Conformal invariance and rationality in an even dimensional quantum field theory

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Abstract

Invariance under finite conformal transformations in Minkowski space and the Wightman axioms imply strong locality (Huygens principle) and rationality of correlation functions, thus providing an extension of the concept of vertex algebra to higher dimensions. Gibbs (finite temperature) expectation values appear as elliptic functions in the conformal time. We survey and further pursue our program of constructing a globally conformal invariant model of a hermitean scalar field $\mathcal{L}$ of scale dimension four in Minkowski space-time which can be interpreted as the Lagrangian density of a gauge field theory.

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1. Introduction

The present paper provides an updated survey of our work [18] [15] [16] [14] aimed at constructing a non-trivial *globally conformal invariant* (GCI) 4-dimensional *quantum field theory* (QFT) model. Our attempt to build such a model is based on the following results of [18].

Invariance under finite conformal transformations in Minkowski space-time M and local commutativity imply the *Huygens principle*: the commutator of two local Bose fields vanishes for non-isotropic separations. The Huygens principle and energy positivity yield rationality of correlation functions (Theorem 3.1 of [18]). These results allow to extend any GCI QFT to compactified Minkowski space $\overline{M}$, which admits the following convenient complex variable realization ([22] [15] [16] [14] [19]):

$$\overline{M} = \Big\{ z \in \mathbb{C}^4 : \, z_\mu = e^{2\pi i \zeta} u_\mu \text{ for } \mu = 1, \dots, D \,, \; \zeta \in \mathbb{R} \,, \\ u \in \mathbb{S}^3 \, (\equiv \big\{ u \in \mathbb{R}^4 : u^2 = \boldsymbol{u}^2 + u_D^2 = 1 \big\}) \Big\} \cong \frac{\mathbb{S}^1 \times \mathbb{S}^{D-1}}{\mathbb{Z}_2} \,. \quad (1.1)$$

The variable ζ plays the role of *conformal time*. Fields $\phi(z)$ are expressed as formal power series of the form

$$\phi\,(z) \; = \sum_{n \,\in\, \mathbb{Z}} \; \sum_{m \,\geq\, 0} \big(z^{\,2} \big)^n \; \phi_{\{n,m\}}\,(z) \;, \quad z^{\,2} := \sum_{\mu\,=\,1}^{D} z_\mu^2 \,, \qquad (1.2)$$

$\phi_{\{n,m\}}\,(z)$ being (in general, multicomponent) operator valued polynomial in z which is *homogeneous and harmonic*,

$$\phi_{\{n,m\}}\,(\lambda z) \; = \; \lambda^m \, \phi_{\{n,m\}}\,(z) \;, \quad \Delta \, \phi_{\{n,m\}}\,(z) \; = \; 0 \,, \quad \Delta \; = \sum_{\mu=1}^{D} \frac{\partial^2}{\partial z_\mu^2} \,. \qquad (1.3)$$

This is unambiguous [14], because, as it is well known, every homogeneous polynomial $p(z)$ of degree m has a *unique decomposition* $p(z) = h(z) + z^{\,2} q(z)$ where h is harmonic (of degree m) and q is homogeneous of degree $m - 2$.

The resulting (*analytic*) *z-picture* provides a higher dimensional generalization [14] of *chiral vertex algebras* (which have been an outgrow of physicists' work on conformal field theory and dual resonance models, formalized by R.E. Borcherds [1] and since subject of numerous studies, including several books – see, *e.g.* [12] [8] and references therein).

The coordinates (1.1) are obtained under the complex conformal transformation (with singularities)

$$g_c : M_{\mathbb{C}} (\ni \xi) \to E_{\mathbb{C}} (\ni z) \,, \quad \boldsymbol{z} = \frac{\boldsymbol{\xi}}{\omega_\xi} \,, \quad z_D = \frac{1 - \xi^2}{\omega_\xi} \,, \\ \xi^2 = \boldsymbol{\xi}^2 - (\xi^0)^2 \,, \quad \omega_\xi = \frac{1}{2}(1 + \xi^2) - i \xi^0 \,, \quad z^{\,2} = \frac{1 + \xi^2 + 2 i \xi^0}{1 + \xi^2 - 2 i \xi^0} \,. \quad (1.4)$$

which is regular on the *Minkowski forward tube domain*

$$\mathcal{T}_+ = \left\{ \xi = x + iy : x, y \in M, y^0 > |\boldsymbol{y}| := \sqrt{y_1^2 + \ldots + y_{D-1}^2} \right\}$$

and maps it onto an open subset

$$T_+ = \left\{ z \in \mathbb{C}^D : \left| z^2 \right| < 1 \,,\; |z|^2 \left(:= \sum_{\mu=1}^{D} |z^\mu|^2 \right) < \frac{1 + \left| z^2 \right|^2}{2} \right\} \tag{1.5}$$

of $\mathbb{C}^D$ with compact closure. (Note that g_c maps the real Minkowski space M onto an open dense subset of $\overline{M}$ (1.1) such that $\boldsymbol{z}^2 + (z_4+1)^2 = 2e^{2\pi i\zeta}(\cos 2\pi\zeta + u_4) \neq 0$.) Recall that Minkowski space *spectral conditions* (including *energy positivity*) imply analyticity of the (Minkowski) vector valued function $\phi(\xi)|0\rangle$ for $\zeta \in \mathcal{T}_+$, where $|0\rangle$ is the conformally invariant vacuum vector, and leads to the *z-picture analyticity* of $\phi(z)|0\rangle$ for $z \in T_+$. It follows, in particular, that no negative powers of z^2 appear in $\phi(z)|0\rangle$:

$$\phi_{\{n,m\}}(z)|0\rangle = 0 \quad \text{for} \quad n < 0 \tag{1.6}$$

In the GCI QFT the natural choice of the conformal group $\mathcal{C}$ is the *D-dimensional (connected) spinorial conformal group Spin* $(D, 2)$ $(\equiv \mathcal{C})$. The coordinates (1.1) correspond to a special *complex* basis in the (complex) conformal Lie algebra $\mathfrak{c}_{\mathbb{C}}$ including:

1) T_μ for $\mu = 1, \ldots, D$ – the generators of the translations $z \mapsto z + \lambda\, e_\mu$;

2) C_μ for $\mu = 1, \ldots, D$ – the generators of the special conformal transformations of the z–coordinates;

3) $\Omega_{\mu\nu} \equiv -\Omega_{\nu\mu}$ for $1 \leq \mu < \nu \leq D$ – the generators of the orthogonal transformations of the z–coordinates.

4) H, the *conformal Hamiltonian* – the generator of the z–dilations $z \mapsto \lambda^H(z) := \lambda\, z$. Note that $e^{2\pi i t H}$ acts on the conformal time variable ζ in (1.1) by a translation $\zeta \mapsto \zeta + t$.

The conjugation corresponding to the real conformal Lie algebra is an antilinear *automorphism* acting on these generators as:

$$T_\mu^\star := C_\mu \,, \quad H^\star = -H \,, \quad \Omega_{\mu\nu}^\star = \Omega_{\mu\nu}. \tag{1.7}$$

(By contrast, the hermitean conjugation is an algebra antiautomorphism acting with an opposite sign on the generators – see (2.16) below.) The generators T_μ, $\Omega_{\mu\nu}$ and H span the subalgebra $\mathfrak{c}_\infty$ of $\mathfrak{c}$, the *stabilizer of the central point at infinity* in the z–chart which is isomorphic to the complex Lie algebra of Euclidean transformations with dilations. Its conjugate $\mathfrak{c}_0$,

$$\mathfrak{c}_0 := Span_{\mathbb{C}} \left\{ C_\mu, \Omega_{\mu\nu}, H \right\}_{\mu,\nu} \,, \tag{1.8}$$

is the stabilizer of $z = 0$ in $\mathcal{c}_{\mathbb{C}}$. The *real counterpart* of the intersection of $\mathcal{c}_\infty$ and $\mathcal{c}_0$ is the Lie algebra of the maximal compact subgroup $\mathcal{K} = Spin(D) \times U(1)/_{\mathbb{Z}_2}$ of the real conformal group $\mathcal{C}$ generated by $\Omega_{\mu\nu}$ and iH. The z-picture domain T_+ is a homogeneous space of the (real) group $\mathcal{C}$ with stabilizer of $z = 0$ equal to its maximal compact subgroup $\mathcal{K}$, so that $T_+ = \mathcal{C}/\mathcal{K}$. The compactified Minkowski space $\overline{M}$ (1.1), appearing as a D–dimensional submanifold of the boundary of T_+, is also a homogeneous space of $\mathcal{C}$. The conformal Hamiltonian H is nothing but the generator of the centre $U(1)$ of $\mathcal{K}$.

Free massless fields and the stress energy tensor only satisfy the Huygens principle for even space-time dimension. We shall therefore assume in what follows that D *is even*.

The D-dimensional vertex algebra of GCI fields with rational correlation functions corresponds to the *algebra of local observables* in Haag's approach [11] to QFT. Its *isotypical* (or *factorial*) *representations* (*i.e.* multiples with a finite multiplicity of an irreducible representation) give rise to the *superselection sectors* of the theory. The intertwiners between the vacuum and other superselection sectors are higher dimensional counterparts of primary fields (which have, as a rule, fractional dimensions and multivalued n-point distributions).

We shall add to the traditional assumption that the conserved (symmetric, traceless) *stress-energy tensor* $T_{\mu\nu}(z)$ is a local observable the requirement that so is the scalar, gauge invariant *Lagrangian density* $\mathcal{L}(z)$ (of dimension $d = D$). Then the construction of a GCI QFT model becomes a rather concrete program of writing down rational (conformally invariant and "crossing symmetric") correlation functions and studying the associated *operator product expansions* (OPE). We are pursuing this program for the $D = 4$ case (of main physical interest) in our work [15] [16] reviewed and continued in [17] (see also secs. 4–6 below).

We begin, in Sect. 2, with a review of properties of z-picture fields (or vertex operators) which follow from GCI and Wightman axioms. Free compact picture fields are shown to have doubly periodic (meromorphic) – *i.e.* elliptic – correlation functions in the conformal time differences ζ_{ij} with periods 1 (in our units, the radius of the universe divided by the velocity of light) and the *modular parameter* τ whose imaginary part is proportional to the inverse absolute temperature (Sect. 3.1). These results are expected to hold beyond the free field examples [19]. In (Sect. 3.2) we compute energy mean values of free fields in an equilibrium state characterized by the Kubo-Martin-Schwinger (KMS) property. The result is expressed as a linear combination of modular forms. The concise survey (in Sects. 2 and 3) of the results of [14] [19] is followed in Sect. 4 with the first step in constructing a (non free) GCI model satisfying Wightman axioms [20]. After a brief review (in Sect. 4.1) of the results of [18], reformulated in the analytic z-picture, we write down (in Sect. 4.2) the general truncated GCI 4-point function w_4^t of a neutral scalar field of (integer) dimension d. This is a homogeneous rational function of degree $-2d$ in the (complex) Euclidean invariant variables

$$\rho_{ij} = z_{ij}^2 \,, \; z_{ij} = z_i - z_j \,, \; z^2 = \boldsymbol{z}^2 + z_4^2 \tag{1.9}$$

of denominator $(\rho_{12}\rho_{13}\rho_{14}\rho_{23}\rho_{24}\rho_{34})^{d-1}$ and numerator, a homogeneous polynomial of degree $4d-6$ (for $d \geq 2$) depending (linearly) on $\left[\!\left[\frac{d^2}{3}\right]\!\right]$ (*i.e.* no more than $d^2/3$) real parameters. It is just $c(\rho_{13}\rho_{24} + \rho_{12}\rho_{34} + \rho_{14}\rho_{23})$ for the simplest candidate for a non-trivial, $d = 2$, model, and involves 5 parameters for the physically interesting case of a $d = 4$ Lagrangian density $\mathcal{L}$. As the model of a $d = 2$ scalar field was proven in [15] to correspond to normal products of free (massless) scalar fields we concentrate in the rest of the paper on the $d = 4$ case. We study in Sect. 5 OPE organized in bilocal fields of fixed twist which provide what could be called a *conformal partial wave expansion* of the 4-point function (a concept, introduced in [2], see also [3], and recently revisited in [6]). The bilocal field $V_1(z_1, z_2)$ of (lowest) dimension $(1, 1)$, which admits a Taylor expansion in z_{12} involving only twist 2 symmetric traceless tensors, is harmonic in each argument allowing to compute (in Sect. 5.2) its (rational) 4-point function. The corresponding (crossing) symmetrized contribution to w_4^t gives rise to a 3-parameter sub-family of the original 5-parameter family of GCI 4-point functions. We give a precise definition of of the symmetrization ansatz in Sect. 6.1.

We argue in Sect. 6.3 (summarizing results of [16]), that the Lagrangian $\mathcal{L}(z)$ of a gauge field theory should have vanishing odd point functions and should not involve a $d = 2$ scalar field in its OPE. This reduces to 3 the 5 parameters in w_4^t. One of the remaining parameters corresponds to the Lagrangian (*i.e.*, the contracted normal square) of a free Maxwell field $F_{\mu\nu}$ giving rise to a bilocal combination of F's, $V_1^{(2)}$. The other bilocal field, $V_1^{(1)}$, contributing to the (restricted) w_4^t, has the properties of a bilinear combination of a free Weyl spinor and its conjugate (Sect. 5.3). This permits the computation of higher point functions as displayed in Sect. 6.2.

