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Abstract

We construct an iterative procedure to compute the vertex operators of the closed superstring in the covariant formalism given a solution of IIA/IIB supergravity. The manifest supersymmetry allows us to construct vertex operators for any generic background in presence of Ramond-Ramond (RR) fields. We extend the procedure to all massive states of open and closed superstrings and we identify two new nilpotent charges which are used to impose the gauge fixing on the physical states. We solve iteratively the equations of the vertex for linear x-dependent RR field strengths. This vertex plays a role in studying non-constant C-deformations of superspace. Finally, we construct an action for the free massless sector of closed strings, and we propose a form for the kinetic term for closed string field theory in the pure spinor formalism.

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Contents

1. Introduction

Motivated by the increasing interest in the covariant techniques for computation of the amplitudes in string theory, we provide a calculation scheme for vertex operators in pure spinor approach string theory in 10 dimensions [1,2]. The covariant methods turned out to be superior in order to derive manifest super-Poincaré invariant effective actions and to handle generic backgrounds (for example with RR fields such as $AdS_5 \times S^5$ and pp-waves) avoiding, for instance, GSO projections, sums over spin structures and light-cone contact terms. However, since the amount of symmetries that are manifest in the covariant formulation increases, also the number of auxiliary fields increases and a useful technique to compute the basic ingredients is needed. Here we provide such a procedure and some applications. First of all we give a brief review of the open superstring formalism, we explain the main idea and we outline the rest of the paper.

In the case of the open superstring, the massless sector is described by a vertex operator $\mathcal{V}^{(1)} = \lambda^\alpha A_\alpha$ at ghost number one where λ^α is a pure spinor (defined in appendix A) and $A_\alpha(x,\theta)$ is the spinorial component of the superconnection. The superfield A_α is completely expressed in terms of the gauge field $a_m(x)$ and the gluino $\psi^\alpha(x)$, for example as

$$A_\alpha(x,\theta) = \frac{1}{2}(\gamma^m\theta)_\alpha a_m(x) + \frac{1}{3}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma\psi^\gamma(x) + \mathcal{O}(\theta^3). \tag{1.1}$$

The vertex operator $\mathcal{V}^{(1)}$ belongs to the cohomology of the BRST charge $Q = \int d\sigma \lambda^\alpha d_{\sigma\alpha}$, where $d_{\sigma\alpha}$ is defined in app. A, if and only if the components of A_α satisfy the linear

Maxwell and Dirac equations

$$\partial^m(\partial_m a_n - \partial_n a_m) = 0\,, \qquad \gamma^m_{\alpha\beta}\partial_m\psi^\beta = 0\,. \tag{1.2}$$

The contributions $\mathcal{O}(\theta^3)$ are given in terms of the derivatives of a_m and ψ and are completely fixed by the equations of motion

$$D_{(\alpha}A_{\beta)} - \gamma^m_{\alpha\beta}A_m = 0 \tag{1.3}$$

given in [3], where A_m is the vectorial part of the superconnection and $D_\alpha = \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$ is the superderivative. The lowest components of A_α in (1.1) are eliminated by a gauge fixing condition.

Even though the computation of all terms in the expansion of A_α seems a straightforward procedure, technically it is rather involved. However, there exists a powerful technique which simplifies the task. The main idea is to choose a suitable gauge fixing such as for instance

$$\theta^\alpha A_\alpha(x,\theta) = 0\,, \tag{1.4}$$

which reduces the independent components in the superfield A_α. This choice[3] fixes part of the super-gauge transformation $\delta A_\alpha = D_\alpha\Omega$, where Ω is a scalar superfield with ghost number zero. To reach the gauge (1.4), we have to impose $\theta^\alpha(A_\alpha + \delta A_\alpha) = 0$, which implies that $\theta^\alpha D_\alpha\Omega = -\theta^\alpha A_\alpha$. Expanding Ω as $\Omega = \sum_{n\geq 0}\Omega_{[\alpha_1\ldots\alpha_n]}\theta^{\alpha_1}\ldots\theta^{\alpha_n}$, all components with $n \geq 1$ are fixed except the lowest component Ω_0, which corresponds to the usual bosonic gauge transformation of Maxwell theory.

Acting with D_α on (1.4) and using the equations of motion (1.3), one gets the recursive relations

$$\begin{aligned}
&(1+\mathbf{D})A_\alpha = (\gamma^m\theta)_\alpha A_m \\
&\mathbf{D}A_m = (\gamma_m\theta)_\gamma W^\gamma \\
&\mathbf{D}W^\alpha = -\frac{1}{4}(\gamma^{mn}\theta)^\alpha F_{mn} \\
&\mathbf{D}F_{mn} = -(\gamma_{[m}\theta)_\gamma\partial_{n]}W^\gamma
\end{aligned} \tag{1.5}$$

where $\mathbf{D} \equiv \theta^\alpha\partial_\alpha$. So, given the zero-order component of A_m, we can compute the order-θ component of A_α. The same can be done for A_m, the spinorial field strength W^α and the bosonic curvature $F_{mn} = \partial_{[m}A_{n]}$ making use of the other three equations. This renders

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[3] The following gauge condition has a counterpart in bosonic string theory: $x^m A_m(x) = 0$. This fixes the gauge invariance under $\delta A_m = \partial_m\omega(x)$ and it coincides with the Lorentz gauge in momentum space $\partial_{p_\mu}\tilde{A}_m = 0$. The gauge fixing yields the equation $(1+x^n\partial_n)A_m = x^n F_{mn}$ which can be solved directly by inverting $(1+x^n\partial_n)$ and obtaining $A_m = \int^x d^{26}y[(1+y^p\partial_p)^{-1}(y^n F_{mn}(y))]$.

the task of computing all components of A_α in terms of initial data $A_m(x) = a_m(x) + \mathcal{O}(\theta)$ and $W^\alpha(x) = \psi^\alpha(x) + \mathcal{O}(\theta)$ a purely algebraic problem ([4] and [5]). Moreover, one is able to compute all components of the superfields appearing in the (descent) ghost-number-zero vertex operator $\mathcal{V}^{(0)}_\sigma$

$$\mathcal{V}^{(0)}_\sigma = \partial_\sigma \theta^\alpha A_\alpha + \Pi^m_\sigma A_m + d_{\sigma\alpha} W^\alpha + \frac{1}{2} N^{mn}_\sigma F_{mn} \,, \tag{1.6}$$

which satisfies the descent equation $[Q, \mathcal{V}^{(0)}_\sigma] = \partial_\sigma \mathcal{V}^{(1)}$. Here σ is the boundary worldsheet coordinate and $N^{mn}_\sigma = \frac{1}{2} w_\sigma \gamma^{mn} \lambda$ is the pure spinor part of the Lorentz current. The operators Π^m_σ and $d_{\sigma\alpha}$ are the supersymmetric line element and the fermionic constraint of the Green-Schwarz superstring [6], respectively.

In the present paper, we apply the same technique to IIA/IIB supergravity. We start from the vertex operators for closed superstrings, we derive the complete set of equations from the BRST cohomology, we define all curvatures and gauge transformations. Then, we impose a set of gauge fixing conditions to remove the lowest components of the superfields and we derive an iterative procedure to compute all components. We show that we need a further gauge fixing to fix the reducible gauge symmetries and we show that all chosen gauges can indeed be reached.

The procedure for closed strings is original by itself, but, more importantly, the present analysis leads to a generalization of (1.4) to all vertex operators, associated to both massless and massive states. Indeed, we will show that the gauge fixing (1.4) can be written in terms of a new nilpotent charge $\mathcal{K}$ (with negative ghost number) as follows

$$\{\mathcal{K}, \mathcal{V}^{(1)}\} = 0 \,. \tag{1.7}$$

This imitates the Siegel gauge in string field theory. When restricted to massless states, this generalized gauge fixing condition reduces to the gauge fixing (1.4) for open superstrings and to the corresponding gauge fixing for closed strings discussed in the text. When applied to massive states, (1.7) also leads to a suitable gauge fixing. In the paper, we explicitly derive the gauge conditions for the first massive state for the open superstring. Again, (1.7) fixes all auxiliary fields in terms of the physical on-shell data and eliminates the lowest components.

As an application of the computation technique we calculate the vertex operator of linear x-dependent RR field strength. This amounts to expanding the superfields for closed strings up to fourth order in powers of both θ and $\hat{\theta}$ (where the hatted quantities refer to the right-moving sector of the theory). This new vertex operator is the starting point for studying x-dependent C-deformations which might give new insight in the superspace structure of supergravity [7].

As pointed out in [8,9,10], in the RNS framework [11] vertex operators in the asymmetric picture are very useful to study the dynamics of D-branes. In fact, in the asymmetric picture the vertex operator is not only expressed in terms of the RR field strength, but it is also parametrized by the RR potential. The analysis in [8] shows that the off-shell vertex operators directly couple the RR potentials to the worldvolume of the D-brane. We present a new method to relax the superspace constraints by adding new auxiliary fields and constructing the corresponding vertex operators. [4] We show that it leads to a deformation of the superspace constraints (that, in 10 dimensions, force the physical fields to be on-shell), generalizing the method presented in [13] for N=1 D=10 SYM theory. The gauge transformations are studied and the vertex operators are expressed in terms of the RR potential.

In the present paper we only consider deformations (vertex operators) at first order in the coupling constant, neglecting the backreaction of background fields. For the complete sigma model for open, closed and heterotic strings see for example [14] and the one-loop computations [15].

The linearized form of supergravity equations written in terms of the BRST charges of the pure spinor sigma model gives us the framework to analyze some aspects of closed string field theory action. As is well-known, the action for closed string field theory has to take into account the presence of selfdual forms (for example the five form in type IIB supergravity). This can be done either by breaking explicitly the Lorentz invariance, or by admitting an infinite number of fields in the action [16]. We show that this action can be indeed constructed mimicking the bosonic closed string field theory action discussed in [17] (and in the references therein). We show that we can easily account for new fields which nevertheless do not propagate, and we check that the action has the correct symmetries leading to the complete BV action for type IIA/IIB supergravity.

In section 2, we derive the complete set of descent equations for vertex operators of closed superstrings and their gauge transformations. Moreover, we derive the consistency conditions for the gauge parameters. In section 3, we compute the complete set of equations in superspace and the equations of motion for the physical fields. In section 4, we construct the gauge transformations in terms of "physical" gauge parameters and we provide the gauge fixing conditions. Section 5 deals with the iterative procedure to extract the relations among supergravity fields and auxiliary fields. In section 6, we generalize the gauge conditions to massive states and we study the first massive state. The form of the vertex for non constant RR fields is given in section 7. Section 8 deals with an off-shell formulation of closed string vertex operators. Finally, in section 9 we make a proposal for the kinetic terms of a string field theory action for closed strings. In appendix A, we give some basic formulas for pure spinor superstring theory. In appendix B, we give the expansion of the auxiliary superfields up to order two in both fermionic coordinates,

[4] The construction of vertices with RR potentials in covariant formulation has been also discussed in [12]. There the authors consider constant RR potentials.

obtained by solving the iterative equations presented in section 5. In appendix C part of the superfields needed in the computation of the vertex for a linear x-dependent RR field strength are given.

2. Vertex Operators for Closed Superstrings

To compute correlation functions we need not only the ghost number $(1,1)$ local vertex $\mathcal{V}^{(1,1)}$, but we also need the integrated vertex operators $\int dz\ \mathcal{V}^{(0,1)}_z$, $\int d\bar z\ \mathcal{V}^{(1,0)}_{\bar z}$, and $\int dz\wedge d\bar z\ \mathcal{V}^{(0,0)}_{z\bar z}$. They satisfy the descent equations which we are going to discuss.

Introducing the notation $\mathcal{O}^{(a,b)}_{c,d}$ for local vertex operators with ghost number $a(b)$ in the left (right) sector and (anti)holomorphic indices $c(d)$, we identify

$$\mathcal{O}^{(1,1)}_{0,0}=\mathcal{V}^{(1,1)}\,,\quad \mathcal{O}^{(0,1)}_{1,0}=\mathcal{V}^{(0,1)}_z dz\,,\quad \mathcal{O}^{(1,0)}_{0,1}=\mathcal{V}^{(1,0)}_{\bar z} d\bar z\,,\quad \mathcal{O}^{(0,0)}_{1,1}=\mathcal{V}^{(0,0)}_{z\bar z} dz\wedge d\bar z\,. \tag{2.1}$$

The descent equations read[5]

$$[Q_L,\mathcal{O}^{(a,b)}_{c,d}]=\partial\,\mathcal{O}^{(a+1,b)}_{c-1,d}\,,\qquad [Q_R,\mathcal{O}^{(a,b)}_{c,d}]=\bar\partial\,\mathcal{O}^{(a,b+1)}_{c,d-1}\,, \tag{2.2}$$

where $\partial=dz\partial_z$ and $\bar\partial=d\bar z\partial_{\bar z}$ are the holomorphic and antiholomorphic differentials. Q_L and Q_R are the BRST charges for holomorphic and antiholomorphic sectors, satisfying the anticommutation relation $[Q_L,Q_R]=0$ (for their explicit form in terms of sigma model fields, see app. A). More explicitly, at the first level we have

$$[Q_L,\mathcal{V}^{(1,1)}]=0\,,\qquad [Q_R,\mathcal{V}^{(1,1)}]=0\,, \tag{2.3}$$

while at the next level we get

$$[Q_L,\mathcal{V}^{(0,1)}_z]=\partial_z\mathcal{V}^{(1,1)}\,,\qquad [Q_R,\mathcal{V}^{(0,1)}_z]=0\,, \tag{2.4}$$

$$[Q_R,\mathcal{V}^{(1,0)}_{\bar z}]=\partial_{\bar z}\mathcal{V}^{(1,1)}\,,\qquad [Q_L,\mathcal{V}^{(1,0)}_{\bar z}]=0\,,$$

and, finally,

$$[Q_L,\mathcal{V}^{(0,0)}_{z\bar z}]=\partial_z\mathcal{V}^{(1,0)}_{\bar z}\,,\qquad [Q_R,\mathcal{V}^{(0,0)}_{z\bar z}]=-\partial_{\bar z}\mathcal{V}^{(0,1)}_z\,. \tag{2.5}$$

The vertex operators $\mathcal{O}^{(a,b)}_{c,d}$ are expanded in powers of ghost fields λ^α and $\hat\lambda^{\hat\alpha}$ or in powers of the supersymmetric holomorphic and antiholomorphic 1-forms

$$\mathbf{X}_z=\left(\partial_z\theta^\alpha,\ \Pi^m_z,\ d_{z\alpha},\ \frac{1}{2}N^{mn}_z\right)\,,\qquad \hat{\mathbf{X}}_{\bar z}=\left(\partial_{\bar z}\hat\theta^{\hat\beta}\ ,\ \hat\Pi^p_{\bar z}\ ,\ \hat d_{\bar z\hat\beta}\ ,\ \frac{1}{2}\hat N^{pq}_{\bar z}\right)\,. \tag{2.6}$$

[5] We use the square backets to denote both commutation and anti-commutation relations. The difference is established by the nature of the operators involved in the relations.

The explicit expression of these 1-form operators in terms of sigma model fields is given in appendix A. The coefficients are superfields of the coordinates x^m, θ^α and $\hat\theta^{\hat\alpha}$. A further relation is obtained by acting from the left on the first equation of (2.5) with Q_R or on the second with Q_L. Using eqs. (2.4) and the commutation relations (A.3), one obtains

$$[Q_R,[Q_L,\mathcal{V}^{(0,0)}_{z\bar z}]] = \partial_z\partial_{\bar z}\mathcal{V}^{(1,1)}\,, \tag{2.7}$$

which turns out to be useful for the explicit computations in sec. 3.

Equations (2.3), (2.4), and (2.5) are invariant under the gauge transformations given by

$$\begin{gathered}\delta\mathcal{V}^{(1,1)} = [Q_L,\Lambda^{(0,1)}] + [Q_R,\Lambda^{(1,0)}]\\ \delta\mathcal{V}^{(1,0)}_{\bar z} = [Q_L,\tau^{(0,0)}_{\bar z}] + \partial_{\bar z}\Lambda^{(1,0)}\,, \qquad \delta\mathcal{V}^{(0,1)}_{z} = [Q_R,\tau^{(0,0)}_{z}] + \partial_{z}\Lambda^{(0,1)}\\ \delta\mathcal{V}^{(0,0)}_{z\bar z} = \partial_z\tau^{(0,0)}_{\bar z} - \partial_{\bar z}\tau^{(0,0)}_{z}\end{gathered} \tag{2.8}$$

where the zero forms $\Lambda^{(0,1)}$ and $\Lambda^{(1,0)}$ have ghost number $(1,0)$ and $(0,1)$ and are proportional to λ^α and $\hat\lambda^{\hat\alpha}$, and the coefficients are superfields. The holomorphic and antiholomorphic 1-forms $\tau^{(0,0)}_z$ and $\tau^{(0,0)}_{\bar z}$ are to be expanded in terms of powers of the 1-forms $\mathbf{X}_z$ and $\hat{\mathbf{X}}_{\bar z}$ given in (2.6) and their coefficients are again superfields.

In addition, the gauge parameters $\Lambda^{(0,1)}$, $\Lambda^{(1,0)}$, $\tau^{(0,0)}_z$ and $\tau^{(0,0)}_{\bar z}$ must satisfy the following consistency conditions

$$[Q_L,\Lambda^{(1,0)}] = 0 \qquad [Q_R,\Lambda^{(0,1)}] = 0\,, \tag{2.9}$$

and

$$[Q_L,\tau^{(0,0)}_z] + \partial_z\Lambda^{(1,0)} = 0 \qquad [Q_R,\tau^{(0,0)}_{\bar z}] + \partial_{\bar z}\Lambda^{(0,1)} = 0\,. \tag{2.10}$$

These equations resemble the descent equations for the open string vertex operator $\mathcal{V}^{(1)} = \lambda^\alpha A_\alpha$, but in that case there are boundary conditions for the fermionic fields: $\theta^\alpha(z) = \hat\theta^{\hat\alpha}(z)$ at $z=\bar z$.

Equations (2.9) and (2.10) are further invariant under the gauge transformations

$$\begin{gathered}\delta\Lambda^{(1,0)} = [Q_L,\Upsilon^{(0,0)}]\,, \qquad \delta\Lambda^{(0,1)} = [Q_R,\hat\Upsilon^{(0,0)}]\,,\\ \delta\tau^{(0,0)}_z = -\partial_z\Upsilon^{(0,0)}\,, \qquad \delta\tau^{(0,0)}_{\bar z} = -\partial_{\bar z}\hat\Upsilon^{(0,0)}\,.\end{gathered} \tag{2.11}$$

where $\Upsilon^{(0,0)}$ and $\hat\Upsilon^{(0,0)}$ are generic superfields. However, consistency with (2.8) imposes $\Upsilon^{(0,0)} = \hat\Upsilon^{(0,0)}$. The superfield $\Upsilon^{(0,0)}$ will be useful to define a suitable gauge fixing procedure and to take into account the reducible gauge symmetry of the NS-NS two form of 10-dimensional supergravity.

