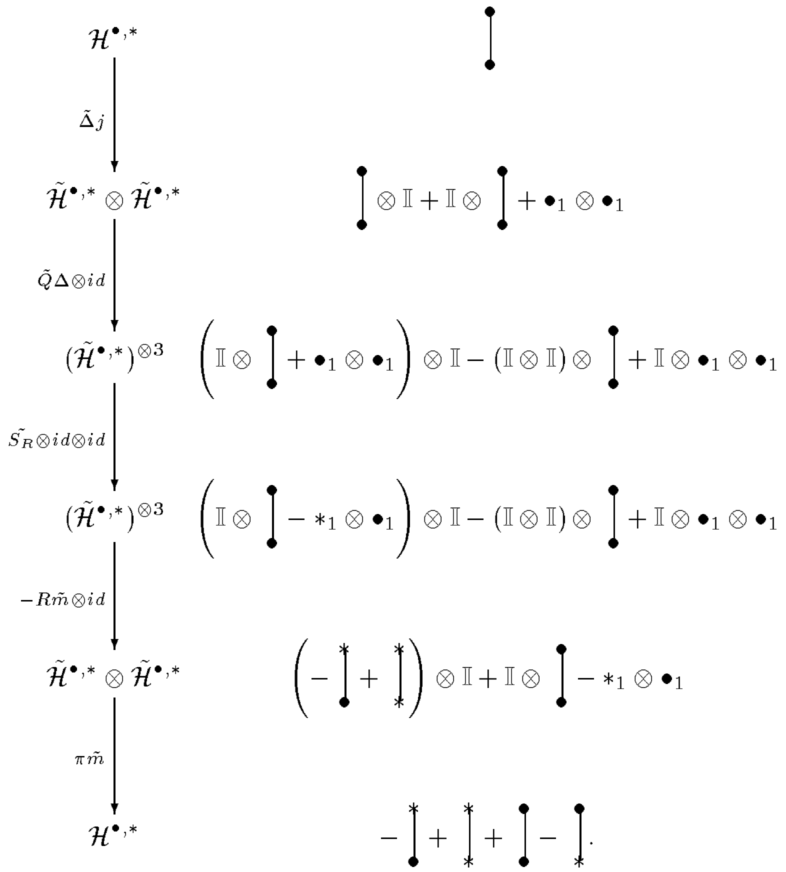
$$\begin{aligned}
(S_R \odot id)B_{+\bullet} &= \pi\tilde{m}(\tilde{S}_R \otimes id)\tilde{\Delta} j B_{+\bullet} = \pi\tilde{m}(\tilde{S}_R \otimes id)\tilde{\Delta}\tilde{B}_{+\bullet} j \\
&= \pi\tilde{m}(\tilde{S}_R \otimes id)(C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} + \tilde{B}_{+\bullet} \otimes \mathbb{I}) j \\
&= \pi\tilde{m}(\tilde{S}_R \otimes id)C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} j + \pi\tilde{S}_R\tilde{B}_{+\bullet} j \\
&= \pi\tilde{m}(\tilde{S}_R \otimes id)C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} j - \pi\tilde{R}\tilde{m}(\tilde{S}_R \otimes id)\tilde{Q}\tilde{\Delta}\tilde{B}_{+\bullet} j \\
&= \pi\tilde{m}(\tilde{S}_R \otimes id)C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} j - \pi\tilde{R}\tilde{m}(\tilde{S}_R \otimes id)C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} j \\
&= (id - R)\pi\tilde{m}(\tilde{S}_R \otimes id)C(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} j \\
&= (id - R)\pi\tilde{m}C(\tilde{S}_R \otimes id)(id \otimes \tilde{B}_{+\bullet})\tilde{\Delta} j \\
&= (id - R)\pi\tilde{m}C(id \otimes \tilde{B}_{+\bullet})(\tilde{S}_R \otimes id)\tilde{\Delta} j \\
&= (id - R)B_+(S_R \odot id).
\end{aligned}$$

where we have used (17), $Q(id \otimes B_{+\bullet}) = id \otimes B_{+\bullet}$, $Q(B_{+\bullet} \otimes \mathbb{I}) = 0$ which are obvious, and $(\tilde{S}_R \otimes id)C = C(\tilde{S}_R \otimes id)$ and $\tilde{m}C(id \otimes B_{+\bullet}) = B_{+\bullet}\tilde{m}$ which follow from the definition of C. This finishes the proof.