The results and the challenging open problems are discussed in Sect. 7.

2. Vertex algebras, strong locality, rationality (a synopsis)

Wightman axioms [20] and GCI can be expressed as *vertex algebra properties of z-picture fields*, [14] [19], which we proceed to sum up.

2.1. Properties of z-picture fields

1) The *state space* $\mathcal{V}$ of the theory is a (*pre-Hilbert*) *inner product space* carrying a (reducible) unitary *vacuum representation* $U(g)$ of the conformal group $\mathcal{C}$, for which:

1a) the corresponding representation of the complex Lie algebra $\mathcal{C}_{\mathbb{C}}$ is such that the spectrum of the $U(1)$ generator H belongs to $\left\{0,\, \frac{1}{2},\, 1,\, \frac{3}{2},\, \dots\right\}$ and has a finite degeneracy:

$$\mathcal{V} = \bigoplus_{\rho = 0, \frac{1}{2}, 1, \dots} \mathcal{V}_\rho\,,\ (H-\rho)\,\mathcal{V}_\rho = 0\,, \quad dim\,\mathcal{V}_\rho < \infty\,, \tag{2.1}$$

each $\mathcal{V}_\rho$ carrying a fully reducible representation of *Spin* (D) (generated by $\Omega_{\mu\nu}$). Moreover, the central element $-\mathbb{I}$ of the subgroup *Spin* (D) is represented on $\mathcal{V}_\rho$ as $(-1)^{2\rho}$.

1b) The *lowest energy space* $\mathcal{V}_0$ is 1-dimensional: it is spanned by the (normalized) *vacuum vector* $|0\rangle$, which is invariant under the full conformal group $\mathcal{C}$.

As a consequence (see [14], Sect. 7) the Lie subalgebra $\mathfrak{c}_0$ (1.8) of $\mathfrak{c}_{\mathbb{C}}$ has locally finite action on $\mathcal{V}$, i.e., every $v \in \mathcal{V}$ belongs to a finite dimensional subrepresentation of $\mathfrak{c}_0$. Moreover, the action of $\mathfrak{c}_0$ is integrable to an action of the complex Euclidean group with dilations $\pi_0\,(g)$ and the function

$$\pi_z\,(g) \;:=\; \pi_0\left(t^{-1}_{g(z)}\,g\,t_z\right) \tag{2.2}$$

is rational in z with values in $End_{\mathbb{C}}\,\mathcal{V}$ (the space endomorphisms of $\mathcal{V}$) and satisfies the cocycle property

$$\pi_z\,(g_1 g_2) \;=\; \pi_{g_2(z)}\,(g_1)\,\pi_z\,(g_2) \quad \text{iff} \quad g_1 g_2\,(z)\,,\; g_2\,(z) \;\in\; \mathbb{C}^D\,. \tag{2.3}$$

2) The fields $\phi\,(z) \equiv \{\phi_a\,(z)\}$ ($\psi\,(z) \equiv \{\psi_b\,(z)\}$, etc.) are represented by infinite power series of type (1.2) and

$$\phi_{n,m}\,(z)\;v \;=\; 0 \tag{2.4}$$

for all $m = 0, 1, \ldots$ if $n > N_v \in \mathbb{Z}$.

3) *Strong locality*: The fields ϕ, ψ, ... are assumed to have $\mathbb{Z}_2$–parities p_ϕ, p_ψ, ... (respectively) such that

$$\rho_{12}^N\,\left\{\phi_a\,(z_1)\,\psi_b\,(z_2) - (-1)^{p_\phi p_\psi}\,\psi_b\,(z_2)\,\phi_a\,(z_1)\right\} \;=\; 0 \quad (\rho_{12} \;:=\; z_{12}^2) \tag{2.5}$$

for sufficiently large N.

The assumption that the field algebra is $\mathbb{Z}_2$ graded, which underlies 3), excludes the so called "Klein transformations" (whose role is discussed e. g. in [20]).

Strong locality implies an analogue of the *Reeh–Schlieder theorem*, the separating property of the vacuum, namely

Proposition 2.1.

(*a*) ([19], Proposition 3.2 (a).) *The series $\phi_a\,(z)\,|0\rangle$ does not contain negative powers of z^2.*

(*b*) ([14], Theorem 3.1.) *Every local field component $\phi_a\,(z)$ is uniquely determined by the vector $v_a = \phi_a\,(0)\,|0\rangle$.*

(*c*) ([14], Theorem 4.1 and Proposition 3.2.) *For every vector $v \in \mathcal{V}$ there exists unique local filed $Y\,(v,\,z)$ such that $Y\,(v,0)\,|0\rangle = v$. Moreover, we have*

$$Y(v,z)\,|0\rangle \;=\; e^{z\cdot T}\,v\,, \quad z\cdot T \;=\; z^1\,T_1 + \cdots + z^D\,T_D\,. \tag{2.6}$$

The part (c) of the above proposition is the higher dimensional analogue of the *state field correspondence.*

6) *Covariance.*

$$[T_\mu\,,\,Y\,(v,\,z)\,]\;=\;\frac{\partial}{\partial z^\mu}\;Y\,(v,\,z)\;,\tag{2.7}$$

$$[H\,,\,Y\,(v,\,z)\,]\;=\;z\cdot\frac{\partial}{\partial z}\;Y\,(v,\,z)\;+\;Y\,(H\,v,\,z)\;,\tag{2.8}$$

$$[\Omega_{\mu\nu}\,,\,Y\,(v,\,z)\,]\;=\;z^\mu\,\frac{\partial}{\partial z^\nu}\,Y\,(v,\,z)\;-\;z^\nu\,\frac{\partial}{\partial z^\mu}\,Y\,(v,\,z)\;+\;Y\,(\Omega_{\mu\nu}\,v,\,z)\;,\tag{2.9}$$

$$\begin{aligned}[C_\mu\,,\,Y\,(v,\,z)\,]\;=\;&\Big(-\,z^2\,\frac{\partial}{\partial z^\mu}\;+2\,z^\mu\,z\cdot\frac{\partial}{\partial z}\Big)\,Y\,(v,\,z)\;+\;2\,z^\mu\,Y\,(H\,v,\,z)\;+\\&+2\sum_{\nu\,=\,1}^{D}\,z^\nu\,Y\,(\Omega_{\nu\mu}\,v,\,z)\;+\;Y\,(C_\mu\,v,\,z)\;.\end{aligned}\tag{2.10}$$

Vectors $v\in\mathcal{V}$ for which $C_\mu v=0$ $(\mu=1,\dots,D)$ are called **quasiprimary**. Their linear span decomposes into irreducible representations of the maximal compact subgroup $\mathcal{K}$, each of them characterized by weights $\big(d;\,j_1,\dots,j_{\frac{D}{2}}\big)$. We assume that our basic fields ϕ_a, ψ_b, $\dots$, correspond to such quasiprimary vectors so that the transformation laws (2.7)–(2.10) give rise to $\mathcal{K}$–induced representations of the conformal group $\mathcal{C}$.

If $v\in\mathcal{V}$ is an eigenvector of H with eigenvalue d_v then Eq. (2.8) implies that the field $Y\,(v,z)$ has dimension d_v:

$$[H\,,\,Y\,(v,\,z)\,]\;=\;z\cdot\frac{\partial}{\partial z}\;Y\,(v,\,z)\;+\;d_v\,Y\,(v,\,z)\;.\tag{2.11}$$

It follows also from the correlation between the dimension and the spin in the property 1a) and, on the other hand, the spin and statistic theorem that the Z_2–parity p_v of v is related with its dimension as $p_v\equiv 2d_v\ \mathit{mod}\,2$ and hence

$$\rho_{12}^{\mu(v_1,v_2)}\,\Big\{Y(v_1,z_1)\,Y(v_2,z_2)-(-1)^{4\,d_{v_1}d_{v_2}}\,Y(v_2,z_2)\,Y(v_1,z_1)\Big\}=0\tag{2.12}$$

where $\mu\,(v_1,v_2)$ depends on the spin and dimensions of v_1 and v_2.

7) *Conjugation.*

$$\big\langle\,v_1\,\big|\,Y\,\big(v^+,\,z\big)\,v_2\,\big\rangle\;=\;\big\langle\,Y\,\big(\pi_{z^*}(J_W)^{-1}\,v,\,z^*\big)\,v_1\,\big|\,v_2\,\big\rangle\tag{2.13}$$

for every $v,v_1,v_2\in\mathcal{V}$, where

$$z^*\;:=\;\frac{\overline{z}}{\overline{z}^{\,2}}\tag{2.14}$$

is the z–picture conjugation (leaving invariant the real space (1.1)) and J_W is the element of $\mathcal{C}_{\mathbb{C}}$ representing the so called *Weyl reflection*

$$J_W\,(z)\;:=\;\frac{R_D\,(z)}{z^{\,2}}\;,\quad R_\mu\,\big(z^1,\,\dots,\,z^D\big)\;:=\;\big(z^1,\,\dots,\,-z^\mu,\,\dots,\,z^D\big)\tag{2.15}$$

(J_W^2 being thus central element of $\mathcal{C}$). Note that the hermitean conjugation X^* of the conformal group generators $X \in \mathcal{c}_{\mathbb{C}}$ is connected by the conjugation (1.7) as

$$X^* \; = \; -X^{\star} \, . \tag{2.16}$$

8) *Borcherds' OPE relation.* The equality

$$Y(v_1, z_1)\, Y(v_2, z_2) \text{ ``$=$'' } Y(Y(v_1, z_{12})\, v_2, z_2) \, , \tag{2.17}$$

is satisfied after applying some transformations to the formal power series on both sides which are not defined on the corresponding series' spaces (see [14], Theorem 4.3). On the other hand, the vector valued function

$$Y(v, z_1)\, Y(v_2, z_2)\, |0\rangle = Y(Y(v_1, z_{12})\, v_2, z_2)\, |0\rangle$$

is analytic with respect to the Hilbert norm topology for $|z_2^2| < |z_1^2| < 1$ and sufficiently small ρ_{12}.

2.2. Free field examples

For a *scalar field* ϕ of dimension d we can write

$$\phi(z) = Y(|d; 0, \ldots, 0\rangle, z) \, , \quad \text{where} \quad |d; 0, \ldots, 0\rangle = \phi_{\{0,0\}}\, |0\rangle \, ; \tag{2.18}$$

here $(j_1, \ldots, j_{\frac{D}{2}})$ stands for the weight of an irreducible $Spin(D)$ representation, $(0, \ldots, 0)$ labeling the trivial (1-dimensional) one. Its conformal invariant two point Wightman function is

$$\langle 0|\, \phi\, (z_1)\, \phi^*\, (z_2)\, |0\rangle \; = \; B_\phi\, \rho_{12}^{-d} \, . \tag{2.19}$$

When d takes its canonical value d_0 for which $\phi = \varphi(z)$ is harmonic

$$d \; = \; d_0 \; := \; \frac{D-2}{2} \qquad \Leftrightarrow \qquad \Delta\, \varphi(z) = 0 \, , \tag{2.20}$$

the expansion (1.2) takes the form

$$\varphi(z) = \sum_{\ell=0}^{\infty} \Big\{ \varphi_{-\ell-d_0}(z) + \big(z^{\,2}\big)^{-\ell-d_0} \;\; \varphi_{\ell+d_0}(z) \Big\} \tag{2.21}$$

where $\varphi_n(z)$ is a homogeneous harmonic polynomial of degree $|n| - d_0$ (≥ 0) and, for a hermitean φ,

$$(\varphi_n(\overline{z}))^* = \varphi_{-n}(z) \, . \tag{2.22}$$

The modes φ_n are related to $\phi_{\{n,m\}}$ of (1.2) by

$$\varphi_{-\ell-d_0}(z) = \phi_{\{0,\ell\}}(z), \quad \varphi_{\ell+d_0}(z) = \phi_{\{-\ell-d_0,\ell\}}(z) \quad (\, [H, \varphi_n(z)] = -n\, \varphi_n(z)\,). \tag{2.23}$$

We proceed to the description of a *free Weyl field* for $D = 4$. Compact picture spinor fields are conveniently studied using the 2×2 matrix realization of the quaternionic algebra. We express the imaginary quaternion units Q_j in terms of the Pauli matrices setting $Q_j = -i\,\sigma_j$ and denote by Q_4 the 2×2 unit matrix. To each (complex) 4-vector z we make correspond a pair of conjugate quaternions

$$\not{z} := \sum_{\mu=1}^{4} z_\mu\, Q_\mu \;=\; z_4 + \boldsymbol{z}\,\boldsymbol{Q}, \quad \not{z}^+ := \sum_{\mu=1}^{4} z_\mu\, Q_\mu^+ \;=\; z_4 - \boldsymbol{z}\,\boldsymbol{Q} \tag{2.24}$$

where

$$\boldsymbol{z} \cdot \boldsymbol{Q} \;=\; \sum_{j=1}^{3} z_j\, Q_j \;=\; -i \begin{pmatrix} z_3 & z_1 - i z_2 \\ z_1 + i z_2 & -z_3 \end{pmatrix} .$$

The basic anticommutation relations for quaternion 4-vectors read:

$$\not{z}\,\not{w}^+ \,+\, \not{w}\,\not{z}^+ \;=\; 2zw\,(=\,2(\boldsymbol{z}\cdot\boldsymbol{w} \,+\, z_4\, w_4)) \;=\; \not{z}^+\,\not{w} + \not{w}^+\,\not{z}\,. \tag{2.25}$$