To derive equations (2.4) we can view the vertex operators $\mathcal{V}_z^{(0,1)}$ and $\mathcal{V}_{\bar z}^{(1,0)}$ as deformations of the BRST charges (A.2)

$$Q_L \to Q_L + \oint d\bar z\, \mathcal{V}_{\bar z}^{(1,0)}\,, \qquad\qquad Q_R \to Q_R + \oint dz\, \mathcal{V}_z^{(0,1)}\,, \tag{2.12}$$

and the vertex operator $\mathcal{V}_{z\bar z}^{(0,0)}$ as the deformation of the action

$$S \to S + \int dz d\bar z\, \mathcal{V}_{z\bar z}^{(0,0)}\,. \tag{2.13}$$

Eqs. (2.3) are derived by requiring the nilpotency of the new charges and their anticommutation relations.

In terms of the vertex operators $\mathcal{O}_{c,d}^{(a,b)}$, the amplitudes on the sphere are defined in [18] as

$$\mathcal{A}_{n+3} = \Big\langle\Big\langle \mathcal{V}^{(1,1)}(z_1,\bar z_1)\mathcal{V}^{(1,1)}(z_2,\bar z_2)\mathcal{V}^{(1,1)}(z_3,\bar z_3)\prod_n \int dz d\bar z \mathcal{V}^{(0,0)}\Big\rangle\Big\rangle \tag{2.14}$$

where the three unintegrated vertex operators are needed to fix the $SL(2,\mathbf{C})$ invariance on the sphere. An unintegrated vertex $\mathcal{V}^{(1,1)}(z_1,\bar z_1)$ can be replaced by a product of $(1,0)$ and $(0,1)$ vertices $\oint dz \mathcal{V}_z^{(0,1)} \oint d\bar z \mathcal{V}_{\bar z}^{(1,0)}$ which has the same total ghost number and the same total conformal spin as the original vertex $\mathcal{V}^{(1,1)}$. In [18] supersymmetry and gauge invariance were proven under the assumption that the prescription for the zero modes is the following

$$\langle\langle \mathcal{V}^{(3,3)}\rangle\rangle = 1 \tag{2.15}$$

where

$$\mathcal{V}^{(3,3)} = (\lambda_0\gamma^m\theta_0\lambda_0\gamma^n\theta_0\lambda_0\gamma^p\theta_0\theta_0\gamma_{mnp}\theta_0)(\hat\lambda_0\gamma^m\hat\theta_0\hat\lambda_0\gamma^n\hat\theta_0\hat\lambda_0\gamma^p\hat\theta_0\hat\theta_0\gamma_{mnp}\hat\theta_0)\,.$$

3. Equations of Motion

In the present section we derive the equations of motion for the massless background fields in superspace from the BRST cohomology of the superstring. Let us start from the simplest equations (2.3) for the vertex $\mathcal{V}^{(1,1)}$ whose general expression is

$$\mathcal{V}^{(1,1)} = \lambda^\alpha A_{\alpha\hat\beta}\hat\lambda^{\hat\beta}\,. \tag{3.1}$$

The superfield $A_{\alpha\hat\beta}(x,\theta,\hat\theta)$ satisfies the equations of motion [3]

$$\gamma^{\alpha\beta}_{mnopq} D_\alpha A_{\beta\hat\beta} = 0\,, \qquad \gamma^{\hat\alpha\hat\beta}_{mnopq} D_{\hat\alpha} A_{\alpha\hat\beta} = 0\,, \tag{3.2}$$

where $\gamma^{\alpha\beta}_{mnopq}$ is the antisymmetrized product of five gamma matrices. The pure spinor conditions imply that only the 5-form parts of the $D_\alpha A_{\beta\hat{\beta}}$ and $D_{\hat{\alpha}} A_{\alpha\hat{\beta}}$ are indeed constrained [1]. By using Bianchi identities, one can show that they yield the type IIA/IIB supergravity equations of motion at the linearized level. All auxiliary fields present in the superfield $A_{\alpha\hat{\beta}}$ are fixed by eqs.(3.2).

As outlined before, one can use different types of vertices to simplify the computations. Integrated vertices are written in terms of a huge number of different superfields, whose components are completely fixed by the equations of motion. As a result, these vertices are quite complicated espressions.

The set of superfields needed to compute $\mathcal{V}^{(0,0)}, \dots, \mathcal{V}^{(1,1)}$ can be grouped into the following matrix

$$\mathbf{A} = \begin{bmatrix} A_{\alpha\hat{\beta}} & A_{\alpha p} & E_\alpha{}^{\hat{\beta}} & \Omega_{\alpha,pq} \\ A_{m\hat{\beta}} & A_{mp} & E_m{}^{\hat{\beta}} & \Omega_{m,pq} \\ E^\alpha{}_{\hat{\beta}} & E^\alpha{}_p & P^{\alpha\hat{\beta}} & C^\alpha{}_{pq} \\ \Omega_{mn,\hat{\beta}} & \Omega_{mn,p} & C_{mn}{}^{\hat{\beta}} & S_{mn,pq} \end{bmatrix} \tag{3.3}$$

The first components of A_{mp}, $E_m{}^{\hat{\beta}}$, $E^\alpha{}_p$ and $P^{\alpha\hat{\beta}}$ are identified with the supergravity fields as follows

$$A_{mp} = g_{mp} + b_{mp} + \eta_{mp}\phi + \mathcal{O}(\theta, \hat{\theta}) \,, \tag{3.4}$$

$$E_m{}^{\hat{\beta}} = \psi_m{}^{\hat{\beta}} + \mathcal{O}(\theta, \hat{\theta}) \,, \qquad E^\alpha{}_p = \psi^\alpha{}_p + \mathcal{O}(\theta, \hat{\theta}) \,,$$

$$P^{\alpha\hat{\beta}} = f^{\alpha\hat{\beta}} + \mathcal{O}(\theta, \hat{\theta}) \,.$$

The fields g_{mn}, b_{mn}, ϕ, $\psi^\alpha{}_p$, $\psi_m{}^{\hat{\beta}}$ and $f^{\alpha\hat{\beta}}$ are the graviton, the NS-NS two-form, the dilaton, the two gravitinos (the gamma-traceless part of $\psi^\alpha{}_p$, $\psi_m{}^{\hat{\beta}}$), the two dilatinos (the gamma-trace part of $\psi^\alpha{}_p$, $\psi_m{}^{\hat{\beta}}$) and the RR field strengths. IIA and IIB differ in the chirality of the two spinorial indices α and $\hat{\alpha}$. This changes the type of RR fields present in the spectrum. The first components of the superfields $\Omega_{m,pq}$ ($\Omega_{mn,p}$), $C_{mn}{}^{\beta}$ ($C^\alpha{}_{pq}$) and $S_{mn,pq}$ are identified with the linearized gravitational connection Γ^t_{rs}, the curvature of the gravitinos and the linearized Riemann tensor, respectively. The remaining superfields are the spinorial partners of the above superfields. In ten dimensions, the superspace constraints together with the Bianchi identities imply the supergravity equations of motion [19]. Those constraints are given in terms of the spinorial components $A_{\alpha\hat{\beta}}$, $A_{\alpha p}$, $E_\alpha{}^{\hat{\beta}}$ and $\Omega_{\alpha,pq}$. The structure of superspace formulation of type IIA and IIB supergravity in the present framework is also discussed in [14].

Given the vectors $\mathbf{X}_z$ and $\hat{\mathbf{X}}_{\bar{z}}$ (see (2.6)) we can explicitly write the vertex operator

$\mathcal{V}^{(0,0)}_{z\bar z} = \mathbf{X}^T_z \mathbf{A} \hat{\mathbf{X}}_{\bar z}$ as

$$
\begin{aligned}
\mathcal{V}^{(0,0)}_{z\bar z} = {}& \partial_z\theta^\alpha\ A_{\alpha\hat\beta}\ \partial_{\bar z}\hat\theta^{\hat\beta} + \partial_z\theta^\alpha\ A_{\alpha p}\ \hat\Pi^p_{\bar z} + \Pi^m_z\ A_{m\hat\beta}\ \partial_{\bar z}\hat\theta^{\hat\beta} + \Pi^m_z\ A_{mp}\ \hat\Pi^p_{\bar z} \\
& + d_{z\alpha}\ E^\alpha{}_{\hat\beta}\ \partial_{\bar z}\hat\theta^{\hat\beta} + d_{z\alpha}\ E^\alpha{}_p\ \hat\Pi^p_{\bar z} + \partial_z\theta^\alpha\ E_\alpha{}^{\hat\beta}\ \hat d_{\bar z\hat\beta} + \Pi^m_z\ E_m{}^{\hat\beta}\ \hat d_{\bar z\hat\beta} + d_{z\alpha}\ P^{\alpha\hat\beta}\ \hat d_{\bar z\hat\beta} \\
& + \frac{1}{2}\ N^{mn}_z\ \Omega_{mn,\hat\beta}\ \partial_{\bar z}\hat\theta^{\hat\beta} + \frac{1}{2}\ N^{mn}_z\ \Omega_{mn,p}\ \hat\Pi^p_{\bar z} + \frac{1}{2}\ \partial_z\theta^\alpha\ \Omega_{\alpha,pq}\hat N^{pq}_{\bar z} + \frac{1}{2}\ \Pi^m_z\ \Omega_{m,pq}\hat N^{pq}_{\bar z} \\
& + \frac{1}{2}\ N^{mn}_z\ C_{mn}{}^{\hat\beta}\ \hat d_{\bar z\hat\beta} + \frac{1}{2}\ d_{z\alpha}\ C^\alpha{}_{pq}\ \hat N^{pq}_{\bar z} + \frac{1}{4}\ N^{mn}_z\ S_{mn,pq}\ \hat N^{pq}_{\bar z}
\end{aligned}
\tag{3.5}
$$

From equations (2.3), (2.4), (2.5) and (2.7) in the previous section we derive the complete set of equations for the background fields

$$
\begin{array}{lll}
(\frac12,\frac12,\frac12) & D_\alpha A_{\beta\hat\gamma} + D_\beta A_{\alpha\hat\gamma} - \gamma^m_{\alpha\beta}A_{m\hat\gamma} = 0 & \hat D_{\hat\alpha} A_{\beta\hat\gamma} + \hat D_{\hat\gamma} A_{\beta\hat\alpha} - \gamma^m_{\hat\alpha\hat\gamma}A_{\beta m} = 0 \\
(\frac12,\frac12,1) & D_\alpha A_{m\hat\beta} - \partial_m A_{\alpha\hat\beta} - \gamma_{m\alpha\gamma}E^\gamma{}_{\hat\beta} = 0 & \hat D_{\hat\alpha} A_{\beta p} - \partial_p A_{\beta\hat\alpha} - \gamma_{p\hat\alpha\hat\gamma}E_\beta{}^{\hat\gamma} = 0 \\
(\frac12,\frac12,1) & D_\alpha A_{\beta p} + D_\beta A_{\alpha p} - \gamma^m_{\alpha\beta}A_{mp} = 0 & \hat D_{\hat\alpha} A_{m\hat\beta} + \hat D_{\hat\beta} A_{m\hat\alpha} + \gamma^p_{\hat\alpha\hat\beta}A_{mp} = 0 \\
(\frac12,\frac12,\frac32) & D_\alpha E^\beta{}_{\hat\gamma} - \frac14(\gamma^{mn})_\alpha{}^\beta\Omega_{mn,\hat\gamma} = 0 & \hat D_{\hat\alpha} E_\beta{}^{\hat\gamma} - \frac14(\gamma^{pq})_{\hat\alpha}{}^{\hat\gamma}\Omega_{\beta,pq} = 0 \\
(\frac12,\frac12,\frac32) & D_\alpha E_\beta{}^{\hat\gamma} + D_\beta E_\alpha{}^{\hat\gamma} - \gamma^m{}_{\alpha\beta}E_m{}^{\hat\gamma} = 0 & \hat D_{\hat\alpha} E^\beta{}_{\hat\gamma} + \hat D_{\hat\gamma} E^\beta{}_{\hat\alpha} - \gamma^p{}_{\hat\alpha\hat\gamma}E^\beta{}_p = 0 \\
(\frac12,1,1) & D_\alpha A_{mp} - \partial_m A_{\alpha p} - \gamma_{m\alpha\gamma}E^\gamma{}_p = 0 & \hat D_{\hat\alpha} A_{mp} + \partial_p A_{m\hat\alpha} + \gamma_{p\hat\alpha\hat\beta}E_m{}^{\hat\beta} = 0 \\
(\frac12,\frac32,1) & D_\alpha E^\beta{}_p - \frac14(\gamma^{mn})_\alpha{}^\beta\Omega_{mn,p} = 0 & \hat D_{\hat\alpha} E_m{}^{\hat\beta} + \frac14\Omega_{m,pq}(\gamma^{pq})_{\hat\alpha}{}^{\hat\beta} = 0 \\
(\frac12,\frac32,1) & D_\alpha E_m{}^{\hat\beta} - \partial_m E_\alpha{}^{\hat\beta} - \gamma_{m\alpha\gamma}P^{\gamma\hat\beta} = 0 & \hat D_{\hat\alpha} E^\beta{}_p - \partial_p E^\beta{}_{\hat\alpha} - \gamma_{p\hat\alpha\hat\gamma}P^{\beta\hat\gamma} = 0 \\
(\frac12,\frac12,2) & D_\alpha\Omega_{\beta,pq} + D_\beta\Omega_{\alpha,pq} - \gamma^m{}_{\alpha\beta}\Omega_{m,pq} = 0 & \hat D_{\hat\alpha}\Omega_{mn,\hat\beta} + \hat D_{\hat\beta}\Omega_{mn,\hat\alpha} + \gamma^p_{\hat\alpha\hat\beta}\Omega_{mn,p} = 0 \\
(\frac12,\frac32,\frac32) & D_\alpha P^{\beta\hat\gamma} - \frac14(\gamma^{mn})_\alpha{}^\beta C_{mn}{}^{\hat\gamma} = 0 & \hat D_{\hat\alpha} P^{\beta\hat\gamma} - \frac14(\gamma^{pq})_{\hat\alpha}{}^{\hat\gamma} C^\beta{}_{pq} = 0 \\
(\frac12,1,2) & D_\alpha\Omega_{m,pq} - \partial_m\Omega_{\alpha,pq} - \gamma_{m\alpha\gamma}C^\gamma{}_{pq} = 0 & \hat D_{\hat\alpha}\Omega_{mn,p} + \partial_p\Omega_{mn,\hat\alpha} + \gamma_{p\hat\alpha\hat\beta}C_{mn}{}^{\hat\beta} = 0 \\
(\frac12,\frac32,2) & D_\alpha C^\beta{}_{pq} - \frac14(\gamma^{mn})_\alpha{}^\beta S_{mn,pq} = 0 & \hat D_{\hat\alpha} C_{mn}{}^{\hat\beta} + \frac14(\gamma^{pq})_{\hat\alpha}{}^{\hat\beta} S_{mnpq} = 0
\end{array}
\tag{3.6}
$$

where the labels (a, b, c) denote the scaling dimensions of the generators of the extended super-Poincaré algebra ([20] and [21]) which the equations belong to.

Moreover, one obtains the following eight equations, which do not provide further information, since they are implied by (3.6) and pure spinor conditions given in app. A

$$
\begin{aligned}
&N^{mn}\lambda^\gamma D_\gamma \Omega_{mn,\hat\beta} = 0 \qquad && \hat\lambda^{\hat\gamma}\hat D_{\hat\gamma}\Omega_{\alpha,mn}\hat N^{mn} = 0\\
&N^{mn}\lambda^\gamma D_\gamma \Omega_{mn,p} = 0 && \hat\lambda^{\hat\gamma}\hat D_{\hat\gamma}\Omega_{m,pq}\hat N^{pq} = 0\\
&N^{mn}\lambda^\gamma D_\gamma C_{mn}{}^{\hat\beta} = 0 && \hat\lambda^{\hat\gamma}\hat D_{\hat\gamma}C^\alpha{}_{mn}\hat N^{mn} = 0\\
&N^{mn}\lambda^\gamma D_\gamma S_{mn,pq}\hat N^{pq} = 0 && N^{mn}\hat\lambda^{\hat\gamma}\hat D_{\hat\gamma}S_{mn,pq}\hat N^{pq} = 0
\end{aligned}
\tag{3.7}
$$

Since we assumed that the superfields $\Omega_{mn,p}, \Omega_{m,pq}, C_{mn}{}^{\hat\beta}, C^\alpha{}_{pq}$ and $S_{mn,pq}$ correspond to the linearized curvatures of the connections, we can derive new equations needed for the iterative procedure outlined in the introduction. By contracting equations (3.6) with respect to the bosonic derivative and antisymmetrizing the bosonic indices, one obtains

$$
\begin{aligned}
&D_\alpha \Omega_{mn,\hat\beta} = \partial_{[m}\gamma_{n]\alpha\gamma}E^\gamma{}_{\hat\beta} \qquad && \hat D_{\hat\beta}\Omega_{\alpha,pq} = \partial_{[p}\gamma_{q]\hat\beta\hat\gamma}E_\alpha{}^{\hat\gamma}\\
&D_\alpha \Omega_{mn,p} = \partial_{[m}\gamma_{n]\alpha\gamma}E^\gamma{}_p && \hat D_{\hat\beta}\Omega_{m,pq} = -\partial_{[p}\gamma_{q]\hat\beta\hat\gamma}E_m{}^{\hat\gamma}\\
&D_\alpha C_{mn}{}^{\hat\beta} = \partial_{[m}\gamma_{n]\alpha\gamma}P^{\gamma\hat\beta} && \hat D_{\hat\beta}C^\alpha{}_{pq} = \partial_{[p}\gamma_{q]\hat\beta\hat\gamma}P^{\alpha\hat\gamma}\\
&D_\alpha S_{mn,pq} = \partial_{[m}\gamma_{n]\alpha\gamma}C^\gamma{}_{pq} && \hat D_{\hat\beta}S_{mn,pq} = -\partial_{[p}\gamma_{q]\hat\beta\hat\gamma}C_{mn}{}^{\hat\gamma}
\end{aligned}
\tag{3.8}
$$

(we define $a_{[m}b_{n]} = a_m b_n - a_n b_m$). The identification of the superfields $\Omega_{mn,p}$, $\Omega_{m,pq}$, $C_{mn}{}^{\hat\beta}$, $C^\alpha{}_{pq}$ and $S_{mn,pq}$ with the linearized curvatures is automatically derived in the formalism [2], and equations (3.8) are the usual Bianchi identities.

