On the Landau Background Gauge Fixing and the IR Properties of YM Green Functions

P.A. GRASSI, T. HURTH and A. QUADRI

On the Landau Background Gauge Fixing and the IR Properties of YM Green Functions

Pietro A. Grassi[a1], Tobias Hurth[b2], Andrea Quadri[c3]

[a] C.N. Yang Institute for Theoretical Physics, SUNY at Stony Brook
Stony Brook, NY 11794-3840, USA,
Dipartimento di Scienze, Università del Piemonte Orientale,
C.so Borsalino 54, I-15100 Alessandria, ITALY, and
IHES, Le Bois-Marie, 35,
route de Chartres, F-91440 Bures-sur-Yvette, FRANCE.

[b] CERN, Theory Division, CH-1211 Geneva 23, SWITZERLAND and
SLAC, Stanford University, Stanford, CA 94309, USA

[c] Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)
Föhringer Ring, 6 - D80805 München, GERMANY

Abstract

We analyse the complete algebraic structure of the background field method for Yang–Mills theory in the Landau gauge and show several structural simplifications within this approach. In particular we present a new way to study the IR behavior of Green functions in the Landau gauge and show that there exists a unique Green function whose IR behaviour controls the IR properties of the gluon and the ghost propagators.

---

[1]pgrassi@insti.physics.sunysb.edu
[2]tobias.hurth@cern.ch
[3]quadri@mppmu.mpg.de

arXiv:hep-th/0405104 v1   11 May 2004

1 Introduction

The use of the background field method (BFM) [1] is an outstanding technique to simplify the computation of Green functions in gauge theories. This assessment should be extended beyond perturbation theory as demonstrated in numerous studies (see for example [2]); in particular, it was shown that the BFM is most suitable in quantum field theoretical computations around classical solutions of the field equations [3].

We have, however, to point out that the complete algebraic structure of the BFM in gauge theories has been scarcely used in the applications. Nevertheless this structure, encoded in the BRST formalism, has led to several important results: the proof of the equivalence of the correlation functions between physical observables computed in the BFM and in the conventional formalism [4], important simplifications in the computations of radiative corrections for the SM [5], some progresses in constructing effective charges and gauge-invariant quantities [6] and the extensions to open algebras and non-trivial manifolds [7].

In the present paper, we exploit the complete algebraic structure of the BFM for Yang–Mills theory in the Landau gauge. We recall that one can prove some non-renormalization theorems for the ghost-gluon vertex and for the ghost two-point functions in the Landau gauge. These properties are consequences of an *integrated* functional equation – *the antighost equation* – which can be shown to hold for the quantum effective action in the Landau gauge [8, 9]. Moreover, as we will show here, by combining the properties of the Landau gauge with the BFM, one can derive a more powerful *local* equation for the antighost fields. Another example of a local antighost equation was presented in [10].

In the case of the Landau gauge, the commutation relation between the integrated antighost equation and the Slavnov–Taylor identities (STI) (which implement the BRST symmetry at the functional level) yields the integrated Ward–Takahashi identities (WTI) for the $SU(N)$ rigid symmetry. In the Landau background gauge, the local antighost equation leads to local WTI. The latter are functional equations for the gauge fixed effective action and they carry all the relevant information on the original gauge symmetry of the ungauged classical theory. In the present context, the role of the WTI partly supersedes that of STI, which essentially implement only the relation between the background fields and the quantum ones.

The implementation of the BRST symmetry at the quantum level requires

the introduction of certain sources (called in the following antifields) to study the renormalization of the composite operators emerging from the BRST variation of the fields. In the same way, the BFM requires new sources to be coupled to the variation of the action under the BRST transformations of the background fields. Those sources are indeed sufficient to implement the full set of symmetries of the theory at the quantum level. Moreover, we also show that some of the Green functions obtained by differentiating the functional generators (for one-particle-irreducible (1-PI) or connected graphs) with respect to these sources can be used to study IR properties of the gluon and ghost propagator. This is an interesting feature that is not usually taken into account for practical applications.

In the paper, after working out the algebraic structure of the Landau-background gauge fixing and its symmetries, we study some applications of the formalism. Here we list them with some comments.

- During the last ten years, some effort has been devoted to study the infrared behaviour of the Green functions of QCD. Understanding the infrared properties is intimately related to grasping some information on the non-perturbative regime of the theory in the realm of confined gluons and quarks.

  One of the most used and promising techniques is lattice QCD, where the natural regularization and infrared cut-off has led to fundamental simulations to very low energies. However, other techniques have been used, such as those based on the renormalization group (RG) equations and the Schwinger–Dyson equations.

  Let us first comment on RG equations. The renewed interest in the Wilson renormalization group stimulated by a paper by Polchinski [11] and the vast literature that followed it (see for example [12]–[18] and references thereof and therein) yielded new applications of these techniques to the study of low-energy effective actions. The aim was to see emerging from the RG evolution the signals of confinements or new phases in the strong regime of QCD. However, in order to look after non-perturbative effects, one has to solve the functional equations non-perturbatively. This amounts to providing suitable truncations and relatively simple ansätze for the Green functions involved in the computations. In the Landau gauge, some of these ansätze can be explicitly implemented and justified. Here we would like to point out that in addition to the Landau gauge, the BFM might give further simplifications

and insights. To show this, we present a simple application and we leave to future publications a real non-perturbative computation in the Landau BFM.

Recently it has been shown in Ref. [19] that within the Wilson RG formalism a regularization that preserves the background gauge invariance can be found. This regularization preserves also all linear identities associated to ghost or antighost equations of motion, but it breaks the STI. However, we show that the number of breaking terms of STI in the background Landau gauge is highly reduced. Moreover, it is easy to solve the fine-tuning problem by computing those counterterms which are needed in order to reabsorb the ST breakings, thanks to the special structure of the STI in the Landau background gauge. In contrast to the usual technique with BFM in non-singular gauges, the gauge symmetry is encoded in the WTI. The present analysis can be extended to chiral gauge theories whenever the anomaly is cancelled.