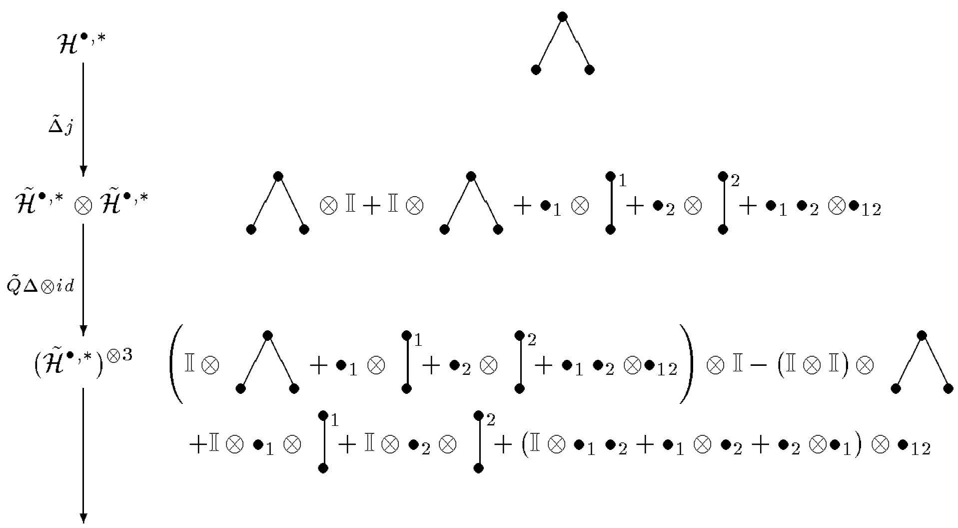
EXAMPLE 9. *We illustrate the action of the map*

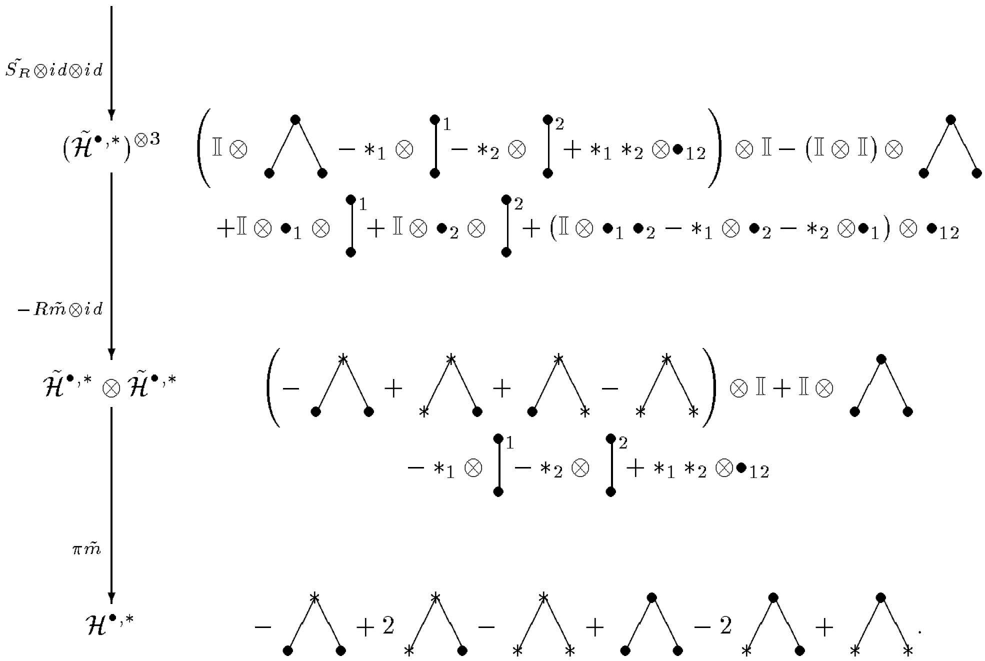
$$S_R \odot id = -\pi R\tilde{m}(-R\tilde{m}(\ldots \otimes id)\tilde{Q}\tilde{\Delta} \otimes id)\tilde{\Delta}j$$