A *free Weyl field* $\psi(z)$ is a complex 2-component spinor field of $\mathcal{K} = S(U(2)\times U(2))$ weight $\left(\frac{3}{2};\frac{1}{2},0\right)$, satisfying the Weyl equation:

$$\not{\partial}\,\psi(z) \;:=\; Q_\mu\, \frac{\partial}{\partial z_\mu}\,\psi(z) \;=\; 0 \;(=\, \frac{\partial}{\partial z_\mu}\,\psi^+(z)\,Q_\mu\,)\,. \tag{2.26}$$

The unique conformal invariant two point Wightman function (up to normalization) is

$$\langle 0|\;\psi\,(z_1)\,\psi^+\,(z_2)\,|0\rangle \;=\; \rho_{12}^{-2}\;\not{z}_{12}^{+}\,. \tag{2.27}$$

The mode expansion reads:

$$\begin{aligned} \psi(z) \;&=\; \sum_{\ell=0}^{\infty} \left\{ \psi_{-\ell-\frac{3}{2}}(z) + \left(z^{\,2}\right)^{-\ell-2}\, \not{z}^+\,\psi_{\ell+\frac{3}{2}}(z) \right\} , \\ \psi^+(z) \;&=\; \sum_{\ell=0}^{\infty} \left\{ \psi^+_{-\ell-\frac{3}{2}}(z) + \left(z^{\,2}\right)^{-\ell-2}\, \psi^+_{\ell+\frac{3}{2}}(z)\,\not{z}^+ \right\} , \end{aligned} \tag{2.28}$$

where $\psi_\rho^{(+)}(z)$ are 2-component spinor-valued homogeneous (harmonic) polynomials of degree $|\rho| - \frac{3}{2}$ (≥ 0) satisfying appropriate Weyl equations,

$$\not{\partial}\,\psi_\rho(z) \;=\; 0 \;=\; \frac{\partial}{\partial z_\mu}\,\psi^+_{-\rho}(z)\,Q_\mu, \quad \not{\partial}^+\,\psi^+_\rho(z) \;=\; 0 \;=\; \frac{\partial}{\partial z_\mu}\,\psi_{-\rho}(z)\,Q^+_\mu, \quad \rho > 0, \tag{2.29}$$

and the conjugation law

$$(\psi^+_{-\rho}(\overline{z}))^* \;=\; \psi_\rho(z)\,. \tag{2.30}$$

The vacuum is annihilated by $\psi_\rho^{(+)}$ for positive ρ and the modes $\phi_{\{n,m\}}(z)$ of (1.2) are related to ψ_ρ by:

$$\psi_{-\ell-\frac{3}{2}}(z) \;=\; \phi_{\{0,\ell\}}(z), \quad \not{z}^+\,\psi_{\ell+\frac{3}{2}}(z) \;=\; \phi_{\{-\ell-2,\ell+1\}}(z)\,. \tag{2.31}$$

We can write

$$\psi^e(z) = Y\left(\left| e; \frac{3}{2}; \frac{1}{2}, 0\right\rangle, z\right) \text{ where } \left| e; \frac{3}{2}; \frac{1}{2}, 0\right\rangle = \psi^e_{-\frac{3}{2}} |0\rangle\,, \;\; e = \pm\,(\psi^- \equiv \psi), \tag{2.32}$$

the minimal energy state vectors $\left| \pm; \frac{3}{2}; \frac{1}{2}, 0\right\rangle$ being conjugate 2-component spinors.

3. Temperature equilibrium states and mean values

3.1. Elliptic Gibbs correlation functions

Finite temperature equilibrium (Gibbs) state correlation functions of 2-dimensional chiral vertex operators are (doubly periodic) elliptic functions in the conformal time variables ζ, while the characters of the corresponding vertex algebra representations exhibit modular invariance properties [25]. We summarize in what follows the results of [19] where similar properties are established for higher even D.

Energy mean values in some higher dimensional models exhibit modular properties (similar to those in a rational 2D CFT) for a "renormalized" (shifted) Hamiltonian

$$\widetilde{H} \,:=\, H \,+\, E_0 \tag{3.1}$$

with appropriate nonzero vacuum energy E_0. Noting that the conformal Hamiltonian $\widetilde{H}$ (as well as H) has a bounded below discrete spectrum of finite degeneracy, it is natural to assume the existence of the *partition function*

$$Z(\tau) \,=\, \mathrm{tr}_{\mathcal{V}}\, q^{\widetilde{H}}\,, \quad q \,=\, e^{2\pi i \tau}\,, \quad \mathrm{Im}\,\tau \,>\, 0 \;(|q| < 1)\,, \tag{3.2}$$

as well as of all traces of the type $\mathrm{tr}_{\mathcal{V}}(A q^{\widetilde{H}})$ where A is any polynomial in the local GCI fields. This assumption is satisfied in the theory of free massless fields as we shall see shortly.

In order to display the periodicity properties of correlation functions it is convenient to use the *compact picture* fields $\phi(\zeta, u)$ (of dimension d_ϕ) related to the above z-picture fields by

$$\phi(\zeta, u) \,=\, e^{2\pi i d_\phi \zeta}\, \phi(e^{2\pi i \zeta}\, u) \quad \Rightarrow \quad q^{\widetilde{H}}\, \phi(\zeta, u)\, |0\rangle \,=\, q^{E_0}\, \phi(\zeta + \tau, u)\, |0\rangle\;; \tag{3.3}$$

in particular,

$$\phi(\zeta + 1, u) \,=\, (-1)^{2d_\phi}\; \phi(\zeta + 1, u)\,. \tag{3.4}$$

We shall rewrite the expansion (1.2) in the compact picture as:

$$\phi(\zeta, u) \,=\, \sum_{n\,\in\,\mathbb{Z}} \phi_{-nm}(u)\, e^{2\pi i n \zeta}\,, \quad [H, \phi_{nm}(u)] \,=\, -n\, \phi_{nm}(u) \tag{3.5}$$

$\phi_{nm}(u)$ being homogeneous harmonic polynomial of degree m.

We shall sketch the main properties of *Gibbs (temperature) correlation functions* (studied in [19]) using the simplest non-trivial example of the 2-point function

$$w_q(\zeta_{12}; u_1, u_2) = \left\langle\, \phi(\zeta_1, u_1)\, \phi^*(\zeta_2, u_2)\,\right\rangle_q := \frac{\mathrm{tr}_{\mathcal{V}}(\phi(\zeta_1, u_1)\, \phi^*(\zeta_2, u_2)\, q^{\widetilde{H}})}{Z(\tau)} \tag{3.6}$$

of the compact picture local field $\phi\,(\zeta, u)$, where, due to $\widetilde{H}$-invariance of the right hand side, w_q only depends on the difference $\zeta_{12} = \zeta_1 - \zeta_2$ of the conformal time variables. Our terminology ("temperature means") is justified by the fact that the (positive!) imaginary part of τ is identified with the inverse absolute temperature. More precisely, as we have defined (according to (2.1) and (2.8)) the conformal Hamiltonian $\widetilde{H}$ to be dimensionless we should set

$$2\pi\, \mathrm{Im}\, \tau = \frac{h\nu}{kT}\,, \tag{3.7}$$

$h\nu$ being the Planck energy quantum, k, the Boltzmann constant.

The cyclic property of the trace and the second equation (3.3) imply the *Kubo-Martin-Schwinger (KMS) boundary condition*:

$$\left\langle\, \phi(\zeta_1, u_1)\, \phi^*(\zeta_2 + \tau, u_2)\,\right\rangle_q \;=\; \left\langle\, \phi^*(\zeta_2, u_2)\, \phi(\zeta_1, u_1)\,\right\rangle_q\,. \tag{3.8}$$

If the 2-point function is meromorphic and $\mathbb{Z}_2$ symmetric under permutation of the factors (according to (3.4) and (2.12)) then w_q (3.6) is elliptic in ζ_{12} satisfying

$$w_q(\zeta_{12}+1; u_1, u_2) \;=\; (-1)^{2d_\phi}\, w_q(\zeta_{12}; u_1, u_2) \;=\; (-1)^{2d_\phi}\; w_q(\zeta_{12}+\tau; u_1, u_2), \tag{3.9}$$

as a consequence of the KMS condition (3.8). If ϕ is a generalized free field then the KMS condition on the modes gives

$$\begin{aligned} &\left\langle\, \phi_{n_1 m_1}(u_1)\, \overset{*}{\phi}_{n_2 m_2}(u_2)\,\right\rangle_q \;= \\ &= \frac{q^{n_2}}{1 - (-1)^{2d_\phi}\, q^{n_2}}\, \langle 0|\, \left[\, \phi_{n_1 m_1}(u_1)\,,\, \overset{*}{\phi}_{n_2 m_2}(u_2)\,\right]_{\mp} |0\rangle \end{aligned} \tag{3.10}$$

($\overset{*}{\phi}_{nm}(u)$ stand for the modes of the conjugated field $\phi^*\,(\zeta, u)$) so that if we expand $\dfrac{q^{n_2}}{1 - (-1)^{2d_\phi}\, q^{n_2}}$ in a progression for $|q| < 1$ and then take the corresponding sums (1.3) over the modes we will obtain ([19] Theorem 4.1):

$$w_q(\zeta_{12}; u_1, u_2) \;=\; \sum_{k\,=\,-\infty}^{\infty} (-1)^{2\,k\,d_\phi}\; w_0\,(\zeta_{12} + k\,\tau\,; u_1, u_2)\;, \tag{3.11}$$

where $w_0(\zeta_{12}; u_1, u_2)$ is the vacuum ($q = 0$) 2-point function. Indeed, the progression term $\left((-1)^{2d_\phi}\, q^{|n_2|}\right)^{\pm k}$ multiplying the vacuum expectations will produce the term $(-1)^{2\,k\,d_\phi}\, w_0\,(\zeta_{12} + k\,\tau\,; u_1, u_2)$ in the sum (3.11) (see Remark 3.1 below).

In particular, for a 4-dimensional massless scalar field $\varphi(\zeta, u)$ with 2-point function

$$\langle 0|\,\varphi(\zeta_1,u_1)\,\varphi(\zeta_2,u_2)\,|0\rangle \;=\; \frac{e^{2\pi i(\zeta_1+\zeta_2)}}{(e^{2\pi i\zeta_1}\,u_1-e^{2\pi i\zeta_2}\,u_2)^2} \;=\; \frac{-1}{4\,\sin\pi\zeta_+\,\sin\pi\zeta_-}\,, \tag{3.12}$$

where

$$\cos 2\pi\alpha \;:=\; u_1\cdot u_2\,, \quad \zeta_\pm = \zeta_{12}\pm\alpha\,, \tag{3.13}$$

the Gibbs correlation function is given by:

$$w_q(\zeta_{12},\alpha) = \big\langle\, \varphi(\zeta_1,u_1)\,\varphi(\zeta_2,u_2)\,\big\rangle_q = \frac{1}{4\pi\sin 2\pi\alpha}\,(p_1(\zeta_+,\tau)-p_1(\zeta_-,\tau))\,, \tag{3.14}$$

$$p_1(\zeta,\tau) \;=\; \lim_{M\to\infty}\,\lim_{N\to\infty}\,\sum_{m=-M}^{M}\,\sum_{n=-N}^{N}\frac{1}{\zeta+m\tau+n}\,. \tag{3.15}$$

(Note that $p_1(\zeta,\tau)$ is not elliptic but the difference (3.14) is.) In order to derive (3.14) from (3.11) one should use the identity

$$\frac{-1}{\sin\pi\zeta_+\,\sin\pi\zeta_-} \;=\; \frac{1}{\sin 2\pi\alpha}\,(\cot\pi\zeta_+ - \cot\pi\zeta_-)\,,$$

and the Euler formula

$$\pi\cot\pi\zeta \;=\; \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{\zeta+n}\,. \tag{3.16}$$

Remark 3.1. The mode expansion (2.21) of the compact picture field $\varphi(\zeta, u)$ reads:

$$\varphi(\zeta,u) \;=\; \sum_{n\in\mathbb{Z}}\varphi_n(u)\,e^{-2\pi i n\zeta} \quad (\varphi_n(u)\,|0\rangle = 0 \quad \text{for} \quad n\geq 0)\,. \tag{3.17}$$

(comparing with (3.5) we have $\phi_{nm}(u)\equiv\delta_{m+1,|n|}\,\varphi_n(u)$); then Eq. (3.10) takes the form

$$\big\langle\, \varphi_{-m}(u_1)\,\varphi_n(u_2)\,\big\rangle_q \;=\; \delta_{mn}\,\frac{q^n}{1-q^n}\,\frac{\sin 2\pi n\alpha}{\sin 2\pi\alpha}\,. \tag{3.18}$$

We thus obtain the following q–expansion of $w_q(\zeta,\alpha)$:

$$w_q(\zeta,\alpha) \;=\; w_0(\zeta,\alpha) \,+\, 2\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\,\frac{\sin 2\pi n\alpha}{\sin 2\pi\alpha}\,\cos 2\pi n\zeta \tag{3.19}$$

which can be derived also from Eq. (3.14) and the q–expansion

$$p_1(\zeta,\tau) \;=\; \pi\cot\pi\zeta + 4\pi\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\,\sin 2\pi n\zeta\,. \tag{3.20}$$

