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Noncentral extension of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superalgebra : supermultiplet of brane charges

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Abstract: We propose an extension of the su$(2,2|4)$ superalgebra to incorporate the $F1/D1$ string charges in type IIB string theory on the $AdS_5\times S^5$ background, or the electro-magnetic charges in the dual super Yang-Mills theory. With the charges introduced, the superalgebra inevitably undergoes a noncentral extension, as noted recently in [1]. After developing a group theoretical method of obtaining the noncentral extension, we show that the charges form a certain nonunitary representation of the original unextended superalgebra, subject to some constraints. We solve the constraints completely and show that, apart from the su$(2,2|4)$ generators, there exist 899 complex brane charges in the extended algebra. Explicitly we present all the super-commutation relations among them.

Keywords: noncentral extension, superalgebra, $AdS_5\times S^5$.

Contents

1. Introduction and summary

D-branes have been the cornerstone to understand the non-perturbative aspects of string/M-theory, and the "central" extensions of super Poincaré algebras provide a useful tool to analyze the possible supersymmetric brane configurations. The identification of the central charge with the magnetic charge of a monopole by Witten and Olive [2] was the first crucial step in discovering many exact results in the supersymmetric gauge theories. Also the celebrated Montonen-Olive duality conjecture [3] received the first support from the analysis on the central charges in four dimensional $\mathcal{N}=4$ super Yang-Mills theory by Osborn [4]. The method has been applied to the M-theory matrix model on the flat background [5] by Banks *et al.* [6], and further to the pp-wave matrix model [7] by Hyun and Shin [8] in order to identify all the extended objects. In the supersymmetric field theories the central charges appear as surface integrals in the expression of the anti-commutator of the supercharges, while in the matrix models they come as traces of a commutator.

Although much effort has been put to obtain the explicit expressions of the brane charges in various theories, it seems that few questions have been addressed to their central property, which can be, in principle, straightforwardly checked by investigating the supersymmetry transformations of them. Historically, the central property was "proven" in a more abstract way by Haag, Lopuszanski and Sohnius [9] studying the general structure of the Z_2-graded symmetry algebras or the superalgebras. The proof was based on the Coleman-Mandula theorem [10] on all the possible *symmetry generators* in the quantum field theories not having trivial scattering amplitudes. Now the essential motivation to question the central property of the brane charges comes from the fact that the brane charges are not symmetry generators nor Noether charges. Rather, they are topological living at the spatial infinity only, and hence free from the constraint by the Haag-Lopuszanski-Sohnius theorem. In fact, some straightforward manipulations indicate that the generic brane charges are not central.[1]

Recently, Peeters and Zamaklar considered some extensions of the AdS superalgebra as well as the pp-wave superalgebra, and noticed that the brane charges are inevitably noncentral [1] (see also [11] for the related work). The AdS superalgebras are superconformal algebras and bigger than the super Poincaré algebras. In particular, the anti-commutator of the supercharges gives rotational generators, M_{ab}, either for the anti-de-Sitter space or for the internal space, under which the brane charges, say Z_a, transform nontrivially. The crucial observation made in [1] follows from the Jacobi identity which contains two supercharges and one brane charge,

$$[\{Q,\,\bar{Q}\},\,Z_a]=\{Q,\,[\bar{Q},\,Z_a]\}+\{\bar{Q},\,[Q,\,Z_a]\}\,. \tag{1.1}$$

By contracting the spinorial indices of the supercharges properly, the left hand side can be set to be an infinitesimal rotation of the brane charge, which do not have any prior reason to vanish. Thus, from the right hand side, one can see the noncentral property of the brane charge. Namely the brane charge do not commute with the supercharges in general.

In the mathematics literature, all the semi-simple superalgebras were classified by Kac [12, 13] (see also a review by Nahm [14]), but the systematic study of the noncentral extensions of them remains an open problem. The primary goal of the present paper is to explore the possible noncentral extensions of the $AdS_5 \times S^5$ superalgebra or $\mathrm{su}(2,2|4)$. There are three types of BPS branes[2] one can add to the anti-commutator of the supercharges, as a starting point for the extension; $F1/D1$ and $D5/NS5$ charges combine into complex charges, while $D3$ charges are real-valued. After developing the general method for the extensions, we focus on the electro-magnetic ($F1/D1$) extension. We show that (i) the corresponding extension is unique, (ii) apart from the $\mathrm{su}(2,2|4)$ generators, there are 899 complex brane charges in the extended algebra, (iii) the brane charges form a supermultiplet of the original unextended superalgebra, and we present all the super-commutation

[1] Nevertheless all the known solitonic objects seem to have the vanishing values for the novel charges.

[2] For the discussion of the branes on the AdS space, see for example [15].

relations of them explicitly. Although in the paper we focus on the $AdS_5 \times S^5$ superalgebra, our method can be straightforwardly applied to other superalgebras.

The organization as well as the summary of the paper is as follows.

Section 2 is to set up our notations to write down the su$(2,2|4)$ superalgebra in a su$(2,2) \oplus$ su(4) covariant way. In section 3, we analyze the root structure of the su$(2,2|4)$ superalgebra and discuss its representations. In particular, we focus on a class of representations which are realized by the adjoint actions of the su$(2,2|4)$ generators. They are nonunitary and have finite dimensions.

Section 4 contains our main results. Motivated by the super Yang-Mills analysis, we define brane charges to be the space integrals of the total derivative terms or the surface integrals. We argue then that the super-commutator involving a brane charge is also a brane charge, and that all the brane charges super-commute with each other. Finally, by investigating all possible Jacobi identities, we find out that the brane charges form a "adjoint representation" of the original unextended superalgebra, su$(2,2|4)$, and that it is subject to some constraints. In subsection 4.2, the constraints are solved completely for the electro-magnetic extension. We identify the explicit structure of the supermultiplet and present all the nontrivial super-commutation relations.

In section 5, we describe how to translate our result to the four dimensional language: first for the extended $\mathcal{N}=4$ superconformal algebra and second for the extended $\mathcal{N}=4$ super Poincaré algebra. We also comment how our extended superalgebra acts on the quantum monopole states in the super Yang-Mills theory. For the purpose of the last section, in Appendix we relate the twelve dimensional gamma matrices to the four and ten dimensional ones.

2. $AdS_5 \times S^5$ superalgebra - unextended

This section is to set up the notations in order to write the $AdS_5 \times S^5$ superalgebra in terms of the $\mathrm{su}(2,2) \oplus \mathrm{su}(4)$ spinorial conventions. The main formulae are (2.18), (2.19), (2.20), (2.21), (2.22), (2.23).

2.1 Gamma matrices and spinors

In order to make the $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ isometry of $AdS_5 \times S^5$ geometry manifest, it is convenient to employ the twelve dimensional gamma matrices of spacetime signature $(--++++++++++)$, and write them in terms of two sets of six dimensional gamma matrices, $\{\gamma^\mu\}$, $\{\gamma^a\}$,

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \gamma^{(7)} \qquad \text{for} \quad \mu = 1,2,3,4,5,6 \\ \Gamma^a &= 1 \otimes \gamma^a \qquad \text{for} \quad a = 7,8,9,10,11,12\,. \end{aligned} \tag{2.1}$$

The two sets of the six dimensional gamma matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,, \qquad \gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}\,, \tag{2.2}$$

where $\eta^{\mu\nu} = \mathrm{diag}(--++++)$. With the choice

$$\gamma^{(7)} = i\gamma^1\gamma^2\cdots\gamma^6 = i\gamma^7\gamma^8\cdots\gamma^{12} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{2.3}$$

all the six dimensional gamma matrices are in the block diagonal form,

$$\gamma^\mu = \begin{pmatrix} 0 & \rho^\mu \\ \bar\rho^\mu & 0 \end{pmatrix}\,, \qquad \gamma^a = \begin{pmatrix} 0 & \rho^a \\ \bar\rho^a & 0 \end{pmatrix}\,, \tag{2.4}$$

satisfying the hermiticity conditions,

$$\bar\rho_\mu = \eta_{\mu\nu}\bar\rho^\nu = (\rho^\mu)^\dagger\,, \qquad \bar\rho^a = (\rho^a)^\dagger\,, \tag{2.5}$$

which ensure that Γ^1, Γ^2 are anti-hermitian and others hermitian.

If we further set all the 4×4 matrices, ρ^μ, $\bar\rho^\mu$ ρ^a, $\bar\rho^a$ to be anti-symmetric [16]

$$\begin{aligned} (\rho^\mu)_{\alpha\beta} &= -(\rho^\mu)_{\beta\alpha}\,, \qquad (\bar\rho^\mu)^{\alpha\beta} = -\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\rho^\mu)_{\gamma\delta}\,, \\ (\rho^a)_{\dot\alpha\dot\beta} &= -(\rho^a)_{\dot\beta\dot\alpha}\,, \qquad (\bar\rho^a)^{\dot\alpha\dot\beta} = -\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta\dot\gamma\dot\delta}(\rho^a)_{\dot\gamma\dot\delta}\,, \end{aligned} \tag{2.6}$$

the relations, $\mathrm{su}(2,2) \equiv \mathrm{so}(2,4)$ and $\mathrm{su}(4) \equiv \mathrm{so}(6)$, become manifest. That is, the indices $\alpha,\beta = 1,2,3,4$ and $\dot\alpha,\dot\beta = 1,2,3,4$ denote the fundamental representations of $\mathrm{su}(2,2)$ and $\mathrm{su}(4)$, respectively.

It follows that $\{\rho^\mu\}$ and $\{\bar\rho^\mu\}$ separately form bases for the anti-symmetric 4×4 matrices with the completeness relation,

$$\mathrm{tr}(\rho^\mu\bar\rho_\nu) = 4\delta^\mu{}_\nu\,, \qquad (\rho^\mu)_{\alpha\beta}(\bar\rho_\mu)^{\gamma\delta} = 2(\delta_\alpha{}^\delta\delta_\beta{}^\gamma - \delta_\beta{}^\delta\delta_\alpha{}^\gamma)\,. \tag{2.7}$$

On the other hand, the choice of chirality matrices in Eq.(2.3) implies that[3]

$$\rho^{[\mu}\bar{\rho}^{\nu}\rho^{\lambda]} = +i\tfrac{1}{6}\epsilon^{\mu\nu\lambda\sigma\tau\kappa}\rho_{[\sigma}\bar{\rho}_{\tau}\rho_{\kappa]}\,, \qquad \bar{\rho}^{[\mu}\rho^{\nu}\bar{\rho}^{\lambda]} = -i\tfrac{1}{6}\epsilon^{\mu\nu\lambda\sigma\tau\kappa}\bar{\rho}_{[\sigma}\rho_{\tau}\bar{\rho}_{\kappa]}\,, \tag{2.8}$$

so each of the sets $\rho^{[\mu}\bar{\rho}^{\nu}\rho^{\lambda]} \equiv \rho^{\mu\nu\lambda}$ or $\bar{\rho}^{[\mu}\rho^{\nu}\bar{\rho}^{\lambda]} \equiv \bar{\rho}^{\mu\nu\lambda}$ has only 10 independent components and forms a basis for symmetric 4×4 matrices,

$$\begin{aligned} \mathrm{tr}(\rho^{\mu\nu\lambda}\bar{\rho}_{\sigma\tau\kappa}) &= -i4\,\epsilon^{\mu\nu\lambda}{}_{\sigma\tau\kappa} - 24\delta^{[\mu}_{\sigma}\delta^{\nu}{}_{\tau}\delta^{\lambda]}_{\kappa}\,, \\ (\rho^{\mu\nu\lambda})_{\alpha\beta}(\bar{\rho}_{\mu\nu\lambda})^{\gamma\delta} &= -24(\delta_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta} + \delta_{\beta}{}^{\gamma}\delta_{\alpha}{}^{\delta})\,. \end{aligned} \tag{2.9}$$

Finally, $\{\rho^{\mu\nu} \equiv \frac{1}{2}(\rho^{\mu}\bar{\rho}^{\nu} - \rho^{\nu}\bar{\rho}^{\mu})\}$ or $\{\bar{\rho}^{\mu\nu} \equiv \frac{1}{2}(\bar{\rho}^{\mu}\rho^{\nu} - \bar{\rho}^{\nu}\rho^{\mu})\}$ forms an orthonormal basis for the general 4×4 traceless matrices,

$$\mathrm{tr}(\rho^{\mu\nu}\rho_{\lambda\kappa}) = 4(\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\lambda} - \delta^{\nu}{}_{\kappa}\delta^{\mu}{}_{\lambda})\,, \quad -\tfrac{1}{8}(\rho^{\mu\nu})_{\alpha}{}^{\beta}(\rho_{\mu\nu})_{\gamma}{}^{\delta} + \tfrac{1}{4}\delta_{\alpha}{}^{\beta}\delta_{\gamma}{}^{\delta} = \delta_{\alpha}{}^{\delta}\delta_{\gamma}{}^{\beta}\,, \tag{2.10}$$

satisfying

$$(\bar{\rho}^{\mu\nu})^{\alpha}{}_{\beta} = -(\rho^{\mu\nu})_{\beta}{}^{\alpha}\,. \tag{2.11}$$

Note that precisely the same equations as (2.7)-(2.11) hold for the so(6) gamma matrices, $\{\rho^{a},\, \bar{\rho}^{b}\}$ after replacing μ,ν, α,β by a,b, $\dot{\alpha},\dot{\beta}$, etc.