on the two trees  *and*  .

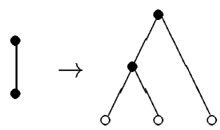


Note that we do not need to go into higher than the third tensor power of $\tilde{\mathcal{H}}^{\bullet,*}$ because $S_R(\mathbb{I}) = \mathbb{I}$ and $S_R(\bullet) = -*$ terminate the recursion. Now the second, less trivial example:



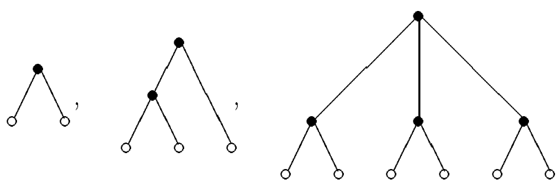


2.4. **An alternative presentation of the Hopf algebra.** In this subsection we will give a somewhat different presentation $\underline{\mathcal{H}}$ of $\mathcal{H}$ which will turn out to be more instructive for Epstein-Glaser renormalization. The basic idea is as follows: We consider a tree t of the preceding subsections as a trunk and let two more branches, called "hair" grow out of each leaf and one more branch out of each binary vertex of the trunk, which yields a tree $\underline{t}$ in the presentation $\underline{\mathcal{H}}$.

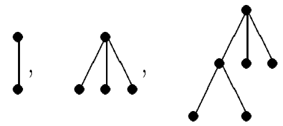


While the trunk will correspond to an abstract nest of subdivergences, the leaves of the hairy tree actually represent (some unordered set of) space-time points to which that particular subdivergence situation applies. For the reader's convenience, we visualize hair by $\circ$ and the trunk vertices by $\bullet$. This is only to make it easier to distinguish between the hairy trees in $\mathcal{H}$ and the bold trees in $\underline{\mathcal{H}}$, so we are not talking about trees with "two types of vertices" here. Now in order to underline the power of the Hochschild 1-cocycle and to illustrate Lemma 6, we will prescribe the cocycle and see what the coproduct looks like then.

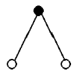
Let $\underline{\mathcal{H}}$ be the free commutative algebra generated by rooted trees the leaves of which descend exclusively from binary vertices. In other words each leaf must have one and only one sibling (which needs not necessarily to be a leaf too). For example, the trees



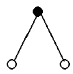
are in $\underline{\mathcal{H}}$ while

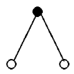
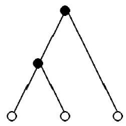


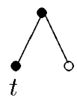
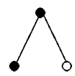
are not. The tree $\bullet$ consisting only of the root is *not* in $\underline{\mathcal{H}}$ by convention, so the most primitive generator is

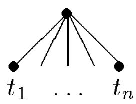


Now we demand $\underline{B}_+$ to act as follows:

$\underline{B}_+(\mathbb{I}) =$ 

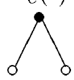
$\underline{B}_+($  $) =$ 

more generally, for a tree t, $\underline{B}_+(t) =$ , so t is grafted to a leaf of 

and for a forest, $\underline{B}_+(t_1 \ldots t_n) =$ 

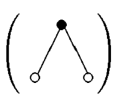
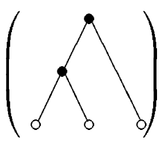
LEMMA 10. *There is a unique Hopf algebra structure* $(\underline{\Delta}, \underline{\epsilon}, \underline{S})$ *on* $\underline{\mathcal{H}}$ *such that* $\underline{B}_+$ *is Hochschild closed.* $\underline{\Delta}$ *is given on trees* $\underline{t}$ *by*

$$\underline{\Delta}(t) = \mathbb{I} \otimes \underline{t} + \underline{t} \otimes \mathbb{I} + \sum_{adm' c} \underline{P}_c(\underline{t}) \otimes \underline{R}_c(\underline{t})$$

where the definition of admissible cuts and $\underline{P}_c$, $\underline{R}_c$ *is as in the preceding subsections with the following differences:*
(i) cuts containing external edges (hair) are not admissible here
(ii) if a vertex v *of* $R_c(t)$ *has no more outgoing edges due to cut edges in* c, *that vertex* v *is to be replaced by*  *in* $\underline{R}_c(t)$.
If a vertex v *of* $R_c(t)$ *is left with only one outgoing edge due to cut edges in* c, *an additional branch is to be adjoint to* v *in* $\underline{R}_c(t)$.

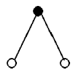
The map $\beta : \underline{\mathcal{H}} \to \mathcal{H}$, *given by removing all leaves and external edges, is an isomorphism of Hopf algebras.* β^{-1} *in turn replaces vertices with fertility* 0 *or* 1 *by binary vertices.*

Proof. First of all we note that whole $\underline{\mathcal{H}} - k\mathbb{I}$ is the iterated image of $\underline{B}_+$ and the multiplication. Moreover, $\underline{\mathcal{H}}$ is graded as an algebra by the augmentation degree, that is the number of times $n(\underline{t})$ that $\underline{B}_+$ has to be applied to $\mathbb{I}$ in order to get a given tree $\underline{t}$. Existence and uniqueness of $(\underline{\Delta}, \underline{\epsilon}, \underline{S})$ is then a consequence of Lemma 6. The remaining statements are easy to check using the map β, in particular

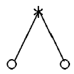
β  $= \bullet$, β  $=$ .

So $\underline{\mathcal{H}}$ is nothing but a somewhat different presentation of $\mathcal{H}$. Using β, we can transfer all notions developped in the preceding subsections to $\underline{\mathcal{H}}$ (underlining everything). Note that in $\underline{\mathcal{H}}^{\bullet,*}$ only *internal* vertices can have type $*$, in $\tilde{\underline{\mathcal{H}}}^{\bullet,*}$ only *internal* vertices are decorated etc.
From now on, we work only in the presentation $\underline{\mathcal{H}}$, $\underline{\mathcal{H}}^{\bullet,*}$, $\tilde{\underline{\mathcal{H}}}^{\bullet,*}$.

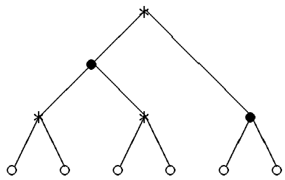
2.5. **Feynman rules and counterterms. Main result.** On the Hopf algebra level, a tree represents a certain subdivergence situation. Internal vertices of type $\bullet$ mean that the unrenormalized Feynman rules have been applied to the corresponding subdivergence, while $*$ denotes the corresponding counterterm. For example,



corresponds to the distribution $\dot{T}_2 = p_{1,2}T_1(x_1)T_1(x_2) + p_{2,1}T_1(x_2)T_1(x_1)$, defined on $M^2 - \Delta_2$. The tree

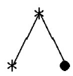


represents the counterterm $-W^*_{12}\dot{T}_2 = p_{1,2}T_1(x_1)T_1(x_2)+p_{2,1}T_1(x_2)T_1(x_1)$. We already know that their sum $(Id - W)\dot{T}_2 = T_2$ is the well defined Epstein-Glaser time-ordered product. In less trivial cases subtrees represent subdivergences, the root represents the overall divergence. For example



yields

$$W^*_{123456}p_{1234,56}p_{12,34}W^*_{12}p_{1,2}W^*_{34}p_{3,4}p_{5,6}\bigotimes_{i=1}^{6}T_1(x_i) + \text{suitable perm. of indices.}$$