- The technique developed for solving the Schwinger–Dyson equations (SDE) in the Landau gauge [20] has also been used in the past to understand the non-perturbative properties of the theory. This gauge is the most suitable to justify the truncations used and also favored due to the non-renormalization theorems. Recently, the role of the Gribov horizon in the dynamical mass generation in euclidean Yang-Mills theories has been investigated in the context of the Landau gauge in [21]. Nevertheless adding the background field method would enhance even more simplifications. As an illustration of these phenomena we rederive the non-renormalization theorems for the Landau background gauge, and in particular we show how the WTI replace the STI to uncover the gauge properties of the Green functions. We point out that the relevant properties of the IR behaviour of gluon and ghost propagators are encoded in a single Green function, obtained by differentiating the generating functional with respect to certain classical sources. As an aside we show the relation between this Green function and the Kugo–Ojima criterion for confinement [22]. Some recent developements where the Landau-Background can be used is the Maldestam's approximation to SDE [23].

- In general, the STI are complicated non-linear functional equations, which are rather difficult to handle outside of perturbation theory[4]. Instead of studying the STI, one can, in the BFM, substitute part of them with linear WTI identities for the background gauge invariance. This simplifies also the study of counterterms at higher orders. This is a drastic simplification for supersymmetric QCD where the Landau gauge has been used for explicit computations.

- Concerning the famous problem of Gribov copies in covariant gauges, the use of BFM has several advantages. We should recall that the Gribov copies are induced by the vanishing of the Faddeev–Popov determinant. However, a convenient way to overcome these difficulties is to use the technique developed in [24], where a suitable patching of the configuration space is used by changing the background field $\hat{A}_\mu$. The Landau gauge indeed has Gribov copies that may affect the computations at low energy. The use of BFM in the Landau gauge might prevent this problem also in numerical simulations where the background can be changed. In [24] the prescription to move from one patch to another is given. The practical implications will be discussed elsewhere.

- The structure of the BRST symmetry with background fields is rather suggestive. As is known, the BRST symmetry, the anti-BRST symmetry, the ghost equations and the antighost equation combine in a symmetric fashion; it can be shown that the commutation relation of the Nakanishi-Lautrup equation for the Lagrangian multiplier and the linearized ST operator or the linearized anti-ST operator leads to the ghost and antighost equations, respectively. Further commutation relations between the linearized ST operator and the linearized anti-ST operator lead to the WTI.

The paper is organized as follows: in Section 2 we present the Landau background gauge fixing, the STI and WTI for two-point functions and the non-renormalization theorem for the ghost-gluon-ghost vertex. In Section 3 we study the following applications: RG equations and the BFM, the IR

---

[4]Here we assume that a perturbative definition of BRST symmetry is possible.

behaviour of Green functions, the relation with Green functions with external classical sources, and the Kugo–Ojima criterion in the present framework. We also comment on some details of the computations. Finally conclusions are presented in Section 4.

2 Landau Background Gauge Fixing

We consider $SU(N)$ Yang–Mills theory and adopt the following conventions: $[T^a, T^b] = if^{abc}T^c$, where T^a are hermitian generators of the corresponding Lie algebra and f^{abc} are the associated real structure constants. The covariant derivative on adjoint fields Φ^a is given by $(\nabla_\mu\Phi)^a = \partial_\mu\Phi^\alpha + f^{abc}A^b_\mu\Phi^c = (\partial_\mu\Phi - i[A_\mu, \Phi])^a$ where $A_\mu = A^a_\mu T^a$ and $\Phi = \Phi^a T^a$. The T^a are normalized in such a way that $\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

We impose the Landau background gauge fixing

$$\hat{\nabla}_\mu(A - \hat{A})^\mu = 0\,, \tag{1}$$

where $\hat{\nabla}_\mu$ is the background covariant derivative, A_μ is the gauge field, and $\hat{A}_\mu$ is the background gauge field. The complete Lagrangian is obtained by the usual BRST prescription:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\text{tr}F^2 + s\,\text{tr}\left(\bar{c}\hat{\nabla}_\mu(A - \hat{A})^\mu - A^*_\mu A^\mu + c^* c\right) = \\
&= \text{tr}\left[-\frac{1}{4}F^2 + b\hat{\nabla}_\mu(A - \hat{A})^\mu - \bar{c}\left(\hat{\nabla}^\mu\nabla_\mu c - \nabla_\mu\Omega^\mu\right)\right. \\
&\qquad \left. + A^*_\mu\nabla^\mu c + ic^* c^2\right],
\end{aligned}
\tag{2}
$$

where s is the generator of the BRST transformations:

$$
\begin{aligned}
s\,A_\mu &= \nabla^\mu c\,, & s\,c &= ic^2\,, & s\,\hat{A}_\mu &= \Omega_\mu\,, & s\,\Omega_\mu &= 0\,, \\
s\,\bar{c} &= b\,, & s\,b &= 0\,, & s\,A^*_\mu &= 0\,, & s\,c^* &= 0\,.
\end{aligned}
\tag{3}
$$

The dimensions of the fields and antifields and their ghost number are summarized in Table 1.