In the above choice of gamma matrices, the twelve dimensional charge conjugation matrices, $\mathcal{C}_{\pm}$, are given by

$$\pm(\Gamma^{M})^{T} = \mathcal{C}_{\pm}\Gamma^{M}\mathcal{C}_{\pm}^{-1}\,, \quad M = 1,2,\cdots,12, \qquad \mathcal{C}_{\pm} = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ \mp 1 & 0 \end{pmatrix}\,, \tag{2.12}$$

while the complex conjugate matrices, $\mathcal{A}_{\pm}$ read

$$\pm(\Gamma^{M})^{\dagger} = \mathcal{A}_{\pm}\Gamma^{M}\mathcal{A}_{\pm}^{-1}\,, \quad \mathcal{A}_{\pm} = \begin{pmatrix} A^{t} & 0 \\ 0 & \mp A \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\,, \quad A = -i\bar{\rho}_{12} = A^{\dagger} = A^{-1}\,. \tag{2.13}$$

In particular, for $\mu = 1,2,\cdots,6$, we have

$$(\rho^{\mu})^{\dagger} = -A\bar{\rho}^{\mu}A^{t} = \bar{\rho}_{\mu}\,, \qquad (\bar{\rho}^{\mu})^{\dagger} = -A^{t}\rho^{\mu}A = \rho_{\mu}\,. \tag{2.14}$$

Now if we define the twelve dimensional chirality operator as $\Gamma^{(13)} \equiv \gamma^{(7)} \otimes \gamma^{(7)}$, then

$$\{\Gamma^{(13)},\, \Gamma^{M}\} = 0\,, \qquad \mathcal{C}_{-} = \Gamma^{(13)}\mathcal{C}_{+}\,, \qquad \mathcal{A}_{-} = \Gamma^{(13)}\mathcal{A}_{+}\,. \tag{2.15}$$

In 2+10 dimensions it is possible to impose the Majorana-Weyl condition on spinors to have sixteen independent complex components which coincides with the number of supercharges in the $AdS_5 \times S^5$ superalgebra, su$(2,2|4)$. Up to the redefinition of the spinor by a phase factor, there are essentially two choices for the Majorana-Weyl condition depending on the chirality,

$$\Psi = \pm\Gamma^{(13)}\Psi\,, \quad \text{and} \quad \bar{\Psi} = \Psi^{\dagger}\mathcal{A}_{+} = \Psi^{t}\mathcal{C}_{+}\,. \tag{2.16}$$

[3]We put $\epsilon^{123456} = 1$ and "[ ]" denotes the standard anti-symmetrization with "strength one".

2.2 The special unitary Lie superalgebra, su$(2,2|4)$

Using the twelve dimensional convention, the special unitary Lie superalgebra, su$(2,2|4)$, reads simply

$$\{\mathcal{Q},\bar{\mathcal{Q}}\}=P_{13}\Big(i\Gamma^{\mu\nu}M_{\mu\nu}-i\Gamma^{ab}M_{ab}\Big)P_{13}\,, \tag{2.17}$$

where $\mathcal{Q}$ satisfies the Majorana-Weyl condition (2.16) and $P_{13}=\frac{1}{2}(1\pm\Gamma^{(13)})$.

Explicitly, the sixteen component supercharges, $Q_{\alpha\dot{\alpha}}$, carry only the chiral indices for su$(2,2)$ and su(4) so that the whole superalgebra, su$(2,2|4)$, reads

$$\{Q_{\alpha\dot{\alpha}}\,,\bar{Q}^{\beta\dot{\beta}}\}=i\delta_{\dot{\alpha}}{}^{\dot{\beta}}(\rho^{\mu\nu})_{\alpha}{}^{\beta}M_{\mu\nu}-i\delta_{\alpha}{}^{\beta}(\rho^{ab})_{\dot{\alpha}}{}^{\dot{\beta}}M_{ab}\,, \tag{2.18}$$

$$\{Q_{\alpha\dot{\alpha}}\,,Q_{\beta\dot{\beta}}\}=0\,,\qquad\qquad \{\bar{Q}^{\alpha\dot{\alpha}},\,\bar{Q}^{\beta\dot{\beta}}\}=0\,, \tag{2.19}$$

$$[M_{\mu\nu},Q_{\alpha\dot{\alpha}}]=(i\tfrac{1}{2}\rho_{\mu\nu})_{\alpha}{}^{\beta}Q_{\beta\dot{\alpha}}\,,\qquad [M_{\mu\nu},\bar{Q}^{\alpha\dot{\alpha}}]=\bar{Q}^{\beta\dot{\alpha}}(-i\tfrac{1}{2}\rho_{\mu\nu})_{\beta}{}^{\alpha}\,, \tag{2.20}$$

$$[M_{ab},Q_{\alpha\dot{\alpha}}]=(i\tfrac{1}{2}\rho_{ab})_{\dot{\alpha}}{}^{\dot{\beta}}Q_{\alpha\dot{\beta}}\,,\qquad [M_{ab},\bar{Q}^{\alpha\dot{\alpha}}]=\bar{Q}^{\alpha\dot{\beta}}(-i\tfrac{1}{2}\rho_{ab})_{\dot{\beta}}{}^{\dot{\alpha}}\,, \tag{2.21}$$

$$[M_{\mu\nu},M_{\kappa\lambda}]=i(\eta_{\mu\kappa}M_{\nu\lambda}-\eta_{\mu\lambda}M_{\nu\kappa}-\eta_{\nu\kappa}M_{\mu\lambda}+\eta_{\nu\lambda}M_{\mu\kappa})\,, \tag{2.22}$$

$$[M_{ab},M_{cd}]=i(\delta_{ac}M_{bd}-\delta_{ad}M_{bc}-\delta_{bc}M_{ad}+\delta_{bd}M_{ac})\,, \tag{2.23}$$

where $\bar{Q}^{\alpha\dot{\alpha}}\equiv A^{\alpha}{}_{\beta}(Q^{\dagger})^{\beta\dot{\alpha}}$, and all the bosonic generators are hermitian, $(M_{\mu\nu})^{\dagger}=M_{\mu\nu}$, $(M_{ab})^{\dagger}=M_{ab}$. A few remarks are in order. The relative sign difference for the so$(2,4)$ and so(6) generators appearing in (2.18) is crucial for consistency, as required from the Jacobi identity involving $[Q_{\alpha\dot{\alpha}},\{Q_{\beta\dot{\beta}},\bar{Q}^{\gamma\dot{\gamma}}\}]$. However, the overall sign as well as the chirality choices, namely whether $\rho^{12}\rho^{34}\rho^{56}$ is $+1$ or -1, are solely matter of conventions.[4] Firstly the over all sign can be flipped by rewriting the superalgebra in terms of the conjugate supercharges, ($Q'=\bar{Q}^{t}$, $\bar{Q}'=Q^{t}=(Q')^{\dagger}A$) [17]. The equivalence between the different so$(2,4)$, so(6) chirality choices becomes clear when we rewrite the superalgebra by the su$(2,2)$, su(4) generators,[5]

$$\begin{aligned}
&T_{(2,2)}=-i\tfrac{1}{4}\bar{\rho}^{\mu\nu}M_{\mu\nu}\,,\qquad T^{\dagger}_{(2,2)}=AT_{(2,2)}A\,,\qquad \mathrm{tr}\,T_{(2,2)}=0\,,\\
&T_{\mathrm{su}(4)}=-i\tfrac{1}{4}\bar{\rho}^{ab}M_{ab}\,,\qquad T^{\dagger}_{\mathrm{su}(4)}=T_{\mathrm{su}(4)}\,,\qquad \mathrm{tr}\,T_{\mathrm{su}(4)}=0\,.
\end{aligned} \tag{2.24}$$

From the completeness relation (2.10) which does not depend on the chiralities, we get the following expression for the su$(2,2|4)$ algebra regardless of the chirality choices,

$$[T_{(2,2)}{}^{\alpha}{}_{\beta}\,,\,Q_{\gamma\dot{\gamma}}]=\delta^{\alpha}{}_{\gamma}Q_{\beta\dot{\gamma}}-\tfrac{1}{4}\delta^{\alpha}{}_{\beta}Q_{\gamma\dot{\gamma}}\,,\quad [T_{(2,2)}{}^{\alpha}{}_{\beta}\,,\,T_{(2,2)}{}^{\gamma}{}_{\delta}]=\delta^{\alpha}{}_{\delta}T_{(2,2)}{}^{\gamma}{}_{\beta}-\delta^{\gamma}{}_{\beta}T_{(2,2)}{}^{\alpha}{}_{\delta}\,, \tag{2.25}$$

[4] The freedom for different chiral choices reflects two different Majorana-Weyl conditions in $2+10$ dimensions, (2.16).

[5] From (2.13), $A=A^{\dagger}$. In fact, as explained in the next section (3.7), one can set $A=\mathrm{diag}(-1,-1,+1,+1)$.

$$[T_{\mathrm{su}(4)}{}^{\dot\alpha}{}_{\dot\beta}\,,\,Q_{\gamma\dot\gamma}]=\delta^{\dot\alpha}{}_{\dot\gamma}Q_{\gamma\dot\beta}-\tfrac{1}{4}\delta^{\dot\alpha}{}_{\dot\beta}Q_{\gamma\dot\gamma}\,,\quad [T_{\mathrm{su}(4)}{}^{\dot\alpha}{}_{\dot\beta}\,,\,T_{\mathrm{su}(4)}{}^{\dot\gamma}{}_{\dot\delta}]=\delta^{\dot\alpha}{}_{\dot\delta}T_{\mathrm{su}(4)}{}^{\dot\gamma}{}_{\dot\beta}-\delta^{\dot\gamma}{}_{\dot\beta}T_{\mathrm{su}(4)}{}^{\dot\alpha}{}_{\dot\delta}\,. \tag{2.26}$$

Essentially the different chiral choices are equivalent to each other up to the redefinition of the so$(2,4)$, so(6) generators through (2.24), and (2.10), i.e. $T_{(2,2)}=-i\frac{1}{4}\bar\rho^{\mu\nu}M_{\mu\nu}=-i\frac{1}{4}\bar\rho'^{\mu\nu}M'_{\mu\nu}$.

3. u$(1)_Y$ extended superalgebra and its root structure

Before we proceed further to obtain the noncentral extensions of the *AdS* superalgebra, here as an intermediate stage we consider the inclusion of an additional or "bonus" u$(1)_Y$ charge into the su$(2,2|4)$ superalgebra which acts as an automorphism of the supergroup. This u$(1)_Y$ symmetry appears both in the IIB supergravity and in the analysis of the four dimensional $\mathcal{N}=4$ superconformal group. In IIB supergravity the u$(1)_Y$ symmetry rotates the two chiral spinors (see e.g. [18]), while on superspace the superconformal group is defined in terms of the superspace coordinate transformations so that the u$(1)_Y$ phase rotation of the odd coordinates is a part of the superconformal transformations [19]. However the stringy α' correction to the supergravity violates the u$(1)_Y$ symmetry [20, 21], and in $\mathcal{N}=4$ super Yang-Mills theory *more than three*-point correlation functions do not respect the u$(1)_Y$ symmetry generically [22, 23, 24, 25]. Nevertheless, in our analysis of the extended superalgebra, the u$(1)_Y$ charge will always act as an automorphism to the superalgebra extended or unextended so that one can safely switch it off any time. The main advantage to include the u$(1)_Y$ charge is to reduce the number of the fermionic simple roots from two to one.

3.1 Inclusion of a u$(1)_Y$ symmetry

The additional u$(1)_Y$ charge assigns quantum numbers $+1/2$, $-1/2$ to the supercharges, $Q_{\alpha\dot\alpha}$, $\bar Q^{\alpha\dot\alpha}$,

$$[T_{\mathrm{u}(1)},Q_{\alpha\dot\alpha}]=+\tfrac{1}{2}\,,\qquad [T_{\mathrm{u}(1)},\bar Q^{\alpha\dot\alpha}]=-\tfrac{1}{2}\,,\qquad T^{\dagger}_{\mathrm{u}(1)}=T_{\mathrm{u}(1)}\,, \tag{3.1}$$

which reflect the u$(1)_Y$ phase rotation of the chiral spinors. One of the bosonic subalgebras, su(4), is now extended to u(4),

$$T_{\mathrm{u}(4)}{}^{\dot\alpha}{}_{\dot\beta}=T_{\mathrm{su}(4)}{}^{\dot\alpha}{}_{\dot\beta}+\tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\,T_{\mathrm{u}(1)}\,, \tag{3.2}$$

satisfying

$$[T_{\mathrm{u}(4)}{}^{\dot\alpha}{}_{\dot\beta}\,,Q_{\gamma\dot\gamma}]=\delta^{\dot\alpha}{}_{\dot\gamma}Q_{\gamma\dot\beta}\,,\qquad [T_{\mathrm{u}(4)}{}^{\dot\alpha}{}_{\dot\beta}\,,\,T_{\mathrm{u}(4)}{}^{\dot\gamma}{}_{\dot\delta}]=\delta^{\dot\alpha}{}_{\dot\delta}T_{\mathrm{u}(4)}{}^{\dot\gamma}{}_{\dot\beta}-\delta^{\dot\gamma}{}_{\dot\beta}T_{\mathrm{u}(4)}{}^{\dot\alpha}{}_{\dot\delta}\,. \tag{3.3}$$

The additional u$(1)_Y$ charge commutes with all the bosonic generators so that the resulting superalgebra is a semi-direct sum of su$(2,2|4)$ and u$(1)_Y$, or su$(2,2|4)\oplus_{\mathrm{semi}}u(1)_Y$.