In order to compute the Gibbs 2-point function of a Weyl field we first write its vacuum Wightman function in the form

$$\langle 0|\, \psi(\zeta_1, u_1)\, \psi^+(\zeta_2, u_2)\, |0\rangle \;=\; \frac{i}{8\sin\pi\zeta_-\, \sin\pi\zeta_+} \left(\frac{\not{v}^+}{\sin\pi\zeta_-} + \frac{\bar{\not{v}}^+}{\sin\pi\zeta_+} \right)$$

$$= \frac{i}{8\sin 2\pi\alpha} \Bigg\{ \not{v}^+ \left(\frac{\cos\pi\zeta_-}{\sin^2\pi\zeta_-} - \frac{\cot 2\pi\alpha}{\sin\pi\zeta_-} + \frac{1}{\sin 2\pi\alpha \sin\pi\zeta_+} \right)$$
$$- \bar{\not{v}}^+ \left(\frac{\cos\pi\zeta_+}{\sin^2\pi\zeta_+} + \frac{\cot 2\pi\alpha}{\sin\pi\zeta_+} - \frac{1}{\sin 2\pi\alpha \sin\pi\zeta_-} \right) \Bigg\} \tag{3.21}$$

where v and $\overline{v}$ are conjugate complex isotropic 4-vectors determined for non-collinear u_1, u_2 from the equations

$$u_1 = e^{i\pi\alpha}\, v + e^{-i\pi\alpha}\, \overline{v}, \quad u_2 = e^{-i\pi\alpha}\, v + e^{i\pi\alpha}\, \overline{v} \quad (u_1^2 = u_2^2 = 1 \;\Rightarrow\; 2v\cdot\overline{v} = 1). \tag{3.22}$$

Then Eq. (3.11) gives

$$\big\langle\, \psi(\zeta_1, u_1)\, \psi^+(\zeta_2, u_2)\, \big\rangle_q \;=\; \frac{i}{8\pi\sin 2\pi\alpha} \Bigg\{ \not{v}^+ \Big(p_2^{11}(\zeta_-, \tau) - \cot 2\pi\alpha\, p_1^{11}(\zeta_-, \tau)$$
$$+ \frac{p_1^{11}(\zeta_+, \tau)}{\sin 2\pi\alpha} \Big) - \bar{\not{v}}^+ \left(p_2^{11}(\zeta_+, \tau) + \cot 2\pi\alpha\, p_1^{11}(\zeta_+, \tau) - \frac{p_1^{11}(\zeta_-, \tau)}{\sin 2\pi\alpha} \right) \Bigg\}. \tag{3.23}$$

Here we are using the functions $p_k^{\kappa,\lambda}(\zeta, \tau)$ given by the Eisenstein series

$$p_k^{\kappa,\lambda}(\zeta, \tau) \;=\; \sum_{m,n\in\mathbb{Z}} \frac{(-1)^{\kappa m + \lambda n}}{(\zeta + m\tau + n)^k}\,, \qquad \kappa, \lambda = 0, 1 \tag{3.24}$$

which are not absolutely convergent for $k \le 2$ and hence require specifying order of limits; in particular,

$$p_1^{11}(\zeta, \tau) \;=\; \pi \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\sin\pi(\zeta + n\tau)}\,, \quad p_2^{11}(\zeta, \tau) \;=\; -\frac{\partial}{\partial\zeta}\, p_1^{11}(\zeta, \tau)\,. \tag{3.25}$$

The functions $p_k^{\kappa,\lambda}$ are defined (in Appendix A of [19]) to satisfy

$$p_k^{\kappa,\lambda}(\zeta + 1, \tau) = (-1)^\lambda\, p_k^{\kappa,\lambda}(\zeta, \tau)\,, \quad p_k^{\kappa,\lambda}(\zeta + \tau, \tau) = (-1)^\kappa\, p_k^{\kappa,\lambda}(\zeta, \tau)$$

for $k + \kappa + \lambda > 1$.

3.2. Energy mean values as (combinations of) modular forms

Knowing the temperature 2-point function one can compute by differentiating in ζ and equating the two points (after subtracting the singularity for coinciding arguments) the energy mean value in the corresponding equilibrium state. This procedure is based on the assumption that the energy momentum tensor occurs in the OPE of the product of two basic fields. We shall follow a more specific procedure (*cf.* [4]) applicable whenever there is a Fock space realization of the state space $\mathcal{V}$ (*e.g.*, in the bosonic case, whenever $\mathcal{V}$ can be viewed as a "symmetric exponent" of a 1-particle space). To this end one only needs the degeneracy of eigenvalues of the conformal Hamiltonian $\widetilde{H}$ in the 1-particle subspace. Rather than describing the general method we shall illustrate it in two typical examples: a free scalar field in dimension $D = 2d_0 + 2$ (*cf.* (2.20)) and the free Weyl field in four dimension.

The 1-particle state space in the theory of a free hermitean scalar field φ in dimension D is the direct sum of spaces of homogeneous harmonic polynomials $\{\varphi_{-n}(z)|0\rangle\}$ of degree $n - d_0$ ($n = d_0, d_0 + 1, \ldots$). The dimension of each such eigenspace of $\widetilde{H}$ is $d(n - d_0, D)$, where

$$d(m, D) \;=\; \binom{m + D - 1}{D - 1} - \binom{m + D - 3}{D - 1},$$
$$i.e. \quad d(n - d_0, D) \;=\; \frac{2}{(2d_0)!} \prod_{i=0}^{d_0 - 1} (n^2 - i^2) \tag{3.26}$$

is the dimension of the space of homogeneous harmonic polynomials of degree m in $\mathbb{C}^D$ which carries an irreducible representation of $Spin(D)$; hence,

$$\langle\, \widetilde{H}\, \rangle_q^{(d_0)} \;=\; E(d_0) + \sum_{n=d_0}^{\infty} \frac{2}{(2d_0)!}\, n^2 \ldots [n^2 - (d_0 - 1)^2]\, \frac{nq^n}{1 - q^n}\,. \tag{3.27}$$

For $D = 4$ ($d_0 = 1$) the energy mean value is a modular form (of weight 4),

$$\langle\, \widetilde{H}\, \rangle_q^{(1)} \;=\; G_4(\tau) \quad \text{for} \quad E(1) \;=\; \langle 0|\, \widetilde{H}\, |0\rangle \;=\; -\frac{B_4}{8} \;=\; \frac{1}{240} \tag{3.28}$$

where we are using the notation and conventions of [24]:

$$G_{2k}(\tau) \;:=\; -\frac{B_{2k}}{4k} + \sum_{n\,=\,1}^{\infty} \frac{n^{2k-1}}{1 - q^n}\, q^n$$
$$\left(= \frac{(2k-1)!}{(2\pi i)^{2k}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} + \sum_{m\,=\,1}^{\infty} \sum_{n \in \mathbb{Z}} (m\tau + n)^{-2k} \right\} \quad \text{for} \quad k \geq 2 \right) \tag{3.29}$$

(the Eisenstein series in the braces being only absolutely convergent for $k \geq 2$).

Remark 3.2. We note that for pure imaginary τ related to the absolute temperature T by (3.7), multiplying $\widetilde{H}$ in (3.28) by $h\nu$ in order to express it in units of energy we find for the n-th term in the expansion of G_4 (3.28) the *Planck distribution law*

$$\frac{n^3\, h\, \nu}{e^{\frac{hn\nu}{kT}} - 1}$$

(up to an overall factor, $\frac{8\pi\nu^2}{c^3}$, independent of n and T).

For $D = 6$ we find from (3.27) a linear combination of modular forms of weight 6 and 4:

$$\langle \widetilde{H} \rangle_q^{(2)} = \frac{1}{12} \left[G_6(\tau) - G_4(\tau) \right] = \frac{-31}{12 \times 7!} + \frac{18\, q^3}{1-q^3} + \frac{80\, q^4}{1-q^4} + \cdots \tag{3.30}$$

(the coefficients to q^n for $n > 0$ being thus positive integers). The functions $G_{2k}(\tau)$ for $k > 1$ are *modular forms of weight* $2k$ (and level 1):

$$(c\tau + d)^{-2k}\, G_{2k} \left(\frac{a\tau + b}{c\tau + d} \right) = G_{2k}(\tau)$$
$$\text{for} \quad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) := SL(2, \mathbb{Z}) \quad (k = 2, 3, \ldots)\,. \tag{3.31}$$

There is no level 1 modular form of weight 2; the function G_2 (given by the first equation (3.29)) transforms inhomogeneously under $\Gamma(1)$:

$$\frac{1}{(c\tau + d)^2}\, G_2 \left(\frac{a\tau + b}{c\tau + d} \right) = G_2(\tau) + \frac{ic}{4\pi(c\tau + d)}\,. \tag{3.32}$$

The 1-particle state space for a complex Weyl field is the direct sum of positive and negative charge states of energy $\left(n + \frac{1}{2} \right)$ $(n = 1, 2, \ldots)$ of equal multiplicity. The Weyl equation (2.29) implies that the positive charge energy eigenspace spanned by $\left\{ \psi^+_{-n-\frac{1}{2}}(z)\, |0\rangle \right\}$ carries the irreducible $Spin(4) = SU(2) \times SU(2)$ representation $\left(\frac{n}{2}, \frac{n-1}{2} \right)$ of dimension $n(n+1)$. The negative charge energy eigenspace spanned by $\left\{ \psi_{-n-\frac{1}{2}}(z)\, |0\rangle \right\}$ has the same dimension. As a result, the mean energy in an equilibrium temperature state is given by

$$\begin{aligned} \langle \widetilde{H} \rangle_q &= E_0 + \sum_{n=1}^{\infty} (2n+1)\, \frac{n(n+1)\, q^{n+\frac{1}{2}}}{1 + q^{n+\frac{1}{2}}} \\ &= \frac{1}{4} \left\{ G_4 \left(\frac{\tau+1}{2} \right) - 8\, G_4(\tau) - G_2 \left(\frac{\tau+1}{2} \right) + 2\, G_2(\tau) \right\}. \end{aligned} \tag{3.33}$$

To prove the first equation one again uses the KMS condition combined with the canonical *anticommutation relations* for the ψ modes. The second is derived from the identities

$$n(n+1) = \frac{1}{4} \left[(2n+1)^2 - 1 \right], \quad q \left(\frac{\tau+1}{2} \right) = -q^{1/2} \quad (\text{for } q \equiv q(\tau))\,. \tag{3.34}$$

We note that although $G_2(\tau)$ is not a modular form the difference

$$F(\tau) := 2\, G_2(\tau) - G_2 \left(\frac{\tau+1}{2} \right), \tag{3.35}$$

equal to the energy distribution of a chiral Weyl field (in 2-dimensional conformal field theory), is a weight 2 form with respect to the index 2 subgroup $\Gamma_\theta \subset \Gamma(1)$ generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

4. General 4-point functions

4.1. Strong locality and energy positivity imply rationality

As already stated in Sect. 2.1 GCI and Wightman axioms imply strong locality (2.12) for vertex operators. In other words, for any pair of conjugate (Bose or Fermi) fields $\psi(z_1)$ and $\psi^*(z_2)$ there is a positive integer N_ψ such that

$$\rho_{12}^N \left\{ \psi(z_1)\, \psi^*(z_2) - \varepsilon_\psi\, \psi^*(z_2)\, \psi(z_1) \right\} \,=\, 0 \quad \text{for} \quad N \geq N_\psi \quad (\rho_{12} \,:=\, z_{12}^2) \tag{4.1}$$

where $\varepsilon_\psi (= \pm 1)$ is the fermion parity of ψ. If ψ transforms under an *elementary local field representation* of the spinor conformal group $SU(2,2)$ (see [13, 21]) – *i.e.* one induced by a $(2j_1+1)(2j_2+1)$ dimensional representation $(d; j_1, j_2)$ of the maximal compact subgroup $S(U(2) \times U(2))$ of $SU(2,2)$ (d being the $U(1)$ character coinciding with the scale dimension), then

$$N_\psi \,=\, d + j_1 + j_2 \,, \quad \varepsilon_\psi \,=\, (-1)^{2j_1+2j_2} \,=\, (-1)^{2d} \,. \tag{4.2}$$

It follows that for any n-point function of GCI local fields and for large enough $N \in \mathbb{N}$ the product

$$F_{1\dots n}(z_1, \dots, z_n) \,:=\, \Bigg(\prod_{1 \,\leq\, i \,<\, j \,\leq\, n} \rho_{ij} \Bigg)^N \langle 0|\, \phi_1(z_1) \dots \phi_n(z_n)\, |0\rangle \tag{4.3}$$

($\rho_{ij} = z_{ij}^2 \equiv (z_i - z_j)^2$), is $\mathbb{Z}_2$ symmetric under any permutation of the factors within the vacuum expectation value. Energy positivity, on the other hand, implies that $\langle 0|\, \phi_1(z_1) \dots \phi_n(z_n)\, |0\rangle$, and hence $F_{1\dots n}(z_1, \dots, z_n)$ do not contain negative powers of z_n^2. It then follows from the symmetry and the homogeneity of $F_{1\dots n}$ that it is a polynomial in all z_i^μ. Thus the (Wightman) correlation functions are rational functions of the coordinate differences. (See for more detail [19]; an equivalent Minkowski space argument based on the support properties of the Fourier transform of (the x-space counterpart of) (4.3) is given in [18].)