At the classical level the theory is characterized by the STI for the action $S = \int d^4 x\,\mathcal{L}$,

$$\mathcal{S}(S) = \int d^4 x\,\text{tr}\left(\frac{\delta S}{\delta A^*_\mu}\frac{\delta S}{\delta A_\mu} + \frac{\delta S}{\delta c^*}\frac{\delta S}{\delta c} + b\frac{\delta S}{\delta \bar{c}} + \Omega_\mu\frac{\delta S}{\delta \hat{A}_\mu}\right) = 0\,, \tag{4}$$

6

	A_μ	b_μ	$\bar{c}$	c	$\hat{A}$	Ω	A_μ^*	c^*
Dim.	1	2	2	0	1	1	3	4
Gh. numb.	0	0	-1	1	0	1	-1	-2

Table 1: Ghost number and dimension of the fields and the antifields

by the equation for the Nakanishi–Lautrup field b

$$\frac{\delta S}{\delta b} = \hat{\nabla}_\mu (A - \hat{A})^\mu \tag{5}$$

and by the antighost equation

$$\frac{\delta S}{\delta c} - \hat{\nabla}_\mu \frac{\delta S}{\delta \Omega_\mu} + i \left[\bar{c}, \frac{\delta S}{\delta b} \right] - \nabla_\mu A^{*\mu} + i[c^*, c] = 0 . \tag{6}$$

The last equation is a consequence of the special gauge fixing we have adopted.

Due to the special choice of the background gauge fixing, all the composite operators obtained by differentiating the action S with respect to the ghost field c can be re-expressed in terms of functional derivatives of the action with respect to the source Ω_μ and with respect to b. This can be shown by using integration by parts. A fundamental property to be used is $[\hat{\nabla}_\mu, \nabla^\mu] = -i\hat{\nabla}_\mu (A - \hat{A})^\mu$.

Equation (6) tells us that the ghost field c will not get an independent renormalization and that, in addition, the dynamics of c is completely fixed by this equation. Notice that in the case of the Landau gauge fixing without background fields only an integrated version of (6) was derived [8].

As a consequence of equations. (4) – (6) we can obtain the following two functional equations: the Faddeev–Popov equation

$$\mathcal{S}_S \left(\frac{\delta S}{\delta b} - \hat{\nabla}_\mu (A - \hat{A})^\mu \right) - \frac{\delta}{\delta b} \mathcal{S}(S) = \mathcal{G}(S) \equiv \frac{\delta S}{\delta \bar{c}} + \hat{\nabla}_\mu \frac{\delta S}{\delta A_\mu^*} - \nabla_\mu \Omega^\mu = 0 , \tag{7}$$

where $\mathcal{S}_S$ is the linearized ST operator, defined by

$$\mathcal{S}_S = \int d^4x \, \mathrm{tr} \left(\frac{\delta S}{\delta A_\mu^*} \frac{\delta}{\delta A_\mu} + \frac{\delta S}{\delta A_\mu} \frac{\delta}{\delta A_\mu^*} + \frac{\delta S}{\delta c^*} \frac{\delta}{\delta c} + \frac{\delta S}{\delta c} \frac{\delta}{\delta c^*} + b \frac{\delta}{\delta \bar{c}} + \Omega^\mu \frac{\delta}{\delta \hat{A}_\mu} \right) , \tag{8}$$

and the WTI

$$\mathcal{S}_S \left(\frac{\delta S}{\delta c} - \hat{\nabla}_\mu \frac{\delta S}{\delta \Omega_\mu} - \nabla_\mu A^{*\mu} + i[c^*, c] + i[\bar{c}, \frac{\delta S}{\delta b}] \right) + \frac{\delta}{\delta c} \mathcal{S}(S) \equiv \mathcal{W}(S) =$$

$$-\nabla_\mu \frac{\delta S}{\delta A_\mu} - \hat{\nabla}_\mu \frac{\delta S}{\delta \hat{A}_\mu} + [A^*_\mu, \frac{\delta S}{\delta A^*_\mu}] + [c^*, \frac{\delta S}{\delta c^*}] + [c, \frac{\delta S}{\delta c}] + [\bar{c}, \frac{\delta S}{\delta \bar{c}}] +$$

$$+[\Omega_\mu, \frac{\delta S}{\delta \Omega_\mu}] + [b, \frac{\delta S}{\delta b}] = 0 \,. \tag{9}$$

In the case of ordinary Landau gauge fixing (without the background), the anticommutation relation between the antighost equation and the STI yields the integrated WTI. On the other hand in the Landau background gauge the local equation for the ghost field (6) leads directly to the local WTI for the background gauge invariance.

Moving to the quantum theory and assuming a renormalization prescription that preserves the identities (4) – (9), we substitute the tree-level action S with the 1-PI generating functional Γ.

In order to simplify the analysis of the 1-PI Green functions in the following sections, we introduce the new functional $\hat{\Gamma}$

$$\hat{\Gamma} = \Gamma - \int d^4x \, \mathrm{tr} \left(b \hat{\nabla}_\mu (A - \hat{A})^\mu + \Omega^\mu \nabla_\mu \bar{c} \right) \,. \tag{10}$$

The introduction of $\hat{\Gamma}$ simplifies equation (5), which in particular implies $\delta \hat{\Gamma}/\delta b = 0$; it also simplifies equation (7). It modifies the ST identities in (4), but it is easy to see that they have the same form, provided that we substitute A^*_μ with $\hat{A}^*_\mu = A^*_\mu + \hat{\nabla}_\mu \bar{c}$. The WTI in (9) are preserved from all manipulations, and the ghost equation (7) is automatically solved. Finally, it is convenient to introduce $\hat{\Omega}_\mu = \Omega_\mu + \hat{\nabla}_\mu c$, in order to simplify the antighost equation (6). In particular, the latter tells us that the only dependence of the functional $\hat{\Gamma}[A, \hat{A}, \hat{A}^*, \hat{\Omega}, c]$ upon the ghost field c is given by

$$\frac{\delta \hat{\Gamma}}{\delta c} = \nabla^\mu \hat{A}^*_\mu - i[c^*, c] \,, \tag{11}$$

and therefore, we can finally introduce the functional

$$\hat{\Gamma}' = \hat{\Gamma} - \int d^4x \, \mathrm{tr} \left(\hat{A}^*_\mu \nabla^\mu c + i c^* c^2 \right) \,, \tag{12}$$