3.2 The root structure of $\mathbf{su}(2,2|4)\oplus_{\rm semi}\mathbf{u}(1)_Y$

In this subsection we analyze the root structure of $\mathrm{su}(2,2|4)\oplus_{\rm semi}\mathrm{u}(1)_Y$. We first start with the following 16×16 representation of the bosonic part, $\mathrm{su}(2,2)\oplus\mathrm{su}(4)\oplus\mathrm{u}(1)_Y$, acting on spinors,

$$\Big(R(M_{\mu\nu})\,,\ R(M_{ab}\,)\,,\ R(T_{\rm u(1)})\Big)=\Big(-i\tfrac{1}{2}\bar{\rho}_{\mu\nu}\otimes 1\,,\ 1\otimes -i\tfrac{1}{2}\bar{\rho}_{ab}\,,\ 1\otimes\tfrac{1}{2}\Big)\,, \qquad (3.4)$$

which are orthonormal and satisfy the reality condition,

$$\mathrm{Tr}(R_I^\dagger R_J)=4\delta_{IJ}\,,\qquad R_I^\dagger=(A\otimes 1)R_I(A\otimes 1)\,,\qquad I,J=1,2,\cdots,31\,. \qquad (3.5)$$

The above representation for $\mathrm{su}(2,2)$ is nonunitary. This is unavoidable in order to have a finite dimensional representation for the noncompact algebra, since any unitary representation of a noncompact algebra is always infinite dimensional.

Our choice of the Cartan subalgebra is

$$\vec{H}=(T_{\rm u(1)},\,M_{12},\,M_{34},\,M_{56},\,M_{78},\,M_{9\,10},\,M_{11\,12})\,. \qquad (3.6)$$

Using the SU(4) symmetry, $\rho_\mu\rightarrow U\rho_\mu U^T$, $UU^\dagger=1$, which preserves the anti-symmetric property (2.6) of ρ_μ, we can take the representation of the Cartan subalgebra in a diagonal form. Adopting the bra and ket notations we set

$$\begin{aligned}
R(M_{12})&=\tfrac{1}{2}\,(\,-|1\rangle\langle1|-|2\rangle\langle2|+|3\rangle\langle3|+|4\rangle\langle4|\,)\otimes 1=\tfrac{1}{2}A\otimes 1\,,\\
R(M_{34})&=\tfrac{1}{2}\,(\,-|1\rangle\langle1|+|2\rangle\langle2|-|3\rangle\langle3|+|4\rangle\langle4|\,)\otimes 1\,,\\
R(M_{56})&=\tfrac{1}{2}\,(\,-|1\rangle\langle1|+|2\rangle\langle2|+|3\rangle\langle3|-|4\rangle\langle4|\,)\otimes 1\,,\\
R(M_{78})&=1\otimes\tfrac{1}{2}\,(\,-|1\rangle\langle1|-|2\rangle\langle2|+|3\rangle\langle3|+|4\rangle\langle4|\,)\,,\\
R(M_{9\,10})&=1\otimes\tfrac{1}{2}\,(\,-|1\rangle\langle1|+|2\rangle\langle2|-|3\rangle\langle3|+|4\rangle\langle4|\,)\,,\\
R(M_{11\,12})&=1\otimes\tfrac{1}{2}\,(\,-|1\rangle\langle1|+|2\rangle\langle2|+|3\rangle\langle3|-|4\rangle\langle4|\,)\,.
\end{aligned} \qquad (3.7)$$

All the bosonic positive roots and their representations are then given by

$$\begin{aligned}
R(\mathcal{E}_x)&=|2\rangle\langle1|\otimes 1\,, & x&=(0,0,1,1,0,0,0)\,,\\
R(\mathcal{E}_s)&=|3\rangle\langle2|\otimes 1\,, & s&=(0,1,-1,0,0,0,0)\,,\\
R(\mathcal{E}_y)&=|4\rangle\langle3|\otimes 1\,, & y&=(0,0,1,-1,0,0,0)\,,\\
R(\mathcal{E}_{s+x})&=|3\rangle\langle1|\otimes 1\,, & s+x&=(0,1,0,1,0,0,0)\,,\\
R(\mathcal{E}_{y+s})&=|4\rangle\langle2|\otimes 1\,, & y+s&=(0,1,0,-1,0,0,0)\,,\\
R(\mathcal{E}_{y+s+x})&=|4\rangle\langle1|\otimes 1\,, & y+s+x&=(0,1,1,0,0,0,0)\,,
\end{aligned} \qquad (3.8)$$

$$
\begin{aligned}
&R(\mathcal{E}_u) = 1\otimes|2\rangle\langle 1|\,, && u=(0,0,0,0,0,1,1)\,,\\
&R(\mathcal{E}_v) = 1\otimes|3\rangle\langle 2|\,, && v=(0,0,0,0,1,-1,0)\,,\\
&R(\mathcal{E}_w) = 1\otimes|4\rangle\langle 3|\,, && w=(0,0,0,0,0,1,-1)\,,\\
&R(\mathcal{E}_{v+u}) = 1\otimes|3\rangle\langle 1|\,, && v+u=(0,0,0,0,1,0,1)\,,\\
&R(\mathcal{E}_{w+v}) = 1\otimes|4\rangle\langle 2|\,, && w+v=(0,0,0,0,1,0,-1)\,,\\
&R(\mathcal{E}_{w+v+u}) = 1\otimes|4\rangle\langle 1|\,, && w+v+u=(0,0,0,0,1,1,0)\,,
\end{aligned}
\tag{3.9}
$$

where x, y, s and u, v, w are respectively the su$(2,2)$ and su(4) simple roots. For a given root, χ, the corresponding negative root and its representation follow simply from

$$
\mathcal{E}_{-\chi} = \mathcal{E}_\chi^\dagger\,, \qquad R(\mathcal{E}_{-\chi}) = (A\otimes 1)\,R(\mathcal{E}_\chi)^\dagger\,(A\otimes 1)\,, \tag{3.10}
$$

so that

$$
R(\mathcal{E}_{-\chi}) = \begin{cases} -R(\mathcal{E}_\chi)^\dagger & \text{for } \ \chi\in\{s,\ s+x,\ y+s,\ y+s+x\} \\ +R(\mathcal{E}_\chi)^\dagger & \text{otherwise} \end{cases}\,. \tag{3.11}
$$

Note that $\{s,\ s+x,\ y+s,\ y+s+x\}$ spans the noncompact directions of su$(2,2)$.

Just like R_I in (3.5), $R(\vec{H})$, $R(\mathcal{E}_+)$, $R(\mathcal{E}_-)$ are also orthonormal. This implies that those two are related by the unitary transformation. In particular, the objects appearing in the anti-commutator, $\{\bar{Q}, Q\}$, read

$$
\begin{aligned}
\tfrac{1}{2}R(M^{\mu\nu})M_{\mu\nu} = \tfrac{1}{2}R(M_{\mu\nu})^\dagger M_{\mu\nu} = {}& R(M_{12})M_{12} + R(M_{34})M_{34} + R(M_{56})M_{56} \\
&+ \sum_{\chi\in\Delta^+_{(2,2)}} \Big(R(\mathcal{E}_\chi)^\dagger\mathcal{E}_\chi + R(\mathcal{E}_{-\chi})^\dagger\mathcal{E}_{-\chi}\Big)\,, \\
\tfrac{1}{2}R(M^{ab})M_{ab} = \tfrac{1}{2}R(M_{ab})^\dagger M_{ab} = {}& R(M_{78})M_{78} + R(M_{9\,10})M_{9\,10} + R(M_{11\,12})M_{11\,12} \\
&+ \sum_{\chi\in\Delta^+_{\mathrm{su}(4)}} \Big(R(\mathcal{E}_\chi)^\dagger\mathcal{E}_\chi + R(\mathcal{E}_{-\chi})^\dagger\mathcal{E}_{-\chi}\Big)\,,
\end{aligned}
\tag{3.12}
$$

where $\Delta^+_{(2,2)}$ and $\Delta^+_{\mathrm{su}(4)}$ denote the sets of all the su$(2,2)$ and su(4) positive roots respectively. In fact, for the given set of orthonormal matrices, $R(\vec{H})$, $R(\mathcal{E}_+)$, $R(\mathcal{E}_-)$, (3.7), (3.8), (3.9), (3.11), the formulae above *define* all the roots, $\mathcal{E}_\pm$, in terms of the hermitian generators, $M_{\mu\nu}, M_{ab}$, and make sure that $R(\mathcal{E}_\pm)$ *are* the representations for them.

In terms of the Cartan subalgebra and su(2,2) ⊕ su(4) roots, $\chi \in \Delta^+_{(2,2)} \cup \Delta^+_{\rm su(4)}$, the $\mathrm{u}(1)_Y$ extended $AdS_5 \times S^5$ superalgebra, $\mathrm{su}(2,2|4) \oplus_{\rm semi} \mathrm{u}(1)_Y$, reads

$$
\begin{array}{l}
[\vec{H}, \mathcal{E}_\chi] = \chi \mathcal{E}_\chi\,, \qquad [\vec{H}, \mathcal{E}_{-\chi}] = -\chi \mathcal{E}_{-\chi}\,, \\[2ex]
[\mathcal{E}_\chi, \mathcal{E}_{-\chi}] = \begin{cases} -\chi\cdot\vec{H} & \text{for } \ \chi \in \{s,\, s+x,\, y+s,\, y+s+x\} \\[2ex] +\chi\cdot\vec{H} & \text{otherwise} \end{cases} \,,
\end{array}
\tag{3.13}
$$

$$
\begin{array}{ll}
[\mathcal{E}_s, \mathcal{E}_x] = \mathcal{E}_{s+x}\,, & [\mathcal{E}_y, \mathcal{E}_s] = \mathcal{E}_{y+s}\,, \\[2ex]
[\mathcal{E}_y, \mathcal{E}_{s+x}] = [\mathcal{E}_{y+s}, \mathcal{E}_x] = \mathcal{E}_{y+s+x}\,, & [\mathcal{E}_x, \mathcal{E}_y] = 0\,, \\[2ex]
[\mathcal{E}_v, \mathcal{E}_u] = \mathcal{E}_{v+u}\,, & [\mathcal{E}_w, \mathcal{E}_v] = \mathcal{E}_{w+v}\,, \\[2ex]
[\mathcal{E}_w, \mathcal{E}_{v+u}] = [\mathcal{E}_{w+v}, \mathcal{E}_u] = \mathcal{E}_{w+v+u}\,, & [\mathcal{E}_u, \mathcal{E}_w] = 0\,,
\end{array}
\tag{3.14}
$$

$$
\begin{array}{ll}
[\vec{H}, Q_{\alpha\dot\alpha}] = Q_{\beta\dot\beta} R(\vec{H})^{\beta\dot\beta}{}_{\alpha\dot\alpha}\,, & [\vec{H}, \bar{Q}^{\alpha\dot\alpha}] = -R(\vec{H})^{\alpha\dot\alpha}{}_{\beta\dot\beta} \bar{Q}^{\beta\dot\beta}\,, \\[2ex]
[\mathcal{E}_{\pm\chi}, Q_{\alpha\dot\alpha}] = Q_{\beta\dot\beta} R(\mathcal{E}_{\pm\chi})^{\beta\dot\beta}{}_{\alpha\dot\alpha}\,, & [\mathcal{E}_{\pm\chi}, \bar{Q}^{\alpha\dot\alpha}] = -R(\mathcal{E}_{\pm\chi})^{\alpha\dot\alpha}{}_{\beta\dot\beta} \bar{Q}^{\beta\dot\beta}\,,
\end{array}
\tag{3.15}
$$

and

$$
\begin{array}{l}
\{\bar{Q}^{\alpha\dot\alpha}, Q_{\beta\dot\beta}\} = 2\delta^{\dot\alpha}{}_{\dot\beta} \begin{pmatrix} f_1 & \mathcal{E}_x & \mathcal{E}_{s+x} & \mathcal{E}_{y+s+x} \\ \mathcal{E}_{-x} & f_2 & \mathcal{E}_s & \mathcal{E}_{y+s} \\ -\,\mathcal{E}_{-s-x} & -\,\mathcal{E}_{-s} & f_3 & \mathcal{E}_y \\ -\mathcal{E}_{-y-s-x} & -\,\mathcal{E}_{-y-s} & \mathcal{E}_{-y} & f_4 \end{pmatrix}^{\alpha}{}_{\beta} \\[6ex]
\qquad\qquad\qquad - 2\delta^{\alpha}{}_{\beta} \begin{pmatrix} f_5 & \mathcal{E}_u & \mathcal{E}_{v+u} & \mathcal{E}_{w+v+u} \\ \mathcal{E}_{-u} & f_6 & \mathcal{E}_v & \mathcal{E}_{w+v} \\ \mathcal{E}_{-v-u} & \mathcal{E}_{-v} & f_7 & \mathcal{E}_w \\ \mathcal{E}_{-w-v-u} & \mathcal{E}_{-w-v} & \mathcal{E}_{-w} & f_8 \end{pmatrix}^{\dot\alpha}{}_{\dot\beta} \,,
\end{array}
\tag{3.16}
$$

where the Cartan subalgebra is organized as

$$
\begin{aligned}
f_1 &= \tfrac{1}{2}(-M_{12}-M_{34}-M_{56})\,, & f_2 &= \tfrac{1}{2}(-M_{12}+M_{34}+M_{56})\,,\\
f_3 &= \tfrac{1}{2}(M_{12}-M_{34}+M_{56})\,, & f_4 &= \tfrac{1}{2}(M_{12}+M_{34}-M_{56})\,,\\
f_5 &= \tfrac{1}{2}(-M_{78}-M_{9\,10}-M_{11\,12})\,, & f_6 &= \tfrac{1}{2}(-M_{78}+M_{9\,10}+M_{11\,12})\,,\\
f_7 &= \tfrac{1}{2}(M_{78}-M_{9\,10}+M_{11\,12})\,, & f_8 &= \tfrac{1}{2}(M_{78}+M_{9\,10}-M_{11\,12})\,.
\end{aligned}
\tag{3.17}
$$

In particular from (3.15), Q_{11} corresponds to the unique fermionic simple root,

$$
\begin{aligned}
&[\vec{H}, Q_{11}] = qQ_{11}\,, \qquad q = (+\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2})\,,\\
&[\mathcal{E}_{-\chi}, Q_{11}] = 0 \qquad \text{for all} \ \ \chi \in \Delta^{+}_{(2,2)} \cup \ \Delta^{+}_{\mathrm{su}(4)}\,.
\end{aligned}
\tag{3.18}
$$

Other fermionic positive roots are $\{q+\chi,\, q+\chi',\, q+\chi+\chi' \,|\, \chi \in \Delta^{+}_{(2,2)},\, \chi' \in \Delta^{+}_{\mathrm{su}(4)}\}$.