Epstein-Glaser renormalization is essentially a binary operation since in each step only products $T(I)T(N - I)$ of *two* operator-valued distributions are considered. Indeed it is impossible to extend a distribution from $M^n - D_n$ onto the thin diagonal in $(M^n - D_n) \cup \Delta_n$ without extending it to the thicker diagonals, e. g. $\{x_i = x_j$ for some $i, j\}$ first. So we will be needing only binary trees.
Now let $\underline{t}$ be a binary tree in $\underline{\mathcal{H}}$, so all vertices are of type $\bullet$. We need a map which changes the types of internal vertices of $\underline{t}$ in all possible combinations and sums the resulting trees up in order to take care of the Bogoliubov recursion. This is essentially done by $\underline{S}_R \odot id$, as we have proved in Lemma 8, but in order to avoid overcounting, we will have to take care of symmetry factors which show up whenever the coproduct is applied. For example, in the second part of Example 9 we got 2  because two cuts, one on the "left", the other on the "right hand side", lead to the same result. We will compensate that by eventually dividing by symmetry factors.
Let $\mathcal{T}_1 = \{\mathbb{I}\}$ and for $n \geq 2$ let $\mathcal{T}_n$ be the subset of $\underline{\mathcal{H}}$ of binary trees with n leaves. Furthermore, let $T\mathcal{W}'(M) := T(Hom_{\mathbb{R}}(\mathcal{W}(M)), End(D))$ be the tensor algebra over $\mathbb{R}$ of $Hom_{\mathbb{R}}(\mathcal{W}(M), End(D))$ where the tensor product is the obvious generalization of the tensor product of scalar distributions:

(18) $$T_1 \otimes T_2 : f_1 \otimes f_2 \mapsto T_1(f_1)T_2(f_2),$$

and $FT\mathcal{W}'(M)$ the free commutative algebra generated by the (linear generators of) of $T\mathcal{W}'(M)$. So we have two different products in $FT\mathcal{W}'(M)$, $\otimes$ and a free multiplication, the latter of which is supposed to be the analogon of the disjoint union of trees. The reader might wish to review the notation for Epstein-Glaser time-ordered products in subsection 1.2 at this point. Note that $FT\mathcal{W}'(M)$ is essentially the tensor algebra of operator-valued distributions on M with an additional dependence on Wick-polynomials and an additional free product.

THEOREM 11 (Main result). *Let $\phi : \underline{\mathcal{H}}^{\bullet,*} \to FT\mathcal{W}'(M)$ be the homomorphism of free commutative algebras such that*

$$\phi(\mathbb{I}) = T_1$$

and for $1 \leq i \leq n-1$, $\underline{t}_i \in \mathcal{T}_i$, $\underline{t}_j \in \mathcal{T}_{n-i}$ and $f_1 \dots f_n \in \mathcal{W}(M)$

$$\phi(B_{+\bullet}(\underline{t}_i\underline{t}_j))(f_1 \otimes \dots \otimes f_n) =$$

$$= \frac{1}{S(\underline{t}_i, \underline{t}_j)} \sum_{I \subset N, |I|=i} p_{I,N-I}\phi(\underline{t}_i)(\otimes_{k\in I} f_k)\phi(\underline{t}_j)(\otimes_{l\in N-I} f_l) +$$

$$+ p_{N-I,I}\phi(\underline{t}_j)(\otimes_{l\in N-I} f_l)\phi(\underline{t}_i)(\otimes_{k\in I} f_k),$$

$$\phi(B_{+*}(\underline{t}_i\underline{t}_j)) \quad = \quad W^*_{1\dots n}\phi(B_{+\bullet}(\underline{t}_i\underline{t}_j)).$$

while $\phi(\underline{t}') = 0$ on non-binary trees $\underline{t}'$. The symmetry factor $S(\underline{t}_i, \underline{t}_j) = 2$ if the root of $\underline{t}_i$ has type $\bullet$ and $\underline{t}_j = \underline{R}(\underline{t}_i)$, and $S(\underline{t}_i, \underline{t}_j) = 1$ otherwise. Then, for $n \geq 2$, the n-th Epstein-Glaser time-ordered product is

$$T_n = \sum_{\underline{t}\in\mathcal{T}_n} \phi(\underline{S}_R \odot id)(\underline{t}). \tag{19}$$