8

which removes the explicit dependence upon c; $\hat{\Gamma}'$ is a functional of A_μ, $\hat{A}_\mu$, c^* and $\hat{\Omega}_\mu$ only. The complete dynamics is constrained by the remaining two functional equations:

$$\mathcal{S}(\hat{\Gamma}') = \int d^4x \, \mathrm{tr} \left(\frac{\delta\hat{\Gamma}'}{\delta\hat{A}_\mu^*} \frac{\delta\hat{\Gamma}'}{\delta A_\mu} + \hat{\Omega}_\mu \frac{\delta\hat{\Gamma}'}{\delta\hat{A}_\mu} \right) = 0 \,, \qquad (13)$$

$$\mathcal{W}(\hat{\Gamma}') = -\nabla_\mu \frac{\delta\hat{\Gamma}'}{\delta A_\mu} - \hat{\nabla}_\mu \frac{\delta\hat{\Gamma}'}{\delta\hat{A}_\mu} + [\hat{A}_\mu^*, \frac{\delta\hat{\Gamma}'}{\delta\hat{A}_\mu^*}] + [c^*, \frac{\delta\hat{\Gamma}'}{\delta c^*}] + [\hat{\Omega}_\mu, \frac{\delta\hat{\Gamma}'}{\delta\hat{\Omega}_\mu}] = 0 \,. \quad (14)$$

The structure of the gauge group is completely encoded in the WTI. The dynamics of the ghost fields is encoded in the functional dependence of $\hat{A}_\mu^*$ and $\hat{\Omega}_\mu$; therefore, the only Green functions with external fields with non-vanishing ghost number have external $\hat{A}_\mu^*, \hat{\Omega}_\mu$ or c^* fields. This is an enormous simplification, because these three fields are classical sources and not quantum fields. In order to clarify this point further, we can observe the following: non-trivial relations between Green functions can be obtained from (4) by differentiation with respect to a field carrying positive ghost number. However, the only positive charged field is $\hat{\Omega}$, so that we have to take at least one derivative with respect to this field. This yields a relation among Green functions with one background field and those Green functions with only quantum gauge fields. So, the only information encoded in (4) is the relation between background Green functions $\hat{\Gamma}'_{\hat{A}...\hat{A}}$ and those with quantum fields $\hat{\Gamma}'_{A...A}$. The rest of the information on the symmetry of the theory is encoded in equations (9) and (14), which are linear identities and thus simpler than equation (4).

Before ending this section, we just want to recall that the existence of the anti-ghost equation in the Landau background gauge can be detected in a different way. As is well-known, the BRST symmetry can be paired with an additional symmetry, the anti-BRST symmetry, where the ghost c and the antighost $\bar{c}$ are exchanged. More precisely, one defines the anti-BRST differential $\bar{s}$ by

$$\bar{s}A_\mu = \nabla_\mu \bar{c} \,, \quad \bar{s}\bar{c} = i\bar{c}^2 \quad \bar{s}c = -b - i[c, \bar{c}] \,, \quad \bar{s}b = -i[b, \bar{c}] \,. \qquad (15)$$

We will not enter in all details here (but a similar analysis has been made in [25]). We only point out that besides the STI operator $\mathcal{S}_S$ (see (8)) one

can correspondingly introduce the paired $\bar{\mathcal{S}}_S$ that generates the anti-BRST transformations. The two BRST operators anticommute and are both nilpotent.

One then finds, in complete analogy with equation (7):

$$\bar{\mathcal{S}}_S\left(\frac{\delta S}{\delta b} - \hat{\nabla}_\mu (A - \hat{A})^\mu\right) - \frac{\delta}{\delta b}\bar{\mathcal{S}}(S) = \bar{\mathcal{G}}\Gamma = 0 \tag{16}$$

where $\bar{\mathcal{G}}$ is the differential functional operator appearing in the antighost equation (6). Hence in the presence of the anti-BRST symmetry the structure of the ghost and antighost equations are direct consequences of the choice of gauge fixing. Furthermore, it is possible to recast the gauge fixing in a BRST–anti-BRST exact expression. Since the details of the derivation of such a result are rather trivial and common in the literature we do not dwell on this point any further. We just observe that the field Ω_μ is the source of $\nabla_\mu \bar{c}$. The latter is the anti-BRST transformation of the gauge field A_μ. Thus Ω_μ is the antifield of A_μ with respect to the anti-BRST symmetry. Moreover, it is natural to view A_μ^* as the anti-BRST transformation of the background field $\hat{A}_\mu$. For additional details we refer the reader to Ref. [25].

2.1 Identities for two-point functions

In this subsection, we use the Green functions generated by Γ without the redefinition discussed above. This leads to relations between the scalar functions appearing in the tensor decomposition of each single Green function. Those can be computed directly by Feynman diagrams stemming from the action (2).

Let us start from the ghost sector. There we have four two-point functions: $\Gamma_{c\bar{c}}, \Gamma_{cA_\nu^*}, \Gamma_{\Omega_\mu \bar{c}}$ and $\Gamma_{\Omega_\mu A_\nu^*}$. By power-counting they are all divergent quantities, but they are related by means of (7) and (6)

$$\begin{aligned}
\Gamma_{c\bar{c}}(p) &= ip_\nu \Gamma_{cA_\nu^*}(p)\,, \\
\Gamma_{\Omega_\mu \bar{c}}(p) &= ip_\nu \Gamma_{\Omega_\mu A_\nu^*}(p) + ip^\mu\,, \\
\Gamma_{\bar{c}c}(p) &= -ip_\mu \Gamma_{\bar{c}\Omega_\mu}(p)\,, \\
\Gamma_{A_\nu^* c}(p) &= -ip_\mu \Gamma_{A_\nu^* \Omega_\mu}(p) + ip^\nu\,.
\end{aligned} \tag{17}$$