In order to show that the above equations imply the supergravity equations of motion we proceed as follows. We first consider the third line of (3.8) and the $(\frac{1}{2}, \frac{3}{2}, \frac{3}{2})$ line of (3.6), that we recall for the reader convenience

$$
\begin{aligned}
&D_\alpha P^{\beta\hat\gamma} - \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta C_{mn}{}^{\hat\gamma} = 0 \qquad && \hat D_{\hat\alpha}P^{\beta\hat\gamma} - \frac{1}{4}(\gamma^{pq})_{\hat\alpha}{}^{\hat\gamma}C^\beta{}_{pq} = 0\\
&D_\alpha C_{mn}{}^{\hat\beta} = \partial_{[m}\gamma_{n]\alpha\gamma}P^{\gamma\hat\beta} && \hat D_{\hat\beta}C^\alpha{}_{pq} = \partial_{[p}\gamma_{q]\hat\beta\hat\gamma}P^{\alpha\hat\gamma}
\end{aligned}
\tag{3.9}
$$

Acting with $\gamma^m_{\alpha\sigma}\partial_m$ on $P^{\sigma\hat\beta}$ and using the commutation relations of the D's, one gets

$$
\gamma^m_{\alpha\sigma}\partial_m P^{\sigma\hat\beta} = (D_\alpha D_\sigma + D_\sigma D_\alpha)P^{\sigma\hat\beta} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\sigma D_\sigma C_{mn}{}^{\hat\beta}\,, \tag{3.10}
$$

$$
= -\frac{1}{2}(\gamma^{mn})_\alpha{}^\sigma \gamma_{m\sigma\gamma}\partial_n P^{\gamma\hat\beta} = \frac{9}{2}\gamma^m_{\alpha\sigma}\partial_m P^{\sigma\hat\beta}
$$

Here we also used the first equation of (3.9) and $D_\alpha P^{\alpha\hat\beta} = 0$ (which follows from (3.9)). In the second line we used the first equation in the second line on (3.9) and the identity $(\gamma^{mn}\gamma_m)_{\alpha\beta} = -9\gamma^n_{\alpha\beta}$. By performing the same manipulations on the hatted quantities we derive the equations

$$
\gamma^m_{\alpha\sigma}\partial_m P^{\sigma\hat\beta} = 0\,, \qquad \gamma^m_{\hat\alpha\hat\sigma}\partial_m P^{\alpha\hat\sigma} = 0\,. \tag{3.11}
$$

Decomposing $P^{\alpha\hat\beta}$ in terms of Dirac matrices, it is straightforward to show that (3.11) implies the equations of motion for the RR fields.

Acting again with $\gamma_n^{\alpha\gamma} D_\alpha$ on (3.11) and using equations (3.9) one gets

$$0 = \gamma_n^{\alpha\gamma}\gamma^m_{\alpha\beta}\partial_m D_\gamma P^{\beta\hat\beta} = (\gamma_n\gamma^m)^\gamma_\beta(\gamma^{pq})^\beta_\gamma \partial_m C_{pq}{}^{\hat\beta} = \partial^m C_{mn}{}^{\hat\beta}\,, \tag{3.12}$$

and analogously for $C^\alpha{}_{pq}$. These equations are the Maxwell equations for the curvature of the gravitinos. They are not enough to describe the dynamic of gravitinos and we have to invoke new equations coming from the second line of (3.8) and the $(\frac{1}{2}, \frac{3}{2}, 1)$ line of (3.8).

Applying $\gamma^m_{\alpha\sigma}\partial_m$ on $E^\sigma{}_p$ and with $\gamma^p_{\hat\alpha\hat\sigma}\partial_p$ on $E_m{}^{\hat\sigma}$, the same algebraic manipulations yield

$$\gamma^m_{\alpha\sigma}\partial_m E^\sigma{}_p = 0\,, \qquad \gamma^p_{\hat\alpha\hat\sigma}\partial_p E_m{}^{\hat\sigma} = 0\,. \tag{3.13}$$

which are the Dirac equations for the gravitinos. These equations are gauge invariant under the gauge transformations discussed in the next section since the gauge parameters have to satisfy a field equation. In addition, as above, we find the equations

$$\partial^m \Omega_{mn,p} = 0\,, \qquad \partial^p \Omega_{m,pq} = 0\,, \tag{3.14}$$

which are, at the lowest component of the superfield $\Omega_{mn,p}$ and $\Omega_{m,pq}$, the equations of motion of the graviton, the dilaton and the NS-NS form

$$\partial^m(\partial_{[m}g_{n]p} + \partial_{[m}b_{n]p} + \eta_{p[n}\partial_{m]}\phi) = 0\,, \tag{3.15}$$

$$\partial^p(\partial_{[p}g_{|m|q]} + \partial_{[p}b_{|m|q]} + \eta_{nm[q}\partial_{p]}\phi) = 0\,.$$

Pursuing this line of reasoning, one can derive similar equations for $E^\beta{}_{\hat\alpha}, E_\alpha{}^{\hat\beta}, \Omega_{mn,\hat\gamma}$ and $\Omega^\alpha{}_{pq}$, which guarantee that the fields are either pure gauge or auxiliary fields. Finally, by studying the last line of (3.8) and the line $(\frac{1}{2}, \frac{3}{2}, 2)$ of (3.6), one derives new equations for $C^\beta{}_{pq}, C_{mn}{}^{\hat\beta}$ and $S_{mn,pq}$, which do not give further information since they are implied by the previous ones.

4. Gauge Transformations and Gauge Fixing

In order to solve the equations of motion (3.6) and (3.8) it is convenient to choose a suitable gauge. Indeed, for supersymmetric theories, the large amount of auxiliary fields can be reduced by choosing the Wess-Zumino gauge. We first discuss the general structure of the gauge transformations (2.8), we then provide a gauge fixing and we finally check that this gauge can be reached. In the present framework, the gauge parameters $\Lambda^{(0,1)}$, $\Lambda^{(1,0)}$, $\tau_z^{(0,0)}$ and $\tau_{\bar z}^{(0,0)}$ satisfy equations (2.9) and (2.10) and they are defined up to the gauge transformation (2.11). This additional gauge invariance is fixed by a further gauge fixing.

The general structure of the gauge parameters $\Lambda^{(0,1)}$, $\Lambda^{(1,0)}$, $\tau_z^{(0,0)}$ and $\tau_{\bar z}^{(0,0)}$ is given by

$$\Lambda^{(1,0)} = \lambda^\alpha \Theta_\alpha \qquad\qquad \Lambda^{(0,1)} = \hat\Theta_{\hat\alpha} \hat\lambda^{\hat\alpha} \,, \tag{4.1}$$

and

$$\begin{aligned}
\tau_z^{(0,0)} &= \partial_z \theta^\alpha \Xi_\alpha + \Pi_z^m \Sigma_m + d_{z\alpha} \Phi^\alpha + \frac{1}{2} N_z^{mn} \Psi_{mn} \\
\tau_{\bar z}^{(0,0)} &= \hat\Xi_{\hat\alpha} \partial_{\bar z} \hat\theta^{\hat\alpha} + \hat\Sigma_p \hat\Pi_{\bar z}^p + \hat\Phi^{\hat\alpha} \hat d_{\bar z \hat\alpha} + \frac{1}{2} \hat\Psi_{pq} \hat N_{\bar z}^{pq} \,.
\end{aligned} \tag{4.2}$$

where $\Theta_\alpha, \dots, \hat\Psi_{mn}$ are superfields in the variables x^m, θ^α, and $\hat\theta^{\hat\alpha}$. In terms of these superfields, eq. (2.9) gives

$$(\gamma^{mnpqr})^{\alpha\beta} D_\beta \Theta_\alpha = 0 \qquad\qquad (\gamma^{mnpqr})^{\hat\alpha\hat\beta} \hat D_{\hat\beta} \hat\Theta_{\hat\alpha} = 0 \,, \tag{4.3}$$

while eq. (2.10) gives

$$\begin{array}{ll}
\Theta_\alpha + \Xi_\alpha = 0 & \hat\Theta_{\hat\alpha} - \hat\Xi_{\hat\alpha} = 0 \\
D_\alpha \Theta_\beta - D_\beta \Xi_\alpha + \gamma^m_{\alpha\beta} \Sigma_m = 0 & \hat D_{\hat\alpha} \hat\Theta_{\hat\beta} + \hat D_{\hat\beta} \hat\Xi_{\hat\alpha} + \gamma^p_{\hat\alpha\hat\beta} \hat\Sigma_p = 0 \\
D_\alpha \Sigma_m + \partial_m \Theta_\alpha - \gamma_{m\alpha\beta} \Phi^\beta = 0 & \hat D_{\hat\alpha} \hat\Sigma_p + \partial_p \hat\Theta_{\hat\alpha} + \gamma_{p\hat\alpha\hat\beta} \hat\Phi^{\hat\beta} = 0 \\
D_\alpha \Phi^\beta - \frac{1}{4} (\gamma^{mn})_\alpha{}^\beta \Psi_{mn} = 0 & \hat D_{\hat\alpha} \hat\Phi^{\hat\beta} + \frac{1}{4} (\gamma^{pq})_{\hat\alpha}{}^{\hat\beta} \hat\Psi_{pq} \\
N^{mn} \lambda^\gamma D_\gamma \Psi_{mn} = 0 & \hat\lambda^{\hat\gamma} \hat D_{\hat\gamma} \hat\Psi_{pq} \hat N^{pq} = 0 \,.
\end{array} \tag{4.4}$$

These equations look like the superspace field equations for SYM theory (cf. sec. 1), however the superfields $\Theta_\alpha, \Sigma_m, \Phi^\alpha$ and Ψ_{mn} and the corresponding hatted quantities depend on x^m, θ^α and $\hat\theta^{\hat\alpha}$. Therefore, the eqs. (4.4) are not sufficient to determine completely the components of those superfields. The free independent components are indeed the gauge parameters. We also notice that the last pair of equations is trivial when the previous equations and the pure spinor conditions are imposed. Finally, because of the similarity with SYM case, it is quite natural to impose the condition that Ψ_{mn} and $\hat\Psi_{pq}$ are the linearized curvatures of Σ_m and $\hat\Sigma_p$. Again, this assumption is automatic in [2]. The gauge transformations of the superfields in $\mathcal{V}^{(0,0)}_{z\bar{z}}$ are given by

$$
\begin{array}{lll}
(\frac{1}{2},\frac{1}{2}) & \delta A_{\alpha\hat\beta} = D_\alpha \hat\Xi_{\hat\beta} + \hat D_{\hat\beta}\Theta_\alpha & \\
(\frac{1}{2},1) & \delta A_{\alpha p} = \partial_p \Theta_\alpha + D_\alpha \hat\Sigma_p; & \delta A_{m\hat\beta} = \partial_m \hat\Theta_{\hat\beta} + \hat D_{\hat\beta}\Sigma_m \\
(1,1) & \delta A_{mp} = \partial_m \hat\Sigma_p - \partial_p \Sigma_m & \\
(\frac{3}{2},\frac{1}{2}) & \delta E^\alpha{}_{\hat\beta} = -\hat D_{\hat\beta}\Phi^\alpha; & \delta E_\alpha{}^{\hat\beta} = D_\alpha \hat\Phi^{\hat\beta} \\
(\frac{3}{2},1) & \delta E^\alpha{}_p = -\partial_p \Phi^\alpha; & \delta E_m{}^{\hat\beta} = \partial_m \hat\Phi^{\hat\beta} \\
(\frac{1}{2},2) & \delta \Omega_{\alpha,pq} = D_\alpha \hat\Psi_{pq}; & \delta\Omega_{mn,\hat\beta} = \hat D_{\hat\beta}\Psi_{mn} \\
(1,2) & \delta\Omega_{m,pq} = \partial_m \hat\Psi_{pq}; & \delta\Omega_{mn,p} = -\partial_p \Psi_{mn} \\
(\frac{3}{2},\frac{3}{2}) & \delta P^{\alpha\hat\beta} = 0 & \\
(\frac{3}{2},2) & \delta C^\alpha{}_{pq} = 0; & \delta C_{mn}{}^{\hat\beta} = 0 \\
(2,2) & \delta S_{mn,pq} = 0\,. &
\end{array}
\tag{4.5}
$$

From these equations, we easily see that the superfields $P^{\alpha\hat\beta}$, $C^\alpha{}_{pq}$, $C_{mn}{}^{\hat\beta}$ and $S_{mn,pq}$ are indeed gauge invariant, as expected, being linearized field strengths. At zero order in θ and $\hat\theta$ eq. (4.5) gives the gauge transformations of supergravity fields. For example, the first components of $\hat\Sigma_p = \zeta_p + \xi_p + \mathcal{O}(\theta,\hat\theta)$ and $\Sigma_m = \zeta_m - \xi_m + \mathcal{O}(\theta,\hat\theta)$ are to be identified with the parameters of diffeomorphisms $\delta g_{mp} = \partial_m \xi_p + \partial_p \xi_m$ and with the gauge transformations of the NS-NS form $\delta b_{mp} = \partial_m \zeta_p - \partial_p \zeta_m$. So, the zero-order terms of the gauge parameter superfields Θ_α, Σ_m, Φ^α and of the corresponding hatted quantities are

$$
\begin{array}{ll}
\Theta_\alpha = \mathcal{O}(\theta,\hat\theta); & \hat\Theta_{\hat\beta} = \mathcal{O}(\theta,\hat\theta) \\
\Sigma_m = \zeta_m - \xi_m + \mathcal{O}(\theta,\hat\theta); & \hat\Sigma_p = \zeta_p + \xi_p + \mathcal{O}(\theta,\hat\theta) \\
\Phi^\alpha = \varphi^\alpha + \mathcal{O}(\theta,\hat\theta); & \hat\Phi^{\hat\beta} = \hat\varphi^{\hat\beta} + \mathcal{O}(\theta,\hat\theta)
\end{array}
\tag{4.6}
$$

Furthermore, the large amount of gauge parameters allows us to choose the gauge

$$
\begin{array}{ll}
\theta^\alpha A_{\alpha\hat\beta} = 0 & A_{\alpha\hat\beta}\,\hat\theta^{\hat\beta} = 0 \\
\theta^\alpha A_{\alpha p} = 0 & A_{m\hat\beta}\,\hat\theta^{\hat\beta} = 0 \\
\theta^\alpha E_\alpha{}^{\hat\beta} = 0 & E^\alpha{}_{\hat\beta}\,\hat\theta^{\hat\beta} = 0 \\
\theta^\alpha\ \Omega_{\alpha,pq} = 0 & \Omega_{mn,\hat\beta}\,\hat\theta^{\hat\beta} = 0\,.
\end{array}
\tag{4.7}
$$

Indeed, we have at our disposal the parameters $\Theta_\alpha, \Sigma_m, \Phi^\alpha$ and Ψ_{mn} and the corresponding hatted quantities to impose the gauge (4.7). Before showing that the gauge can be reached we have to notice that the transformations (4.5) and the equations (4.3) are invariant under the residual gauge transformations (2.11)

$$\begin{aligned} &\delta\Theta_\alpha = D_\alpha\Omega && \delta\hat{\Xi}_{\hat\beta} = \hat{D}_{\hat\beta}\Omega \\ &\delta\Sigma_m = -\partial_m\Omega && \delta\hat{\Sigma}_p = -\partial_p\Omega \\ &\delta\Phi^\alpha = 0 && \delta\hat{\Phi}^{\hat\beta} = 0 \\ &\delta\Psi_{mn} = 0 && \delta\hat{\Psi}_{pq} = 0\,, \end{aligned} \tag{4.8}$$

depending on the scalar superfield $\Upsilon^{(0,0)} = \hat{\Upsilon}^{(0,0)} \equiv \Omega$. This requires an additional gauge fixing

$$\theta^\alpha\Theta_\alpha + \hat\theta^{\hat\beta}\hat{\Xi}_{\hat\beta} = 0\,. \tag{4.9}$$

To show that the gauge choice (4.7) can be reached by the gauge transformations (4.5), we have to solve, for instance, the equations

$$\theta^\alpha(A_{\alpha\hat\beta} + \delta A_{\alpha\hat\beta}) = 0\,, \qquad (A_{\alpha\hat\beta} + \delta A_{\alpha\hat\beta})\hat\theta^{\hat\beta} = 0\,, \tag{4.10}$$

and analogously for all other gauge conditions (4.7). By using the properties of the superderivative, gauge fixing (4.9), consistency conditions (4.4), and by defining the operators

$$\mathbf{D} \equiv \theta^\alpha D_\alpha = \theta^\alpha\frac{\partial}{\partial\theta^\alpha}\,, \qquad \hat{\mathbf{D}} \equiv \hat\theta^{\hat\beta}\hat{D}_{\hat\beta} = \hat\theta^{\hat\beta}\frac{\partial}{\partial\hat\theta^{\hat\beta}}\,, \tag{4.11}$$

we get the following recursive equations

$$\begin{aligned} &(1+\mathbf{D}+\hat{\mathbf{D}})\Theta_\alpha = -A_{\alpha\hat\beta}\hat\theta^{\hat\beta} - (\gamma^m\theta)_\alpha\Sigma_m && (1+\mathbf{D}+\hat{\mathbf{D}})\hat{\Theta}_{\hat\beta} = -\theta^\alpha A_{\alpha\hat\beta} - (\gamma^p\hat\theta)_{\hat\beta}\hat{\Sigma}_p \\ &(\mathbf{D}+\hat{\mathbf{D}})\Sigma_m = A_{m\hat\beta}\hat\theta^{\hat\beta} + (\gamma_m\theta)_\beta\Phi^\beta && (\mathbf{D}+\hat{\mathbf{D}})\hat{\Sigma}_p = -\theta^\alpha A_{\alpha p} - (\gamma_p\hat\theta)_{\hat\gamma}\hat{\Phi}^{\hat\gamma} \\ &(\mathbf{D}+\hat{\mathbf{D}})\Phi^\alpha = E^\alpha{}_{\hat\beta}\hat\theta^{\hat\beta} - \frac{1}{4}(\gamma^{mn}\theta)^\alpha\Psi_{mn} && (\mathbf{D}+\hat{\mathbf{D}})\hat{\Phi}^{\hat\beta} = -\theta^\alpha E_\alpha{}^{\hat\beta} + \frac{1}{4}(\gamma^{pq}\hat\theta)^\beta\hat{\Psi}_{pq} \\ &(\mathbf{D}+\hat{\mathbf{D}})\Psi_{mn} = \Omega_{mn,\hat\beta}\hat\theta^{\hat\beta} - (\gamma_{[m}\theta)_\gamma\partial_{n]}\Phi^\gamma && (\mathbf{D}+\hat{\mathbf{D}})\hat{\Psi}_{pq} = -\theta^\alpha\Omega_{\alpha,pq} + (\gamma_{[p}\hat\theta)_{\hat\gamma}\partial_{q]}\hat{\Phi}^{\hat\gamma}\,. \end{aligned} \tag{4.12}$$

The operator $(\mathbf{D}+\hat{\mathbf{D}})$ acts on homogeneous polynomials in θ^α and $\hat\theta^{\hat\alpha}$ by multiplication by the degree of homogeneity and it does not change its degree. Therefore, the relations (4.12) are recursive in powers of θ and $\hat\theta$. They can be solved algebraically given $A_{\alpha\hat\beta}, \ldots, \Omega_{\alpha,pq}$ order by order in θ and $\hat\theta$ and this proves that the gauge can indeed be imposed. Of course, to reconstruct the gauge-parameter superfields by means of the recursive equations (4.12), we also need lowest order data for them. These are the zero order supergravity gauge parameters (4.6). To obtain the last couple of equations we used the additional condition that Ψ_{mn} and $\hat{\Psi}_{pq}$ are the linearized curvatures of Σ_m and $\hat{\Sigma}_p$.