Rationality of correlation functions implies that all dimensions of GCI fields should be integer or half integer depending on their spin, more precisely, that sums like N_ψ (4.2) should be integer. This condition is, however, not sufficient for rationality even of 3-point functions.

Observation 4.1. *The necessary and sufficient condition for the existence of a GCI 3-point function $\langle 0|\, \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3)\, |0\rangle$ of elementary conformal fields $\phi_i(z)$ of $S(U(2) \times U(2))$ weight $(d_i; j_{i1}, j_{i2})$ is:*

$$N_i \,:=\, d_i + j_{i1} + j_{i2} \in \mathbb{N} \,, \quad \frac{1}{2} \sum_{i=1}^{3} N_i \,\in\, \mathbb{N} \,, \quad \sum_{i=1}^{3} d_i \,\in\, \mathbb{N} \,. \tag{4.4}$$

The statement follows from the explicit knowledge of 3-point functions (see for reviews [23] [21] [10]).

In particular, there is no Yukawa type rational conformal 3-point function of a pair of conjugate canonical ($d = \frac{3}{2}$) spinor fields and a canonical ($d = 1$) scalar field. Similarly, one observes that 2-point functions of free massless fields in odd space-time dimensions are not rational and hence cannot be GCI.

It is important for the feasibility of constructing a GCI model that the singularities of n-point functions (the integer N in (4.3)) is majorized by that of the 2-point function (N_ψ in (4.2)) whenever Wightman positivity is satisfied.

4.2. General truncated 4-point function of a GCI scalar field

Infinitesimal (or Euclidean) conformal invariance is sufficient to determine 2- and 3-point functions (see, *e.g.* [23]). One can construct, however, two independent conformally invariant cross-ratios out of four points,

$$s \;=\; \frac{\rho_{12}\rho_{34}}{\rho_{13}\rho_{24}}\,, \quad t \;=\; \frac{\rho_{14}\rho_{23}}{\rho_{13}\rho_{24}}\,, \tag{4.5}$$

so that a simple minded symmetry argument does not determine the 4-point functions. GCI, on the other hand, combined with Wightman axioms, yields rationality and thus allows to construct higher point correlation functions involving just a finite number of free parameters. In particular, the truncated 4-point function of a hermitean scalar field ϕ of (integer) dimension d can be written in the form ([15] Sect. 1):

$$\begin{aligned} w_4^t \;\equiv\; & w^t(z_1,z_2,z_3,z_4) \;:=\; \langle 1234\rangle - \langle 12\rangle\langle 34\rangle - \langle 13\rangle\langle 24\rangle - \langle 14\rangle\langle 23\rangle \\ =\; & \frac{(\rho_{13}\rho_{24})^{d-2}}{(\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{d-1}}\, P_d(s,t)\,, \quad P_d(s,t) \;=\; \sum_{i\,\geq\,0,j\,\geq\,0}^{i+j\,\leq\,2d-3} c_{ij}s^it^j\,, \end{aligned} \tag{4.6}$$

where $\langle 1 \ldots n\rangle$ is a short-hand for the n-point function of ϕ:

$$\langle 1\ldots n\rangle \;=\; \langle 0|\,\phi(z_1)\ldots\phi(z_n)\,|0\rangle \quad (\,\langle 12\rangle \;=\; B_\phi\,\rho_{12}^{-d}\,)\,. \tag{4.7}$$

In writing down (4.6) we have used the fact that for space-time dimensions $D > 2$ Hilbert space positivity implies that the degree of singularities of the truncated n-point function ($n \geq 4$) is strictly smaller than the degree of the 2-point function.

Furthermore, crossing symmetry (which is a manifestation of local commutativity) implies an $\mathcal{S}_3$ symmetry of P_d:

$$\begin{aligned} & s_{i\,i+1}P_d(s,t) \;=\; P_d(s,t),\; i=1,2\,, \\ & s_{12}\,P_d(s,t) \;:=\; t^{2d-3}P_d\left(\frac{s}{t},\frac{1}{t}\right)\,, \quad s_{23}\,P_d(s,t) \;:=\; s^{2d-3}P_d\left(\frac{1}{s},\frac{t}{s}\right)\,, \end{aligned} \tag{4.8}$$

s_{ij} being the substitution exchanging the arguments z_i and z_j. The number of independent crossing symmetric polynomials P_d is $\left[\!\left[\frac{d^2}{3}\right]\!\right]$ (the integer part of $d^2/3$: $\left[\!\left[\frac{d^2}{3}\right]\!\right] = n(2d-3n)$ for $3n-1 \leq d \leq 3n+1$, $n=0,1,2,\ldots$).

The 1-parameter family of crossing symmetric polynomials for $d = 2$ is $P_2(s,t) = c\,(1+s+t)$, *i.e.*

$$w_4^t \,=\, c\,\Big\{ (\rho_{12}\rho_{23}\rho_{34}\rho_{14})^{-1} + (\rho_{13}\rho_{23}\rho_{24}\rho_{14})^{-1} + (\rho_{12}\rho_{13}\rho_{24}\rho_{34})^{-1} \Big\}\,, \qquad (4.9)$$

thus corresponding to the sum of three 1-loop diagrams for a sum of normal products of free massless fields:

$$\phi(z) \,=\, \frac{1}{2}\sum_{i=1}^{N} :\varphi_i^2(z): \quad ([\varphi_i(z_1),\varphi_j(z_2)] \,=\, 0 \quad \text{for} \quad i\neq j\,, \quad \Delta\,\varphi_i(z) \,=\, 0)\,. \qquad (4.10)$$

Indeed, it was proven in [15] that $\phi(z)$ generates under commutations a central extension of the infinite symplectic algebra $sp(\infty,\mathbb{R})$ for $d = 2$ and that the unitary vacuum representations of this algebra correspond to integer central charge $c = N\,(\in \mathbb{N}\,)$. Thus, (4.10) is the general form of a $d = 2$ GCI field satisfying Wightman axioms (including Hilbert space positivity) and involving a unique rank 2 symmetric traceless tensor of dimension 4 in its OPE algebra.

The physically most attractive example, corresponding to a $d = 4$ scalar field $\mathcal{L}(z)$ that can be interpreted as a QFT Lagrangian density, gives rise to a 5-parameter truncated 4-point function [16] of type (4.6) with

$$P_4(s,t) \,=\, \textstyle\sum_{\nu=0}^{2} a_\nu J_\nu(s,t) + st\,\Big[b(Q_1(s,t) - 2Q_2(s,t)) + c\,Q_2(s,t)\Big], \qquad (4.11)$$

$$J_0(s,t) \,:=\, s^2(1+s) + t^2(1+t) + s^2t^2(s+t), \qquad (4.12)$$

$$J_1(s,t) := s(1-s)(1-s^2) + t(1-t)(1-t^2) + st\left[(s-t)(s^2-t^2) - 2Q_1\right], \qquad (4.13)$$

$$J_2(s,t) := (1+t)^3[(1+s-t)^2 - s] - 3s\,(1-t) + s^3\,\left[(1+t-s)^2 - t\right], \qquad (4.14)$$

$$Q_1(s,t) \,:=\, 1+s^2+t^2\,, \quad Q_2(s,t) := s+t+st\,,$$
$$Q_1(s,t) - 2Q_2(s,t) \,=\, (1-s-t)^2 - 4st\,. \qquad (4.15)$$

As we shall see in the next section, the J_ν polynomials correspond to the twist 2 fields' contribution to the OPE, symmetrized in an appropriate way, while the terms $st\,Q_j$ correspond to twist 4 and higher contributions.

The above choice of basic $\mathcal{S}_3$ symmetric polynomials is not accidental: it is essentially determined by its relation to the "partial wave" expansion of w_4^t to be displayed in the next section.

5. OPE in terms of bilocal fields

5.1. Fixed twist fields. Conformal partial wave expansion

From now on we shall study the GCI theory of the above $d = 4$ hermitean scalar field $\mathcal{L}(z)$ (in $D = 4$ space-time dimensions).

The infinite series of local tensor fields appearing in the OPE of $\mathcal{L}(z_1)\,\mathcal{L}(z_2)$ can be organized into an infinite sum of scalar fields depending on both arguments z_1 and z_2:

$$\mathcal{L}(z_1)\,\mathcal{L}(z_2) \;=\; \frac{B}{\rho_{12}^4} \;+\; \sum_{\kappa\,=\,1}^{\infty} \rho_{12}^{\kappa-4}\, V_\kappa(z_1,z_2)\,. \tag{5.1}$$

By definition, the field $V_\kappa\,(z_1,z_2)$ is regular for coinciding arguments (see Remark 5.1 below) and its Taylor expansion in z_{12} involves only *twist* 2κ *tensor fields* and their derivatives. More precisely, it can be written in the form (*cf.* [9] [2] [23] [5] [15])

$$V_\kappa(z_1,z_2) \;=\; \sum_{\ell\,=\,0}^{\infty} C_{\kappa\ell}\, K_{\kappa\ell}(z_{12}\cdot\partial',\; \rho_{12}\,\Delta')\, O_{2\kappa,\ell}(z_2;z_{12})\, d\alpha\,. \tag{5.2}$$

Here $O_{2\kappa,\ell}(z;w)$ are (contracted) symmetric traceless tensor fields,

$$O_{2\kappa,\ell}(z;w) = O_{2\kappa}^{\mu_1\ldots\mu_\ell}(z)\, w_{\mu_1}\ldots w_{\mu_\ell} \quad (\text{tracelessness} \;\Leftrightarrow\; \Delta_w\, O_{2\kappa,\ell}(z;w)=0), \tag{5.3}$$

of scale dimension $d_{\kappa,\ell} = 2\kappa+\ell$ (*i.e.* of *fixed twist* $d_{\kappa\ell}-\ell = 2\kappa$); $K_{\kappa\ell}\,(t_1,t_2)$ is the Taylor series in $t_{1,2}$ of the analytic function

$$K_{\kappa,\ell}\,(t_1,t_2) \;:=\; \sum_{n\,=\,0}^{\infty} \int_0^1 \frac{[\alpha\,(1-\alpha)]^{\ell+\kappa+n-1}\, e^{\alpha\, t_1}\; (-t_2)^n}{4^n\, B(\ell+\kappa,\ell+\kappa)\, n!\,(2\ell+2\kappa-1)_n}\; d\alpha \tag{5.4}$$

($(a)_n := \dfrac{\Gamma\,(a+n)}{\Gamma\,(a)}$) and t_1, t_2 are substituted in (5.2) by the operators $z_{12}\cdot\partial'$ and $\rho_{12}\Delta'$ (resp.), where

$$\partial'_\mu\, O_{2\kappa,\,\ell}(z_2;z_{12}) \;:=\; \frac{\partial O_{2\kappa,\ell}(z;w)}{\partial z^\mu}\,\Big|_{\,z\,=\,z_2,\; w\,=\,z_{12}}\,, \qquad \Delta' \;:=\; \partial'\cdot\partial'\,. \tag{5.5}$$

The integrodifferential operator with kernel $K_{k\ell}$ in the right hand side of (5.2) is defined to transform the 2-point function of $O_{2\kappa,\ell}$ into the 3-point function

$$\langle 0|\, V_\kappa(z_1,z_2) O_{2\kappa,\ell}(z_3;w)\,|0\rangle = A_{\kappa\ell}\, \frac{1}{(\rho_{13}\rho_{23})^\kappa} \left(\left(\frac{z_{13}}{\rho_{13}} - \frac{z_{23}}{\rho_{23}}\right)\cdot w\right)^{\ell} \;\text{ for }\; w^2=0 \tag{5.6}$$

(see [5] and [16] Sect. 3, where more general OPE – for any scale dimension d and for complex fields are considered, and the kernel $K_{\kappa\ell}$ is written down). For real $\mathcal{L}$, due to the strong locality, $V_\kappa\,(z_1,z_2)$ should be symmetric:

$$V_\kappa(z_1,z_2) \;=\; V_\kappa(z_2,z_1)\,; \tag{5.7}$$

it then follows that only even rank tensors (even ℓ) appear in the expansion (5.2). Since every field $V_\kappa\,(z_1,z_2)$ corresponds to a conformally invariant part of the OPE of $\mathcal{L}\,(z_1)\,\mathcal{L}\,(z_2)$ then V_κ should be a conformally invariant scalar field of dimension (κ,κ), because of the factor $\rho_{12}^{\kappa-4}$ in (5.1).

Remark 5.1. The condition of regularity of a field $V(z,w)$ for coinciding arguments in terms of formal series is stated as follows: for every state $\Psi\in\mathcal{V}$ there exists $N \in \mathbb{N}$ such that $\left(z^{2} w^{2}\right)^{N_\Psi} V(z,w)\,\Psi$ does not contain negative powers of z^{2} and z^{2} (see [14] Definition 2.2). It follows then that $V(z,z)$ is a correctly defined series in z of type (1.2) (containing, in general, negative powers of z^{2}) by setting $V(z,z)\,\Psi := \left(z^{2}\right)^{-2N} \left\{ \left[\left(z^{2} w^{2}\right)^{N} V(z,w)\ \Psi \right] \Big|_{z\,=\,w} \right\}$.