The second order Casimir operator, $\mathcal{C}_{AdS}$, reads

$$
\mathcal{C}_{AdS} = \mathcal{C}_{(2,2)} - \mathcal{C}_{\mathrm{su}(4)} - \tfrac{1}{2} Q_{\alpha\dot{\alpha}} \bar{Q}^{\alpha\dot{\alpha}}\,,
\tag{3.19}
$$

where $\mathcal{C}_{(2,2)}$ and $\mathcal{C}_{\mathrm{su}(4)}$ are the $\mathrm{su}(2,2)$ and $\mathrm{su}(4)$ Casimirs respectively. With the $\mathrm{su}(2,2)$ roots for the noncompact directions, $\Delta^{+}_{s} = \{s, s+x, y+s, y+s+x\}$, they are

$$
\begin{aligned}
\mathcal{C}_{(2,2)} &= \tfrac{1}{2} M^{\mu\nu} M_{\mu\nu}\\
&= M_{12}^2 + M_{34}^2 + M_{56}^2 + \{\mathcal{E}_x\,, \mathcal{E}_{-x}\} + \{\mathcal{E}_y\,, \mathcal{E}_{-y}\} - \sum_{\chi \in \Delta^{+}_{s}} \{\mathcal{E}_\chi\,, \mathcal{E}_{-\chi}\}\,,\\
\mathcal{C}_{\mathrm{su}(4)} &= \tfrac{1}{2} M^{ab} M_{ab} = M_{78}^2 + M_{9\,10}^2 + M_{11\,12}^2 + \sum_{\chi \in \Delta^{+}_{4}} \{\mathcal{E}_\chi\,, \mathcal{E}_{-\chi}\}\,.
\end{aligned}
\tag{3.20}
$$

3.3 Nonunitary finite representations

Starting with an eigenstate of $T_{\mathrm{u}(1)}$, by acting the negative fermionic roots, $\bar{Q}^{\alpha\dot{\alpha}}$, as many as possible - maximally sixteen times surely - one can obtain a state which is annihilated by all the $\bar{Q}^{\alpha\dot{\alpha}}$'s. Now under the action of the bosonic operators, the state opens up an irreducible representation of $\mathrm{u}(1)_Y \oplus \mathrm{su}(2,2) \oplus \mathrm{su}(4)$ or the zeroth floor multiplet. Further from (3.15), any state in the multiplet is annihilated by all the fermionic negative roots.

Generic unitary representations of the noncompact Lie algebra, $\mathrm{su}(2,2)$, are infinite dimensional. However unitary representations are not of our interest. In the present paper we focus on the nonunitary finite representations of $\mathrm{su}(2,2|4) \oplus_{\mathrm{semi}} \mathrm{u}(1)_Y$, denoted by $\mathcal{R}$, satisfying

$$
\begin{aligned}
&\mathcal{R}_{\vec{H}} = (\mathcal{R}_{\vec{H}})^{\dagger}\,,\\
&\mathcal{R}_{-\chi} = \begin{cases} -(\mathcal{R}_\chi)^{\dagger} & \text{for} \ \ \chi \in \{s,\, s+x,\, y+s,\, y+s+x\} \\ +(\mathcal{R}_\chi)^{\dagger} & \text{otherwise} \end{cases}\,.
\end{aligned}
\tag{3.21}
$$

Namely, just like $R(\mathcal{E}_\chi)$ (3.11), the representations of the roots for the su(2,2) noncompact directions are anti-hermitian. This makes the su(2,2) and su(4) Casimirs (3.20) nonnegative definite and ensures finiteness of the representation. Essentially, one can regard $\{\mathcal{R}_\chi,\,(\mathcal{R}_\chi)^\dagger\,|\,\chi\in\Delta^+_{(2,2)}\cup\Delta^+_{\mathrm{su}(4)}\}$ as a unitary representation of su(4) $\oplus$ su(4), since, as an alternative to (3.13), we have

$$[\mathcal{R}_{\vec{H}},\mathcal{R}_\chi]=\chi\mathcal{R}_\chi\,,\qquad [\mathcal{R}_{\vec{H}},(\mathcal{R}_\chi)^\dagger]=-\chi(\mathcal{R}_\chi)^\dagger\,,\qquad [\mathcal{R}_\chi,(\mathcal{R}_\chi)^\dagger]=\chi\cdot\mathcal{R}_{\vec{H}}\,. \tag{3.22}$$

Consequently for any such irreducible representation there exists a unique *superlowest weight*, $|\Lambda_L\rangle$, being annihilated by all the negative roots,

$$\bar{Q}^{\alpha\dot{\alpha}}|\Lambda_L\rangle=0\,,\qquad \mathcal{E}_{-\chi}|\Lambda_L\rangle=0\,,\quad \chi\in\Delta^+_{(2,2)}\cup\Delta^+_{\mathrm{su}(4)}\,. \tag{3.23}$$

The superlowest weight vector is specified by an arbitrary real number, r and six non-negative integers or the Dynkin labels, J_x,J_s,J_y,J_u,J_v,J_w,

$$\begin{aligned}\Lambda_L=\Big(r\,,\,&-\tfrac{1}{2}(J_x+2J_s+J_y)\,,\,-\tfrac{1}{2}(J_x+J_y)\,,\,-\tfrac{1}{2}(J_x-J_y)\,,\\ &-\tfrac{1}{2}(J_u+2J_v+J_w)\,,\,-\tfrac{1}{2}(J_u+J_w)\,,\,-\tfrac{1}{2}(J_u-J_w)\Big)\,,\end{aligned} \tag{3.24}$$

satisfying for the su(2,2) $\oplus$ su(4) simple roots, $\chi=x,s,y,u,v,w$ in (3.8) and (3.9),

$$-2\frac{\chi\cdot\Lambda_L}{\chi^2}=J_\chi\,,\qquad (\mathcal{E}_\chi)^{J_\chi+1}|\Lambda_L\rangle=0\,. \tag{3.25}$$

All the other states are generated by repeated applications of the positive roots on $|\Lambda_L\rangle$, and without loss of generality one can safely work with the simple roots only, Q_{11}, $\mathcal{E}_\chi$, $\chi=x,s,y,u,v,w$. Using the commutator relations, $[\mathcal{E}_\chi,Q]\sim Q$ in (3.15), one can always move all the Q_{11}'s appearing to either far right or far left allowing other fermionic positive roots. Therefore the whole supermultiplet is spanned by

$$\mathcal{E}_{\chi m}\cdots\mathcal{E}_{\chi 1}Q_{\alpha_n\dot{\alpha}_n}\cdots Q_{\alpha_1\dot{\alpha}_1}|\Lambda_L\rangle\,, \tag{3.26}$$

or equivalently

$$Q_{\alpha_n\dot{\alpha}_n}\cdots Q_{\alpha_1\dot{\alpha}_1}\mathcal{E}_{\chi m}\cdots\mathcal{E}_{\chi 1}|\Lambda_L\rangle\,. \tag{3.27}$$

The latter form makes clear that the whole multiplet is built on the zeroth floor by repeated application of the fermionic positive roots. As the zeroth floor multiplet has dimension [26]

$$\begin{aligned}d_0=&\Big[\tfrac{1}{12}(J_x+1)(J_s+1)(J_y+1)(J_x+J_s+2)(J_s+J_y+2)(J_x+J_s+J_y+3)\Big]\\ &\times\Big[\tfrac{1}{12}(J_u+1)(J_v+1)(J_w+1)(J_u+J_v+2)(J_v+J_w+2)(J_u+J_v+J_w+3)\Big]\,,\end{aligned} \tag{3.28}$$

Eq.(3.27) implies that the supermultiplet has a finite dimension, d_s,

$$d_s\leq 2^{16}\times d_0\,. \tag{3.29}$$

The application of a $Q_{\alpha\dot{\alpha}}$ changes the $\mathrm{u}(1)_Y \oplus \mathrm{su}(2,2) \oplus \mathrm{su}(4)$ multiplets, jumping from one irreducible representation to another. In particular, the number of the applied fermionic positive roots determines the floor number, zero to sixteen at most. Each floor is specified by the $\mathrm{u}(1)_Y$ charge,

$$r_N = r + \tfrac{1}{2}N \,, \qquad N = 0, 1, 2, \cdots, 16. \tag{3.30}$$

Each of the zeroth and the highest floors forms an irreducible representation of $\mathrm{u}(1)_Y \oplus \mathrm{su}(2,2) \oplus \mathrm{su}(4)$, while other floors are in general reducible and decompose into irreducible ones. All the irreducible representations for $\mathrm{u}(1)_Y \oplus \mathrm{su}(2,2) \oplus \mathrm{su}(4)$ are specified by their own lowest weights, λ_L, annihilated by all the bosonic negative roots,

$$\begin{aligned} \lambda_L = \Big(& r + \tfrac{1}{2}N \,,\, -\tfrac{1}{2}(j_x + 2j_s + j_y) \,,\, -\tfrac{1}{2}(j_x + j_y) \,,\, -\tfrac{1}{2}(j_x - j_y) \,, \\ & -\tfrac{1}{2}(j_u + 2j_v + j_w) \,,\, -\tfrac{1}{2}(j_u + j_w) \,,\, -\tfrac{1}{2}(j_u - j_w) \Big) \,. \end{aligned} \tag{3.31}$$

The corresponding highest weight is then [27]

$$\begin{aligned} \lambda_H = \Big(& r + \tfrac{1}{2}N \,,\, \tfrac{1}{2}(j_y + 2j_s + j_x) \,,\, \tfrac{1}{2}(j_y + j_x) \,,\, \tfrac{1}{2}(j_y - j_x) \,, \\ & \tfrac{1}{2}(j_w + 2j_v + j_u) \,,\, \tfrac{1}{2}(j_w + j_u) \,,\, \tfrac{1}{2}(j_w - j_u) \Big) \,, \end{aligned} \tag{3.32}$$

while the dimension is given by (3.28) with $J \leftrightarrow j$.

In general, different orderings in the multiplications of the positive roots on the superlowest weight may result in degeneracy for states of the same weight vector. To verify the possible degeneracy one should check whether a state can be rewritten as the other through changes of orderings using the super-commutation relations of the superalgebra. Especially for irreducible representations, if a state is annihilated by all the negative simple roots - hence by all the negative roots - the state must be either the superlowest weight or trivial. This provides an alternative criteria to distinguish or identify any given two states of the same weight vector in a representation.

The particular representation we have in mind for the noncentral extension of $\mathrm{su}(2,2|4)$ superalgebra to be carried out in the next section is a kind of *adjoint representation* where $\mathrm{su}(2,2|4)$ generators act in the adjoint manner on brane charges which carry finite number of $\mathrm{su}(2,2) \oplus \mathrm{su}(4)$ spinor indices, e.g. $Z_{\alpha_1 \cdots \alpha_k \dot{\alpha}_1 \cdots \dot{\alpha}_l}{}^{\beta_1 \cdots \beta_m \dot{\beta}_1 \cdots \dot{\beta}_n}$. Naturally the dimension of the representation is finite and $\mathcal{R}_{\vec{H}}$, $\mathcal{R}_{\pm\chi}$ satisfy the condition (3.21), since they are essentially given by $R(\vec{H})$, $R(\mathcal{E}_{\pm\chi})$, $-R(\vec{H})^t$, $-R(\mathcal{E}_{\pm\chi})^t$, depending on whether the spinor indices are lower or upper ones. Acting the fermionic positive roots, $Q_{\alpha\dot{\alpha}}$, on the ground floor as in (3.27), all possible states in the supermultiplet are built up, which in fact by definition gives representations of the fermionic positive roots, $\mathcal{R}_{\alpha\dot{\alpha}}$. On the other hand, the representations of the fermionic negative roots, $\mathcal{R}^{\alpha\dot{\alpha}}$, should be read off from explicit manipulation of their actions on all the existing states utilizing the anti-commutation relation of the superalgebra until it hits the superlowest weight to terminate the procedure. For the adjoint representation we have[6]

$$
\begin{aligned}
\{\mathcal{R}^{\alpha\dot{\alpha}},\mathcal{R}_{\beta\dot{\beta}}\} = {}& 2\delta^{\dot{\alpha}}{}_{\dot{\beta}}\begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_x & \mathcal{R}_{s+x} & \mathcal{R}_{y+s+x} \\ \mathcal{R}_x^\dagger & \mathcal{R}_2 & \mathcal{R}_s & \mathcal{R}_{y+s} \\ \mathcal{R}_{s+x}^\dagger & \mathcal{R}_s^\dagger & \mathcal{R}_3 & \mathcal{R}_y \\ \mathcal{R}_{y+s+x}^\dagger & \mathcal{R}_{y+s}^\dagger & \mathcal{R}_y^\dagger & \mathcal{R}_4 \end{pmatrix}^{\alpha}{}_{\beta} \\
& - 2\delta^{\alpha}{}_{\beta}\begin{pmatrix} \mathcal{R}_5 & \mathcal{R}_u & \mathcal{R}_{v+u} & \mathcal{R}_{w+v+u} \\ \mathcal{R}_u^\dagger & \mathcal{R}_6 & \mathcal{R}_v & \mathcal{R}_{w+v} \\ \mathcal{R}_{v+u}^\dagger & \mathcal{R}_v^\dagger & \mathcal{R}_7 & \mathcal{R}_w \\ \mathcal{R}_{w+v+u}^\dagger & \mathcal{R}_{w+v}^\dagger & \mathcal{R}_w^\dagger & \mathcal{R}_8 \end{pmatrix}^{\dot{\alpha}}{}_{\dot{\beta}} .
\end{aligned}
\tag{3.33}
$$

Note that, compared to (3.16), there is no minus sign for the generators of the noncompact directions in su(2, 2).