5. Iterative Procedure and Superfield Reconstruction

The next step is the derivation of the recursion equations for supergravity superfields. Acting with D_α and $\hat{D}_{\hat{\beta}}$ on the gauge fixing conditions (4.7), and using the definition (4.11), it is straightforward to derive the recursion relations from eq. (3.6)

$$
\begin{aligned}
&(1+\mathbf{D})A_{\alpha\hat{\beta}} = (\gamma^m\theta)_\alpha A_{m\hat{\beta}} && (1+\hat{\mathbf{D}})A_{\alpha\hat{\beta}} = (\gamma^p\hat{\theta})_{\hat{\beta}} A_{\alpha p}\\
&\mathbf{D}A_{m\hat{\beta}} = (\gamma_m\theta)_\gamma E^\gamma{}_{\hat{\beta}} && \hat{\mathbf{D}}A_{\alpha p} = (\gamma_p\hat{\theta})_{\hat{\gamma}} E_\alpha{}^{\hat{\gamma}}\\
&\mathbf{D}E^\alpha{}_{\hat{\beta}} = -\frac{1}{4}(\gamma^{mn}\theta)^\alpha \Omega_{mn,\hat{\beta}} && \hat{\mathbf{D}}E_\alpha{}^{\hat{\beta}} = -\frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}\Omega_{\alpha,pq}\\
&\mathbf{D}\Omega_{mn,\hat{\beta}} = -(\gamma_{[m}\theta)_\gamma \partial_{n]} E^\gamma{}_{\hat{\beta}} && \hat{\mathbf{D}}\Omega_{\alpha,pq} = -(\gamma_{[p}\hat{\theta})_{\hat{\gamma}}\partial_{q]} E_\alpha{}^{\hat{\gamma}}
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&(1+\mathbf{D})A_{\alpha p} = (\gamma^m\theta)_\alpha A_{mp} && (1+\hat{\mathbf{D}})A_{m\hat{\beta}} = -(\gamma^p\hat{\theta})_{\hat{\beta}} A_{mp}\\
&\mathbf{D}A_{mp} = (\gamma_m\theta)_\beta E^\beta{}_p && \hat{\mathbf{D}}A_{mp} = -(\gamma_p\hat{\theta})_{\hat{\beta}} E_m{}^{\hat{\beta}}\\
&\mathbf{D}E^\alpha{}_p = -\frac{1}{4}(\gamma^{mn}\theta)^\alpha \Omega_{mn,p} && \hat{\mathbf{D}}E_m{}^{\hat{\beta}} = \frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}\Omega_{m,pq}\\
&\mathbf{D}\Omega_{mn,p} = -(\gamma_{[m}\theta)_\gamma \partial_{n]} E^\gamma{}_p && \hat{\mathbf{D}}\Omega_{m,pq} = (\gamma_{[p}\hat{\theta})_{\hat{\gamma}}\partial_{q]} E_m{}^{\hat{\gamma}}
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&(1+\mathbf{D})E_\alpha{}^{\hat{\beta}} = (\gamma^m\theta)_\alpha E_m{}^{\hat{\beta}} && (1+\hat{\mathbf{D}})E^\alpha{}_{\hat{\beta}} = (\gamma^p\hat{\theta})_{\hat{\beta}} E^\alpha{}_p\\
&\mathbf{D}E_m{}^{\hat{\beta}} = (\gamma_m\theta)_\gamma P^{\gamma\hat{\beta}} && \hat{\mathbf{D}}E^\alpha{}_p = (\gamma_p\hat{\theta})_{\hat{\gamma}} P^{\alpha\hat{\gamma}}\\
&\mathbf{D}P^{\alpha\hat{\beta}} = -\frac{1}{4}(\gamma^{mn}\theta)^\alpha C_{mn}{}^{\hat{\beta}} && \hat{\mathbf{D}}P^{\alpha\hat{\beta}} = -\frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}} C^\alpha{}_{pq}\\
&\mathbf{D}C_{mn}{}^{\hat{\beta}} = -(\gamma_{[m}\theta)_\gamma \partial_{n]} P^{\gamma\hat{\beta}} && \hat{\mathbf{D}}C^\alpha{}_{pq} = -(\gamma_{[p}\hat{\theta})_{\hat{\gamma}}\partial_{q]} P^{\alpha\hat{\gamma}}
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&(1+\mathbf{D})\Omega_{\alpha,pq} = (\gamma^m\theta)_\alpha \Omega_{m,pq} && (1+\hat{\mathbf{D}})\Omega_{mn,\hat{\beta}} = -(\gamma^p\hat{\theta})_{\hat{\beta}} \Omega_{mn,p}\\
&\mathbf{D}\Omega_{m,pq} = (\gamma_m\theta)_\beta C^\beta{}_{pq} && \hat{\mathbf{D}}\Omega_{mn,p} = -(\gamma_p\hat{\theta})_{\hat{\beta}} C_{mn}{}^{\hat{\beta}}\\
&\mathbf{D}C^\alpha{}_{pq} = -\frac{1}{4}(\gamma^{mn}\theta)^\alpha S_{mn,pq} && \hat{\mathbf{D}}C_{mn}{}^{\hat{\beta}} = \frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}} S_{mn,pq}\\
&\mathbf{D}S_{mn,pq} = -(\gamma_{[m}\theta)_\gamma \partial_{n]} C^\gamma{}_{pq}, && \hat{\mathbf{D}}S_{mn,pq} = (\gamma_{[p}\hat{\theta})_{\hat{\gamma}}\partial_{q]} C_{mn}{}^{\hat{\gamma}}
\end{aligned}
\tag{5.4}
$$

A given superfield appears in two groups of equations in order that both its θ and $\hat{\theta}$ components are fixed. Inside each group there is an iterative structure (see [4] and [5]) which allows us to solve those equations recursively given the initial conditions and there is a hierarchical structure among the different groups of equations which allows us to solve them subsequently. To provide the initial data, we identify the lowest-components of the matrix superfield $\mathbf{A}$ in (3.3) with supergravity fields

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & g_{mp} + b_{mp} + \eta_{mp}\phi & \psi_m{}^{\hat\beta} & \omega_{m,pq} \\ 0 & \psi^\alpha{}_p & f^{\alpha\hat\beta} & c^\alpha{}_{pq} \\ 0 & \omega_{mn,p} & c_{mn}{}^{\hat\beta} & s_{mn,pq} \end{bmatrix} + \mathcal{O}(\theta,\hat\theta)\,, \tag{5.5}$$

where the linearized gravitational connection and curvatures are given by

$$\begin{aligned} \omega_{m,pq} &= (\partial_p g_{mq} - \partial_q g_{mp}) + (\partial_p b_{mq} - \partial_q b_{mp}) + (\eta_{mq}\partial_p - \eta_{mp}\partial_q)\phi\,, \\ c_{mn}{}^{\hat\beta} &= (\partial_m \psi_n{}^{\hat\beta} - \partial_n \psi_m{}^{\hat\beta})\,, \\ s_{mn,pq} &= (\partial_m \omega_{n,pq} - \partial_n \omega_{m,pq})\,, \end{aligned} \tag{5.6}$$

and, analogously, for $\omega_{mn,p}$ and $c^\alpha{}_{pq}$.

In the following we give the component-expansion for the physical superfields A_{mp}, $E_m{}^{\hat\beta}$, $E^\alpha{}_p$ and $P^{\alpha\hat\beta}$, up to second order in both θ and $\hat\theta$. The corresponding curvatures can be easily computed from the defining equations (5.6). The component-expansion of the auxiliary superfields is given in appendix B.

$$\begin{aligned} A_{mp} = {} & (g+b+\eta\phi)_{mp} + (\gamma_m\theta)_\beta \psi^\beta{}_p - (\gamma_p\hat\theta)_{\hat\beta}\psi_m{}^{\hat\beta} + (\gamma_m\theta)_\beta(\gamma_p\hat\theta)_{\hat\gamma} f^{\beta\hat\gamma} \\ & - \frac{1}{8}(\gamma_m\theta)_\beta(\gamma^{nr}\theta)^\beta \omega_{nr,p} - \frac{1}{8}(\gamma_p\hat\theta)_{\hat\beta}(\gamma^{qr}\hat\theta)^{\hat\beta}\omega_{m,qr} \\ & + \frac{1}{8}(\gamma_m\theta)_\beta(\gamma^{nr}\theta)^\beta(\gamma_p\hat\theta)_{\hat\gamma} c_{nr}{}^{\hat\gamma} - \frac{1}{8}(\gamma_m\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}(\gamma^{qr}\hat\theta)^{\hat\beta} c^\gamma{}_{qr} \\ & + \frac{1}{64}(\gamma_m\theta)_\beta(\gamma^{nr}\theta)^\beta(\gamma_p\hat\theta)_{\hat\gamma}(\gamma^{qs}\hat\theta)^{\hat\gamma} s_{nr,qs} + \ldots \end{aligned}$$

$$\begin{aligned} E_m{}^{\hat\beta} = {} & \psi_m{}^{\hat\beta} + (\gamma_m\theta)_\gamma f^{\gamma\hat\beta} + \frac{1}{4}(\gamma^{pq}\hat\theta)^{\hat\beta}\omega_{m,pq} - \frac{1}{4}(\gamma_m\theta)_\gamma(\gamma^{pq}\hat\theta)^{\hat\beta} c^\gamma{}_{pq} \\ & - \frac{1}{8}(\gamma_m\theta)_\gamma(\gamma^{nr}\theta)^\gamma c_{nr}{}^{\hat\beta} + \frac{1}{4}(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\partial_q\psi_m{}^{\hat\gamma} \\ & - \frac{1}{32}(\gamma_m\theta)_\gamma(\gamma^{nr}\theta)^\gamma(\gamma^{pq}\hat\theta)^{\hat\beta} s_{nr,pq} + \frac{1}{4}(\gamma_m\theta)_\gamma(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\partial_q f^{\gamma\hat\gamma} \\ & - \frac{1}{32}(\gamma_m\theta)_\gamma(\gamma^{nr}\theta)^\gamma(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\partial_q c_{nr}{}^{\hat\gamma} + \ldots \end{aligned} \tag{5.7}$$

$$
\begin{aligned}
E^{\alpha}{}_{p} &= \psi^{\alpha}{}_{p} - \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}\omega_{mn,p} + (\gamma_p\hat{\theta})_{\hat{\gamma}} f^{\alpha\hat{\gamma}} + \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}(\gamma_p\hat{\theta})_{\hat{\beta}} c_{mn}{}^{\hat{\beta}} \\
&+ \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}\partial_n\psi^{\gamma}{}_{p} - \frac{1}{8}(\gamma_p\hat{\theta})_{\hat{\gamma}}(\gamma^{qr}\hat{\theta})^{\hat{\gamma}} c^{\alpha}{}_{qr} \\
&+ \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}(\gamma_p\hat{\theta})_{\hat{\beta}}\partial_n f^{\gamma\hat{\beta}} + \frac{1}{32}(\gamma^{mn}\theta)^{\alpha}(\gamma_p\hat{\theta})_{\hat{\gamma}}(\gamma^{qr}\hat{\theta})^{\hat{\gamma}} s_{mn,qr} \\
&+ \frac{1}{32}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}(\gamma_p\hat{\theta})_{\hat{\beta}}(\gamma^{qr}\hat{\theta})^{\hat{\beta}}\partial_n c^{\gamma}{}_{qr} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
P^{\alpha\hat{\beta}} &= f^{\alpha\hat{\beta}} - \frac{1}{4}(\gamma^{mn}\theta)^{\alpha} c_{mn}{}^{\hat{\beta}} - \frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}} c^{\alpha}{}_{pq} - \frac{1}{16}(\gamma^{mn}\theta)^{\alpha}(\gamma^{pq}\hat{\theta})^{\hat{\beta}} s_{mn,pq} \\
&+ \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}\partial_n f^{\gamma\hat{\beta}} + \frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}\partial_q f^{\alpha\hat{\gamma}} \\
&+ \frac{1}{16}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}\partial_n c^{\gamma}{}_{pq} - \frac{1}{16}(\gamma^{mn}\theta)^{\alpha}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}\partial_q c_{mn}{}^{\hat{\gamma}} \\
&+ \frac{1}{16}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}\partial_n\partial_q f^{\gamma\hat{\gamma}} + \ldots
\end{aligned}
$$

It is easy to verify that this expansion satisfies the gauge conditions (4.7) and that all auxiliary fields have been eliminated and reexpressed in terms of derivatives of physical supergravity fields.

The next step is to insert the expansion (5.7) into the definition of the vertex operator (3.5) and recombine the worldsheet one-forms $\mathbf{X}_z$ and $\mathbf{X}_{\bar{z}}$ in order to get a more manageable expression. However, it makes sense to provide such expression for an interesting example in sec. 7.

We have to notice that the vertex operator $\mathcal{V}^{(1,1)}$ contains only the superfield $A_{\alpha\hat{\beta}}$ which encodes all the needed information regarding the supergravity fields, which however appear at higher orders in θ's and $\hat{\theta}$'s. This is sufficient for amplitudes computations, even though the measure factor on zero modes in the correlation functions has to soak up plenty of θ's and $\hat{\theta}$'s ([22], [18], and [23]).

6. Gauge Fixing for Massive States

In the previous sections, we explored the gauge fixing for the massless sector of open and closed string theory. However, the spectrum of string theory contains infinitely many massive states defined, in the closed string case, by the equations

$$
\left[Q_L, \mathcal{V}^{(1,1)}_n\right] = 0\,, \qquad \left[Q_R, \mathcal{V}^{(1,1)}_n\right] = 0\,, \qquad \left[L_{0,L} + L_{0,R} - n, \mathcal{V}^{(1,1)}_n\right] = 0\,, \tag{6.1}
$$

where $L_{0,L} = \oint dz\, z\, T_{zz}$ and $L_{0,R} = \oint d\bar{z}\, \bar{z}\, \bar{T}_{\bar{z}\bar{z}}$. The index n denotes the mass of the state. Even if these equations can be solved by expanding the vertex operators $\mathcal{V}^{(1,1)}_n$ in terms of the building-blocks $\partial\theta^\alpha$, $\bar\partial\hat\theta^{\hat\alpha}$, Π^m, $\bar\Pi^m$,..., it is convenient to fix a gauge as in the massless case and then solve the equations by an iterative construction as shown in the previous section. However, since we cannot explore the complete set of vertices and provide a gauge fixing for each of them, we propose a definition of gauge fixing based on new anticommuting and nilpotent charges to be imposed on the physical states. This resembles the Siegel gauge (where the corresponding charges are $b_{L,0} = \oint dz\, z\, b_{zz}$ and $b_{L,0} = \oint d\bar z\, \bar z\, \hat b_{\bar z\bar z}$ where b_{zz} and $\hat b_{\bar z\bar z}$ are the left- and right-moving antighosts) used in string field theory to eliminate all auxiliary fields and to define the propagator for the string field.

We introduce the following charges "dual" to the BRST operators

$$\mathcal{K}_L = \oint dz\, \theta^\alpha w_\alpha\,, \qquad \mathcal{K}_R = \oint d\bar z\, \hat\theta^{\hat\alpha} \hat w_{\hat\alpha}\,. \tag{6.2}$$

They are nilpotent and anti-commute. They are not supersymmetry invariant as can be directly seen by the presence of θ^α and $\hat\theta^{\hat\alpha}$. This in fact implies that we are choosing a non symmetric gauge which can be viewed as a generalization of the Wess-Zumino gauge condition in 10 dimensions. It eliminates the lowest non physical component of the superfields and it fixes the auxiliary fields – appearing at higher order in the superspace expansion – in terms of the physical fields and their derivatives. In addition, $\mathcal{K}_{L/R}$ are not invariant under the gauge transformations (A.4) , but their gauge variations are BRST invariant because of the pure spinor conditions

$$\{Q_L, \Delta_L \mathcal{K}_L\} = 0\,, \qquad \{Q_R, \Delta_R \mathcal{K}_R\} = 0\,, \tag{6.3}$$

Moreover, $\mathcal{K}_{L/R}$ have the following commutation relations with the BRST operators

$$\{Q_L, \mathcal{K}_L\} = \mathcal{D} + J_L\,, \qquad \{Q_R, \mathcal{K}_L\} = 0\,, \tag{6.4}$$

$$\{Q_R, \mathcal{K}_R\} = \hat{\mathcal{D}} + J_R\,, \qquad \{Q_L, \mathcal{K}_R\} = 0\,,$$

where

$$\mathcal{D} = \oint dz : \theta^\alpha d_\alpha :\,, \qquad J_L = \oint dz : \lambda^\alpha w_\alpha : \tag{6.5}$$

$$\hat{\mathcal{D}} = \oint d\bar z : \hat\theta^{\hat\alpha} \hat d_{\hat\alpha} :\,, \qquad J_R = \oint d\bar z : \lambda^{\hat\alpha} \hat w_{\hat\alpha} :$$

Acting on superfields $F(x,\theta,\hat\theta)$, we have that $\{\mathcal{D}, F\} = \mathbf{D}F$ and $\{\hat{\mathcal{D}}, F\} = \hat{\mathbf{D}}F$. The ordering of fields in the operators $\mathcal{D}, \hat{\mathcal{D}}, J_L$ and J_R is needed to define the corresponding currents. The operators are gauge invariant under (A.4) because of (6.3). The main difference with respect to Siegel gauge fixing in string field theory is that in that case b_{zz} and $\hat b_{\hat z\hat z}$ are holomorphic and antiholomorphic anticommuting currents of spin 2.

In the case of the open superstring, denoting by Q and by $\mathcal{K}$ the BRST and gauge fixing operators, the gauge condition on the massless vertex operator $\mathcal{V}^{(1)} = \lambda^\alpha A_\alpha$ is given by

$$\{\mathcal{K}, \mathcal{V}^{(1)}\} = \oint dw \Big(\theta^\alpha w_\alpha\Big)(w) \Big(\lambda^\alpha A_\alpha(x,\theta)\Big)(z) = \theta^\alpha A_\alpha = 0\,. \tag{6.6}$$

We notice that the field θ^α in $\mathcal{K}$ is harmless for massless vertices, but it will give a nontrivial contribution in the massive case. In the latter case one has to add a compensating non-gauge invariant contribution on the r.h.s. of (6.6) in order to compensate the fact that $\mathcal{K}$ is not gauge invariant under (A.4).

Applying Q on the left hand side of (6.6), applying $\mathcal{K}$ on the equation $\{Q, \mathcal{V}^{(1)}\} = \lambda\gamma^m\lambda A_m(x,\theta) = 0$ and using the commutation relations (6.4), we obtain

$$(\mathbf{D}+1)\mathcal{V}^{(1)} = \lambda\gamma^m\theta A_m\,. \tag{6.7}$$

Eliminating the ghost λ^α, we end up with equation (1.5) for the superfields A_α and A_m. This procedure can be clearly generalized to massive states. First, we discuss the closed string case, then we show an example for the first massive state for open superstrings and, finally, we show that the zero momentum cohomology satisfies the same equations generalized to zero modes.