The fields $V_\kappa(z_1,z_2)$ are determined by their correlation functions. Note that

$$\langle 0|\, V_\kappa(z_1,z_2)\,|0\rangle = 0 = \langle 0|\, V_\kappa(z_1,z_2)\, V_\lambda(z_3,z_4)\,|0\rangle \quad \text{for} \quad 0 < \kappa \neq \lambda \tag{5.8}$$

since fields of different twists are mutually orthogonal under vacuum expectation values. For equal dimensions we can write

$$\begin{aligned} \langle 0|\, V_\kappa(z_1,z_2)\, V_\kappa(z_3,z_4)\,|0\rangle \ &= \ (\rho_{13}\rho_{24})^{-\kappa} f_\kappa(s,t) \\ (\, s_{12}\, f_\kappa(s,t) \ &= \ t^{-\kappa} f_\kappa\left(\frac{s}{t},\frac{1}{t}\right) \ = \ f_\kappa(s,t)\,) \end{aligned} \tag{5.9}$$

thus obtaining the expansion of the 4-point function $\langle 1234\rangle - \langle 12\rangle\langle 34\rangle$ (see (4.6)):

$$t^{-3}\, P_4(s,t) + B^2\, s^4 \left(1+\frac{1}{t^4}\right) \ = \ \sum_{\kappa\,=\,1}^{\infty} s^{\kappa-1}\, f_\kappa(s,t) \tag{5.10}$$

B being the 2-point normalization constant in (4.7). Moreover, as it is shown in [5] and [6], the OPE (5.2) implies that $f_\kappa(s,t)$ should have the form:

$$\begin{aligned} f_\kappa(s,t) = \ \frac{1}{u-v} \ &\Big\{ F(\kappa-1,\ \kappa-1;\ 2\kappa-2;\ v)\ g_\kappa(u) \\ &- F(\kappa-1,\ \kappa-1;\ 2\kappa-2;\ u)\ g_\kappa(v) \Big\}\ , \end{aligned} \tag{5.11}$$

$F(\alpha,\beta;\gamma;x) = \sum\limits_{n\,=\,0}^{\infty} \dfrac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}\, x^n = 1 + \dfrac{\alpha\beta}{\gamma}\, x + \ldots$ being Gauss' hypergeometric function and $F(0,0;0;u) \equiv 1$; u and v are the "chiral variables" of [5] (see also [7]):

$$s \ = \ uv\,, \quad t \ = \ (1-u)(1-v)\,. \tag{5.12}$$

The functions $g_\kappa(u)$ should satisfy

$$g_\kappa(0) \ = \ 0 \tag{5.13}$$

and they determine the $O_{2\kappa,\ell}$ contributions in (5.2) via the expansion

$$g_\kappa(u) = u\, f_\kappa(0,1-u) = u \sum_{\ell=0}^{\infty} B_{\kappa\ell}\, u^{2\ell}\, F(2\ell+\kappa, 2\ell+\kappa; 4\ell+2\kappa; u) \tag{5.14}$$

(the first equality follows from Eqs. (5.13), (5.11) and (5.12)). The *structure constants* $B_{\kappa\ell}$, unlike those appearing in (5.2) and (5.6), are invariant under rescaling of $O_{2\kappa 2\ell}$:

$$B_{\kappa\ell} \ := \ A_{\kappa 2\ell}\, C_{\kappa 2\ell} \quad (O_{2\kappa 2\ell} \mapsto \lambda\, O_{2\kappa 2\ell} \ \Rightarrow \ A_\kappa \mapsto \lambda\, A_{\kappa 2\ell},\ C_{\kappa 2\ell} \mapsto \frac{1}{\lambda}\, C_{\kappa 2\ell})\,. \tag{5.15}$$

Eqs. (5.10) and (5.11) allow one to find recurrently the functions $f_\kappa(s,t)$ as follows:

$$f_1(0,t) = t^{-3}P_4(0,t), \quad f_\kappa(0,t) = \lim_{s\to 0}\left\{s^{1-\kappa}\left[t^{-3}P_4(s,t) - \sum_{\nu=1}^{\kappa-1} s^{\nu-1} f_\nu(s,t)\right]\right\}, \tag{5.16}$$

for $\kappa = 2,3$, $f_\kappa(s,t)$ being obtained from the first equation (5.14) and from Eq. (5.11). For $\kappa > 3$ second equation (5.16) should be replaced by the left hand side of Eq. (5.10). It is important to realize that the kernels $K_{\kappa\ell}$ and the hypergeometric functions defining the conformal partial waves (5.14) are *universal*; only the structure constants $B_{\kappa\ell}$ depend on $\mathcal{L}(z)$ and are, in fact, determined by its 4-point function.

Note that the above algorithm always gives rational solution for $f_1(s,t)$. Indeed, by Eq. (5.11)

$$f_1(s,t) = \frac{g(u) - g(v)}{u - v} \tag{5.17}$$

(for $g(t) = (1-t)^{-3} P_4(0, 1-t)$) is a rational *symmetric* function in u and v which are, on the other hand, the roots of the second order polynomial equation

$$u^2 + (t-s-1)\,u + s \;=\; 0 \quad (v^2 + (t-s-1)\,v + s \;=\; 0) \tag{5.18}$$

implied by (5.12). This motivates us to consider the field $V_1(z,w)$ as a (*strongly*) *bilocal* field. For higher twist contributions $V_\kappa(z,w)$, the function $f_\kappa(s,t)$ is not rational; for instance,

$$f_2(s,t) \;=\; \frac{\ln(1-u)\,g(v)\,v \;-\; \ln(1-v)\,g(u)\,u}{u\,v\,(v-u)} \tag{5.19}$$

and this expression contains a log term for any nonzero function $g(u)$.

5.2. Symmetrized contribution of twist 2 (conserved) tensors

The general conformal invariant 3–point function $\langle 0|\, V_1(z_1,z_2)\, O_{2,\ell}(z_3;w)\, |0\rangle$ (5.6) is harmonic in both z_1 and z_2 for all ℓ which combined with Wightman positivity and locality (in particular, the Reeh-Schilder theorem) implies that:

$$\Delta_1\, V_1(z_1,z_2) \;=\; 0 \;=\; \Delta_2\, V_1(z_1,z_2) \quad (\,\Delta_j \;=\; \sum_{\mu=1}^{4} \frac{\partial^2}{(\partial z_j^\mu)^2}\,)\,. \tag{5.20}$$

Apart from the assumptions of Wightman positivity and locality it is shown in [16] that under the expansion (5.2) and symmetry (5.7) the equations (5.20) are equivalent to the conservation laws:

$$\partial_z \cdot \partial_w\, O_{2,\ell}(z,w) \;=\; 0\,. \tag{5.21}$$

At the level of the 4–point function $\langle 0|\, V_2(z_1,z_2)\, V_2(z_3,z_4)\, |0\rangle$ the harmonicity is always satisfied. Indeed, the Laplace equation in z_1 implies what may be called the *conformal Laplace equation*[1] for f_1 (5.9):

$$\Delta_{st} f_1(s,t) = 0\,, \;\Delta_{st} := s\,\frac{\partial^2}{\partial s^2} + t\,\frac{\partial^2}{\partial t^2} + (s+t-1)\,\frac{\partial^2}{\partial s\,\partial t} + 2\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\,. \quad (5.22)$$

Its general solution is expressed in terms of u, v (5.12) by Eq. (5.17) which is a special case of (5.11) for $\kappa = 1$ (and $F(0,0;0;u) \equiv 1$).

We now proceed to writing down the general rational solution $f_1\,(s,t)$ for our $d = 4$ model. Accordingly to the first equation (5.16) and Eq. (4.6) it follows that

$$p\,(t) \;:=\; t^3 f_1\,(0,t) \quad (5.23)$$

is a polynomial of degree not exceeding 5 and then the symmetry condition (5.9) implies:

$$t^5\,p\left(\frac{1}{t}\right) \;=\; p\,(t) \quad\Rightarrow\quad p\,(t) \;=\; \alpha_0(1+t^5) + \alpha_1\left(t+t^4\right) + \alpha_2(t^2+t^3)\,. \quad (5.24)$$

Thus the general form of $f_1\,(s,t)$ is a linear combination of three basic functions constructed by the basic polynomials $p_\nu\,(t) = t^\nu + t^{5-\nu}$ for $\nu = 0,1,2$ and the algorithm of the previous subsection.

There is a more convenient basis of functions $f_1\,(s,t) = j_\nu\,(s,t)$, again indexed by $\nu = 0,1,2$, satisfying the requirement that the $\mathcal{S}_3$–symmetrizations of $t^3\,j_\nu$ are "*eigenfunctions*" for the following equations:

$$\lambda_\nu\,(1+s_{23}+s_{13})\left[t^3 j_\nu\,(s,t)\right] \;-\; t^3 j_\nu\,(s,t) \;=\; s^{\sigma_\nu} q_\nu\,(s,t) \quad (5.25)$$

for $\sigma_\nu \geq 1$ where $q_\nu\,(s,t)$ are polynomials (of overall degree $5-\sigma_\nu$) such that $q_\nu\,(0,t) \neq 0$, and s_{23}, s_{13} are the $\mathcal{S}_3$ generators (4.8). The solutions are given by

$$\begin{aligned} j_0(s,t) \;&=\; j_0(0,t) = 1 + t^{-1}\,, \\ j_1(s,t) \;&=\; \left(\frac{1-t}{t}\right)^2 (1+t-s) - 2\frac{s}{t} \;=\; j_1(0,t) - s(1+t^{-2})\,, \\ j_2(s,t) \;&=\; (1+t^{-3})[(1+s-t)^2 - s] - 3s(1-t)\,t^{-3} \end{aligned} \quad (5.26)$$

with "eigenvalues" $\lambda_0 = \lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$, and $\sigma_0 = 2$, $\sigma_1 = 1$, $\sigma_2 = 3$. The J_ν polynomials (4.12)–(4.14) are related to j_ν as

$$J_\nu\,(s,t) \;=\; \lambda_\nu\,(1+s_{23}+s_{13})\left[t^3 j_\nu\,(s,t)\right]\,. \quad (5.27)$$

Thus the equalities (5.25) mean that a 4–point function (4.6) obtained by $P_4\,(s,t) = J_\nu\,(s,t)$, for $\nu = 0,1,2$, corresponds to OPE of $\mathcal{L}\,(z_1)\,\mathcal{L}\,(z_2)$ containing twist 2 contribution determined by $f_1\,(s,t) = j_\nu\,(s,t)$ and the higher twist contributions start with σ_ν.

[1]The operator (5.22) has appeared in various contexts in [7] [5] and [16].

Putting everything together we can, in principle, determine all structure constants $B_{\kappa\ell}$. It follows from (5.6) (5.15) and from the relation $A_{\kappa\ell} = N_{\kappa\ell}\, C_{\kappa\ell}$ where $N_{\kappa\ell}(>0)$ stands for the normalization of the 2-point function of $O_{2\kappa\ell}$ that $B_{\kappa\ell} = N_{\kappa 2\ell}\, C^2_{\kappa 2\ell}$ should be positive if Hilbert space (or Wightman) positivity holds. (The full argument uses the classification [13] of unitary positive energy representations of $SU(2,2)$ according to which the state spaces spanned by $O_{2\kappa 2\ell}(z,w)|0\rangle$, for $\kappa = 1, 2, \ldots,$ $\ell \geq 0$, belong to the unitary series.) Thus, such a calculation will restrict the admissible values of the parameters a_ν, b, c and B in (4.11) and (5.1), providing a non-trivial positivity check for the 4-point function of $\mathcal{L}$. We shall display the corresponding equations and their solution for $\kappa = 1, 2, 3$ (the twists for which the 2-point normalization B does not contribute).

Inserting in the left hand side of (5.14) for $\kappa = 1$ the expression $f(0, 1-u) = \sum\limits_{\nu=0}^{2} a_\nu\, j_\nu(0, 1-u)$ we can solve with respect to $B_{1\ell}$ with the result

$$B_{1\ell} \;=\; \binom{4\ell}{2\ell}^{-1} \Big\{ 2a_0 + 2\ell(2\ell+1)[2a_1 + (2\ell-1)(\ell+1)\, a_2] \Big\} \,. \tag{5.28}$$

The equation for $\kappa = 2$ involves a_1, b and c:

$$\begin{aligned} f_2(0, 1-u) =&\; a_1 u \left(\frac{1}{(1-u)^3} - 1 \right) + \frac{bu^2}{(1-u)^2} + \frac{c}{1-u} \\ =&\; \sum_{\ell=0}^{\infty} B_{2\ell}\, u^{2\ell} F(2\ell+2, 2\ell+2; 4\ell+4; u) \,. \end{aligned} \tag{5.29}$$

Its solution is

$$B_{2\ell} \;=\; \binom{4\ell+1}{2\ell}^{-1} \Big\{ \ell(2\ell+3)[(\ell+1)(2\ell+1)\, a_1 + 2b] + c \Big\} \,. \tag{5.30}$$

For $\kappa = 3$ we have to use the expression (5.11) for $f_2(s,t)$ which involves a log term as $F(1,1;2;v) = \sum\limits_{n=1}^{\infty} \frac{v^{n-1}}{n} = \frac{1}{v}\, \log \frac{1}{1-v}$. The result is

$$\begin{aligned} f_3(0, 1-u) \;=&\; \Big(a_0 + \frac{a_1}{2} \Big) (1 + (1-u)^{-3}) - \frac{3}{2}\, \frac{b}{1-u} \left(1 + \frac{1}{1-u} \right) \\ &+ \frac{c}{2} \left\{ \frac{2u-1}{u(1-u)} \left(1 + \frac{2}{u(1-u)} \right) - 2\, \frac{\log(1-u)}{u^3} \right\} \\ =&\; \sum_{\ell=0}^{\infty} B_{3\ell}\, u^{2\ell}\, F(2\ell+3, 2\ell+3; 4\ell+6; u) \,. \end{aligned} \tag{5.31}$$

A computer aided calculation (using Maple) gives in this case

$$B_{3\ell} \;=\; \frac{1}{2} \binom{4\ell+3}{2\ell+1}^{-1} \Big\{ (\ell+1)(2\ell+3)\, [(\ell+2)(2\ell+1)(2a_0 + a_1) - 6b + 4c] - c \Big\} \,. \tag{5.32}$$

The positivity of $B_{j\ell}$, $j = 1, 2, 3$ implies

$$a_\nu \geq 0, \;\; \nu = 0,1,2; \;\; 3a_1 + b \geq 0\,, \;\; c \geq 0; \;\; 6(2a_0 + a_1 - 3b) + 11c \geq 0\,. \quad (5.33)$$

This leaves a nonempty domain in the space of (4-point function) parameters in which positivity holds.