REMARK 12 (Abuse of notation). *The map ϕ is not really well-defined because its image is not necessarily a distribution on M^n but rather on $M^n - D$, with $D \subset D_n$ a subset of the thick diagonal. But as a consequence of Theorem 4, the whole sum (19) is well-defined on M^n. It is possible though cumbersome to define $\phi_\epsilon := h_\epsilon\phi$ instead with h_ϵ a smooth cutoff function which vanishes in a euclidean ϵ-neighborhood of the thick diagonals D_n and is 1 outside, then taking the limit $T_n = \lim_{\epsilon\to 0}\sum_{\underline{t}}\phi_\epsilon(S_R \odot id)(\underline{t})$ to formalize the extension procedure.*

Proof of theorem. For $n = 2$ the statement is obviously true (take $t_1 = t_2 = \mathbb{I}$). Now for $\underline{t} \in \mathcal{T}_n$ it is easy to see that $(\phi\underline{R})(\underline{t}) = (W^*_{1\dots n}\phi)(\underline{t})$ (note that W^* is idempotent as well) and $\phi\underline{B}_{+\bullet}$ is the very sum of causal partitions times lower order time-ordered products that shows up in the equation

$$T_n = (Id - W^*_{1\dots n}) \sum_{\emptyset \subsetneq I \subsetneq N} p_{I,N-I}T(I)T(N-I) \tag{20}$$

which defines the time-ordered product T_n by Corollary 5. More precisely, the diagrams

$$\begin{array}{ccc} \underline{\mathcal{H}}^{\bullet,*} & \xrightarrow{\phi} & FT\mathcal{W}'(M) \\ \downarrow \underline{B}_{+\bullet} & & \downarrow \times\sum p_{I,N-I}\dots \\ \underline{\mathcal{H}}^{\bullet,*} & \xrightarrow{\phi} & FT\mathcal{W}'(M) \end{array} \qquad \begin{array}{ccc} \underline{\mathcal{H}}^{\bullet *} & \xrightarrow{\phi} & FT\mathcal{W}'(M) \\ \downarrow \underline{R} & & \downarrow W^* \\ \underline{\mathcal{H}}^{\bullet *} & \xrightarrow{\phi} & FT\mathcal{W}'(M) \end{array}$$

commute. This can be seen as follows:

$$\mathcal{T}_n = \underline{B}_+(\mathcal{T}_{n-1}) \cup \bigcup_{i=2}^{n-2} \underline{B}_+(\mathcal{T}_i\mathcal{T}_{n-i}) = \bigcup_{i=1}^{n-1} \underline{B}_+(\mathcal{T}_i\mathcal{T}_{n-i})$$

where we are overcounting since $\underline{\mathcal{H}}$ is commutative. So, using the Hochschild closedness of B_+, Lemma 8, and the fact that $\underline{S}_R \odot id$ is an algebra homomorphism, we get by induction on n, using the symmetry factor $S'(\underline{t}_i,\underline{t}_j) = 2$ if $\underline{t}_i = \underline{t}_j$ and $S'(\underline{t}_i,\underline{t}_j) = 1$ otherwise:

$$\begin{aligned}
T_n &= \sum_{\underline{t}\in\mathcal{T}_n} \phi(\underline{S}_R \odot id)(\underline{t}) \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{\underline{t}_i\in\mathcal{T}_i}\sum_{\underline{t}_j\in\mathcal{T}_{n-i}} S'(\underline{t}_i,\underline{t}_j)\phi(\underline{S}_R \odot id)\underline{B}_+(\underline{t}_i\underline{t}_j) \\
&= \frac{1}{2}\sum_{i=1}^{n-1}\sum_{\underline{t}_i\in\mathcal{T}_i}\sum_{\underline{t}_j\in\mathcal{T}_{n-i}} S'(\underline{t}_i,\underline{t}_j)\phi(id-\underline{R})\underline{B}_{+\bullet}(\underline{S}_R \odot id)(\underline{t}_i\underline{t}_j) \\
&= \frac{1}{2}(id - W^*_{1\ldots n})\sum_{i=1}^{n-1}\phi\underline{B}_{+\bullet}\left(\sum_{\underline{t}_i\in\mathcal{T}_i}(\underline{S}_R \odot id)(\underline{t}_i)\sum_{\underline{t}_j\in\mathcal{T}_{n-i}}(\underline{S}_R \odot id)(\underline{t}_j) + C\right) \\
&= \frac{1}{2}(id - W^*_{1\ldots n})\sum_{i=1}^{n-1}\sum_{I\subset N,|I|=i} p_{I,N-I}T(I)T(N-I) + p_{N-I,I}T(N-I)T(I) \\
&= (id - W^*_{1\ldots n})\sum_{\emptyset\subsetneq I\subsetneq N} p_{I,N-I}T(I)T(N-I)
\end{aligned}$$

where C is eventually $C = \sum_{\underline{t}}(\underline{S}_R \odot id)(\underline{t})(\underline{S}_R \odot id)(\underline{t})$ (for each $\underline{t}$ such that $\underline{t}_i = \underline{t}_j = \underline{t}$ has occurred in the sum, so in particular for all $\underline{t} \in \mathcal{T}_{n/2}$ if n is even) which cancels the symmetry factor $S(\underline{t}_i,\underline{t}_j)$ in the statement of the theorem. This finishes the proof.

While the preceding theorem just defines ϕ inductively by pushing it forward along $\underline{B}_+$, which is a perfectly natural way of doing so, one might also work out a non-recursive formula for ϕ as follows: Draw the tree, scan it from the top to the bottom and wherever you see an $*$, apply W^*. Then symmetrize in all possible ways.
Since $\underline{\mathcal{H}}$ is nothing but a different presentation of $\mathcal{H}$, one could also have stated the theorem in terms of trees of $\mathcal{H}$ from the very beginning, which would have required a grading on $\mathcal{H}$ that is isomorphic to the grading of $\underline{\mathcal{H}}$ by the number of external (hairy) vertices.
We encourage the reader to check that one could obtain the same result in *complete* analogy to momentum space renormalization (BPHZ, dimensional regularisation, etc.) [3, 4, 9, 12] as reviewed in subsection 2.2 by the following approach: Define the (unrenormalized) Feynman rules $\phi : \underline{H} \to FT\mathcal{W}'(M)$ as in Theorem 11, but let $R : FT\mathcal{W}'(M) \to FT\mathcal{W}'(M)$ be the idempotent free algebra endomorphism $T \mapsto W^*T$. Note that R is a Rota-Baxter operator. Then replace the cut product $\odot$ by the usual convolution product $\star$ again, and Theorem 11 yields

$$T_n = \sum_{\underline{t}\in\mathcal{T}_n}(\underline{S}_R^\phi \star \phi)(\underline{t})$$

which should be compared to (14).
The reason why we preferred the method of letting R act in the Hopf algebra $\mathcal{H}^{\bullet,*}$ and using $\odot$ is that like this we achieved a complete decoupling of the combinatorics (which happen in $\mathcal{H}^{\bullet,*}$) and the analysis (which happens in $T\mathcal{W}'(M)$), making it easier to see how the essential work is being done on the Hopf algebra side while the Feynman rules and counterterms $\phi : \underline{\mathcal{H}}^{\bullet,*} \to FT\mathcal{W}'(M)$ is a rather trivial map translating abstract subdivergence situations into the appropriate operator valued distributions.