We decompose the above Green functions in terms of scalar invariants in the

following way (we omit the colour indices unless they are strictly necessary):

$$
\begin{aligned}
\Gamma_{c\bar{c}}(p) &= -p^2 G^{-1}(p)\,, \\
\Gamma_{A^*_\nu c}(p) &= ip^\nu A(p)\,, \\
\Gamma_{\Omega_\mu \bar{c}}(p) &= ip^\mu B(p)\,, \\
\Gamma_{A^*_\nu \Omega_\mu}(p) &= -(L^{\mu\nu} C^L(p) + T^{\mu\nu} C^T(p))\,,
\end{aligned}
\tag{18}
$$

where $L^{\mu\nu} = p^\mu p^\nu / p^2$ and $T^{\mu\nu} = (g^{\mu\nu} - p^\mu p^\nu / p^2)$. The divergent part of C^T is equal to the divergent part of C^L. Equations (17) yield

$$
A(p) = B(p) = G^{-1}(p) = C^L(p) + 1\,,
\tag{19}
$$

while $C^T(p)$ remains unconstrained.

In the bosonic sector we have to study the Green functions

$$
\Gamma_{bb}(p)\,,\ \Gamma_{bA_\mu}(p)\,,\ \Gamma_{A_\mu A_\nu}(p)\,,\ \Gamma_{b\hat{A}_\mu}(p)\,,\ \Gamma_{\hat{A}_\mu A_\nu}(p)\,,\ \Gamma_{A_\mu A_\nu}(p)\,.
$$

From the Nakanishi–Lautrup equation (5) we get

$$
\Gamma_{bb}(p) = 0\,,\qquad \Gamma_{A^b_\mu b^a}(p) = i\delta^{ab} p^\mu\,,\qquad \Gamma_{\hat{A}^b_\mu b^a}(p) = -i\delta^{ab} p^\mu\,,
\tag{20}
$$

to all orders. For the two-point functions with gauge fields and their backgrounds, we adopt the following decomposition

$$
\begin{aligned}
\Gamma_{A_\mu A_\nu}(p) &= L^{\mu\nu} E^L(p) + T^{\mu\nu} E^T(p)\,, \\
\Gamma_{\hat{A}_\mu A_\nu}(p) &= L^{\mu\nu} F^L(p) + T^{\mu\nu} F^T(p)\,, \\
\Gamma_{\hat{A}_\mu \hat{A}_\nu}(p) &= L^{\mu\nu} H^L(p) + T^{\mu\nu} H^T(p)\,.
\end{aligned}
\tag{21}
$$

They have to satisfy the STI

$$
\Gamma_{\Omega_\nu A^*_\rho}(p)\Gamma_{A_\rho A_\mu}(p) + \Gamma_{\hat{A}_\nu A_\mu}(p) = 0\,,
\tag{22}
$$

$$
\Gamma_{\Omega_\nu A^*_\rho}(p)\Gamma_{A_\rho \hat{A}_\mu}(p) + \Gamma_{\hat{A}_\nu \hat{A}_\mu}(p) = 0\,,
\tag{23}
$$

and the WTI

$$
p_\mu \Gamma_{A_\mu A_\nu}(p) + p_\mu \Gamma_{\hat{A}_\mu A_\nu}(p) = 0\,,
$$

$$
p_\mu \Gamma_{\hat{A}_\mu A_\nu}(p) + p_\mu \Gamma_{\hat{A}_\mu \hat{A}_\nu}(p) = 0\,,
\tag{24}
$$

11

leading to the equations

$$H^L = -F^L = E^L \,,$$
$$H^L = -C^L F^L \,, \qquad F^L = -C^L E^L$$
$$H^T = -C^T F^T \,, \qquad F^T = -C^T E^T \,. \qquad (25)$$

From the Nakanishi–Lautrup equation (5), there follows the transversality of the gluon propagator $(\Gamma_{A_\mu A_\nu})^{-1}$. Hence $E^L = 0$ and therefore from the first of equations (25) $H^L = F^L = E^L = 0$. Moreover from equations (25) we get

$$H^T = (C^T)^2 E^T \,.$$

To conclude, we note that all relevant Green functions are completely fixed in terms of three scalar invariants C^T, E^T and C^L. The last one is given in terms of the two-point function of the ghost field and can be computed from the corresponding SDE. The two-point function E^T is the transverse contribution to the gluon propagator and can also be computed by the corresponding SDE; finally, the scalar function C^T can be computed by the two-point function of the background propagator. However, both C^L and C^T are the scalar functions appearing in the Green function $\Gamma_{\Omega_\mu A_\nu^*}(p)$; they might therefore be computed non-perturbatively by solving the corresponding SDE (which we are going to discuss in the next section) or from the RGE.

2.2 Non-renormalization of the ghost–gluon vertex $\Gamma_{c\bar{c}A_\mu}$

In order to illustrate our method, we derive the non-renormalization of the gluon-ghost vertex and other useful relations between Green functions in the same sector. We are interested in the following functions[5]:

$$\Gamma_{c\bar{c}A_\mu}(p,q) \,, \; \Gamma_{\Omega_\nu \bar{c}A_\mu}(p,q) \,, \; \Gamma_{cA_\rho^* A_\mu}(p,q) \,, \; \Gamma_{\Omega_\nu A_\rho^* A_\mu}(p,q) \,.$$