5.3. Free field realizations

Every solution $j_\nu(s,t)$ has a free field realization by a composite field $V_1^{(\nu)}(z,w)$, for $\nu = 0,1,2$, constructed as follows. Let $\varphi(z)$ and $\psi(z)$ be the canonical massless scalar and Weyl fields, respectively, introduced in Sect. 2 and $F_{\mu\nu}(z)$ be the free electromagnetic (Maxwell) field, characterized by the two point function

$$\langle 0|\, F_{\mu_1\nu_1}(z_1)\, F_{\mu_2\nu_2}(z_2)\,|0\rangle \;=\; R_{\mu_1\mu_2}(z_{12})\, R_{\nu_1\nu_2}(z_{12}) - R_{\mu_1\nu_2}(z_{12})\, R_{\nu_1\mu_2}(z_{12}) \quad (5.34)$$

where $R_{\mu\nu}$ is related to the vector representation of the conformal inversion:

$$R_{\mu\nu}(z) \;=\; \frac{r_{\mu\nu}(z)}{z^{\,2}}\,, \quad r_{\mu\nu}(z) \;=\; \delta_{\mu\nu} - 2\,\frac{z_\mu z_\nu}{z^{\,2}}\,, \quad (r^2)_{\mu\nu} \;=\; \delta_{\mu\nu}\,. \quad (5.35)$$

Then the bilocal fields:

$$V_1^{(0)}(z_1,z_2) = \,:\varphi(z_1)\,\varphi(z_2):\,, \quad (5.36)$$

$$V_1^{(1)}(z_1,z_2) = \,:\psi^+(z_1)\,\not{z}_{12}\,\psi(z_2): \,-\, :\psi^+(z_2)\,\not{z}_{12}\,\psi(z_1):\,, \quad (5.37)$$

$$V_1^{(2)}(z_1,z_2) = \;\frac{1}{4}\,\rho_{12} :F^{\sigma\tau}(z_1)F_{\sigma\tau}(z_2): -\, \delta^{\sigma\tau}\, z_{12}^{\mu}\, z_{12}^{\nu} :F_{\sigma\mu}(z_1)F_{\tau\nu}(z_2): \quad (5.38)$$

(: _ : being the standard free field's normal product) or the sum of commuting copies of such expressions, are harmonic, symmetric under exchange of z_1 and z_2, and give an operator realization of the corresponding dimensionless 4-point functions $j_\nu(s,t)$ (5.26). Indeed, the cases of $\nu = 0$ and $\nu = 2$ have been already proved in [15] and [16], and the corresponding calculation for $\nu = 1$ is done in Appendix A.

Moreover, for every $\nu = 0,1,2$ there is a composite field realization of the scalar field $\mathcal{L}(z) \equiv \mathcal{L}^{(\nu)}(z)$ reproducing the OPE (5.1) with twist two part presented by the bilocal field $V_1^{(\nu)}(z,w)$ as well as the truncated 4-point function corresponding to $J_\nu(s,t)$ (4.12)–(4.14) up to normalization.

For $\nu = 2$ it was shown in [16] that the free Maxwell Lagrangian

$$\mathcal{L}_0(x) \;=\; -\frac{1}{4} :F_{\mu\nu}(z)\, F^{\mu\nu}(z): \quad (5.39)$$

does the job. (The calculation proving that Eqs. (5.39), (5.34) and (5.35) yield a w_4^t proportional to $\rho_{13}^2\, \rho_{24}^2\, (\rho_{12}\, \rho_{23}\, \rho_{34}\, \rho_{14})^{-3}\; J_2(s,t)$ is given in Appendix B of [15].)

For $\nu = 0$ and $\nu = 1$ one has to introduce an additional (independent) generalized free field[2]. Let $\phi(z)$ be a generalized free scalar neutral field of dimension 3 commuting with $\varphi(z)$. Then the composite scalar field of dimension 4:

$$\mathcal{L}^{(0)}(z) \;:=\; \varphi(z)\,\phi(z)\,(\equiv\, :\varphi(z)\,\phi(z):) \tag{5.40}$$

has the OPE:

$$\mathcal{L}^{(0)}(z_1)\,\mathcal{L}^{(0)}(z_2) \;=\; \frac{B_\varphi B_\phi}{\rho_{12}^4} \;+\; \frac{B_\phi}{\rho_{12}^3}\,V_1^{(0)}(z_1,z_2) \;+\; \frac{B_\varphi}{\rho_{12}}\,:\phi(z_1)\,\phi(z_2): \\ +\; :\mathcal{L}^{(0)}(z_1)\,\mathcal{L}^{(0)}(z_2): \tag{5.41}$$

(using the standard Wick normal product). Similarly, let $\chi(z)$ be a generalized free Weyl field of dimension $\frac{5}{2}$, with 2-point function

$$\langle 0|\,\chi\,(z_1)\,\chi^+\,(z_2)\,|0\rangle \;=\; \rho_{12}^{-3}\,\not{z}_{12} \tag{5.42}$$

and anticommuting with $\psi(z)$. Then the composite field:

$$\mathcal{L}^{(1)}(z) \;:=\; \psi^+(z)\,\chi(z) \;+\; \chi^+(z)\,\psi(z) \tag{5.43}$$

has the OPE:

$$\mathcal{L}^{(1)}(z_1)\,\mathcal{L}^{(1)}(z_2) \;=\; \frac{1}{\rho_{12}^4} \;+\; \frac{1}{\rho_{12}^3}\,V_1^{(1)}(z_1,z_2) \;+\; \frac{1}{\rho_{12}^2}\,\Big\{:\chi^+(z_1)\,\not{z}_{12}^{+}\,\chi(z_2): \\ -\; :\chi^+(z_2)\,\not{z}_{12}^{+}\,\chi(z_1):\Big\} \;+\; :\mathcal{L}^{(1)}(z_1)\,\mathcal{L}^{(1)}(z_2): \;. \tag{5.44}$$

It follows then that the $V_1^{(\nu)}$ correspond to the twist two parts in the OPE of $\mathcal{L}^{(\nu)}$ ($\nu = 0, 1$). Note that the third terms in the OPE's (5.41) and (5.44) do not correspond to $\rho_{12}^{-1} V_3$ and $\rho_{12}^{-2} V_2$, respectively, since they have rational correlation functions, while the 4-point functions of V_κ for $\kappa \geq 2$ involve log terms (cf. (5.19)).

A straightforward computations show that the 4-point functions of $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ are proportional to J_0 and J_1 (resp.). For J_1 an explicit calculation is made in Appendix A. We will give also in the next section a general argument (see Proposition 6.1) that all the $2n$–point functions of $\mathcal{L}^{(\nu)}$ for $\nu = 0, 1, 2$ are reproduced, up to a multiplicative constant, by the $2n$-point function of the corresponding $V_1^{(\nu)}$ using a generalization of the symmetrization procedure of (5.25).

6. Towards constructing nontrivial GCI QFT models

6.1. The symmetrization ansatz

We observe that the twist two contribution to the $2n$-point function,

$$w_1(1,2;3,4;\ldots;2n-1,2n) \;:= \\ \Big(\prod_{i=1}^{n} \rho_{2i-1,2i}^{-3}\Big)\,\langle 0|\,V_1(z_1,z_2)\,V_1(z_3,z_4)\ldots V_1(z_{2n-1},z_{2n})\,|0\rangle\;, \tag{6.1}$$

[2] K.-H. Rehren, private communication.

combined with locality, implies the existence of higher twist terms. The question arises is there a possibility to generate the full truncated correlation functions of the model by an appropriate symmetrization of such a bilocal field contribution. The difficulty in making this idea precise is that after the permutation symmetrization of $w_1(1,2;3,4;\ldots;2n-1,2n)$ its twist two part may not be represented by the initial w_1. Indeed, we have already seen in Sect. 5.2 that for the dimensionless 4-point function $j_\nu\,(s,t)$ we should use permutation symmetrizations with different normalization (λ_ν, see Eq. (5.25)) in order to obtain the truncated (dimensionless) function P_4 whose twist two part is again j_ν. Moreover, for higher point functions it may happen that the generalized "eigenvalue" problem (5.25) does not have a solution spanning the whole space of possible twist two contributions. We shall say that $V_1\,(z,w)$ is symmetrizable if its correlation functions belong to the resulting subspace.

To be more precise, instead of the functions w_1 (6.1) we will use what may be called *truncated* $2n$-point function of bilocal fields, setting

$$\begin{aligned} w_1^t(1,2;\ldots;2n-1,2n) &= w_1(1,2;\ldots;2n-1,2n) \quad \text{for} \quad n<4\,, \\ w_1^t(1,2;\ldots;7,8) &= w_1(1,2;\ldots;7,8) - w_1(1,2;3,4)\,w_1(5,6;7,8) \\ &\quad - w_1(1,2;5,6)\,w_1(3,4;7,8) \\ &\quad - w_1(1,2;7,8)\,w_1(3,4;5,6) \end{aligned} \tag{6.2}$$

(and similar expressions involving symmetric subtractions for $n>4$). We say that V_1 is *symmetrizable* if for any $n=2,3,\ldots$, there is a λ_n such that for all $i=1,\ldots,n-1$:

$$\lim_{\rho_{2i-1,2i}\to 0} \left\{\rho_{2i-1,2i}^3(w^t(z_1,\ldots,z_{2n}) - w_1^t(1,2;\ldots;2n-1,2n))\right\} = 0\,, \tag{6.3}$$

$$w^t(1,2,\ldots,2n) = \lambda_n {\sum}' w_1^t(1,i_2;\ldots;i_{2n-1},i_{2n})\,, \tag{6.4}$$

where the sum $\sum'$ is spread over all $(2n-1)!!$ permutations $(1,2,\ldots,2n)\mapsto(1,i_2,\ldots,i_{2n})$ whose entries satisfy the inequalities

$$1\equiv i_1<i_2\,,\ \ldots\ ,\ i_{2n-1}<i_{2n} \ \text{ and } \ i_1<i_3<\cdots<i_{2n-1}. \tag{6.5}$$

The odd point truncated functions $w_1^t(1,\ldots,2n-1)$ are assumed to be zero. (This assumption will be justified for a gauge field theory Lagrangian in Sect. 6.3 below.) Eqs. (6.3) and (6.4) tell us that $w^t(1,2,\ldots,2n)$ involves the same twist two contribution for any pair of arguments.

It can be demonstrated, using the free fields realizations, that all bilocal fields $V_1^{(\nu)}$ are symmetrizable.

Proposition 6.1. *The bilocal fields $V_1^{(\nu)}(z,w)$ of the models of $\mathcal{L}^{(\nu)}(z)$, introduced in Sect. 5.3, are symmetrizable.*

Sketch of the proof. By the Wick theorem, it is obvious that the odd point correlation functions of $\mathcal{L}^{(0)}(z)$ and $\mathcal{L}^{(1)}(z)$ vanish. For $\mathcal{L}^{(2)}(z)$ this is implied by the "electric-magnetic" (or Hodge) duality (see for more details Sect. 6.3).

Observing that the fields $\mathcal{L}^{(\nu)}(z)$ and $V_1^{(\nu)}(z_1,z_2)$ have the following general structure:

$$
\begin{aligned}
\mathcal{L}^{(\nu)}(z) \; &= \; A_\nu \sum_a \Big(:\vartheta_a^*(z)\,\sigma_a(z): \, + \, :\sigma_a^*(z)\,\vartheta_a(z): \Big)\,, \\
V_1^{(\nu)}(z_1,z_2) \; &= \; B_\nu\,\rho_{12}^3 \sum_{a,\,b} \Big(\langle 0|\,\sigma_a(z_1)\,\sigma_b^*(z_2)\,|0\rangle \, :\vartheta_a^*(z_1)\,\vartheta_b(z_2): \\
&\qquad + \langle 0|\,\sigma_a^*(z_1)\,\sigma_b(z_2)\,|0\rangle \, :\vartheta_a(z_1)\,\vartheta_b^*(z_2): \Big) \\
&\big(\text{for } \nu = 0: \;\; \vartheta = \varphi,\;\; \sigma = \phi\,;\;\; \text{for } \nu = 1: \;\; \{\vartheta_a\} = \psi,\;\; \{\sigma_a\} = \chi\,; \\
&\qquad \text{for } \nu = 2: \;\; \{\vartheta_a\} = \{\sigma_a\} = \{F_{\mu\nu}\}\,\big)\,,
\end{aligned}
\tag{6.6}
$$

where A_ν and B_ν are constants, it then follows, by the Wick theorem, that the truncated $2n$–point function of $\mathcal{L}^{(\nu)}(z)$ is a sum of 1–loop contributions which is proportional to the symmetrization of the type of (6.4) of the $2n$–point function w_1^t (6.2). $\Box$

Remark 6.1. We note that the electro-magnetic Lagrangian $\mathcal{L}^{(2)}$ has an additional symmetry under the exchange of the fields $F_{\mu\nu}$ in comparison with $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$. This leads to the fact that the eigenvalues λ_n for $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ are equal and twice bigger than those for $\mathcal{L}^{(2)}$.