For closed strings, we reproduce the gauge fixing (4.7) by the following conditions

$$\{\mathcal{K}_L, \mathcal{V}^{(1,1)}\} = 0\,, \qquad \{\mathcal{K}_R, \mathcal{V}^{(1,1)}\} = 0 \tag{6.8}$$

and, for the gauge parameters $\Lambda^{(1,0)}$ and $\Lambda^{(0,1)}$ in eq. (4.1), by the gauge condition

$$\{\mathcal{K}_L, \Lambda^{(1,0)}\} + \{\mathcal{K}_R, \Lambda^{(0,1)}\} = 0\,. \tag{6.9}$$

which coincides with (4.9). Applying the BRST charge on the left hand sides of (6.8), acting with $\mathcal{K}_L$ and $\mathcal{K}_R$ on equations (2.3), and finally using the commutation relations (6.4), we derive the conditions for the iterative equations given in the previous section.

Let us show that the gauge fixing (6.6) also fixes the gauge transformations in a suitable way for the first massive state of the open superstring $\mathcal{V}^{(1)}_1$, leading to a recursive procedure to compute the vertex operator in term of the initial data, a multiplet of on-shell fields containing a massive spin 2 field [6].

A general decomposition of $\mathcal{V}^{(1)}_1$ in terms of fundamental building-blocks is given by

$$\mathcal{V}^{(1)}_1 = \partial\lambda^\alpha A_\alpha + \lambda^\alpha\partial\theta^\beta B_{\alpha\beta} + \lambda^\alpha : \Pi^m C_{\alpha m} : + \lambda^\alpha : d_\beta D^\beta{}_\alpha : \tag{6.10}$$

$$+ : \lambda^\alpha N^{mn} : E_{\alpha mn} + : \lambda^\alpha w_\beta \lambda^\beta : F_\alpha$$

and its gauge transformation is generated by

$$\delta\mathcal{V}_1^{(1)} = \left[Q, \Omega_1^{(0)}\right], \tag{6.11}$$

with

$$\Omega_1^{(0)} = \partial\theta^\beta\Omega_\beta + :\Pi^m\Omega_m: + :d_\beta\Omega^\beta: + :N^{mn}:\Omega_{mn} + :w_\beta\lambda^\beta:\Omega\,.$$

The decompositions are based on the requirement that the vertex operator should be invariant under the gauge transformation Δ given in (A.4). A further gauge transformation of $\Omega_1^{(0)}$ would be a variation of a negative ghost number field. The only one is the antighost w_α, but there is no gauge invariant operator only with w_α without λ^α. Notice that we have to add a (BRST-invariant) compensating term of the form $w\gamma^{mnpq}\lambda$ in order to reabsorb the non-invariance of $\mathcal{K}$.

Imposing (6.6), we get

$$A_\alpha + \theta^\beta B_{\beta\alpha} = 0\,, \qquad \theta^\alpha C_{\alpha m} = 0\,, \qquad \theta^\beta D_\beta{}^\alpha = 0\,, \tag{6.12}$$

$$\theta^\beta E_{\beta mn} + \frac{1}{1440}\left[(\gamma_{mn})^\alpha_{\ \gamma} D^\gamma{}_\alpha - (\gamma_{mn})^\alpha_{\ \gamma}\theta^\gamma F_\alpha\right] = 0\,, \qquad \frac{1}{2}(\gamma^{mn}\theta)^\beta E_{\beta mn} - D^\alpha{}_\alpha + 2\theta^\alpha F_\alpha = 0\,.$$

This gauge fixing can be reached by adjusting the parameters $\Omega_\alpha, \Omega_m, \Omega^\alpha, \Omega_{mn}$ and Ω. Using equations (6.12) and applying the operator $\mathbf{D}$, we obtain the iterative relations to compute the vertex. The gauge fixing (6.12) fixes only the supergauge part of the gauge transformation. This gauge does not fix the physical gauge transformation of the massive spin 2 system [24].

Finally, we show that the measure for zero modes satisfies the gauge fixing proposed above. In fact, by restricting the attention to zero momentum cohomology, we supersede $\mathcal{K}$ with the differential

$$\mathcal{K}_0 = \theta_0^\alpha \frac{\partial}{\partial\lambda_0^\alpha} \tag{6.13}$$

which acting on $\mathcal{V}^{(3)} = (\lambda_0\gamma^m\theta_0)(\lambda_0\gamma^n\theta_0)(\lambda_0\gamma^p\theta_0)(\theta_0\gamma_{mnp}\theta_0)$, yields

$$\mathcal{K}_0\mathcal{V}^{(3)} = 0\,. \tag{6.14}$$

Similarly, for the closed superstring, the ghost number $(3,3)$ cohomology representative satisfies the corresponding gauge fixing.

Even if the gauge fixing is not manifestly supersymmetric, the supersymmetry of the target space theory is still a symmetry. As usual, in the Wess-Zumino gauge, a supersymmetry transformation must be accompanied by a gauge transformation to bring the vertex to the original gauge. This means that

$$\delta_\epsilon[\mathcal{K}, \mathcal{V}] + [\mathcal{K}, \delta\mathcal{V}] = 0 \tag{6.15}$$

where $\delta\mathcal{V} = [Q, \Omega_\epsilon]$, $\delta_\epsilon \mathcal{V} = [\epsilon^\alpha Q_\alpha, \mathcal{V}]$ and $Q_\alpha = \oint dz\, q_\alpha$ (the supersymmetry generator q_α is given in (A.8)). As an example, we show that Ω_ϵ can be indeed found for the massless sector of the open superstring and the extension is similar for the other cases. Equation (6.15) reduces to

$$\epsilon^\alpha A_\alpha + \theta^\alpha \epsilon^\beta Q_\beta A_\alpha + \theta^\alpha D_\alpha \Omega_\epsilon = 0\,, \tag{6.16}$$

which yields

$$\mathbf{D}\Omega_\epsilon = 0\,. \tag{6.17}$$

Again, this equation can be solved iteratively in powers of θ's and it follows that $\Omega = \Omega_0(x)$. (6.17) can be checked explicitly on the solutions (5.7).

7. Non-Constant RR Field-Strength

In [25] sigma models for superstrings in the presence of constant RR field strengths have been studied. It has been verified that non-(anti)commutative superspaces [26] naturally appear in the presence of that background. A series of applications, from topological strings to deformed supersymmetric instanton analysis, has then been considered [27]. In [25], it has also been conjectured that from non-constant RR field strengths one can derive new equal-time commutation relations between coordinates x^m and θ^α living on the boundaries such as

$$\{\theta^\alpha, \theta^\beta\} = \gamma_m^{\alpha\beta} x^m\,, \tag{7.1}$$

generalizing the construction of Lie-algebraic non-commutative geometries to supermanifolds [7] (for a different example of a Lie-algebraic geometry in superspace see [28]).

The vertex operator for non-constant RR fields strengths is the basic ingredient of this kind of analysis. Applying our method, we compute the vertex for linearly x-dependent RR field strengths

$$P^{\alpha\hat\beta} = f^{\alpha\hat\beta} + \mathcal{C}_m{}^{\alpha\hat\beta} x^m \tag{7.2}$$

where $\mathcal{C}_m{}^{\alpha\hat\beta}$ is constant. $P^{\alpha\hat\beta}$ must satisfy eqs. (3.11), which become $\gamma^m_{\alpha\beta}\mathcal{C}_m^{\beta\hat\gamma} = \gamma^m_{\hat\alpha\hat\beta}\mathcal{C}^{\gamma\hat\alpha} = 0$ for the specific ansatz (7.2)[6].

[6] Equations (3.11) can be rewritten in terms of forms by decomposing $P^{\alpha\hat\beta}$ according to Dirac equations. For example, for type IIB we have the 1-form P_m, the 3-form $P_{[mnp]}$ and the 5-form $F_{[mnpqr]}$. Solving the Bianchi identities we get $P_m = \partial_m A$, $P_{[mnp]} = \partial_{[m} A_{np]}$,... and the field equations are $\partial^m P_m = \partial^2 A = 0$, $\partial^m P_{[mnp]} = \partial^m \partial_{[m} A_{np]}$,... These can be solved in terms of quadratic polynomials $A(x) = (10\, a_{(mn)} - a_r{}^r \eta_{mn}) x^m x^n$, $A_{[mn]} = (10\, a_{[mn],(rs)} - a_{[mn,t]}^{\;\;t} \eta_{rs}) x^r x^s$,... where $a_{(mn)}$, $a_{[mn],(rs)}$,... are constant background fields.

We remind the reader that, in the constant field strength case, $\mathcal{V}^{(0,0)}_{z\bar{z}} = q_\alpha f^{\alpha\hat{\beta}}\hat{q}_{\hat{\beta}}$, where q_α and $\hat{q}_{\hat{\beta}}$ are the supersymmetry currents given in (A.8). So it is easy to see that equation (2.7) is verified with $\mathcal{V}^{(1,1)} = \chi_\alpha f^{\alpha\hat{\beta}}\hat{\chi}_{\hat{\beta}}$, which is clearly BRST invariant (see (A.9) and (A.10)). Since in the θ and $\hat{\theta}$ expansions of the auxiliary and physical superfields $A_{\alpha\hat{\beta}}$,...,$P^{\alpha\hat{\beta}}$ (see eqs. (5.7) and appendix B) the number of bosonic derivatives acting on physical zero-order components grows with growing order in θ and $\hat{\theta}$, it is clear that the ansatz (7.2) will correspond to only a few non-zero terms in the expansion. Actually, the highest-order contributions are $\theta^4\hat{\theta}^2$ and $\theta^2\hat{\theta}^4$ terms. Here we give the explicit expressions for $A_{\alpha\hat{\beta}}$

$$\begin{aligned} A_{\alpha\hat{\beta}} = & \frac{1}{9}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta(\gamma^p\hat{\theta})_{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}(f^{\beta\hat{\gamma}} + \mathcal{C}_n{}^{\beta\hat{\gamma}}x^n) \\ & + \frac{1}{180}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^p\hat{\theta})_{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}(\gamma^{qr}\hat{\theta})^{\hat{\gamma}}(\gamma_q\hat{\theta})_{\hat{\delta}}\mathcal{C}_r{}^{\gamma\hat{\delta}} \\ & + \frac{1}{180}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\delta(\gamma^{nr}\theta)^\delta(\gamma_n\theta)_\gamma(\gamma_p\hat{\theta})_{\hat{\beta}}(\gamma^p\hat{\theta})_{\hat{\gamma}}\mathcal{C}_r{}^{\gamma\hat{\gamma}}\,, \end{aligned} \tag{7.3}$$

The remaining superfields are given in app. C. To obtain the vertices $\mathcal{V}^{(1,1)}$ and $\mathcal{V}^{(0,0)}_{z\bar{z}}$ for the linearly x-dependent RR field strength we have to insert (7.3) and the superfields given in app. C back into (3.1) and (3.5).

For the unintegrated vertex operator we find

$$\begin{aligned} \mathcal{V}^{(1,1)} = & \chi_\alpha f^{\alpha\hat{\beta}}\hat{\chi}_{\hat{\beta}} \\ & + \chi_\alpha\left[\left(x^m\delta^\alpha_\gamma\delta^{\hat{\beta}}_{\hat{\gamma}} + \frac{1}{20}(\gamma^{qm}\hat{\theta})^{\hat{\beta}}(\gamma_q\hat{\theta})_{\hat{\gamma}}\delta^\alpha_\gamma + \frac{1}{20}(\gamma^{nm}\theta)^\alpha(\gamma_n\theta)_\gamma\delta^{\hat{\beta}}_{\hat{\gamma}}\right)\mathcal{C}_m{}^{\gamma\hat{\gamma}}\right]\hat{\chi}_{\hat{\beta}} \end{aligned} \tag{7.4}$$

while for the integrated vertex operator $\mathcal{V}^{(0,0)}_{z\bar{z}}$ we obtain

$$\begin{aligned} \mathcal{V}^{(0,0)}_{z\bar{z}} = & q_\alpha f^{\alpha\hat{\beta}}q_{\hat{\beta}} \\ & + q_\alpha\left[x^s\delta^\alpha_\gamma\delta^{\hat{\beta}}_{\hat{\gamma}} + \frac{1}{4}(\gamma^{rs}\theta)^\alpha(\gamma_r\theta)_\gamma\delta^{\hat{\beta}}_{\hat{\gamma}} + \frac{1}{4}(\gamma^{ps}\hat{\theta})^{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}\delta^\alpha_\gamma\right]\mathcal{C}_s{}^{\gamma\hat{\gamma}}\hat{q}_{\hat{\beta}} \\ & + \left[-\frac{1}{6}(\partial x^m + \frac{1}{10}\theta\gamma^m\partial\theta)(\theta\gamma_m\gamma^{rs}\theta) - N^{rs}\right](\gamma_r\theta)_\alpha\mathcal{C}_s{}^{\alpha\hat{\beta}}\hat{q}_{\hat{\beta}} \\ & + q_\alpha\mathcal{C}_s{}^{\alpha\hat{\beta}}(\gamma_r\hat{\theta})_{\hat{\beta}}\left[-\frac{1}{6}(\bar{\partial}x^p + \frac{1}{10}\hat{\theta}\gamma^p\bar{\partial}\hat{\theta})(\hat{\theta}\gamma_p\gamma^{rs}\hat{\theta}) - \hat{N}^{rs}\right] \end{aligned} \tag{7.5}$$

The complicated structure of $\mathcal{V}^{(0,0)}_{z\bar{z}}$ prevents from a simple analysis of superspace deformations as in [25], and this will be discussed in a separate publication [29].

8. Vertex Operators with RR Gauge Potentials

In presence of D-branes, one can ask which states couple to them and which vertex operators describe such interaction. As it was discussed in [8] in the framework of RNS formalism, one has to construct the vertex operators for RR fields in the asymmetric picture.[7] In addition, a propagating closed string (*i.e.* with non vanishing momentum) emitted from a disk or a D-brane, has to be off-shell. Therefore, one needs to break the BRST invariance by allowing a non vanishing commutator with $Q_{L,0}+Q_{R,0}$ where $Q_{L/R,0}$ are the picture conserving parts of BRST charges in the RNS formalism. In particular in [9] the authors construct a solution of $[Q_{L,1}+Q_{R,1},W]=0$, where W is the vertex operator in the asymmetric picture.

In the present section, we construct analogous vertices for closed superstrings which do not satisfy the classical supergravity equations of motion, but modified superfield constraints. They allow a description of the RR gauge potentials, in contradistinction to the on-shell formalism case, where only the field strengths $P^{\alpha\hat\beta}$ appear. First of all, there are some important differences. The two BRST charges Q_L and Q_R contain a single term and therefore the decomposition used in [9] is not viable. In addition, there are no different pictures (in the usual sense) for a given vertex since there are no superghosts associated to local worldsheet supersymmetry. There is, however, the possibility of constructing two operators which resemble the picture lowering and raising operator, as suggested by Berkovits [30], but the implications of this new idea have not been explored yet.

Nevertheless we can construct an off-shell formalism by considering the following combination of vertices with different ghost numbers:

$$\mathcal{V}^{(2)}=\mathcal{V}^{(2,0)}+\mathcal{V}^{(1,1)}+\mathcal{V}^{(0,2)}\,, \tag{8.1}$$

where the notation $\mathcal{V}^{(a,b)}$ stands for vertex operators with the left ghost number a and with the right ghost number b. The ghost number of the l.h.s. is just the sum of the ghost numbers. Notice that if we insert a single $\mathcal{V}^{(2)}$ vertex in an amplitude where all the other vertices have definite ghost number, only the central part $\mathcal{V}^{(1,1)}$ of the vertex $\mathcal{V}^{(2)}$ does contribute. In principle, we have to add all the possible terms $\sum_{i=-\infty}^{\infty}\mathcal{V}^{(1+i,1-i)}$. However, only the antighosts $w_{z\alpha}$ and $\hat w_{\bar z\hat\alpha}$, which have to be gauge invariant under (A.4), carry a negative ghost number and higher conformal spin. This is not the case in the RNS formalism where the presence of a worldsheet index is compensated by the worldsheet ghosts that carry conformal spin -1.

The general form of $\mathcal{V}^{(2)}$ is given by

$$\mathcal{V}^{(2)}=\lambda^\alpha\lambda^\beta G_{(\alpha\beta)}+\lambda^\alpha\hat\lambda^{\hat\beta}A_{\alpha\hat\beta}+\hat\lambda^{\hat\alpha}\hat\lambda^{\hat\beta}\hat G_{(\hat\alpha\hat\beta)}\,. \tag{8.2}$$

where $G_{(\alpha\beta)}$, $A_{\alpha\hat\beta}$ and $\hat G_{(\hat\alpha\hat\beta)}$ are superfields of the variables x^m,θ^α and $\hat\theta^{\hat\alpha}$. The superfields $G_{(\alpha\beta)}$ and $\hat G_{(\hat\alpha\hat\beta)}$ are symmetric in the spinorial indices and they are defined up to the

[7] We thank M. Bianchi for suggesting the present analysis.

algebraic gauge transformations

$$\delta G_{(\alpha\beta)} = \gamma^m_{\alpha\beta}\Sigma_m \,, \qquad \delta \hat{G}_{(\hat{\alpha}\hat{\beta})} = \gamma^m_{\hat{\alpha}\hat{\beta}}\hat{\Sigma}_m \,, \tag{8.3}$$

which leave only the 5-form parts unfixed.

Instead of imposing the equations (2.3) separately for the left and the right sector, we then impose the modified condition

$$[Q_L + Q_R, \mathcal{V}^{(2)}] = 0 \,, \tag{8.4}$$

which leads to the equations

$$\begin{aligned}
&D_\alpha A_{\beta\hat{\beta}} + D_\beta A_{\alpha\hat{\beta}} - \gamma^m_{\alpha\beta} A_{m\hat{\beta}} = -\hat{D}_{\hat{\beta}} G_{(\alpha\beta)} \,, \\
&\hat{D}_{\hat{\alpha}} A_{\alpha\hat{\beta}} + \hat{D}_{\hat{\beta}} A_{\alpha\hat{\alpha}} - \gamma^m_{\hat{\alpha}\hat{\beta}} A_{\alpha m} = -D_\alpha \hat{G}_{(\hat{\alpha}\hat{\beta})} \,, \\
&D_\alpha G_{(\beta\gamma)} + D_\beta G_{(\gamma\alpha)} + D_\gamma G_{(\alpha\beta)} = \gamma^m_{(\alpha\beta} G_{\gamma)m} \,, \\
&\hat{D}_{\hat{\alpha}} \hat{G}_{(\hat{\beta}\hat{\gamma})} + \hat{D}_{\hat{\beta}} \hat{G}_{(\hat{\gamma}\hat{\alpha})} + \hat{D}_{\hat{\gamma}} \hat{G}_{(\hat{\alpha}\hat{\beta})} = \gamma^m_{(\hat{\alpha}\hat{\beta}} \hat{G}_{\hat{\gamma})m} \,,
\end{aligned} \tag{8.5}$$

The first two equations are consistent generalizations of the first two equations in (3.6). The other two equations emerge from the ghost number three part of (8.4). They are equations of motion for the auxiliary fields $G_{(\alpha\beta)}$ and $\hat{G}_{(\hat{\alpha}\hat{\beta})}$. It has been shown in the context of spinorial cohomology [13] that the third and fourth equations have non-trivial solutions. Therefore, the first and second equations are deformations of the usual superfield constraints. In the limit $G, \hat{G} \to 0$ one recovers the usual on-shell supergravity. It is interesting to note that the way supergravity equations and/or on-shell superspace constraints are relaxed in order to go off-shell follows very closely the case of N=1 SYM theory presented in [13]. In fact, in those papers the authors considered an extension of d=10 N=1 SYM theory by relaxing the superspace constraint $F_{(\alpha\beta)} \equiv D_{(\alpha}A_{\beta)} - \gamma^m_{\alpha\beta}A_m = 0$ (cf. the introduction for the notation) by introducing a self-dual 5-form $F_{(\alpha\beta)} = J^{[5]}_{(\alpha\beta)}$. In the present case, the relaxation of the on-shell constraints is again due to two 5-forms $G_{(\alpha\beta)}$ and $\hat{G}_{(\hat{\alpha}\hat{\beta})}$. The 5-form $J^{[5]}_{(\alpha\beta)}$ satisfies the constraint $D_{(\alpha}J^{[5]}_{\beta\gamma)} - \gamma^m_{(\alpha\beta}J_{\gamma)m} = 0$ due to the Bianchi identities. The solution of these identities by means of the spinorial cohomology shows that the 5-form is not completely constrained, but it allows extensions of the superspace equation of motions. In the same way, the conditions for $G_{(\alpha\beta)}$ and $\hat{G}_{(\hat{\alpha}\hat{\beta})}$ constrain them, but allow a certain useful modifications of the supergravity equations of motion.