They satisfy the identities

$$\Gamma_{c^a \bar{c}^b A_\mu^c}(p,q) = -i(p+q)_\nu \Gamma_{\Omega_\nu^a \bar{c}^b A_\mu^c}(p,q) + f^{abd}\Gamma_{b^d A_\mu^c}(q) \,,$$
$$\Gamma_{c\bar{c}A_\mu}(p,q) = ip_\rho \Gamma_{cA_\rho^* A_\mu}(p,q)$$
$$\Gamma_{\Omega_\nu^a \bar{c}^b A_\mu^c}(p,q) = ip_\rho \Gamma_{\Omega_\nu^a A_\rho^{*,b} A_\mu^c}(p,q) + f^{abc}g_{\mu\nu} \,,$$
$$\Gamma_{c^a A_\nu^{*,b} A_\mu^c}(p,q) = -i(p+q)_\rho \Gamma_{\Omega_\rho^a A_\nu^{*,b} A_\mu^c}(p,q) + f^{abc}g_{\mu\nu} \,. \qquad (26)$$

_____

[5]We use the notation $\Gamma_{\Phi_1 \ldots \Phi_{n-1}\Phi_n}(p_2, \ldots p_{n-1}, p_n)$ where p_i is the ingoing momentum of the field Φ_i. For the field Φ_1 we always assume the momentum conservation $p_1 = -\sum_{i\neq 1} p_i$ and we omit it in the notation.

By power counting the Green function $\Gamma_{\Omega_\nu^a A_\rho^{*,b} A_\mu^c}(p,q)$ is finite (we recall that Ω_μ^a has dimension 1, the gauge field A_ν^a has dimension 1 and the antifield $A_\rho^{*,a}$ has dimension 3). By equation (26) this implies that also the Green functions $\Gamma_{\Omega_\nu \bar{c} A_\mu}(p,q)$ and $\Gamma_{cA_\rho^* A_\mu}(p,q)$ are finite. Hence we conclude that the ghost-gluon vertex $\Gamma_{c\bar{c}A_\mu}(p,q)$ is also finite. This is the well-known non-renormalization property of the ghost-gluon vertex in the Landau gauge.

In addition, we can also prove that the ghost-background gluon vertex is finite in the Landau background gauge. This amounts to deriving the above equations for the Green functions

$$\Gamma_{c\bar{c}\hat{A}_\mu}(p,q)\,,\ \Gamma_{\Omega_\nu \bar{c}\hat{A}_\mu}(p,q)\,,\ \Gamma_{cA_\rho^*\hat{A}_\mu}(p,q)\,,\ \Gamma_{\Omega_\nu A_\rho^*\hat{A}_\mu}(p,q)\,.$$

and repeating the same argument.

Furthermore, some useful identities are worth mentioning among the background and quantum Green functions. Some of them are obtained from the STI (4) and are given (in condensed notation) by

$$\begin{aligned}
\Gamma_{c\bar{c}\hat{A}_\mu} &= -\Gamma_{cA_\rho^*}\Gamma_{A_\rho\bar{c}\Omega_\mu} - \Gamma_{\Omega_\mu A_\rho^*}\Gamma_{A_\rho c\bar{c}} + \Gamma_{c\Omega_\mu c^*}\Gamma_{c\bar{c}}\,,\\
\Gamma_{cA_\nu^*\hat{A}_\mu} &= -\Gamma_{cA_\rho^*}\Gamma_{A_\rho A_\nu^*\Omega_\mu} - \Gamma_{\Omega_\mu A_\rho^*}\Gamma_{A_\rho cA_\nu^*} + \Gamma_{c\Omega_\mu c^*}\Gamma_{A_\nu^* c}\,,\\
\Gamma_{\Omega_\nu \bar{c}\hat{A}_\mu} &= -\Gamma_{\Omega_{[\nu} A_\rho^*}\Gamma_{A_\rho\bar{c}\Omega_{\mu]}} + \Gamma_{\Omega_\nu\Omega_\mu c^*}\Gamma_{\bar{c}c}\,,\\
\Gamma_{\Omega_\nu A_\sigma^*\hat{A}_\mu} &= -\Gamma_{\Omega_{[\nu} A_\rho^*}\Gamma_{A_\rho A_\sigma^*\Omega_{\mu]}} + \Gamma_{\Omega_\nu\Omega_\mu c^*}\Gamma_{A_\sigma^* c}
\end{aligned} \qquad (27)$$

and from the WTI (9) with the following form (again in condensed notation):

$$\begin{aligned}
-i(q+p)_\mu(\Gamma_{A_\mu c\bar{c}} + \Gamma_{\hat{A}_\mu c\bar{c}}) + (\Gamma_{c\bar{c}}(p) - \Gamma_{c\bar{c}}(q)) &= 0\,,\\
-i(q+p)_\mu(\Gamma_{A_\mu\Omega_\nu\bar{c}} + \Gamma_{\hat{A}_\mu\Omega_\nu\bar{c}}) + (\Gamma_{\Omega_\nu\bar{c}}(p) - \Gamma_{\Omega_\nu\bar{c}}(q)) &= 0\,,\\
-i(q+p)_\mu(\Gamma_{A_\mu cA_\nu^*} + \Gamma_{\hat{A}_\mu cA_\nu^*}) + \big(\Gamma_{cA_\nu^*}(p) - \Gamma_{cA_\nu^*}(q)\big) &= 0\,,\\
-i(q+p)_\mu(\Gamma_{A_\mu\Omega_\rho A_\nu^*} + \Gamma_{\hat{A}_\mu\Omega_\rho A_\nu^*}) + \big(\Gamma_{\Omega_\rho A_\nu^*}(p) - \Gamma_{\Omega_\rho A_\nu^*}(q)\big) &= 0\,. \quad (28)
\end{aligned}$$

The last terms in the R.H.S. of equation (27) are given by superficially convergent functions involving one c^* and at least one Ω_μ times two-point functions $\Gamma_{\bar{c}c}$ or $\Gamma_{A_\nu^* c}$. These two-point functions are in turn characterized by the parametrization given in equation (18), fulfilling the constraints in equation (19).

These equations show that the trilinear background Green functions with the insertion of an antighost or an antifield A^* are fixed in terms of the quantum Green functions (and vice versa). In particular the above relations constrain the form of the ghost-gluon vertex.