As usual, for some small irreducible representations of the su(4) algebra, we may denote them simply by their dimensions, instead of the Dynkin labels,

$$
\begin{array}{llll}
4 \sim (1,0,0)\,, & \bar{4} \sim (0,0,1)\,, & 6 \sim (0,1,0)\,, & 10 \sim (2,0,0)\,, \\
15 \sim (1,0,1)\,, & 20 \sim (3,0,0)\,, & 20' \sim (1,1,0)\,, & 20'' \sim (0,2,0)\,, \\
35 \sim (4,0,0)\,, & 36 \sim (2,0,1)\,, & 45 \sim (2,1,0)\,, & 60 \sim (1,2,0)\,, \\
70 \sim (1,0,3)\,, & 84 \sim (3,1,0)\,. & &
\end{array}
\tag{3.34}
$$

[6]Note that if we assumed $\mathcal{R}^{\alpha\dot{\alpha}} = \mathcal{R}^\dagger_{\alpha\dot{\alpha}}$, then our representation would coincide with the unitary representation of the su(4|4) superalgebra. However, then, from $\{\mathcal{R}^{\alpha\dot{\alpha}},\mathcal{R}_{\alpha\dot{\alpha}}\} = 0$ and its positive definite property, the representation should have been trivial. In fact, the precise relation of $\mathcal{R}^{\alpha\dot{\alpha}}$ to $\mathcal{R}^\dagger_{\alpha\dot{\alpha}}$ can be obtained only when we complete the vector space of the representation by the complex conjugate.

4. Noncentral extensions of the $AdS_5 \times S^5$ superalgebra

One possible way to obtain the noncentral extension of the su$(2,2|4)$ superalgebra is to perform the Witten-Olive type analysis on the four dimensional $\mathcal{N}=4$ super Yang-Mills theory [2, 4]. Namely starting with the explicit expressions for the supercharges, including the special superconformal charges too, one may evaluate the anti-commutators of them to see what kinds of surface terms appear. In principle, one gets

$$\begin{aligned} \{\bar{Q}^{\alpha\dot{\alpha}}, Q_{\beta\dot{\beta}}\} &= 4\delta^{\dot{\alpha}}{}_{\dot{\beta}} T_{(2,2)}{}^{\alpha}{}_{\beta} - 4\delta^{\alpha}{}_{\beta} T_{\mathrm{su}(4)}{}^{\dot{\alpha}}{}_{\dot{\beta}} + \mathcal{H}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}}\,, \\ \{Q_{\alpha\dot{\alpha}}\,, Q_{\beta\dot{\beta}}\} &= \mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}\,. \end{aligned} \tag{4.1}$$

Here $\mathcal{H}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}}$ and $\mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}$ correspond to the possible surface integrals or the brane charges, and they can further decompose into $(15,15)\oplus(1,15)\oplus(15,1)\oplus(1,1)$ and $(6,6)\oplus(10,10)$,

$$\begin{aligned} \mathcal{H}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} &= H^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} + \tfrac{1}{4}\delta^{\dot{\alpha}}{}_{\dot{\beta}} H^{\alpha}{}_{\beta} - \tfrac{1}{4}\delta^{\alpha}{}_{\beta} H^{\dot{\alpha}}{}_{\dot{\beta}} + \tfrac{1}{16}\delta^{\alpha}{}_{\beta}\delta^{\dot{\alpha}}{}_{\dot{\beta}} H\,, \\ \mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}} &= Z^{(6,6)}_{\alpha\beta\dot{\alpha}\dot{\beta}} + Z^{(10,10)}_{\alpha\beta\dot{\alpha}\dot{\beta}}\,, \end{aligned} \tag{4.2}$$

satisfying the traceless and symmetric properties,

$$\begin{aligned} & H^{\alpha}{}_{\alpha} = 0\,, \quad H^{\dot{\alpha}}{}_{\dot{\alpha}} = 0\,, \quad H^{\alpha\dot{\alpha}}{}_{\alpha\dot{\beta}} = 0\,, \quad H^{\alpha\dot{\alpha}}{}_{\beta\dot{\alpha}} = 0\,, \\ & Z^{(6,6)}_{\alpha\beta\dot{\alpha}\dot{\beta}} = Z^{(6,6)}_{[\alpha\beta][\dot{\alpha}\dot{\beta}]}\,, \qquad Z^{(10,10)}_{\alpha\beta\dot{\alpha}\dot{\beta}} = Z^{(10,10)}_{(\alpha\beta)(\dot{\alpha}\dot{\beta})}\,. \end{aligned} \tag{4.3}$$

Using the 4×4 matrices, ρ^{μ}, ρ^{a}, they can be rewritten as

$$\begin{aligned} & H^{\alpha}{}_{\beta} = -i\tfrac{1}{2}(\bar{\rho}^{\mu\nu})^{\alpha}{}_{\beta} H_{\mu\nu}\,, \quad H^{\dot{\alpha}}{}_{\dot{\beta}} = -i\tfrac{1}{2}(\bar{\rho}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} H_{ab}\,, \quad H^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} = \tfrac{1}{4}(\bar{\rho}^{\mu\nu})^{\alpha}{}_{\beta}(\bar{\rho}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} H_{\mu\nu ab}\,, \\ & Z^{(6,6)}_{\alpha\beta\dot{\alpha}\dot{\beta}} = (\rho^{\mu})_{\alpha\beta}(\rho^{a})_{\dot{\alpha}\dot{\beta}} Z_{\mu a}\,, \qquad Z^{(10,10)}_{\alpha\beta\dot{\alpha}\dot{\beta}} = \tfrac{1}{144}(\rho^{\mu\nu\lambda})_{\alpha\beta}(\rho^{abc})_{\dot{\alpha}\dot{\beta}} Z^{-}_{\mu\nu\lambda abc}\,, \end{aligned} \tag{4.4}$$

where $H_{\mu\nu ab}$, $H_{\mu\nu}$, H_{ab}, H are all hermitian, and from (2.8), $Z^{-}_{\mu\nu\lambda abc}$ is anti-self-dual for each so$(2,4)$ and so(6) indices,

$$Z^{-}_{\mu\nu\lambda abc} = -i\tfrac{1}{6}\epsilon_{\mu\nu\lambda}{}^{\kappa\sigma\tau} Z^{-}_{\kappa\sigma\tau abc} = -i\tfrac{1}{6}\epsilon_{abc}{}^{def} Z^{-}_{\mu\nu\lambda def}\,. \tag{4.5}$$

Physically, $H_{\mu\nu ab}$, $Z_{\mu a}$, $Z^{-}_{\mu\nu\lambda abc}$ correspond to the $D3$, $D1$, $D5$ branes having the codimension four to the $D3$ branes on which the super Yang-Mills theory lives. On the other hand, the result of Osborn [4] shows that there appears no $H_{\mu\nu}$, H_{ab}, H terms in the expression of the anti-commutator between the two ordinary supercharges. This implies that, at least, some components of $H_{\mu\nu}, H_{ab}$ are identically vanishing in the extended superalgebra. Then the covariance under the so$(2,4)\oplus$so(6) rotation makes sure that all of them are indeed absent. Hence we conclude

$$H_{\mu\nu} = 0\,, \qquad H_{ab} = 0\,, \qquad H = 0\,, \qquad \mathcal{H}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} = H^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}}\,. \tag{4.6}$$

As noted by Peeters and Zamaklar [1], due to the Jacobi identity involving Q, $\bar{Q}$ and a brane charge, the commutators between the brane charges and supercharges should not vanish, e.g. $[H_{\mu\nu ab}\,,\,Q_{\alpha\dot{\alpha}}]\neq 0$ if $H_{\mu\nu ab}\neq 0$. Naturally this leads to a noncentral extension of the superalgebra, su$(2,2|4)$. In the rest of the present paper, we study the noncentral extension in a group theoretical manner, rather than pursuing the Witten-Olive type analysis on the four dimensional $\mathcal{N}=4$ super Yang-Mills theory.

4.1 Generic features of the extended superalgebra

In our terminology, *brane charges* are, by definition, the space integrals of the total derivative terms or the surface integrals. In particular, they are not symmetry generators of the corresponding field theory, and hence they are not forbidden by the Coleman-Mandula theorem [10]. Some immediate important consequences are as follows. Firstly **the super-commutator involving a brane charge is also a brane charge**, since whatever comes out should remain as a surface integral. Furthermore, **all the brane charges super-commute with each other**, since one can take the two radii of the spatial infinite spheres, S^2, to be finitely different so that the two surfaces have no contact point.[7] As a result all the bosonic brane charges can be diagonalized simultaneously and provide good quantum numbers. Schematically we have[8]

$$
\begin{aligned}
&[\mathcal{O}_A\,,\,\mathcal{O}_B\}=c_{AB}{}^{C}\mathcal{O}_C\,+\,d_{AB}{}^{I}B_I\,,\\
&[B_I\,,\,B_J\}=0\,,\\
&[\mathcal{O}_A\,,\,B_I\}=f_{AI}{}^{J}B_J\,,
\end{aligned}
\tag{4.7}
$$

where $\mathcal{O}_A$ denotes the old generators in the unextended superalgebra, su$(2,2|4)$, with the structure constant, $c_{AB}{}^{C}$, while B_I corresponds to the brane charges.

For consistency, it is necessary and sufficient to require the extended superalgebra to satisfy the Jacobi identity, as the structure constants which are ordinary c-numbers will then realize *a* representation or the adjoint representation. In our case, the Jacobi identities involving more than one brane charges are trivial so that there exist essentially two types of Jacobi identities to consider :

$$
[\mathcal{O}_A\,,[\mathcal{O}_B\,,B_I\}\}-(-1)^{\#_A\#_B}[\mathcal{O}_B\,,[\mathcal{O}_A\,,B_I\}\}=[[\mathcal{O}_A\,,\mathcal{O}_B\}\,,B_I\}\,,\tag{4.8}
$$

$$
[\mathcal{O}_A\,,[\mathcal{O}_B\,,\mathcal{O}_C\}\}-(-1)^{\#_A\#_B}[\mathcal{O}_B\,,[\mathcal{O}_A\,,\mathcal{O}_C\}\}=[[\mathcal{O}_A\,,\mathcal{O}_B\}\,,\mathcal{O}_C\}\,.\tag{4.9}
$$

The first identity clearly shows that **the brane charges form a representation realized by the adjoint actions of the generators in the original unextended superalgebra**, while the second one indicates that **the adjoint representation is subject to**

[7]One exceptional case is the square of a fermionic brane charge, which diverges in general. Either we can take again two different radii at spatial infinities and set it vanish as a kind of regularization scheme, or leave them undetermined. In any case, our main results are not affected by this subtlety.

[8]The super-commutator is defined to be $[\mathcal{O}_A\,,\,\mathcal{O}_B\}=\mathcal{O}_A\mathcal{O}_B-(-1)^{\#_A\#_B}\mathcal{O}_B\mathcal{O}_A$, where $\#_A$ is zero or one depending whether $\mathcal{O}_A$ is bosonic or fermionic.

some constraints. In particular, the dimension of the adjoint representation is finite, meaning that there are only finitely many brane charges.

Requiring that the brane charges transform covariantly for the $\mathrm{su}(2,2)\oplus\mathrm{su}(4)$ generators, as described in subsection (3.3), any Jacobi identity involving the $\mathrm{su}(2,2)\oplus\mathrm{su}(4)$ generators holds automatically. Therefore the only nontrivial constraints come from Jacobi identities containing either three Q's or two Q's and one $\bar{Q}$,

$$[Q_{\alpha\dot{\alpha}}\,,\,\mathcal{Z}_{\beta\dot{\beta}\gamma\dot{\gamma}}]+[Q_{\beta\dot{\beta}},\mathcal{Z}_{\gamma\dot{\gamma}\alpha\dot{\alpha}}]+[Q_{\gamma\dot{\gamma}}\,,\,\mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}]\equiv\Psi_{\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}=0\,, \tag{4.10}$$

$$[Q_{\alpha\dot{\alpha}}\,,\,H^{\gamma\dot{\gamma}}{}_{\beta\dot{\beta}}]+[Q_{\beta\dot{\beta}}\,,\,H^{\gamma\dot{\gamma}}{}_{\alpha\dot{\alpha}}]+[\bar{Q}^{\gamma\dot{\gamma}}\,,\,\mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}]=0\,. \tag{4.11}$$

To obtain the extended superalgebra, one needs to look for adjoint representations of the original unextended superalgebra such that it contains $\mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}$, $H^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}}$ and satisfies the constraints above. However, this group theoretically well defined problem does not lead to a unique solution, essentially because the relevant superlowest weights are not specified yet, and due to the nonunitary property of the adjoint representation, the states which can decouple may not decouple. In fact, we expect the "correctly" extended superalgebra, which can be in principle uniquely obtained from the Witten-Olive type analysis on the $\mathcal{N}=4$ super Yang-Mills theory, leads to a *reducible* adjoint representation for the brane charges, containing more than one irreducible supermultiplets. The physical reason is that the $D1$, $D3$, $D5$ branes should be able to exist separately, not necessarily weaved by one another.

The filtering of the reducible representation into each irreducible one can be done by restricting the full Hilbert space of the Yang-Mills theory in a suitable way, and this will enable us to obtain the physically relevant noncentral extensions.