Eqs. (8.5) are invariant under the gauge transformations

$$\delta\mathcal{V}^{(2)} = [Q_L + Q_R, \Omega^{(1)}] \,, \qquad \Omega^{(1)} = \lambda^\alpha \Theta_\alpha + \hat{\lambda}^{\hat{\alpha}} \hat{\Theta}_{\hat{\alpha}} \,. \tag{8.6}$$

which imply

$$\delta A_{\alpha\hat{\beta}} = D_\alpha \hat{\Theta}_{\hat{\beta}} + \hat{D}_{\hat{\beta}} \Theta_\alpha \,, \tag{8.7}$$

$$\delta G_{(\alpha\beta)} = \gamma^m_{\alpha\beta}\Sigma_m + D_{(\alpha}\Theta_{\beta)}\,, \qquad \delta\hat G_{(\hat\alpha\hat\beta)} = \gamma^m_{\hat\alpha\hat\beta}\hat\Sigma_m + \hat D_{(\hat\alpha}\hat\Theta_{\hat\beta)}\,.$$

In the last equations we have summed (8.3) to (8.6). The gauge parameters Θ_α and $\hat\Theta_{\hat\alpha}$ are not forced to satisfy consistency conditions as in (2.9) and (2.10). Nevertheless, eqs. (8.5) are gauge invariant because of the second and the third equations of (8.7).

Following the previous section, we can require a suitable gauge fixing to reduce the auxiliary fields by imposing

$$\{\mathcal{K}_L + \mathcal{K}_R, \mathcal{V}^{(2)}\} = 0. \tag{8.8}$$

This gauge condition yields

$$2\theta^\beta G_{(\alpha\beta)} + \hat\theta^\beta A_{\alpha\hat\beta} = 0\,, \qquad \theta^\alpha A_{\alpha\hat\beta} + 2\hat\theta^{\hat\alpha}\hat G_{(\hat\alpha\hat\beta)} = 0\,, \tag{8.9}$$

which partially fix (8.7). Notice that the lowest component of $A_{\alpha\hat\beta}$ is chosen to be kept unfixed and (8.9) can be regarded as a condition on G and $\hat G$. In this gauge the lowest component of $A_{\alpha\hat\beta}$ is not gauged away and for example we can identify the θ^2 and $\hat\theta^2$ component of $A_{\alpha\hat\beta}$ as the RR potentials

$$A_{\alpha\hat\beta} = \ldots + a_\alpha{}^{\hat\gamma}(\gamma_m\hat\theta)_{\hat\gamma}(\gamma^m\hat\theta)_{\hat\beta} + \hat a^\gamma{}_{\hat\beta}(\gamma_m\theta)_\gamma(\gamma^m\theta)_\alpha + \ldots \tag{8.10}$$

where $a_\alpha{}^{\hat\gamma}$ and $\hat a^\alpha{}_{\hat\gamma}$ are the RR potentials. Those potentials can also be found in the lowest components of the superfields $E_\alpha{}^{\hat\gamma}$ and $E^\alpha{}_{\hat\gamma}$.

As a consistency condition we observe that

$$\{Q_L + Q_R, \mathcal{K}_L + \mathcal{K}_R\} = (\mathcal{D} + \hat{\mathcal{D}}) + (J_L + J_R)\,, \tag{8.11}$$

and by using (8.6) and (8.8) we obtain a consistency condition for $\mathcal{V}^{(2)}$. It can be checked that the vertex $\mathcal{V}^{(2)}$ satisfies this new constraint and this condition does not fix the physical gauge transformations.

The natural question is how to extend the analysis of the previous sections in the present case. We have to rederive the complete set of descent equations in order to provide all the equations of motion. The generalized vertex operators are given by the collection

$$\mathcal{V}^{(1)}_{\bar z} = \mathcal{V}^{(2,-1)}_{\bar z} + \mathcal{V}^{(1,0)}_{\bar z} + \mathcal{V}^{(0,1)}_{\bar z}\,, \tag{8.12}$$

$$\mathcal{V}^{(1)}_{z} = \mathcal{V}^{(1,0)}_{z} + \mathcal{V}^{(0,1)}_{z} + \mathcal{V}^{(-1,2)}_{z}\,,$$

$$\mathcal{V}^{(0)}_{z\bar z} = \mathcal{V}^{(1,-1)}_{z\bar z} + \mathcal{V}^{(0,0)}_{z\bar z} + \mathcal{V}^{(-1,1)}_{z\bar z}\,.$$

It is easy to show that all equations (3.6) are deformed by the presence of new fields. By setting those fields to zero we recover the on-shell supergravity equations.

9. Antifields and the Kinetic Terms for Closed String Field Theory

In the present section, we derive the set of antifields for the massless sector of closed string theory. We discuss the couplings of the fields to the antifields for a closed string field theory action and, finally, we propose a kinetic term which leads to the correct equations of motion taking into account the presence of selfdual forms. This section is structured as follows. We first recall some basic facts about the field theory of open superstrings, mainly focusing on the relation between fields and antifields, their coupling and the kinetic terms yielding the linearized equations of motion. Following the analogy with closed bosonic string field theory, we then construct the antifields and the kinetic terms for closed superstrings. Since in the present paper we never dealt with non-linear extensions of supergravity equations, we do not discuss generalizations of Witten's string field $\star$-product for open superstring. Similarly, it is outside of the scope of the present paper to construct a full-fledged closed string field theory and we limit ourselves to a specific sector of the theory.

9.1. Open Superstring (Antifields and the Kinetic Term)

One of the most important ingredients in the construction of an off-shell extension of superstring theory is the Batalin-Vilkovisky formalism. Associated to each field of the theory φ_s (the index s denotes a collection of fields with the same ghost number $G(s)$), there is an antifield φ^*_s, whose BRST variation corresponds to the equations of motion of φ^*. The set of antifields for open superstring theory at the massless level has been discussed in [1] (see also [31]). The generalization at the massive level is straightforward, and the ghost-for-ghosts for higher massive spin fields has to be taken into account.

Following the notation of the previous sections, we introduce the string field $\Phi^{(1)}_o$ which has the general decomposition (at the massless level)

$$\Phi^{(1)}_o = C + \lambda^\alpha A_\alpha + \lambda^\alpha \lambda^\beta A^*_{(\alpha\beta)} + \lambda^\alpha \lambda^\beta \lambda^\gamma C^*_{(\alpha\beta\gamma)} \, . \tag{9.1}$$

The truncation at order three in the ghost fields is justified by the absence of any cohomology at ghost number greater than three[8] and the only cohomology at ghost number three is the zero momentum cohomology constructed in terms of

$$C^*_{\alpha\beta\gamma} = C^* (\gamma^m \theta)_\alpha (\gamma^n \theta)_\beta (\gamma^r \theta)_\gamma (\theta \gamma_{mnr} \theta) \tag{9.2}$$

where C^* is constant. The expansion of $\Phi^{(1)}_o$ into powers of λ^α accounts for different target space fields: the ghost superfield C, the spinorial part of the connection A_α, the antifields

[8] There is a simple way to establish the absence of cohomology at ghost number higher than three by using supergeometrical arguments discussed in [32] and references therein.

$A^*_{(\alpha\beta)}$ and the antifield $C^*_{(\alpha\beta\gamma)}$ of the ghost fields. The interpretation as target space fields with different ghost and antifield numbers is discussed in [33] and as rigid symmetries of the target space in [34]. In order to assign a total (worldsheet plus target space) ghost number to $\Phi^{(1)}_0$, the target space ghost numbers must be $+1$ for the field C, 0 for A_α, -1 for $A^*_{\alpha\beta}$, and -2 for $C^*_{\alpha\beta\gamma}$. They coincide precisely with the usual assignment of ghost and antifield number for theories with irreducible gauge symmetries. The ghost number zero superfield A_α contains the gauge field $a_m(x)$ and the gluino $\psi^\alpha(x)$, the ghost number -1 superfield $A^*_{\alpha\beta}$ contains the antifields $a^{*,m}$ of $a_m(x)$ and ψ^*_α of $\psi^\alpha(x)$.

The main properties are the following:

i) The string field $\Phi^{(1)}_o = \Phi^+_o + \Phi^-_o$ is decomposed into fields Φ^+_o and antifields Φ^-_o which have the expansions

$$\Phi^+_o = \sum_{s, G(s)\geq 0} \varphi_s \Phi^+_{o,s}\,, \qquad \Phi^-_o = \sum_{s, G(s)<0} \varphi^*_s \Phi^-_{o,s}\,. \tag{9.3}$$

where $\Phi^+_{o,s}$ and $\Phi^-_{o,s}$ form two basis for fields and antifields, respectively.

If the inner product $\langle A, B\rangle = \int d\mu^{(-3)} AB$ is defined by using the measure[9]

$$\int d\mu^{(-3)} \mathcal{V}^{(3)}_0 = 1\,, \qquad \mathcal{V}^{(3)}_0 = (\lambda_0\gamma^m\theta_0)(\lambda_0\gamma^n\theta_0)(\lambda_0\gamma^r\theta_0)(\theta_0\gamma_{mnr}\theta_0)\,, \tag{9.4}$$

the antifields Φ^-_o are dual to the fields Φ^+_o according to the relation

$$\int d\mu^{(-3)} \left(\Phi^+_o \Phi^-_o\right) = \sum_{s,s'} \int d\mu^{(-3)} (\varphi_s \Phi^+_{o,s})(\varphi^*_{s'} \Phi^-_{o,s'}) = \sum_s \int d^{10}x\, \varphi_s \varphi^*_s\,. \tag{9.5}$$

The integration over the Grassman coordinates is prescribed by the measure $d\mu^{(-3)}$.

ii) The antifields Φ^-_o couple in the action to the BRST variation of fields. Therefore, the unique choice is

$$S^* = \int d\mu^{(-3)} \left(\Phi^-_o Q \Phi^+_o\right)\,. \tag{9.6}$$

It is easy to check that the antifields φ^*_s correctly couple to the variation of the fields φ_s. A term of the form $S^* = \int d\mu^{(-3)} \left(\Phi^-_o Q \Phi^-_o\right)$ vanishes because of the ghost number of Φ^-_s.

iii) Finally, a kinetic term for the fields can be constructed. Since the inner product defined above is non-degenerate, one can check that the kinetic term

$$S^K = \int d\mu^{(-3)} \left(\Phi^+_o Q \Phi^+_o\right)\,, \tag{9.7}$$

yields the correct linearized equations of motion. It has been verified that, for massless fields, the above equation leads to the correct component action for N=1 SYM in $d = (9,1)$. It would be very interesting to check if (9.7) also leads to an action for free massive higher spin fields [35](see also [36]).

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[9] An expression for $d\mu^{(-3)}$ is given in [23].

9.2. Closed Superstrings (Antifields and the Kinetic Term)

We base our construction on the bosonic closed string field theory. We follow the notations and the definitions given in [17] and we provide a translation in terms of our formulation.

In closed string theory a generic string field $\Phi^{(1,1)}$ has ghost number $(1,1)$ and the usual form for the kinetic term $\int d\mu^{(-3,-3)}\,\Phi Q \Phi$ given above cannot work. In fact the BRST operator $Q_L + Q_R$ has ghost number $(1,0)+(0,1)$, the measure has ghost number $(-3,-3)$ and it is not possible to saturate the ghost number properly. In [17], it was proven that the kinetic term for bosonic closed string theory

$$S_c = \langle \Phi^{(1,1)} | c_0^- (Q_L + Q_R) | \Phi^{(1,1)} \rangle \,, \tag{9.8}$$

where $c_0^- = c_{L,0} - c_{R,0}$ ($c_{L/R,0}$ are the zero modes of diffeomorphisms ghosts) leads to the correct equations of motion. Moreover, string field theory actions have the property to reproduce not only the action for the physical fields (in the case of bosonic closed string field theory, eq. (9.8) yields the action for the graviton, for the NS-NS two form and for the dilaton), but also the full BV action with antifields and gauge transformations.

As explained before, removing the restriction of the ghost number of the string field Φ_c [10] we can decompose it into $\Phi_c = \Phi_c^+ + \Phi_c^-$ where $\Phi_c^+ = \sum_{s,G(s)\geq 0} \varphi_s \Phi_s^+$ with $G(s)$ denoting the ghost number of field φ_s. The other components of Φ_c, namely Φ_c^-, should contain the antifields.

The antifields should be dual with respect to the inner product given in (9.8) $\langle A | c_0^- | B \rangle$, where A and B are two generic vertex operators.[11] Since the fields, the ghosts and the ghost-for-ghosts are contained into Φ_c^+ with ghost number $0, 1$ and 2 respectively, the corresponding antifields Φ_0^- should have ghost number $3, 4$ and 5 to be dual to Φ_c^+. In fact for bosonic closed strings, in order to get a non zero result from the zero mode prescription

$$\langle c_{L,0} c_{L,1} c_{L,-1} c_{R,0} c_{R,1} c_{R,-1} \rangle = 1 \,, \tag{9.9}$$

for tree level amplitudes, the total ghost number should be 6. (On the sphere, one has to compensate the anomaly of the left- and right-moving ghost current anomaly). However, the naive BPZ conjugation of a string field maps Φ_c into $\tilde{\Phi}_c = \sum_{s,G(s)<0} \varphi_s^* \tilde{\Phi}_s$. So, in addition, one has to act with the operator b_0^- (where $b_{L/R,0}$ are the zero modes of the antighosts)

$$\Phi_c^- = \Big\{ b_0^- , \sum_{s,G(s)<0} \varphi_s^* \tilde{\Phi}_s \Big\} = \sum_{s,G(s)<0} \varphi_s^* \{ b_0^- , \tilde{\Phi}_s \} = \sum_{s,G(s)<0} \varphi_s^* \Phi_s^- \,, \tag{9.10}$$

[10] We add the subscript c to distinguish it from the open case and we remove the ghost number.

[11] In [17], it has been proved that the inner product is not degenerate. However, the gauge invariance is achieved only if the string field Φ_c satisfies the two conditions: $\{b_0^-, \Phi_c\} = \{L_0^-, \Phi_c\} = 0$.

The b_0^- operation has the virtue to correctly reduce the ghost number of the conjugated string field. In particular, Φ_s^+, defined above, forms a basis and Φ_s^- forms the dual basis paired with Φ_s^+ as

$$\langle \Phi_s^- | c_0^- | \Phi_{s'}^+ \rangle = \delta_{s,s'} \,. \tag{9.11}$$

Removing the restrictions on ghost numbers, one can show that there is always an element of the basis Φ_c^- corresponding to an element of Φ_c^+.

In the present framework, although we have neither the operator b_0^- nor the operator c_0^-, we can still construct the antifields $\sum_s \varphi_s^* \Phi_s^-$ paired to the fields $\sum_s \varphi_s \Phi_s^+$. In particular for the massless sector Φ_s^+ has the general expansion

$$\Phi_c^+ = \Omega + \lambda^\alpha \Theta_\alpha + \hat\lambda^{\hat\alpha} \hat\Theta_{\hat\alpha} + \lambda^\alpha \hat\lambda^{\hat\beta} A_{\alpha\hat\beta} \,. \tag{9.12}$$

where $A_{\alpha\hat\beta}$ contains the fields, Θ_α and $\hat\Theta_{\hat\alpha}$ contain the target space ghosts and Ω the ghost-for-ghosts. [12] Being the measure for zero modes of $\lambda^\alpha, \hat\lambda^{\hat\alpha}, \theta^\alpha$ and $\hat\theta^{\hat\alpha}$ given by the following equations

$$\int d\mu_c^{(-3,-3)} \mathcal{V}^{(3,3)} = 1 \,, \tag{9.13}$$

$$\mathcal{V}^{(3,3)} = (\lambda_0 \gamma^m \theta_0 \lambda_0 \gamma^n \theta_0 \lambda_0 \gamma^p \theta_0 \theta_0 \gamma_{mnp} \theta_0)(\hat\lambda_0 \gamma^m \hat\theta_0 \hat\lambda_0 \gamma^n \hat\theta_0 \hat\lambda_0 \gamma^p \hat\theta_0 \hat\theta_0 \gamma_{mnp} \hat\theta_0) \,,$$

it follows that the components of the expression

$$\begin{aligned} \Phi_c^- = {} & \lambda^\alpha \lambda^\beta \hat\lambda^{\hat\alpha} \hat\lambda^{\hat\beta} A^*_{(\alpha\beta)(\hat\alpha\hat\beta)} + \lambda^\alpha \lambda^\beta \lambda^\gamma \hat\lambda^{\hat\alpha} \hat\lambda^{\hat\beta} \Theta^*_{(\alpha\beta\gamma)(\hat\alpha\hat\beta)} \\ & + \lambda^\alpha \lambda^\beta \hat\lambda^{\hat\alpha} \hat\lambda^{\hat\beta} \hat\lambda^{\hat\gamma} \hat\Theta^*_{(\alpha\beta)(\hat\alpha\hat\beta\hat\gamma)} + \lambda^\alpha \lambda^\beta \lambda^\gamma \hat\lambda^{\hat\alpha} \hat\lambda^{\hat\beta} \hat\lambda^{\hat\gamma} \Omega^*_{(\alpha\beta\gamma)(\hat\alpha\hat\beta\hat\gamma)} \,, \end{aligned} \tag{9.14}$$

are indeed paired by (9.13) to the different components Φ_c^+ (in the massless sector). The superfields $A^*_{(\alpha\beta)(\hat\alpha\hat\beta)}, \Theta^*_{(\alpha\beta\gamma)(\hat\alpha\hat\beta)}, \hat\Theta^*_{(\alpha\beta)(\hat\alpha\hat\beta\hat\gamma)}$ and $\Omega^*_{(\alpha\beta\gamma)(\hat\alpha\hat\beta\hat\gamma)}$ are defined up to the algebraic gauge transformations

$$\begin{aligned} \delta A^*_{(\alpha\beta)(\hat\alpha\hat\beta)} &= \gamma^m_{\alpha\beta} A^*_{m,(\hat\alpha\hat\beta)} + \gamma^m_{\hat\alpha\hat\beta} A^*_{\alpha\beta,m} \,, \\ \delta\Theta^*_{(\alpha\beta\gamma)(\hat\alpha\hat\beta)} &= \gamma^m_{(\alpha\beta} \Theta^*_{\gamma)m,(\hat\alpha\hat\beta)} + \gamma^m_{(\hat\alpha\hat\beta} \Theta^*_{|\alpha\beta|,\hat\gamma)m} \,, \\ \delta\Theta^*_{(\alpha\beta\gamma)(\hat\alpha\hat\beta\hat\gamma)} &= \gamma^m_{(\alpha\beta} \Theta^*_{\gamma)m,(\hat\alpha\hat\beta\hat\gamma)} + \gamma^m_{(\hat\alpha\hat\beta} \Theta^*_{|\alpha\beta\gamma|,\hat\gamma)m} \,, \end{aligned} \tag{9.15}$$

which reduce the number of independent components in order to match the number of fields present in Φ_c^+. A long, but straightforward computation shows that

$$\int d\mu^{(-3,-3)} \left(\Phi_c^- \Phi_c^+ \right) = \int d^{10}x \sum_s \varphi_s^* \varphi_s \,, \tag{9.16}$$