3   Applications

In this section we review some applications of the Landau gauge fixing in the study of the IR properties of Yang–Mills theory and discuss the simplifications arising from the background Landau gauge and its local antighost equation on some specific issues.

3.1   Renormalization Group Technique and BFM

Recently, Bonini et al. discussed in [19] the application of the BFM to the Wilson renormalization group technique. We can list two important advantages of that application: *i)* the WTI (9) are preserved by the regularization technique and they do not have to be restored order by order, and *ii)* one can provide a mass term for the quantum gauge field without destroying the background gauge invariance (whereas the STI will be dramatically broken by this choice). These important features can be used to pursue the computation of the effective action beyond one loop in the present context. However, in order to construct the full effective action one has to take into account the renormalization of all other identities such as (6), (5), (7) and (4). For what concerns the linear identities, it is easy to check that they are indeed preserved by the regularization procedure and therefore the algebraic manipulations performed in Section 2 remain valid beyond tree level. If one of these equations is broken, an explicit counterterm given by Sorella and Piguet [9] (as a function of the regularization-dependent breaking terms) is used to restore it order by order, and therefore we assume that all equations but (4) are preserved by the regularization procedure.

The Wilson–Polchinski regularization method introduces explicit breaking of the STI in the form

$$\mathcal{S}(\hat{\Gamma}') = \Delta \cdot \hat{\Gamma}', \tag{29}$$

where the R.H.S. denotes the insertion of the breaking term Δ into the Green functions generated by $\hat{\Gamma}'$. The functional dependence of Δ is restricted by the equations (5) – (9) to depend only on background gauge-invariant combinations of $A_\mu, \hat{A}_\mu, \hat{\Omega}_\mu$ and c^*. In addition, the local approximation of the breaking term Δ is restricted by power counting to be a sum of local operators with dimension 4. On general grounds, there are additional breakings in the R.H.S. of equation (29), but they are given by irrelevant operators vanishing in the physical limit of infinite UV cut-off. It can be proved (see

[12] for a complete analysis within gauge theories) that at the physical fixed point the STI are indeed restored, provided that the breaking terms of dimension 4 associated with the local approximation of Δ have been removed order by order in the loop expansion.

The most general polynomial Δ with dimension 4 and ghost-number 1 is

$$\Delta_4^1 = \int d^4x \, \hat{\Omega}_\mu F^\mu(A, \hat{A}) \tag{30}$$

where F is a function of A_μ and of $\hat{A}_\mu$ only. The function F should have dimension 3. Since we can use integration by parts, we assume that $\hat{\Omega}_\mu$ in the first term is undifferentiated.

Notice that there is no room in Δ_4^1 for the representative of the ABJ anomaly. However, this is not surprising since if this anomaly appears, it occurs in (9) and (29) with the same coefficient. Therefore, if we assume that the (9) are preserved by the regularization, we also assume that there is no anomaly in (29). Δ_4^1 is constrained to obey the following Wess–Zumino consistency condition

$$\mathcal{S}_{\hat{\Gamma}'^{(0)}}(\Delta_4^1) = 0 \,, \tag{31}$$

where

$$\mathcal{S}_{\hat{\Gamma}'^{(0)}} = \int d^4x \, \mathrm{tr} \left(\frac{\delta \hat{\Gamma}'^{(0)}}{\delta \hat{A}_\mu^*} \frac{\delta}{\delta A_\mu} + \frac{\delta \hat{\Gamma}'^{(0)}}{\delta A_\mu} \frac{\delta}{\delta \hat{A}_\mu^*} + \hat{\Omega}_\mu \frac{\delta}{\delta \hat{A}_\mu} \right) \tag{32}$$

and $\hat{\Gamma}'^{(0)}$ is the coefficient of order zero in the $\hbar$-expansion of the functional $\hat{\Gamma}'$ (i.e. its classical approximation). Furthermore, $\mathcal{S}_{\hat{\Gamma}'^{(0)}}$ is nilpotent.

The variables $(\hat{A}_\mu, \hat{\Omega}_\mu)$ have special transformation properties with respect to $\mathcal{S}_{\hat{\Gamma}'^{(0)}}$. They form a set of coupled doublets, i.e.

$$\mathcal{S}_{\hat{\Gamma}'^{(0)}}(\hat{A}_\mu) = \hat{\Omega}_\mu \,, \qquad \mathcal{S}_{\hat{\Gamma}'^{(0)}}(\hat{\Omega}_\mu) = 0 \,, \tag{33}$$

while the counting operator

$$\mathcal{N} = \int d^4x \, \mathrm{tr} \left(\hat{A}_\mu \frac{\delta}{\delta A_\mu} + \hat{\Omega}_\mu \frac{\delta}{\delta \Omega_\mu} \right) \tag{34}$$

does not commute with $\mathcal{S}_{\hat{\Gamma}'^{(0)}}$. The cohomological properties of doublet pairs have been widely discussed in the literature [9, 26, 27, 28]. In particular,

coupled doublets have been analysed in [29]. It turns out that if Y is a functional belonging to the kernel of $\mathcal{S}_{\hat{\Gamma}'(0)}$ and that $Y|_{\hat{\Omega}=\hat{A}=0} = 0$, then it can be expressed as a $\mathcal{S}_{\hat{\Gamma}'(0)}$-variation of some suitable functional X, so that $Y = \mathcal{S}_{\hat{\Gamma}'(0)}(X)$. Since Δ_4^1 is in the kernel of $\mathcal{S}_{\hat{\Gamma}'(0)}$ and vanishes at $\hat{\Omega} = 0$, it can be written as

$$\Delta_4^1 = \mathcal{S}_{\hat{\Gamma}'(0)}(\Xi[A, \hat{A}]) \tag{35}$$

for some local counterterm $\Xi[A, \hat{A}]$ of dimension 4, which reabsorbs the breaking. As a consequence of the local antighost equation the cohomology associated to the STI has been trivialized. This property is a distinctive feature of the Landau BFM gauge. It does not hold in the ordinary Landau gauge. A detailed analysis of the structure of the counterterms in Yang–Mills theory in the ordinary Landau gauge has been given in [30].