3. Conclusions and outlook

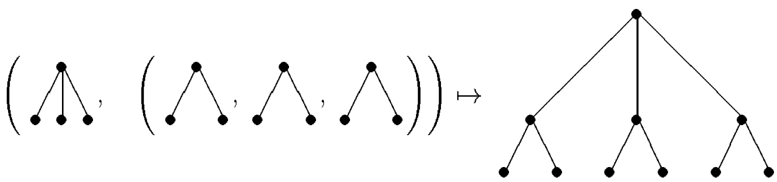
We have seen how Hopf algebras of rooted trees take care of the combinatorics of Epstein-Glaser renormalization. It is the twisted antipode S_R which provides a complete set of counterterms and formally solves the Bogoliubov recursion thanks to the fact that B_+ is Hochschild 1-coclosed. Although we do not claim that the statement of Theorem 11 makes actual calculations easier, it closes the gap between the mathematically rigorous Epstein-Glaser approach and the Hopf algebra picture so useful in momentum space.
We would like to mention another issue which seems to be intimately related to rooted trees in position space renormalization: to construct an analogy between extension of distributions from $M^n - D_n$ to M^n and *compactication* of the configuration space $M^n - D_n$ of n points in M. Indeed, we can already see how this leads to rooted trees if we look at the Fulton-MacPherson compactification of configuration spaces [7, 1, 8] defined as follows: Let M be a smooth manifold without boundaries. There is an obvious inclusion of the configuration space into a product of blowups,

$$M^n - D_n \hookrightarrow M^n \times \prod_{I \subset N, |I| \geq 2} Bl(M^{|I|}, \Delta_{|I|}) \tag{21}$$

where $Bl(M^i, \Delta_i)$ is the (differential-geometric) blowup of M^i along the thin diagonal Δ_i of M^i, i. e. the sphere bundle of the normal bundle of Δ_i. For the details, the reader is referred to [1]. The *Fulton-MacPherson compactification* $M[n]$ of $M^n - D_n$ is then the closure of $M^n - D_n$ upon this inclusion. Obviously $M[n]$ has only a chance to be compact if M is compact. Now a closer look at what happens in the right hand side of (21) when a sequence or rather a smooth curve approaches the thin diagonal in M^n leads to a nice description of $M[n]$ in terms of nested *screens* [7, 1]. In particular, it can be shown that there is a stratification of the manifold with corners $M[n]$,

$$M[n] = \bigcup_{S \in \mathcal{S}} M(S)$$

where $\mathcal{S}$ is the set of all nests of subsets of $N = \{1 \ldots n\}$ with at least 2 elements. Now nested sets are perfectly described by the rooted trees in $\mathcal{H}$. Moreover, if one restricts to $M = \mathbb{R}^k$ and replaces $M^n - D_n$ by the moduli space $\dot{F}_n := (M^n - D_n)/\mathrm{Aff}$ where Aff is the group generated by translations by $\mathbb{R}^k$ and dilatations by $\mathbb{R}^{\geq 0}$, it is rather easy to see that there is an *operad* structure behind the Fulton-MacPherson compactification $F[n]$ of $\dot{F}_n$ [8]. Operads arise in a natural way when rooted trees are grafted to each other:



So it seems tempting to explore possible relations between the operad μ_{FM} of Fulton-MacPherson compactification of the moduli spaces $\dot{F}_n$, the operad μ_{EG} which arises when the trees in $\mathcal{H}$ we used for Epstein-Glaser renormalization are grafted to each other, and finally the operad of Feynman graph insertions μ_{FG} [11]. μ_{FG} is closely related to the pre-Lie structure of Feynman graphs which is dual in a certain sense to the coproduct in $\mathcal{H}$. This might establish a true analogy between the Fulton-MacPherson compactification $M[n]$ of $M^n - D_n$ and the renormalization of time-ordered products in the sense of Epstein-Glaser.

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