[12] For massive states, we have also to take into account the expansion in the antighost w_α as in (6.10).

where the Grassman integration is performed according to the measure given in (9.13).

ii) In the next step, we use the above definition to construct a term which relates the antifields to the BRST sources of the physical fields. Since the gauge transformations are generated by the BRST charge $Q_L + Q_R$, the term of the action which couples antifields and BRST variations is

$$S^* = \int d\mu^{(-3,-3)} \Big(\Phi_c^- (Q_L + Q_R) \Phi_c^+ \Big) \,. \tag{9.17}$$

The BRST operator raises the ghost number of the string field Φ_c^+ of one unity, therefore

$$S^* = \sum_s \int d^{10}x \varphi_s^* \delta\varphi_s \tag{9.18}$$

where $\delta\varphi_s$ is the BRST variation of the field φ_s which depends on the fields φ_{s-1}.

iii) As a last step, we have to construct the kinetic term of the closed string field theory. Since we do not have the operators b_0^- and c_0^-, we cannot follow the bosonic string field theory analogy. However, in our context we have a new vertex operator with ghost number 3 of the form

$$\Phi^{(3)} = \lambda^\alpha \lambda^\beta \hat{\lambda}^{\hat{\alpha}} \mathcal{Q}_{(\alpha\beta)\hat{\alpha}} + \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} \hat{\mathcal{Q}}_{\alpha\hat{\alpha}\hat{\beta}} \,, \tag{9.19}$$

which has been neglected so far. $\mathcal{Q}_{(\alpha\beta)\hat{\alpha}}$ and $\hat{\mathcal{Q}}_{\alpha(\hat{\alpha}\hat{\beta})}$ are defined up to the gauge transformations $\delta\mathcal{Q}_{(\alpha\beta)\hat{\alpha}} = \gamma^m_{\alpha\beta}\Omega_{m\hat{\alpha}}$ and $\delta\hat{\mathcal{Q}}_{\alpha(\hat{\alpha}\hat{\beta})} = \gamma^m_{(\hat{\alpha}\hat{\beta})}\Omega_{\alpha m}$ which allow us to take into account the pure spinor constraints.

We can finally write a kinetic term of the form

$$S^K = \int d\mu^{(-3,-3)} \Big(\Phi_0^{(3)} (Q_L + Q_R) \Phi_c^+ + \sum_{n\geq 0} (-1)^n \Phi_n^{(3)} \Phi_{n+1}^{(3)} \Big) \,, \tag{9.20}$$

where $\Phi_n^{(3)}$ is an infinite collection of fields of the form (9.19) needed to escape the no-go theorem of the existence of an action for selfdual antisymmetric forms with a finite number of fields and covariant [16]. To write action (9.20) we followed the technique developed in [37] and it has similarity with action proposed in [34] where all the pictures are taken simultaneously into account introducing new commuting fields. The expansion of the string fields in terms of powers of these new fields leads to a target space action with infinite number of fields [38].

The action (9.20) is invariant under the gauge transformations

$$\delta\Phi_c^+ = (Q_L + Q_R)\Omega + \Delta \,, \tag{9.21}$$

$$\delta\Phi_{2n}^{(3)} = (Q_L + Q_R)\Gamma \,, \qquad \delta\Phi_{2n+1}^{(3)} = (Q_L + Q_R)\Delta \,, \quad \forall\, n.$$

where Ω has total ghost number 1, whereas Δ and Γ have ghost number 2. Notice that there is no coupling between $\Phi^{(3)}_n$ and the antifields, because of the ghost number of $\Phi^{(3)}_n$.

The presence of commuting ghosts λ^α and $\hat\lambda^{\hat\alpha}$ is usually the source of a well-known problem, an infinite number of equivalent copies of the cohomology, identified by a new quantum number known as picture. In the context of topological theory the Picture Changing Operator (PCO) and its inverse have been constructed on the basis of supergeometry and singular forms [32]. In pure spinor string theory, Berkovits has recently suggested that the PCO and its inverse can be indeed constructed [30]. This would motivate the introduction of an infinite number of fields $\Phi^{(3)}_n$ necessary to construct a string field theory action for closed superstrings.

From the action S^K one can derive the equations of motion

$$\{Q_L + Q_R, \Phi^+_c\} + \Phi^{(3)}_1 = 0\,, \qquad \{Q_L + Q_R\, \Phi^{(3)}_0\} = 0\,, \tag{9.22}$$

$$\Phi^{(3)}_0 = \Phi^{(3)}_2 = \Phi^{(3)}_4 = \dots\,,$$

$$\Phi^{(3)}_1 = \Phi^{(3)}_3 = \Phi^{(3)}_5 = \dots\,.$$

and for a solution with a finite number of string fields, one gets

$$\{Q_L + Q_R, \Phi^+_c\} = 0\,, \tag{9.23}$$

$$\Phi^{(3)}_0 = \Phi^{(3)}_2 = \Phi^{(3)}_4 = \dots = 0\,,$$

$$\Phi^{(3)}_1 = \Phi^{(3)}_3 = \Phi^{(3)}_5 = \dots = 0\,.$$

which are the equations of motion discussed in the previous sections. This also confirms the fact that the fields $\Phi^{(3)}_n$ are not propagating and they are only Lagrange multipliers. For a solution with finite number of fields, the equations are invariant under the gauge transformations with $\Delta = 0$ and $(Q_L + Q_R)\Gamma = 0$. This coincides with the correct gauge invariance of the theory.

The action given summing (9.18) and (9.20) has several properties which justify its form: *i)* it leads to the correct equations of motion in the space of solutions with a finite number of fields, *ii)* it has the correct gauge transformations and it is explicitly supersymmetric, *iii)* the relation between fields and antifields is realized, *iv)* the new operators $\Phi^{(3)}_n$ play only an auxiliary role and they do not propagate, *v)* the action avoids the no-go theorems about selfdual forms, *vi)* the action suggests a Chern-Simons-like action in a higher "dimension" (that can be reduced to the form in (9.18) by discretizing the new dimension) and can be interpreted as a Chern-Simons action for the supermembrane where one has to reabsorb 7 ghosts (see [39]).

10. Outlook

We collected, analysed and studied some of the fundamental ingredients for amplitude computations in covariant superstring formalism. We showed the power of the present framework, where we were able to construct a systematic procedure to compute the vertex operators for closed covariant superstrings. We also found a way to relax superspace contraints and we proposed a tentative closed string field theory action. However, there are several open questions which were not addressed in this paper: *a)* the extension of amplitude computations beyond tree level, *b)* the computation of deformed superspaces associated to non constant RR fields [29], *c)* the analysis of T-duality (it is rather simple to check T-duality for vertex operators, but a detailed analysis should be done at the level of sigma model [3], *d)* the role played by conformal invariance and worldsheet diffeomorphisms, last but not least *e)* a full-fledged field theory for open and closed strings. To answer some of these questions one should probably follow a more geometrical approach based on WZW actions [40,41,42].

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Appendix A. BRST Symmetry, Gauge Invariance and the Sigma Model

The field content is x^m where $m = 0, \ldots, 9$, two Majorana-Weyl spinors $\theta^\alpha, \hat\theta^{\hat\alpha}$ with $\alpha = \hat\alpha = 1, \ldots, 16$ and their conjugate momenta ∂x_m, p_α and $\hat p_{\hat\alpha}$. The Dirac matrices $\gamma^m_{\alpha\beta}$ and $\gamma^m_{\hat\alpha\hat\beta}$ are the 16×16 off-diagonal blocks of $Spin(9,1)$ Dirac matrices. They are real and symmetric and they satisfy the Fierz identities $\gamma^m_{\alpha(\beta}\gamma_{m\gamma\delta)} = 0$. We introduce the commuting Weyl spinors λ^α and $\hat\lambda^{\hat\alpha}$, which satisfy the pure spinor conditions

$$\lambda\gamma^m\lambda = 0\,, \qquad \hat\lambda\gamma^m\hat\lambda = 0\,,$$

and their conjugate momenta $w_\alpha, \hat w_{\hat\alpha}$. The solution of the pure spinor constraints can be only achieved by breaking Lorentz invariance, however we do not need to solve them in the present paper. It is very important to introduce the supersymmetric invariant composite operators

$$d_\alpha = p_\alpha - \frac{1}{2}\partial x^m(\gamma_m\theta)_\alpha - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)\,, \tag{A.1}$$

$$\hat{d}_{\hat{\alpha}} = \hat{p}_{\bar{\alpha}} - \frac{1}{2}\bar{\partial}x^m(\gamma_m\hat{\theta})_{\hat{\alpha}} - \frac{1}{8}(\gamma^m\hat{\theta})_{\hat{\alpha}}(\hat{\theta}\gamma_m\bar{\partial}\hat{\theta})\,,$$

Following Berkovits, we define the BRST operators

$$Q_L = \oint dz\lambda^\alpha d_\alpha\,, \qquad Q_R = \oint d\bar{z}\hat{\lambda}^{\hat{\alpha}}\hat{d}_{\hat{\alpha}}\,. \tag{A.2}$$

which satisfy

$$Q_L^2 = -\oint dz\,\lambda\gamma^m\lambda\Pi_m\,, \qquad [Q_L, Q_R] = 0\,, \quad Q_R^2 = -\oint d\bar{z}\,\hat{\lambda}\gamma^m\hat{\lambda}\hat{\Pi}_m\,, \tag{A.3}$$

where $\Pi^m_z = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta$ and $\hat{\Pi}^m_{\bar{z}} = \bar{\partial}x^m + \frac{1}{2}\hat{\theta}\gamma^m\bar{\partial}\hat{\theta}$.

Due to pure spinor constraints, they are nilpotent up to gauge transformations of $w_\alpha, \hat{w}_{\hat{\alpha}}$, given by

$$\Delta_L w_\alpha = \Lambda_m(\gamma^m\lambda)_\alpha\,, \qquad \Delta_R\hat{w}_\alpha = \hat{\Lambda}_m(\gamma^m\hat{\lambda})_\alpha\,. \tag{A.4}$$

with the local parameters Λ_m and $\hat{\Lambda}_m$ generated by the pure spinor constraints. These gauge transformations remove the degrees of freedom from the covariant w_α and $\hat{w}_{\hat{\alpha}}$ the independent dof of the pure spinors λ^α and $\hat{\lambda}^{\hat{\alpha}}$. Gauge invariant operators are

$$\mathcal{J}_L =: w_\alpha\lambda^\alpha :\,, \qquad \mathcal{J}_R =: \hat{w}_\alpha\hat{\lambda}^\alpha :\,, \tag{A.5}$$

$$N_L = \frac{1}{2} : w\gamma^{mn}\lambda :\,, \qquad N_R = \frac{1}{2} : \hat{w}\gamma^{mn}\hat{\lambda} :\,,$$

Following the usual prescription of the BRST quantization rules, we can define the quantum action as follows

$$S_0 = S_{GS} + Q_L\int d^2z w_\alpha\bar{\partial}\theta^\alpha + Q_R\int d^2z\hat{w}_{\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}}\,. \tag{A.6}$$

where S_{GS} is the Green-Schwarz action in the conformal gauge [6]. Even if this looks like the usual BRST procedure, we have to notice that the BRST-like operators Q_L and Q_R are nilpotent up to gauge transformations (A.4). This compensates the fact that the Green-Schwarz action is not invariant under BRST transformations. In addition, we can always add BRST invariant terms to the action. However, there is no procedure to get (A.6) from an honest gauge fixing of the Green-Schwarz action (a suggestion is given in [43]).

By exploiting the different contributions in (A.6), we obtain

$$S_0 = \int d^2z\Big(\frac{1}{2}\partial x^m\bar{\partial}x_m + p_\alpha\bar{\partial}\theta^\alpha + \hat{p}_{\hat{\alpha}}\partial\hat{\theta}^{\hat{\alpha}} + w_\alpha\bar{\partial}\lambda^\alpha + \hat{w}_{\hat{\alpha}}\partial\hat{\lambda}^{\hat{\alpha}}\Big)\,, \tag{A.7}$$

which is BRST invariant and invariant under the gauge transformation (A.4) if the spinors $\lambda^\alpha, \hat\lambda^{\hat\alpha}$ are pure. The action is also invariant under supersymmetry transformations generated by $Q_\epsilon = \epsilon^\alpha \oint dz q_\alpha + \hat\epsilon^{\hat\alpha} \oint d\bar z \hat q_{\hat\alpha}$ where the explicit expressions for the supersymmetry currents are

$$q_\alpha = p_\alpha + \frac{1}{2}\partial x^m(\gamma_m\theta)_\alpha + \frac{1}{24}(\theta\gamma^m\partial\theta)(\gamma_m\theta)_\alpha \,, \tag{A.8}$$
$$\hat q_{\hat\alpha} = \hat p_{\bar\alpha} + \frac{1}{2}\bar\partial x^m(\gamma_m\hat\theta)_{\hat\alpha} + \frac{1}{24}(\hat\theta\gamma^m\bar\partial\hat\theta)(\gamma_m\hat\theta)_{\hat\alpha} \,.$$

These do not anticommute with the BRST operators Q_L and Q_R, since

$$[Q_L, q_\alpha] = \partial\chi_\alpha \,, \qquad [Q_R, \hat q_{\hat\beta}] = \bar\partial\hat\chi_{\hat\beta} \tag{A.9}$$

where χ_α and $\hat\chi_{\hat\beta}$ are the BRST-invariant quantities

$$\chi_\alpha \equiv \frac{1}{3}(\lambda\gamma^m\theta)(\gamma_m\theta)_\alpha \,, \qquad \hat\chi_{\hat\beta} = \frac{1}{3}(\hat\lambda\gamma^p\hat\theta)(\gamma_p\hat\theta)_{\hat\beta} \tag{A.10}$$

We also introduce the Lorentz currents

$$L^{mn} = \frac{1}{2} :\partial x^{[m}x^{n]}: + \frac{1}{2} :(p\gamma^{mn}\theta): + :N^{mn}: \,, \tag{A.11}$$
$$\hat L^{pq} = \frac{1}{2} :\bar\partial x^{[p}x^{q]}: + \frac{1}{2} :(\hat p\gamma^{pq}\hat\theta): + :\hat N^{pq}: \,,$$

which satisfy the following commutation relations with the BRST charges

$$[Q_L, L^{mn}] = \partial\mathcal{G}^{mn}; \qquad [Q_R, \hat L^{pq}] = \bar\partial\hat{\mathcal{G}}^{pq} \tag{A.12}$$

where

$$\mathcal{G}^{mn} = \frac{1}{4}(\theta\gamma^r\lambda)\left(\delta_r^{[m}x^{n]} + \frac{1}{4}(\theta\gamma_r\gamma^{mn}\theta)\right); \qquad \hat{\mathcal{G}}^{pq} = \frac{1}{4}(\hat\theta\gamma^r\hat\lambda)\left(\delta_r^{[p}x^{q]} + \frac{1}{4}(\hat\theta\gamma_r\gamma^{pq}\hat\theta\right) \tag{A.13}$$

are BRST invariant. By using the equations of motion from (A.7) it is easy to show that the currents q_α, $\hat q_{\hat\beta}$, $\lambda^\alpha d_\alpha$, $\hat\lambda^{\hat\beta}\hat d_\beta$, L^{mn} and $\hat L^{pq}$ are holomophic and anti-holomorphic, respectively.

The energy-momentum tensor is given by

$$T_{zz} = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha - w_\alpha\partial\lambda^\alpha \,, \qquad \hat T_{\bar z\bar z} = -\frac{1}{2}\hat\Pi^m\hat\Pi_m - \hat d_{\hat\alpha}\bar\partial\hat\theta^{\hat\alpha} - \hat w_{\hat\alpha}\bar\partial\hat\lambda^{\hat\alpha} \,. \tag{A.14}$$

where the last term in both expressions is invariant under the gauge transformations (A.4) which allow us to rewrite T and $\hat T$ in terms of independent components of pure spinors.