6.2. Elementary contributions to the truncated $2n$-point functions

The $2n$-point function of the composite field $V_1^{(1)}$ has a simple structure, verified for $n = 2, 3, 4$ (see Appendix) and conjectured for all n.

We begin by illustrating this structure for $n = 2$ and 3. First note that the general $2n$-point function of a bilocal field $V_1(z_1,z_2)$ should have an $2^n n!$–element symmetry group of permutations of the arguments. This group is the $(\mathbb{Z}_2)^{\times n} \times \mathcal{S}_n$ subgroup of $\mathcal{S}_{2n}$ consisting of exchanging the arguments of each individual $V_1(z_{2k-1}, z_{2k})$ and of permuting the V's.

The 4-point function of $V_1^{(1)}$ (presented by $j_1(s,t)$) is a sum of two terms having different *pole structure*. These are

$$
\mathcal{W}(12;34) \; = \; 2\, \frac{\rho_{13}\rho_{24} - \rho_{12}\rho_{34} - \rho_{14}\rho_{23}}{\rho_{14}^2\rho_{23}^2}
\tag{6.7}
$$

and its permutation of the arguments $\mathcal{W}(12;43)$. We will call *elementary* such contributions. For the 6-point function we have 8 different pole structures, i.e. elementary contributions, forming a single orbit under the action of the group $(\mathbb{Z}_2)^{\times 3} \times \mathcal{S}_3$. One of these structures is:

$$
\begin{aligned}
&\mathcal{W}(12;34;56) = (\rho_{16}\rho_{23}\rho_{45})^{-2}\big\{\rho_{12}(\rho_{34}\rho_{56} - \rho_{35}\rho_{46} + \rho_{36}\rho_{45}) \\
&- \rho_{13}(\rho_{24}\rho_{56} - \rho_{25}\rho_{46} + \rho_{26}\rho_{45}) + \rho_{14}(\rho_{23}\rho_{56} - \rho_{25}\rho_{36} + \rho_{26}\rho_{35}) \\
&- \rho_{15}(\rho_{23}\rho_{46} - \rho_{24}\rho_{36} + \rho_{26}\rho_{34}) + \rho_{16}(\rho_{23}\rho_{45} - \rho_{24}\rho_{35} + \rho_{25}\rho_{34})\big\}
\end{aligned}
\tag{6.8}
$$

and it has a $\mathbb{Z}_3 \times \mathbb{Z}_2$ symmetry generated by the cyclic permutation (1, 2, 3, 4, 5, 6) $\mapsto$ (3, 4, 5, 6, 1, 2) and the inversion (1, 2, 3, 4, 5, 6) $\mapsto$ (6, 5, 4, 3, 2, 1). Thus the number of the orbit's elements is indeed $8 = \dfrac{2^3 \times 3!}{2 \times 3} = 2^2 2!$.

Now we observe that the numerators of the expressions (6.7) and (6.8) have a *Wick structure* of fermionic correlation functions with a "propagator" $[12] := \rho_{12} \equiv [21]$. This observation is confirmed also for the 8-point function of $V_1^{(1)}$ and we conjecture it for all $2n$.

Multiplying each elementary contribution to the $2n$–point truncated correlation function of $V_1^{(1)}$ by the prefactor of (6.1) we obtain an elementary contribution to the correlation function of $\mathcal{L}^{(1)}$. For instance, the 6–point truncated function of $\mathcal{L}^{(1)}$ we have 120 elementary contributions organized in 15 sums of 8 element contributions of type w_1^t (6.1).

6.3. Is there a non-trivial gauge field theory model? Restrictions on the parameters in the 4-point function

We now address the question how to characterize the local gauge invariant Lagrangian, which gives rise to a 4-form

$$\mathcal{L}(z)\, dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 = \mathrm{tr}\, ({}^*F(z) \wedge F(z))\,, \tag{6.9}$$

where F is the (Maxwell, Yang-Mills) curvature 2-form and ${}^*F$ is its Hodge dual:

$$F(z) = \frac{1}{2}\, F_{\mu\nu}(z)\, dz^\mu \wedge dz^\nu\,,\ \ {}^*F(z) = \frac{1}{4}\, \varepsilon_{\kappa\lambda\mu\nu}\, F^{\kappa\lambda}\, dz^\mu \wedge dz^\nu\,, \tag{6.10}$$

without introducing gauge dependent quantities like F (in the non-abelian case) or the connection 1-form (of the gauge potential) A. We first note that a pure gauge Lagrangian of type (6.9) (*i.e.* a Lagrangian without matter fields) should not allow for a scalar of dimension (= twist) 2 in the OPE of $\mathcal{L}(z_1)\, \mathcal{L}(z_2)$. In view of (5.28) this implies

$$a_0 = 0\,. \tag{6.11}$$

Furthermore, assuming invariance of the theory under "electric-magnetic" (or Hodge) duality[3], and noting that ${}^*({}^*F) = -F$ in Minkowski space, we deduce that the theory should be invariant under a change of sign of $\mathcal{L}$. Hence all odd-point functions of $\mathcal{L}$ should vanish. We make the stronger assumption that *no* scalar field of dimension 4 should appear in the OPE of two $\mathcal{L}$'s. According to (5.30) this implies

$$c = 0\,. \tag{6.12}$$

(The vanishing of the 3-point function of the Maxwell Lagrangian is verified by a direct calculation.) We are thus left with the 3 parameters a_1, a_2, and b in the truncated 4-point function, the positivity restrictions (5.33) implying

$$a_1 \geq 0\,, \quad a_2 \geq 0\,, \quad a_1 + a_2 > 0\,, \quad -3a_1 \leq b \leq \frac{1}{3}\, a_1\,. \tag{6.13}$$

[3]We thank Dirk Kreimer for a discussion on this point.

Clearly for $a_1 = 0$ we shall also have $b = 0$ and the truncated 4-point function will be a multiple of that of the free electromagnetic Lagrangian (5.39) (Sect. 5.3). In order to go beyond the free field theory we shall assume $a_1 > 0$.

It appears that any "*minimal model*" – generated by the bilocal fields V_1 according to the symmetrization ansatz of Sect. 6.1 – corresponds to an $\mathcal{L}$ that is a sum of normal products of generalized free fields. (It would be interesting to give a proof of this conjecture.) A careful analysis shows that there is no free field realization of the 4-point function with a nonvanishing b (but $a_0 = 0 = c$). It is the resulting 3-parameter family of models that is most attractive from our point of view and deserves a systematic study.

7. Concluding remarks

Global conformal invariance [18] opens the way of constructing 4- (or higher) dimensional QFT models satisfying all Wightman axioms (except for asymptotic completeness). It also allows to construct elliptic correlation functions for finite temperature equilibrium states and to display modular properties of energy mean values.

Experience with gauge field theory suggests that the simplest local gauge invariant observable is the Lagrangian density $\mathcal{L}$. The present update of our effort to construct a non-perturbative GCI gauge QFT [15] [16] displays some new features and suggests new questions (or new ways of approaching old ones).

– We emphasize that the main tool for attacking the difficult problem of Wightman positivity are the *conformal partial wave expansions* of 4-point functions. They should be extended to 4-point functions of composite (tensor) fields or, alternatively, to higher point functions of $\mathcal{L}(z)$. OPE provide just a means to derive such expansions with invariant under rescaling structure constants (like (5.15)).

– The notion of a symmetrizable strongly bilocal field $V_1(z_1, z_2)$, which is harmonic in each argument, is introduced (in Sect. 6).

– It is demonstrated that all twist two contributions to the 4-point function can be realized as normal products of free fields.

It seems possible – and it would be worthwhile the effort of proving – that $\mathcal{L}(z)$ for a "minimal model" would be itself a sum of normal products of free fields. In this case one should concentrate on studying a theory with truncated 4–point function of $\mathcal{L}$ given by (4.6) (4.11) with $a_0 = 0 = c$ but with a nonzero b. Such a conformal model still has a chance to describe a gauge field Lagrangian that is not a part of a free field theory.

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Appendix A. Computing correlation functions of $V_1^{(1)}(z_1, z_2)$

The 4-point function $j_1(s,t)$ for $V_1^{(1)}$ given by (5.37) and the 2-point function of ψ normalized according to (2.27) is expressed as a symmetric combination of traces:

$$\langle 0|\, V_1^{(1)}(z_1,z_2)\, V_1^{(1)}(z_3,z_4)\,|0\rangle = \frac{\mathrm{tr}\{\not{z}_{12}(\not{z}_{23}^{+}\not{z}_{34}\not{z}_{14}^{+} + \not{z}_{14}^{+}\not{z}_{34}\not{z}_{23}^{+})\}}{\rho_{14}^2\,\rho_{23}^2} - \frac{\mathrm{tr}\{\not{z}_{12}(\not{z}_{24}^{+}\not{z}_{34}\not{z}_{13}^{+} + \not{z}_{13}^{+}\not{z}_{34}\not{z}_{24}^{+})\}}{\rho_{13}^2\,\rho_{24}^2}\,. \tag{A.1}$$

It is sufficient to compute the first term since the second can be obtained from it by the substitution $z_3 \rightleftarrows z_4$. To do that we shall use the following trace formula for the product of any four 4-vectors a, b, c, d written as quaternions:

$$\begin{aligned}\mathrm{tr}(\not{a}\,\not{b}^{+}\,\not{c}\,\not{d}^{+}) &= 2[(ab)(cd) - (ac)(bd) + (ad)(bc) + \det(a,b,c,d)]\,,\\ 2(ab) &= \mathrm{tr}\,\not{a}\,\not{b}^{+}\,,\end{aligned} \tag{A.2}$$

$\det(a,b,c,d)$, the determinant of the 4×4 matrix of the components of the four (column) vectors, changing sign under transposition of any two arguments. It follows that

$$\begin{aligned}&\mathrm{tr}(\not{z}_{12}\not{z}_{23}^{+}\not{z}_{34}\not{z}_{14}^{+} + \not{z}_{12}\not{z}_{14}^{+}\not{z}_{34}\not{z}_{23}^{+})\\ &= 4[(z_{12}z_{23})(z_{34}z_{14}) - (z_{12}z_{34})(z_{14}z_{23}) + (z_{12}z_{14})(z_{23}z_{34})]\,.\end{aligned} \tag{A.3}$$

To reproduce (6.7) one uses the relations

$$\begin{aligned}2\,z_{12}z_{23} = \rho_{13} - \rho_{12} - \rho_{23}\,,\ 2\,z_{34}z_{14} = \rho_{34} + \rho_{14} - \rho_{13},\ \text{etc.}\\ 2\,z_{12}z_{34} = \rho_{14} + \rho_{23} - \rho_{13} - \rho_{24}\,,\ 2\,z_{14}z_{23} = \rho_{13} + \rho_{24} - \rho_{12} - \rho_{34},\ \text{etc.}\end{aligned} \tag{A.4}$$

Similarly, the (polynomial) *elementary contribution* (6.8) to the 6-point function

$$\rho_{16}^2\rho_{23}^2\rho_{45}^2\ \langle 0|\, V_1^{(1)}(z_1,z_2)\, V_1^{(1)}(z_3,z_4)\, V_1^{(1)}(z_5,z_6)\,|0\rangle$$

is given by

$$\begin{aligned}P(12;34;56) &:= \mathrm{tr}\{\not{z}_{12}(\not{z}_{23}^{+}\not{z}_{34}\not{z}_{45}^{+}\not{z}_{56}\not{z}_{16}^{+} + \not{z}_{16}^{+}\not{z}_{56}\not{z}_{45}^{+}\not{z}_{34}\not{z}_{23}^{+})\}\\ &= 4\{(z_{12}z_{23})[(z_{34}z_{45})(z_{56}z_{16}) - (z_{34}z_{56})(z_{45}z_{16}) + (z_{34}z_{16})(z_{45}z_{56})]\\ &\quad -(z_{12}z_{34})[(z_{23}z_{45})(z_{56}z_{16}) - (z_{23}z_{56})(z_{45}z_{16}) + (z_{23}z_{16})(z_{45}z_{56})] + \ldots\\ &\quad +(z_{12}z_{16})[(z_{23}z_{39})(z_{45}z_{56}) - (z_{23}z_{45})(z_{34}z_{56}) + (z_{23}z_{56})(z_{34}z_{45})]\}\end{aligned} \tag{A.5}$$

(5×3 terms). Applying to this expression the relations of type (A.4) (and using Maple to simplify the result) we recover (6.8).

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