Firstly we raise the question, 'what is the relevance of the strictly unextended superalgebra, $\mathrm{su}(2,2|4)$, to the Yang-Mills theory, if the "correct" superalgebra of the theory is an extended one not the unextended one?' The answer is simple. Consider a subspace of the full Hilbert space which is annihilated by all the brane charges. Clearly such a subspace forms an invariant subspace for the extended superalgebra, and on the subspace the brane charges have the trivial representations. In other words, the unextended superalgebra is only for the elementary particles in the theory not for the branes, as one can naturally expect.

Now we consider a less restricted subspace of the full Hilbert space. Namely, we focus on the subspace, V, which satisfies the following two properties. First it is annihilated by the $D3$ brane charges, $H_{\mu\nu ab}$, and second it is invariant under the action of all the supercharges,

$$H_{\mu\nu ab}V=0\,,\qquad Q_{\alpha\dot{\alpha}}V\subset V\,,\qquad \bar{Q}^{\alpha\dot{\alpha}}V\subset V\,. \tag{4.12}$$

It follows that V is in fact an invariant subspace for the fully extended superalgebra, since all other generators can be constructed from the supercharges. Furthermore, we get

$$[Q_{\alpha\dot{\alpha}}\,,\,H^{\gamma\dot{\gamma}}{}_{\beta\dot{\beta}}]\,V=0\,. \tag{4.13}$$

Clearly on the subspace, V, the representations of $H^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}}$ and $[Q_{\alpha\dot{\alpha}}\,,\,H^{\gamma\dot{\gamma}}{}_{\beta\dot{\beta}}]$ are trivial, and Eq.(4.11) gets simplified to show that $\mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}$ forms the ground floors of the adjoint representations we are looking for,

$$[\,\bar{Q}^{\gamma\dot{\gamma}}\,,\,\mathcal{Z}_{\alpha\dot{\alpha}\beta\dot{\beta}}\,]=0\,. \tag{4.14}$$

Moreover, as it decomposes into $(6,6)$ and $(10,10)$, there exist two superlowest weights, and hence two irreducible adjoint representations. They can be treated separately, and we only need to impose the remaining constraint, $\Psi_{\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}=0$, (4.10).

Direct calculation, using (4.1) and (4.14) only, shows that

$$\{\,\bar{Q}^{\kappa\dot{\kappa}},\Psi_{\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}\,\}=0\,,\qquad\quad\text{identically}\,. \tag{4.15}$$

Surely this is a necessary condition for the consistent decoupling of $\Psi_{\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}$ in the adjoint representation.

4.2 Electro-magnetic extension

The aim of the present subsection is to obtain the noncentral extension of the superalgebra, su$(2,2|4)$, which contains the $F1/D1$ or the electro-magnetic charge, $Z_{\mu a}$, in the anti-commutator of the supercharges,

$$\{Q_{\alpha\dot{\alpha}}\,,\,Q_{\beta\dot{\beta}}\}=\tfrac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}B^{\gamma\delta\dot{\gamma}\dot{\delta}}\,,\qquad\{\bar{Q}^{\alpha\dot{\alpha}},Q_{\beta\dot{\beta}}\}=4\delta^{\dot{\alpha}}_{\dot{\beta}}T_{(2,2)}{}^{\alpha}{}_{\beta}-4\delta^{\alpha}{}_{\beta}T_{\mathrm{su}(4)}{}^{\dot{\alpha}}{}_{\dot{\beta}}\,, \tag{4.16}$$

where, for the later convenience, we have raised the spinor indices of the electro-magnetic charge by the totally anti-symmetric four form tensors,

$$B^{\alpha\beta\dot{\alpha}\dot{\beta}}=\tfrac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}Z^{(6,6)}_{\gamma\delta\dot{\gamma}\dot{\delta}}=(\bar{\rho}^{\mu})^{\alpha\beta}(\bar{\rho}^{a})^{\dot{\alpha}\dot{\beta}}Z_{\mu a}\,. \tag{4.17}$$

As the brane charge, $B^{\alpha\beta\dot{\alpha}\dot{\beta}}$, can not be central, the superalgebra, su$(2,2|4)$, gets a noncentral extension inevitably. The extension will be uniquely determined, and the corresponding extended superalgebra can be regarded as the superalgebra of the $\mathcal{N}=4$ super Yang-Mills theory restricted on the '$D3$, $D5$ free' Hilbert space or $H^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}}\equiv 0$, $Z^{(10,10)}_{\alpha\beta\dot{\alpha}\dot{\beta}}\equiv 0$.

From the decomposition of the tensor product,

$$(4,4)\otimes(6,6)=(20,20)\oplus(\bar{4},20)\oplus(20,\bar{4})\oplus(\bar{4},\bar{4})\,, \tag{4.18}$$

we write, for the first floor of the adjoint representation,

$$\begin{array}{ll}[Q_{\alpha\dot{\alpha}},B^{\beta\gamma\dot{\beta}\dot{\gamma}}]=N_{\alpha\dot{\alpha}}{}^{\beta\gamma\dot{\beta}\dot{\gamma}}+\frac{1}{9}\delta^{\beta}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}}N^{\gamma\dot{\gamma}}-\frac{1}{9}\delta^{\beta}_{\alpha}\delta^{\dot{\gamma}}_{\dot{\alpha}}N^{\gamma\dot{\beta}}-\frac{1}{9}\delta^{\gamma}_{\alpha}\delta^{\dot{\beta}}_{\dot{\alpha}}N^{\beta\dot{\gamma}}+\frac{1}{9}\delta^{\gamma}_{\alpha}\delta^{\dot{\gamma}}_{\dot{\alpha}}N^{\beta\dot{\beta}} \\ \qquad -\frac{1}{3}\delta^{\beta}_{\alpha}B^{\gamma}{}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}+\frac{1}{3}\delta^{\gamma}_{\alpha}B^{\beta}{}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}-\frac{1}{3}\delta^{\dot{\beta}}_{\dot{\alpha}}B^{\beta\gamma\dot{\gamma}}{}_{\alpha}+\frac{1}{3}\delta^{\dot{\gamma}}_{\dot{\alpha}}B^{\beta\gamma\dot{\beta}}{}_{\alpha}\,,\end{array} \tag{4.19}$$

where each tensor belongs to different $\mathrm{su}(2,2)\oplus\mathrm{su}(4)$ irreducible representation as they are traceless and anti-symmetric,

$$
\begin{aligned}
&N_{\alpha\dot{\alpha}}{}^{\alpha\gamma\dot{\beta}\dot{\gamma}}=0\,, &&N_{\alpha\dot{\alpha}}{}^{\beta\gamma\dot{\beta}\dot{\gamma}}=N_{\alpha\dot{\alpha}}{}^{[\beta\gamma][\dot{\beta}\dot{\gamma}]}\,, &&:\ (20,20)\,,\\
&B^{\gamma}{}_{\dot{\alpha}}{}^{\dot{\alpha}\dot{\gamma}}=0\,, &&B^{\gamma}{}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}=B^{\gamma}{}_{\dot{\alpha}}{}^{[\dot{\beta}\dot{\gamma}]}\,, &&:\ (\bar{4},20)\,,\\
&B^{\alpha\gamma\dot{\gamma}}{}_{\alpha}=0\,, &&B^{\beta\gamma\dot{\gamma}}{}_{\alpha}=B^{[\beta\gamma]\dot{\gamma}}{}_{\alpha}\,, &&:\ (20,\bar{4})\,.
\end{aligned}
\tag{4.20}
$$

In terms of the decomposition, the six form tensor reads

$$
\Psi_{\alpha\beta\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}}=\tfrac{1}{4}\epsilon_{\alpha\beta\rho\varepsilon}\epsilon_{\dot{\alpha}\dot{\beta}\dot{\rho}\dot{\varepsilon}}N_{\gamma\dot{\gamma}}{}^{\rho\varepsilon\dot{\rho}\dot{\varepsilon}}+\tfrac{1}{4}\epsilon_{\beta\gamma\rho\varepsilon}\epsilon_{\dot{\beta}\dot{\gamma}\dot{\rho}\dot{\varepsilon}}N_{\alpha\dot{\alpha}}{}^{\rho\varepsilon\dot{\rho}\dot{\varepsilon}}+\tfrac{1}{4}\epsilon_{\gamma\alpha\rho\varepsilon}\epsilon_{\dot{\gamma}\dot{\alpha}\dot{\rho}\dot{\varepsilon}}N_{\beta\dot{\beta}}{}^{\rho\varepsilon\dot{\rho}\dot{\varepsilon}}+\tfrac{1}{3}\epsilon_{\alpha\beta\gamma\varepsilon}\epsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\varepsilon}}N^{\varepsilon\dot{\varepsilon}}\,.
\tag{4.21}
$$

In particular,

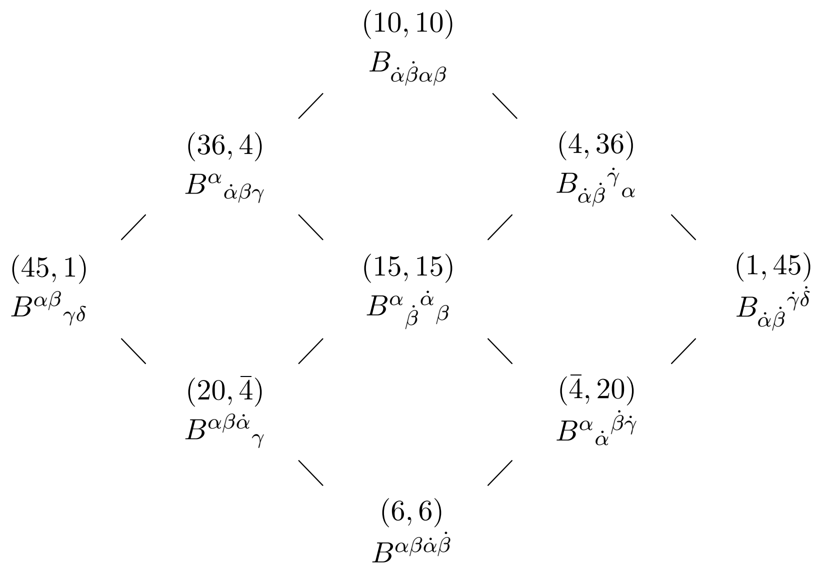
$$
N^{\alpha\dot{\alpha}}=\tfrac{1}{12}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\Psi_{\beta\gamma\delta\dot{\beta}\dot{\gamma}\dot{\delta}}\,.
\tag{4.22}
$$

Hence the constraint, $\Psi\equiv 0$, is equivalent to

$$
N_{\alpha\dot{\alpha}}{}^{\beta\gamma\dot{\beta}\dot{\gamma}}\equiv 0\,,\qquad N^{\alpha\dot{\alpha}}\equiv 0\,,
\tag{4.23}
$$

which imply only the $(\bar{4},20)$ and $(20,\bar{4})$ tensors survive and others decouple.

Consequently the commutation relation for the first floor, (4.19), becomes simplified, and other higher floors can be constructed recurrently. It turns out that the construction terminates on the fourth floor, and the resulting adjoint representation is of the following unique form,



where the diagonal lines link the two neighboring $\mathrm{su}(2,2)\oplus\mathrm{su}(4)$ multiplets which are connected by the supercharges. The complex dimension of the supermultiplet is 899. Below we explicitly present all the super-commutation relations of the extended superalgebra,