Our conventions for superspace covariant derivatives and supersymmetry charges are

$$D_\alpha = \partial_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m \,, \qquad Q_\alpha = \partial_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m \,,$$
$$\hat D_{\hat\alpha} = \partial_{\hat\alpha} + \frac{1}{2}(\gamma^m\hat\theta)_{\hat\alpha}\partial_m \,, \qquad \hat Q_{\hat\alpha} = \partial_{\hat\alpha} - \frac{1}{2}(\gamma^m\theta)_{\hat\alpha}\partial_m \,, \tag{A.15}$$

which satisfy

$$\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m \,, \qquad \left\{\hat D_{\hat\alpha}, \hat D_{\hat\beta}\right\} = \gamma^m_{\hat\alpha\hat\beta}\partial_m \,, \qquad \left\{D_\alpha, \hat D_{\hat\beta}\right\} = 0 \tag{A.16}$$
$$\{D_\alpha, Q_\beta\} = 0 \,, \qquad \{\hat D_{\hat\alpha}, \hat Q_{\hat\beta}\} = 0 \,.$$

Appendix B. Solution of the Iterative Equations

We list here the solution up to second order in both θ^α and $\hat\theta^{\hat\alpha}$ for the superfields $A_{\alpha\hat\beta}, A_{\alpha p}, A_{m\hat\beta}, E_\alpha{}^{\hat\beta}$ and $E^\alpha{}_{\hat\beta}$.

$$
\begin{aligned}
A_{\alpha\hat\beta} = & -\frac{1}{4}(\gamma^m\theta)_\alpha(\gamma^p\hat\theta)_{\hat\beta}(g+b+\eta\phi)_{mp} \\
& + \frac{1}{6}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^p\hat\theta)_{\hat\beta}\psi^\gamma{}_p + \frac{1}{6}(\gamma^m\theta)_\alpha(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\psi_m{}^{\hat\gamma} \\
& + \frac{1}{9}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}f^{\beta\hat\gamma} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
A_{\alpha p} = & \frac{1}{2}\theta^\beta\gamma^m_{\beta\alpha}(g+b+\eta\phi)_{mp} - \frac{1}{2}(\gamma^m\theta)_\alpha(\gamma_p\hat\theta)_{\hat\gamma}\psi_m{}^{\hat\gamma} + \frac{1}{3}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta\psi^\beta{}_p \\
& + \frac{1}{3}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta(\gamma_p\hat\theta)_{\hat\gamma}f^{\beta\hat\gamma} - \frac{1}{16}(\gamma^m\theta)_\alpha(\gamma_p\hat\theta)_{\hat\gamma}(\gamma^{qr}\hat\theta)^{\hat\gamma}\omega_{m,qr} \\
& - \frac{1}{24}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}(\gamma^{qr}\hat\theta)^{\hat\beta}c^\gamma{}_{qr} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
A_{m\hat\beta} = & -\frac{1}{2}\hat\theta^{\hat\gamma}\gamma^p_{\hat\gamma\hat\beta}(g+b+\eta\phi)_{mp} + \frac{1}{2}(\gamma_m\theta)_\gamma(\gamma^p\hat\theta)_{\hat\beta}\psi^\gamma{}_p + \frac{1}{3}(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\psi_m{}^{\hat\gamma} \\
& + \frac{1}{16}(\gamma_m\theta)_\gamma(\gamma^{nr}\theta)^\gamma(\gamma^p\hat\theta)_{\hat\beta}\omega_{nr,p} + \frac{1}{3}(\gamma_m\theta)_\beta(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}f^{\beta\hat\gamma} \\
& - \frac{1}{24}(\gamma_m\theta)_\gamma(\gamma^{nr}\theta)^\gamma(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}c_{nr}{}^{\hat\gamma} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
E_\alpha{}^{\hat\beta} = & \frac{1}{2}\theta^\gamma\gamma^m_{\gamma\alpha}\psi_m{}^{\hat\beta} + \frac{1}{8}(\gamma^m\theta)_\alpha(\gamma^{pq}\hat\theta)^{\hat\beta}\omega_{m,pq} + \frac{1}{3}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma f^{\gamma\hat\beta} \\
& - \frac{1}{12}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^{pq}\hat\theta)^{\hat\beta}c^\gamma{}_{pq} + \frac{1}{8}(\gamma^m\theta)_\alpha(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\partial_q\psi_m{}^{\hat\gamma} \\
& + \frac{1}{12}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\partial_q f^{\gamma\hat\gamma} + \ldots
\end{aligned}
$$

$$
\begin{aligned}
E^\alpha{}_{\hat\beta} = & \frac{1}{2}\hat\theta^{\hat\gamma}\gamma^p_{\hat\gamma\hat\beta}\psi^\alpha{}_p + \frac{1}{8}(\gamma^{mn}\theta)^\alpha(\gamma^p\hat\theta)_{\hat\beta}\omega_{mn,p} + \frac{1}{3}(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}f^{\alpha\hat\gamma} \\
& - \frac{1}{8}(\gamma^{mn}\theta)^\alpha(\gamma_m\theta)_\gamma(\gamma^p\hat\theta)_{\hat\beta}\partial_n\psi^\gamma{}_p - \frac{1}{12}(\gamma^{mn}\theta)^\alpha(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}c_{mn}{}^{\hat\gamma} \\
& + \frac{1}{12}(\gamma^{mn}\theta)^\alpha(\gamma_m\theta)_\gamma(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\partial_n f^{\gamma\hat\gamma} + \ldots
\end{aligned}
$$

Appendix C. Solution of the Iterative Equations for Non-Constant RR Field-Strength

Here we give the rest of the superfields for non linear x-dependent RR fields strengths.

$$
\begin{aligned}
A_{\alpha p} = {} & \frac{1}{3}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta(\gamma_p\hat\theta)_{\hat\gamma}(f^{\beta\hat\gamma} + \mathcal{C}_n{}^{\beta\hat\gamma}x^n) \\
& + \frac{1}{36}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}(\gamma^{qr}\hat\theta)^{\hat\beta}(\gamma_q\hat\theta)_{\hat\gamma}\mathcal{C}_r{}^{\gamma\hat\gamma} \\
& + \frac{1}{60}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta(\gamma^{nr}\theta)^\beta(\gamma_n\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}\mathcal{C}_r{}^{\gamma\hat\beta}
\end{aligned}
$$

$$
\begin{aligned}
A_{m\hat\beta} = {} & \frac{1}{3}(\gamma_m\theta)_\beta(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}(f^{\beta\hat\gamma} + \mathcal{C}_n{}^{\beta\hat\gamma}x^n) \\
& + \frac{1}{36}(\gamma_m\theta)_\alpha(\gamma^{nr}\theta)^\alpha(\gamma_n\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}(\gamma^p\hat\theta)_{\hat\gamma}\mathcal{C}_r{}^{\gamma\hat\gamma} \\
& + \frac{1}{60}(\gamma_m\theta)_\gamma(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}(\gamma^{rs}\hat\theta)^{\hat\gamma}(\gamma_r\hat\theta)_{\hat\delta}\mathcal{C}_s{}^{\gamma\hat\delta}
\end{aligned}
$$

$$
\begin{aligned}
E_\alpha{}^{\hat\beta} = {} & \frac{1}{3}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(f^{\gamma\hat\beta} + \mathcal{C}_n{}^{\gamma\hat\beta}x^n) \\
& + \frac{1}{12}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\gamma(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\mathcal{C}_q{}^{\gamma\hat\gamma} + \frac{1}{60}(\gamma^m\theta)_\alpha(\gamma_m\theta)_\beta(\gamma^{nr}\theta)^\beta(\gamma_n\theta)_\gamma\mathcal{C}_r{}^{\gamma\hat\beta}
\end{aligned}
$$

$$
\begin{aligned}
E^\alpha{}_{\hat\beta} = {} & \frac{1}{3}(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}(f^{\alpha\hat\gamma} + \mathcal{C}_m{}^{\alpha\hat\gamma}x^m) \\
& + \frac{1}{12}(\gamma^{mn}\theta)^\alpha(\gamma_m\theta)_\gamma(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\mathcal{C}_n{}^{\gamma\hat\gamma} + \frac{1}{60}(\gamma^p\hat\theta)_{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}(\gamma^{qr}\hat\theta)^{\hat\gamma}(\gamma_q\hat\theta)_{\hat\delta}\mathcal{C}_r{}^{\alpha\hat\delta}
\end{aligned}
$$

$$
\begin{aligned}
A_{mp} = {} & (\gamma_m\theta)_\beta(\gamma_p\hat\theta)_{\hat\gamma}(f^{\beta\hat\gamma} + \mathcal{C}_n{}^{\beta\hat\gamma}x^n) \\
& + \frac{1}{12}(\gamma_m\theta)_\beta(\gamma^{nr}\theta)^\beta(\gamma_n\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}\mathcal{C}_r{}^{\gamma\hat\beta} + \frac{1}{12}(\gamma_m\theta)_\gamma(\gamma_p\hat\theta)_{\hat\beta}(\gamma^{rs}\hat\theta)^{\hat\beta}(\gamma_r\hat\theta)_{\hat\gamma}\mathcal{C}_s{}^{\gamma\hat\gamma}
\end{aligned}
$$

$$
\begin{aligned}
E_m{}^{\hat\beta} = {} & (\gamma_m\theta)_\gamma(f^{\gamma\hat\beta} + \mathcal{C}_n{}^{\gamma\hat\beta}x^n) \\
& + \frac{1}{4}(\gamma_m\theta)_\gamma(\gamma^{pq}\hat\theta)^{\hat\beta}(\gamma_p\hat\theta)_{\hat\gamma}\mathcal{C}_q{}^{\gamma\hat\gamma} + \frac{1}{12}(\gamma_m\theta)_\alpha(\gamma^{nr}\theta)^\alpha(\gamma_n\theta)_\gamma\mathcal{C}_r{}^{\gamma\hat\beta}
\end{aligned}
$$

$$
\begin{aligned}
E^{\alpha}{}_{p} &= (\gamma_p\hat{\theta})_{\hat{\gamma}}(f^{\alpha\hat{\gamma}} + \mathcal{C}_m{}^{\alpha\hat{\gamma}}x^m) \\
&\quad + \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}(\gamma_p\hat{\theta})_{\hat{\beta}}\mathcal{C}_n{}^{\gamma\hat{\beta}} + \frac{1}{12}(\gamma_p\hat{\theta})_{\hat{\beta}}(\gamma^{qr}\hat{\theta})^{\hat{\beta}}(\gamma_q\hat{\theta})_{\hat{\gamma}}\mathcal{C}_r{}^{\alpha\hat{\gamma}}
\end{aligned}
$$

$$
\begin{aligned}
P^{\alpha\hat{\beta}} &= (f^{\alpha\hat{\beta}} + \mathcal{C}_m{}^{\alpha\hat{\beta}}x^m) \\
&\quad + \frac{1}{4}(\gamma^{mn}\theta)^{\alpha}(\gamma_m\theta)_{\gamma}\mathcal{C}_n{}^{\gamma\hat{\beta}} + \frac{1}{4}(\gamma^{pq}\hat{\theta})^{\hat{\beta}}(\gamma_p\hat{\theta})_{\hat{\gamma}}\mathcal{C}_q{}^{\alpha\hat{\gamma}}\,.
\end{aligned}
$$

References

[1] N. Berkovits, JHEP **0004**, 018 (2000); N. Berkovits, Int. J. Mod. Phys. A **16**, 801 (2001) [hep-th/0008145]; N. Berkovits, JHEP 0009, 046 (2000) [hep-th/0006003]; N. Berkovits, *ICTP lectures on covariant quantization of the superstring,* [hep-th/0209059].

[2] P. A. Grassi, G. Policastro, M. Porrati and P. van Nieuwenhuizen, JHEP **10** (2002) 054, [hep-th/0112162]; P. A. Grassi, G. Policastro, and P. van Nieuwenhuizen, JHEP **11** (2002) 004, [hep-th/0202123]; P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, Adv. Theor. Math. Phys. **7**, 499 (2003) [hep-th/0206216].

[3] P.S. Howe, Phys. Lett. B258 (1991) 141, Addendum-ibid.B259 (1991) 511;
P.S. Howe, Phys. Lett. B273 (1991) 90.

[4] J. P. Harnad and S. Shnider, Commun. Math. Phys. **106**, 183 (1986).

[5] H. Ooguri, J. Rahmfeld, H. Robins and J. Tannenhauser, JHEP **0007**, 045 (2000) [hep-th/0007104].

[6] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory,* vol. 1, chapter 5 (Cambridge U. Press, 1987). Our x^m differs by a sign, and our $\theta^\alpha = \sqrt{2}\,\theta^\alpha_{\rm GSW}$.

[7] J. H. Schwarz and P. Van Nieuwenhuizen, Lett. Nuovo Cim. **34**, 21 (1982);
P. van Niewenhuizen, private communication.

[8] M. Bianchi, G. Pradisi and A. Sagnotti, Nucl. Phys. B **376**, 365 (1992).

[9] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, Nucl. Phys. B **507**, 259 (1997) [hep-th/9707068]; M. Billo, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, Nucl. Phys. B **526**, 199 (1998) [hep-th/9802088].

[10] N. Berkovits, Phys. Rev. Lett. **79**, 1813 (1997) [hep-th/9706024]; N. Berkovits, Nucl. Phys. B **507**, 731 (1997) [hep-th/9704109]; N. Berkovits and B. Zwiebach, Nucl. Phys. B **523**, 311 (1998) [hep-th/9711087];

[11] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.

[12] L. Cornalba, M. S. Costa and R. Schiappa, hep-th/0209164.

[13] M. Cederwall, B. E. W. Nilsson and D. Tsimpis, JHEP **0202**, 009 (2002) [hep-th/0110069]; M. Cederwall, B. E. W. Nilsson and D. Tsimpis, JHEP **0211**, 023 (2002) [hep-th/0205165].

[14] N. Berkovits and P. S. Howe, Nucl. Phys. B **635**, 75 (2002) [hep-th/0112160].

[15] O. Chandia and B. C. Vallilo, hep-th/0401226.

[16] N. Marcus and J. H. Schwarz, Phys. Lett. B **115**, 111 (1982); B. McClain, F. Yu and Y. S. Wu, Nucl. Phys. B **343**, 689 (1990); C. Wotzasek, Phys. Rev. Lett. **66**, 129 (1991); J. H. Schwarz and A. Sen, Nucl. Phys. B **411**, 35 (1994) [hep-th/9304154]; P. Pasti, D. P. Sorokin and M. Tonin, Phys. Rev. D **52**, 4277 (1995) [hep-th/9506109]; F. P. Devecchi and M. Henneaux, Phys. Rev. D **54**, 1606 (1996) [hep-th/9603031];

[17] B. Zwiebach, Nucl. Phys. B **390**, 33 (1993) [hep-th/9206084].

[18] N. Berkovits and B. C. Vallilo, JHEP 0007, 015 (2000) [hep-th/0004171].

[19] L. Brink and P. S. Howe, Phys. Lett. B **88**, 81 (1979). P. S. Howe, Nucl. Phys. B **199**, 309 (1982). P. S. Howe and P. C. West, Nucl. Phys. B **238**, 181 (1984); J. L. Carr, J.S.James Gates,Jr., R. N. Oerter Phys.Lett. B **189**, 68 (1987).

[20] M. B. Green, Phys. Lett. B **223**, 157 (1989).

[21] W. Siegel, Nucl. Phys. B263 (1986) 93; W. Siegel, Phys. Rev. D 50 (1994), 2799.

[22] N. Berkovits, JHEP 0004, 018 (2000) [hep-th/0001035].

[23] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, hep-th/0402122.

[24] N. Berkovits and O. Chandia, JHEP **0208**, 040 (2002) [hep-th/0204121].

[25] J. de Boer, P. A. Grassi and P. van Nieuwenhuizen, Phys. Lett.B. [hep-th/0302078]; H. Ooguri and C. Vafa, [hep-th/0302109]; H. Ooguri and C. Vafa, hep-th/0303063.

[26] S. Ferrara, M.A. Lledo, JHEP 0005:008,2000 [hep-th/0002084]; D. Klemm, S. Penati and L. Tamassia, Class.Quant.Grav.20:2905-2916,2003 [hep-th/0104190].

[27] S. Ferrara, M. A. Lledo, and O. Macia, JHEP **0309**, 068 (2003), hep-th/0307039; M. Hatsuda, S. Iso, and H. Umetsu, hep-th/0306251; J.-H. Park, JHEP **0309**, 046 (2003), hep-th/0307060; R. Britto, B. Feng, and S.-J. Rey, JHEP **0307**, 067 (2003), hep-th/0306215; R. Britto, B. Feng, and S.-J. Rey, JHEP **0308**, 001 (2003), hep-th/0307091; M. T. Grisaru, S. Penati, and A. Romagnoni, JHEP **0308**, 003 (2003), hep-th/0307099; R. Britto and B. Feng, hep-th/0307165; A. Romagnoni, JHEP **0310**, 016 (2003), hep-th/0307209; O. Lunin and S.-J. Rey, JHEP **0309**, 045 (2003), hep-th/0307275; D. Berenstein and S.-J. Rey, hep-th/0308049; R. Abbaspur, hep-th/0308050; A. Imaanpur, JHEP **0309**, 077 (2003), hep-th/0308171; A. Imaanpur, hep-th/0311137; P. A. Grassi, R. Ricci, and D. Robles-Llana, hep-th/0311155; R. Britto, B. Feng, and S.-J. Lunin, O. Rey, hep-th/0311275; M. Billo, M. Frau, I. Pesando, and A. Lerda, hep-th/0402160; N. Berkovits and N. Seiberg, JHEP **0307**, 010 (2003), hep-th/0306226; S. Terashima and J.-T. Yee, hep-th/0306237; T. Araki, K. Ito, and A. Ohtsuka, hep-th/0307076; M. Chaichian and A. Kobakhidze, hep-th/0307243; E. Ivanov, O. Lechtenfeld, and B. Zupnik, hep-th/0308012; S. Ferrara and E. Sokatchev, hep-th/0308021; M. Alishahiha, A. Ghodsi, and N. Sadooghi, hep-th/0309037; A. Sako and T. Suzuki, hep-th/0309076; B. Chandrasekhar and A. Kumar, hep-th/0310137; T. Araki, K. Ito, and A. Ohtsuka, hep-th/0401012; C. Saemann and M. Wolf, hep-th/0401147; T. Inami and H. Nakajima, hep-th/0402137; A. Imaanpur and S. Parvizi, hep-th/0403174; D. Mikulović, hep-th/0310065; D. Mikulović, hep-th/0403290

[28] P. Kosinski, J. Lukierski, P. Maslanka and J. Sobczyk, J.Phys.A27:6827-6838,1994 [hep-th/9405076]

[29] P.A. Grassi and L. Tamassia, in preparation.

[30] N. Berkovits, private communication.

[31] M. Chesterman, JHEP **0402**, 011 (2004) [hep-th/0212261]; M. Chesterman, hep-th/0404021.

[32] A. Belopolsky, Phys. Lett. B **403**, 47 (1997) [hep-th/9609220]; A. Belopolsky, hep-th/9703183; A. Belopolsky, hep-th/9706033.

[33] W. Siegel, and B. Zwiebach, Nucl. Phys. **B**299:206,1988; W. Siegel, Phys. Lett. B **142**, 276 (1984), Phys. Lett B **149**, 157 (1984), Phys. Lett. B **151**, 391 (1985), Phys. Lett. B **149**, 162 (1984), Phys. Lett. B **151**, 396 (1985).

[34] N. Berkovits, M. T. Hatsuda and W. Siegel, Nucl. Phys. B **371**, 434 (1992) [hep-th/9108021].

[35] N. Berkovits and M. M. Leite, Phys. Lett. B **454**, 38 (1999) [hep-th/9812153].

[36] W. Siegel, *Introduction To String Field Theory,* Adv. Ser. Math. Phys. **8**, 1 (1988). hep-th/0107094.

[37] N. Berkovits and C. M. Hull, JHEP **9802**, 012 (1998) [hep-th/9712007].

[38] N. Berkovits, Phys. Lett. B **388**, 743 (1996) [hep-th/9607070].

[39] N. Berkovits, JHEP **0209**, 051 (2002) [hep-th/0201151].

[40] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, Nucl. Phys. B **676**, 43 (2004) [hep-th/0307056].

[41] P. A. Grassi and P. van Nieuwenhuizen, hep-th/0403209.

[42] S. Guttenberg, J. Knapp and M. Kreuzer, hep-th/0405007.

[43] I. Oda and M. Tonin, Phys. Lett. B520 (2001) 398 [hep-th/0109051];
M. Matone, L. Mazzucato, I. Oda, D. Sorokin and M. Tonin, Nucl. Phys. B **639**, 182 (2002) [hep-th/0206104].