$$\begin{aligned}
&[Q_{\kappa\dot{\kappa}}, B^{\alpha\beta\dot{\alpha}\dot{\beta}}] = -\tfrac{1}{3}\delta_{\kappa}^{\alpha} B^{\beta}{}_{\dot{\kappa}}{}^{\dot{\alpha}\dot{\beta}} + \tfrac{1}{3}\delta_{\kappa}^{\beta} B^{\alpha}{}_{\dot{\kappa}}{}^{\dot{\alpha}\dot{\beta}} - \tfrac{1}{3}\delta_{\dot{\kappa}}^{\dot{\alpha}} B^{\alpha\beta\dot{\beta}}{}_{\kappa} + \tfrac{1}{3}\delta_{\dot{\kappa}}^{\dot{\beta}} B^{\alpha\beta\dot{\alpha}}{}_{\kappa}\,,\\
&\{Q_{\kappa\dot{\kappa}}, B^{\alpha}{}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}\} = \tfrac{3}{8}\delta_{\dot{\kappa}}^{\dot{\beta}} B^{\alpha}{}_{\dot{\alpha}}{}^{\dot{\gamma}}{}_{\kappa} - \tfrac{3}{8}\delta_{\dot{\kappa}}^{\dot{\gamma}} B^{\alpha}{}_{\dot{\alpha}}{}^{\dot{\beta}}{}_{\kappa} - \tfrac{1}{8}\delta_{\dot{\alpha}}^{\dot{\beta}} B^{\alpha}{}_{\dot{\kappa}}{}^{\dot{\gamma}}{}_{\kappa} + \tfrac{1}{8}\delta_{\dot{\alpha}}^{\dot{\gamma}} B^{\alpha}{}_{\dot{\kappa}}{}^{\dot{\beta}}{}_{\kappa} + \tfrac{1}{4}\delta_{\kappa}^{\alpha} B_{\dot{\kappa}\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}\,,\\
&\{Q_{\kappa\dot{\kappa}}, B^{\alpha\beta\dot{\alpha}}{}_{\gamma}\} = -\tfrac{3}{8}\delta_{\kappa}^{\alpha} B^{\beta}{}_{\dot{\kappa}}{}^{\dot{\alpha}}{}_{\gamma} + \tfrac{3}{8}\delta_{\kappa}^{\beta} B^{\alpha}{}_{\dot{\kappa}}{}^{\dot{\alpha}}{}_{\gamma} + \tfrac{1}{8}\delta_{\gamma}^{\alpha} B^{\beta}{}_{\dot{\kappa}}{}^{\dot{\alpha}}{}_{\kappa} - \tfrac{1}{8}\delta_{\gamma}^{\beta} B^{\alpha}{}_{\dot{\kappa}}{}^{\dot{\alpha}}{}_{\kappa} + \tfrac{1}{4}\delta_{\dot{\kappa}}^{\dot{\alpha}} B^{\alpha\beta}{}_{\gamma\kappa}\,,\\
&[Q_{\kappa\dot{\kappa}}, B^{\alpha}{}_{\dot{\beta}}{}^{\dot{\alpha}}{}_{\beta}] = \tfrac{4}{15}\delta_{\kappa}^{\alpha} B_{\dot{\kappa}\dot{\beta}}{}^{\dot{\alpha}}{}_{\beta} - \tfrac{1}{15}\delta_{\beta}^{\alpha} B_{\dot{\kappa}\dot{\beta}}{}^{\dot{\alpha}}{}_{\kappa} - \tfrac{4}{15}\delta_{\dot{\kappa}}^{\dot{\alpha}} B^{\alpha}{}_{\dot{\beta}\kappa\beta} + \tfrac{1}{15}\delta_{\dot{\beta}}^{\dot{\alpha}} B^{\alpha}{}_{\dot{\kappa}\kappa\beta}\,,\\
&[Q_{\kappa\dot{\kappa}}, B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}}] = \tfrac{2}{5}\delta_{\dot{\kappa}}^{\dot{\delta}} B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}{}_{\kappa} - \tfrac{2}{5}\delta_{\dot{\kappa}}^{\dot{\gamma}} B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\delta}}{}_{\kappa} + \tfrac{1}{10}\delta_{\dot{\alpha}}^{\dot{\gamma}} B_{\dot{\kappa}\dot{\beta}}{}^{\dot{\delta}}{}_{\kappa} + \tfrac{1}{10}\delta_{\dot{\beta}}^{\dot{\gamma}} B_{\dot{\kappa}\dot{\alpha}}{}^{\dot{\delta}}{}_{\kappa} - \tfrac{1}{10}\delta_{\dot{\alpha}}^{\dot{\delta}} B_{\dot{\kappa}\dot{\beta}}{}^{\dot{\gamma}}{}_{\kappa} - \tfrac{1}{10}\delta_{\dot{\beta}}^{\dot{\delta}} B_{\dot{\kappa}\dot{\alpha}}{}^{\dot{\gamma}}{}_{\kappa},\\
&[Q_{\kappa\dot{\kappa}}, B^{\alpha\beta}{}_{\gamma\delta}] = -\tfrac{2}{5}\delta_{\kappa}^{\alpha} B^{\beta}_{\dot{\kappa}\gamma\delta} + \tfrac{2}{5}\delta_{\kappa}^{\beta} B^{\alpha}_{\dot{\kappa}\gamma\delta} + \tfrac{1}{10}\delta_{\gamma}^{\alpha} B^{\beta}_{\dot{\kappa}\delta\kappa} + \tfrac{1}{10}\delta_{\delta}^{\alpha} B^{\beta}_{\dot{\kappa}\gamma\kappa} - \tfrac{1}{10}\delta_{\gamma}^{\beta} B^{\alpha}_{\dot{\kappa}\delta\kappa} - \tfrac{1}{10}\delta_{\delta}^{\beta} B^{\alpha}_{\dot{\kappa}\gamma\kappa}\,,\\
&\{Q_{\kappa\dot{\kappa}}, B^{\alpha}{}_{\dot{\alpha}\beta\gamma}\} = \tfrac{5}{18}\delta_{\kappa}^{\alpha} B_{\dot{\kappa}\dot{\alpha}\beta\gamma} - \tfrac{1}{18}\delta_{\beta}^{\alpha} B_{\dot{\kappa}\dot{\alpha}\kappa\gamma} - \tfrac{1}{18}\delta_{\gamma}^{\alpha} B_{\dot{\kappa}\dot{\alpha}\kappa\beta}\,,\\
&\{Q_{\kappa\dot{\kappa}}, B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}{}_{\alpha}\} = \tfrac{5}{18}\delta_{\dot{\kappa}}^{\dot{\gamma}} B_{\dot{\alpha}\dot{\beta}\kappa\alpha} - \tfrac{1}{18}\delta_{\dot{\alpha}}^{\dot{\gamma}} B_{\dot{\kappa}\dot{\beta}\kappa\alpha} - \tfrac{1}{18}\delta_{\dot{\beta}}^{\dot{\gamma}} B_{\dot{\kappa}\dot{\alpha}\kappa\alpha}\,,\\
&[Q_{\kappa\dot{\kappa}}, B_{\dot{\alpha}\dot{\beta}\alpha\beta}] = 0\,.
\end{aligned} \tag{4.24}$$

All the brane charges are traceless, anti-symmetric for the upper indices, and symmetric for the lower indices if they belong to the same species. The statistics of the brane charges depends whether the number of the upper indices is even or odd. Furthermore, the upper index can be lowered and converted to the different species using the positive supercharges, $Q_{\alpha\dot{\alpha}}$, from right to left. For example,

$$\begin{aligned}
&[Q_{\gamma\dot{\varepsilon}}\,,\,B^{\alpha\beta\dot{\alpha}\dot{\varepsilon}}] = B^{\alpha\beta\dot{\alpha}}{}_{\gamma} = B^{[\alpha\beta]\dot{\alpha}}{}_{\gamma}\,, && [Q_{\varepsilon\dot{\gamma}}\,,\,B^{\alpha\varepsilon\dot{\alpha}\dot{\beta}}] = B^{\alpha}{}_{\dot{\gamma}}{}^{\dot{\alpha}\dot{\beta}} = B^{\alpha}{}_{\dot{\gamma}}{}^{[\dot{\alpha}\dot{\beta}]}\,,\\
&\{Q_{\varepsilon\dot{\beta}}\,,\,B^{\alpha\varepsilon\dot{\alpha}}{}_{\beta}\} = B^{\alpha}{}_{\dot{\beta}}{}^{\dot{\alpha}}{}_{\beta}\,, && \{Q_{\varepsilon\dot{\alpha}}\,,\,B^{\varepsilon}{}_{\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}}\} = B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}} = B_{(\dot{\alpha}\dot{\beta})}{}^{[\dot{\gamma}\dot{\delta}]}\,,\\
&\{Q_{\gamma\dot{\varepsilon}}\,,\,B^{\alpha\beta\dot{\varepsilon}}{}_{\delta}\} = B^{\alpha\beta}{}_{\gamma\delta} = B^{[\alpha\beta]}{}_{(\gamma\delta)}\,, && [Q_{\varepsilon\dot{\alpha}}\,,\,B^{\alpha\varepsilon}{}_{\beta\gamma}] = B^{\alpha}{}_{\dot{\alpha}\beta\gamma} = B^{\alpha}{}_{\dot{\alpha}(\beta\gamma)}\,,\\
&[Q_{\varepsilon\dot{\alpha}}\,,\,B^{\varepsilon}{}_{\dot{\beta}}{}^{\dot{\gamma}}{}_{\alpha}] = B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}{}_{\alpha} = B_{(\dot{\alpha}\dot{\beta})}{}^{\dot{\gamma}}{}_{\alpha}\,, && \{Q_{\varepsilon\dot{\alpha}}\,,\,B^{\varepsilon}{}_{\dot{\beta}\alpha\beta}\} = B_{\dot{\alpha}\dot{\beta}\alpha\beta} = B_{(\dot{\alpha}\dot{\beta})(\alpha\beta)}\,.
\end{aligned} \tag{4.25}$$

Note that the tracelessness follows from (4.23).

The super-commutators between the negative supercharges and the brane charges can be also obtained recurrently, floor by floor, using the above expressions for the brane charges and the superalgebra itself, (4.16). They are

$$
\begin{aligned}
&[\bar{Q}^{\kappa\dot{\kappa}}, B_{\dot{\alpha}\dot{\beta}\alpha\beta}] = \tfrac{72}{5}\delta^{\kappa}_{\alpha}B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\kappa}}{}_{\beta} + \tfrac{72}{5}\delta^{\kappa}_{\beta}B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\kappa}}{}_{\alpha} - \tfrac{72}{5}\delta^{\dot{\kappa}}_{\dot{\alpha}}B^{\kappa}{}_{\dot{\beta}\alpha\beta} - \tfrac{72}{5}\delta^{\dot{\kappa}}_{\dot{\beta}}B^{\kappa}{}_{\dot{\alpha}\alpha\beta}\,,\\
&\{\bar{Q}^{\kappa\dot{\kappa}}, B^{\alpha}{}_{\dot{\alpha}\beta\gamma}\} = 10\delta^{\dot{\kappa}}_{\dot{\alpha}}B^{\kappa\alpha}{}_{\beta\gamma} - 15\delta^{\kappa}_{\beta}B^{\alpha}{}_{\dot{\alpha}}{}^{\dot{\kappa}}{}_{\gamma} - 15\delta^{\kappa}_{\gamma}B^{\alpha}{}_{\dot{\alpha}}{}^{\dot{\kappa}}{}_{\beta} + 3\delta^{\alpha}_{\beta}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\kappa}}{}_{\gamma} + 3\delta^{\alpha}_{\gamma}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\kappa}}{}_{\beta}\,,\\
&\{\bar{Q}^{\kappa\dot{\kappa}}, B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}{}_{\alpha}\} = 10\delta^{\kappa}_{\alpha}B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\kappa}} - 15\delta^{\dot{\kappa}}_{\dot{\alpha}}B^{\kappa}{}_{\dot{\beta}}{}^{\dot{\gamma}}{}_{\alpha} - 15\delta^{\dot{\kappa}}_{\dot{\beta}}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\gamma}}{}_{\alpha} + 3\delta^{\dot{\gamma}}_{\dot{\alpha}}B^{\kappa}{}_{\dot{\beta}}{}^{\dot{\kappa}}{}_{\alpha} + 3\delta^{\dot{\gamma}}_{\dot{\beta}}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\kappa}}{}_{\alpha}\,,\\
&[\bar{Q}^{\kappa\dot{\kappa}}, B^{\alpha}{}_{\dot{\beta}}{}^{\dot{\alpha}}{}_{\beta}] = -\tfrac{32}{3}\delta^{\dot{\kappa}}_{\dot{\beta}}B^{\alpha\kappa\dot{\alpha}}{}_{\beta} - \tfrac{32}{3}\delta^{\kappa}_{\beta}B^{\alpha}{}_{\dot{\beta}}{}^{\dot{\alpha}\dot{\kappa}} + \tfrac{8}{3}\delta^{\dot{\alpha}}_{\dot{\beta}}B^{\alpha\kappa\dot{\kappa}}{}_{\beta} + \tfrac{8}{3}\delta^{\alpha}_{\beta}B^{\kappa}{}_{\dot{\beta}}{}^{\dot{\alpha}\dot{\kappa}}\,,\\
&[\bar{Q}^{\kappa\dot{\kappa}}, B_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}}] = -16\delta^{\dot{\kappa}}_{\dot{\alpha}}B^{\kappa}{}_{\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}} - 16\delta^{\dot{\kappa}}_{\dot{\beta}}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\gamma}\dot{\delta}} - 4\delta^{\dot{\gamma}}_{\dot{\alpha}}B^{\kappa}{}_{\dot{\beta}}{}^{\dot{\delta}\dot{\kappa}} - 4\delta^{\dot{\gamma}}_{\dot{\beta}}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\delta}\dot{\kappa}} + 4\delta^{\dot{\delta}}_{\dot{\alpha}}B^{\kappa}{}_{\dot{\beta}}{}^{\dot{\gamma}\dot{\kappa}} + 4\delta^{\dot{\delta}}_{\dot{\beta}}B^{\kappa}{}_{\dot{\alpha}}{}^{\dot{\gamma}\dot{\kappa}}\,,\\
&[\bar{Q}^{\kappa\dot{\kappa}}, B^{\alpha\beta}{}_{\gamma\delta}] = 16\delta^{\kappa}_{\gamma}B^{\alpha\beta\dot{\kappa}}{}_{\delta} + 16\delta^{\kappa}_{\delta}B^{\alpha\beta\dot{\kappa}}{}_{\gamma} - 4\delta^{\alpha}_{\gamma}B^{\kappa\beta\dot{\kappa}}{}_{\delta} - 4\delta^{\alpha}_{\delta}B^{\kappa\beta\dot{\kappa}}{}_{\gamma} + 4\delta^{\beta}_{\gamma}B^{\kappa\alpha\dot{\kappa}}{}_{\delta} + 4\delta^{\beta}_{\delta}B^{\kappa\alpha\dot{\kappa}}{}_{\gamma}\,,\\
&\{\bar{Q}^{\kappa\dot{\kappa}}, B^{\alpha}{}_{\dot{\gamma}}{}^{\dot{\alpha}\dot{\beta}}\} = -12\delta^{\dot{\kappa}}_{\dot{\gamma}}B^{\alpha\kappa\dot{\alpha}\dot{\beta}} - 4\delta^{\dot{\alpha}}_{\dot{\gamma}}B^{\alpha\kappa\dot{\beta}\dot{\kappa}} + 4\delta^{\dot{\beta}}_{\dot{\gamma}}B^{\alpha\kappa\dot{\alpha}\dot{\kappa}}\,,\\
&\{\bar{Q}^{\kappa\dot{\kappa}}, B^{\alpha\beta\dot{\alpha}}{}_{\gamma}\} = 12\delta^{\kappa}_{\gamma}B^{\alpha\beta\dot{\alpha}\dot{\kappa}} + 4\delta^{\alpha}_{\gamma}B^{\beta\kappa\dot{\alpha}\dot{\kappa}} - 4\delta^{\beta}_{\gamma}B^{\alpha\kappa\dot{\alpha}\dot{\kappa}}\,,\\
&[\bar{Q}^{\kappa\dot{\kappa}}, B^{\alpha\beta\dot{\alpha}\dot{\beta}}] = 0\,.
\end{aligned}
\tag{4.26}
$$

Note that the $D1$ brane charge, $Z_{\mu a}$, as well as the top floor brane charge, $B_{\dot{\alpha}\dot{\beta}\alpha\beta}$, are annihilated by eight real supercharges, which shows that the adjoint supermultiplet formed by the brane charges is "8/32 BPS multiplet".

5. Comments

5.1 Translation to the $\mathcal{N}=4$ superalgebra in four dimensions

In terms of the twelve dimensional conventions introduced in Section 2, the fully extended $AdS_5 \times S^5$ superalgebra, (4.1), reads

$$
\begin{aligned}
\{\mathcal{Q}, \bar{\mathcal{Q}}\} = P_{13}\Big[\, & i\Gamma^{\mu\nu}M_{\mu\nu} - i\Gamma^{ab}M_{ab} + \tfrac{1}{4}\Gamma^{\mu\nu ab}\Gamma^{(7)}H_{\mu\nu ab} + P_7\Gamma^{\mu a}Z_{\mu a} - \Gamma^{\mu a}P_7 Z_{\mu a}{}^{\dagger}\\
&+ \tfrac{1}{144}P_7\Gamma^{\mu\nu\lambda abc}Z^{-}_{\mu\nu\lambda abc} - \tfrac{1}{144}\Gamma^{\mu\nu\lambda abc}P_7 Z^{-}_{\mu\nu\lambda abc}{}^{\dagger}\Big]P_{13}\,,
\end{aligned}
\tag{5.1}
$$

where $\Gamma^{(7)} = i\Gamma^{123456}$ and $P_7 = \frac{1}{2}(1+\Gamma^{(7)})$.

In order to translate our results to the four dimensional language, we need to write all the higher dimensional objects in terms of the four dimensional conventions. For the gamma matrices we refer (A.1) in Appendix. For the so(2, 4) generators we decompose them into the four dimensional Lorentz generators, $\hat{M}_{mn}$, momenta, P_m, special conformal transformation generators, K_m and Dilation, D, with $m,n=0,1,2,3$,

$$\hat{M}_{mn}=M_{2+m\,2+n}\,,\quad P_m=-M_{1\,m+2}+M_{m+2\,6}\,,\quad K_m=M_{1\,m+2}+M_{m+2\,6}\,,\quad D=M_{1\,6}\,. \tag{5.2}$$

The twelve dimensional Majorana-Weyl supercharge, $\mathcal{Q}$, consists of the four dimensional ordinary supercharges, q, $\bar{q}=q^\dagger$, and the conformal supercharges, s, $\bar{s}=s^\dagger$. As they have the opposite mass dimensions, each of them can be singled out by the projection operator, $\frac{1}{2}(1\mp\Gamma_{16})$. In our choice of the gamma matrices (A.1), $Q_{1\dot{\alpha}},Q_{2\dot{\alpha}},\bar{Q}^{3\dot{\alpha}},\bar{Q}^{4\dot{\alpha}}$ correspond to the ordinary supercharges so that

$$Q_{\alpha\dot{\alpha}}=(\,q_{1\dot{\alpha}}\,,\,q_{2\dot{\alpha}}\,,\,-i\bar{s}^1{}_{\dot{\alpha}}\,,\,-i\bar{s}^2{}_{\dot{\alpha}}\,)^t\,,\qquad \bar{Q}^{\alpha\dot{\alpha}}=(\,s^{1\dot{\alpha}}\,,\,s^{2\dot{\alpha}}\,,\,i\bar{q}_1{}^{\dot{\alpha}}\,,\,i\bar{q}_2{}^{\dot{\alpha}}\,)\,. \tag{5.3}$$

Provided the above dictionary, our extended $AdS_5\times S^5$ superalgebra, (4.16), (4.24), (4.26), leads to a noncentral extension of the four dimensional $\mathcal{N}=4$ superconformal algebra.[9]

5.2 On super Yang-Mills theory and more

In the standard approach to the $\mathcal{N}=4$ super Yang-Mills theory, different vacuum expectation values (vev) of the Higgs correspond to the different theory. Especially for the nonzero values, the conformal symmetry is spontaneously broken, and the Hilbert space parameterized by the Higgs vevs is not invariant under the conformal generators.[10] The truncation of our extended su$(2,2|4)$ superalgebra to an extended four dimensional $\mathcal{N}=4$ super Poincaré algebra can be achieved by the projection operator, $\frac{1}{2}(1-\Gamma_{16})$. Essentially the extended super Poincaré algebra reads, in terms of the ten dimensional gamma matrices, (A.5), and Majorana-Weyl supercharge, (A.9),

$$\begin{aligned}\{\hat{Q},\bar{\hat{Q}}\}=2P_+\Big[\,&\hat{\Gamma}^mP_m+\tfrac{1}{2}\hat{\Gamma}_{(5)}\hat{\Gamma}^{mab}H_{mab}+\hat{\Gamma}^aT^e_a+\hat{\Gamma}_{(5)}\hat{\Gamma}^aT^g_a\\&+\tfrac{1}{24}\hat{\Gamma}^{mnabc}T^{e-}_{mnabc}+\tfrac{1}{24}\hat{\Gamma}_{(5)}\hat{\Gamma}^{mnabc}T^{g-}_{mnabc}\Big]P_-\,,\end{aligned} \tag{5.4}$$

where $\bar{\hat{Q}}=\hat{Q}^\dagger\hat{\Gamma}_0$, $P_\pm=\frac{1}{2}(1\pm\hat{\Gamma}^{(11)})$, $\hat{\Gamma}_{(5)}=\hat{\Gamma}_{0123}$, and all the brane charges are real having the origin,

$$\begin{aligned}Z_{1a}+Z_{6a}=2(T^g_a-iT^e_a)\,,\qquad Z^-_{1\,m+2\,n+2\,abc}=2(T^{g-}_{mnabc}-iT^{e-}_{mnabc})\,,\\ H_{mab}=\tfrac{1}{2}(H_{1\,m+2\,ab}+H_{6\,m+2\,ab})\,.\end{aligned} \tag{5.5}$$

In particular, Osborn identified T^g_a and T^e_a as the electric[11] and magnetic charges by investigating the supersymmetry transformation of the super-current in $\mathcal{N}=4$ super

[9] Our conventions have been chosen to agree with [19] for the unextended sector.

[10] Strictly speaking, this is for the super-Yang-Mills theory on $R^{3,1}$. For the theories on compact spaces, one should integrate over different vevs of the Higgs due to the normalizability of the zero modes.

[11] This electric charge should not be confused as the gauge symmetry Noether charge. The latter is given by the Gauss' law or the equation of motion for A_0.

Yang-Mills theory [4]

$$T^g_a = \int \mathrm{d}\vec{S}\cdot\mathrm{tr}(\vec{B}\Phi_a)\,, \qquad\qquad T^e_a = \int \mathrm{d}\vec{S}\cdot\mathrm{tr}(\vec{E}\Phi_a)\,. \tag{5.6}$$

Straightforward manipulation can show that the *ordinary* supersymmetry transformation of the electro-magnetic charges do not vanish even at the on-shell level.[12] Our results, (4.24) and (4.26), also confirm this, since the brane charges on the ground floor of the adjoint supermultiplet are annihilated by eight real supercharges out of 32. Surprisingly this means the noncentral property of the electro-magnetic charge, in contrast to the conventional wisdom due to the Haag-Lopuszanski-Sohnius theorem [9]. The original argument for the electro-magnetic charge to be central is based on the Coleman-Mandula theorem [10] on all the possible *symmetry generators* in the quantum field theories. The point for the brane charges we discuss in the paper is that they are not symmetry generators nor Noether charges. Rather, they are topological living at the spatial infinity only, and hence free from the constraint by the Haag-Lopuszanski-Sohnius theorem.

Nevertheless, for the ordinary supersymmetric monopole configurations, our new brane charges, at least for those coming from the ordinary supercharges, annihilate the corresponding quantum states. Although the classical monopole or solitons are given by the bosonic configurations only, at the quantum level the fermions act nontrivially on the quantum states essentially to respect the second quantization of them. In other words, there is no quantum state which is annihilated by all the fermions, and one should always keep in mind the fermions. Now for the supersymmetric monopoles, the fermionic zero modes are given by the broken ordinary supersymmetry transformations of the gauginos, $\lambda \sim F_{AB}\hat{\Gamma}^{AB}\varepsilon$. The expressions for the new brane charges coming from the ordinary supercharges contain the gauginos, the field strengths, and the derivatives of the Higgs, but not the Higgs itself, so that, from the asymptotic behavior, one can expect that the corresponding new brane charges annihilate the monopole states.

It will be very interesting to find out novel configurations which have nontrivial realization of the new brane charges, either on the super Yang-Mills side or on the supergravity side. In the former case, the full expressions for the brane charges coming from all the ordinary as well as the conformal supercharges are desirable, which deserves a separate analysis. Another thing to be done is to classify the representations of the extended AdS superalgebra as in [28, 29]. Perhaps a more detailed study of the extended superalgebra may shed light on the nonperturbative aspects of the string/M-theory on the $AdS_5 \times S^5$ background.

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[12] Even Eq.(4.14) does not hold in general. This seems to imply that the expression of $Z_{\mu a}$ further decomposes into several sectors which belong to different irreducible representations corresponding to various configurations, $(D1, D3)$, $(D1, D3, D5)$, $(D1, D3, D3)$, etc.

A. Decomposition of the gamma matrices for lower dimensions

A.1 For the four dimensional $\mathcal{N}=4$ superconformal algebra

In order to translate our results to the four dimensional language, we need to write all the higher dimensional objects in terms of the four dimensional conventions. First we let the six dimensional gamma matrices satisfying (2.3) and (2.5) be

$$\begin{aligned} \rho_1 &= \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \qquad \rho_{m+2} = \begin{pmatrix} \sigma_m & 0 \\ 0 & \bar{\sigma}_m \end{pmatrix}, \qquad \rho_6 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \\ \bar{\rho}_1 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \bar{\rho}_{m+2} = \begin{pmatrix} \bar{\sigma}_m & 0 \\ 0 & \sigma_m \end{pmatrix}, \qquad \bar{\rho}_6 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{A.1}$$

where the 2×2 matrices, $\sigma_m = (+1, \vec{\tau})$, $\bar{\sigma}_m = (-1, \vec{\tau})$, $m = 0,1,2,3$, satisfy the Clifford algebra of the four dimensional spacetime on which the super Yang-Mills exists,

$$\sigma_m \bar{\sigma}_n + \sigma_n \bar{\sigma}_m = 2\hat{\eta}_{mn}\,, \qquad \hat{\eta} = \mathrm{diag}(-+++)\,. \tag{A.2}$$

The four dimensional gamma matrices are then

$$\hat{\gamma}_m = \begin{pmatrix} 0 & \sigma_m \\ \bar{\sigma}_m & 0 \end{pmatrix} = B_4{}^{-1}(\hat{\gamma}_m)^* B_4\,, \qquad B_4 = \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix}, \tag{A.3}$$

where ϵ is the usual 2×2 anti-symmetric matrix satisfying $\sigma_m^t = \sigma_m^* = \epsilon \bar{\sigma}_m \epsilon$, $\epsilon_{12} = 1$.

The above ρ_μ matrices are not anti-symmetric, and to make them so one needs to take some transformations such as

$$\rho_\mu \;\rightarrow\; \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix} \rho_\mu \;\rightarrow\; U \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix} \rho_\mu U^t\,, \qquad U = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \tau_1 & -i\tau_1 \\ \tau_1 & i\tau_1 \end{pmatrix}. \tag{A.4}$$

The first transformation makes ρ_μ's anti-symmetric, while the next similarity transformation involving the unitary matrix, U, further ensures that the representation of the Cartan subalgebra is diagonal, exactly as (3.7).

A.2 For the four dimensional super Poincaré algebra

For the truncation of our extended $AdS_5 \times S^5$ superalgebra to an extended super Poincaré algebra in four dimensions, we write the twelve dimensional gamma matrices in terms of the ten dimensional ones, $\hat{\Gamma}^A$, $A = m, a$,

$$\begin{aligned} \Gamma^1 &= \epsilon \otimes 1\,, \qquad & \Gamma^6 &= \tau^1 \otimes 1\,, \\ \Gamma^{m+2} &= \tau^3 \otimes \hat{\Gamma}^m\,, \qquad & \Gamma^a &= \tau^3 \otimes \hat{\Gamma}^a\,. \end{aligned} \tag{A.5}$$

Further the $10D$ gamma matrices decompose into the $4D$ and $6D$ ones,

$$\hat{\Gamma}^m = \hat{\gamma}^m \otimes \gamma^{(7)}\,, \qquad \hat{\Gamma}^a = 1 \otimes \gamma^a\,, \tag{A.6}$$

satisfying

$$(\hat{\Gamma}^{\hat{M}})^* = -B_{10}\hat{\Gamma}^{\hat{M}}{B_{10}}^{-1}\,, \qquad B_{10} = B_4 \otimes B_6\,, \qquad B_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.7}$$

$10D$ chirality matrix reads

$$\hat{\Gamma}^{(11)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \gamma^{(7)}\,. \tag{A.8}$$

Majorana-Weyl supercharge carries the $4D$ and $6D$ chiral indices of the same chirality,

$$\hat{Q} = \hat{\Gamma}^{(11)}\hat{Q} = B^{-1}\hat{Q}^* = \left(q_{1\dot{\alpha}}\,,\ q_{2\dot{\alpha}}\,,\ (\epsilon^{-1}\bar{q}^t)^{1\dot{\alpha}}\,,\ (\epsilon^{-1}\bar{q}^t)^{2\dot{\alpha}}\right)^t\,. \tag{A.9}$$

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