The background gauge invariance also imposes restrictions on the number of independent polynomials that can appear in F_μ. Indeed by imposing background gauge invariance we find that F_μ must have the form (here we use the notation $Q_\mu = A_\mu - \hat{A}_\mu$ for simplicity)

$$\begin{aligned}
F_d^\mu &= m_{dabc,\nu\rho\sigma}^\mu Q_\nu^a Q_\rho^b Q_\sigma^c + n_{dab,\nu\rho\sigma}^\mu Q_\nu^a (\hat{\nabla}_\rho Q_\sigma)^b + p_{dab,\nu\rho\sigma}^\mu Q_\nu^a \hat{F}_{\rho\sigma}^b + \\
&+ p_{da,\nu\rho\sigma}^\mu \hat{\nabla}_{(\nu} \hat{\nabla}_{\rho)} Q_\sigma^a + r_{da,\nu\rho\sigma}^\mu (\hat{\nabla}_\nu \hat{F}_{\rho\sigma})^a \,,
\end{aligned} \tag{36}$$

where m, n, p, q and r are constant Lorentz- and Lie algebra tensors (independent of the fields and their derivatives). They have to be computed directly from the Feynman diagrams in a perturbative expansion or in terms of the effective action for non-perturbative evaluation. They depend on the regulator and they are finite quantities at the fixed point. Once F_d^μ is known, the functional Ξ in equation (35) can be explicitly obtained by making use of the inversion formulae given in [29].

The BFM of the Landau gauge permits such enormous simplifications to all orders, reducing the effort to restore the STI. Notice that in the absence of a good regularization procedure (for example for chiral gauge theory on the continuum), one can always find the regularization that preserves the WTI identities and few breaking terms of the STI have to be computed order by order.

We wish to conclude this section by stressing that the combination of Landau gauge fixing, covariantized with respect to the background fields, provides a way of significantly simplifying the renormalization of the STI by

trivializing the cohomology of the relevant linearized ST operator in equation (32). In the presence of a background gauge-invariant regularization procedure, such as the regularization adopted in [19], the number of breaking terms (and consequently the number of counterterms) of the STI is henceforth highly reduced, resulting in a much simpler construction of the quantum effective action.

3.2 Fixed points and relation between the IR behaviour of $E^T(p)$ and $C^L(p)$

Following Ref. [31], we derive the relation between the two-point functions $E^T(p)$ and $C^L(p)$ by assuming only the existence of a fixed point. The present derivation is based on the properties of the Landau-BFM.

From the STI (4), by differentiation with respect to A_μ, A_ν and c, we have

$$\Gamma_{cA^{*\rho}}(-p-q)\Gamma_{A_\rho A_\mu A_\nu}(p,q) +$$
$$+\Gamma_{cA^{*\rho}A_\mu}(-q,p)\Gamma_{A_\rho A_\nu}(q) + \Gamma_{cA^{*\rho}A_\nu}(-p,q)\Gamma_{A_\rho A_\mu}(p) = 0. \quad (37)$$

The Green function $\Gamma_{cA^{*\rho}A_\mu}(-q,p)$ is finite as shown in the previous section. By using the transversality of $\Gamma_{A_\rho A_\nu}(q)$ and the Lorentz decomposition of $\Gamma_{cA^{*\rho}}$ given in (18), we get

$$i(C^L(-p-q)+1)\left[(-p-q)_\rho\Gamma_{A_\rho A_\mu A_\nu}(p,q)\right] +$$
$$+ \left[\Gamma_{cA^{*\rho}A_\mu}(-q,p)T_{\rho\nu}(q)\right]E^T(q) + \left[\Gamma_{cA^{*\rho}A_\nu}(-p,q)T_{\rho\mu}(p)\right]E^T(p) = 0. \quad (38)$$

By contracting the above equation with the tensor $p^\mu p^\nu/p^2$, the third term vanishes because it is transverse, and we have

$$(C^L(-p-q;\mu)+1)S(p,q;\mu) + R(p,q;\mu)E^T(q;\mu) = 0, \quad (39)$$

where we have defined the following scalar functions

$$S(p,q;\mu) = \frac{i}{p^2}\left[(-p-q)_\rho\Gamma_{A_\rho A_\mu A_\nu}(p,q;\mu)p^\mu p^\nu\right]$$
$$R(p,q;\mu) = \frac{1}{p^2}\left[\Gamma_{cA^{*\rho}A_\mu}(-q,p;\mu)T_{\rho\nu}(q)p^\mu p^\nu\right] \quad (40)$$

and μ is the renormalization scale. Since the function $S(p,q;\mu)$ is finite, its variation under a variation of the renormalization scale μ can be given entirely in terms of the functions $\partial_\mu E^T(p;\mu)$ and $\partial_\mu C^L(p;\mu)$:

$$\partial_\mu S(\mu^*;\mu) = X_E(\mu^*;\mu)\partial_\mu E^T(\mu^*;\mu) + X_C(\mu^*;\mu)\partial_\mu C^L(\mu^*;\mu) \quad (41)$$

17

computed at the symmetric point $p^2 = q^2 = (\mu^*)^2$. Assuming the existence of a fixed point for the three-point function $R(\mu^*; \mu) = 0$ at a given scale μ^* we find that

$$\partial_\mu E^T(\mu^*; \mu) Y_E + \partial_\mu C^L(\mu^*; \mu) Y_C = 0 \,, \tag{42}$$

which relates the μ-dependence of the two-point functions. The functions Y_E and Y_C are finite and completely fixed by the normalization conditions. The existence of a fixed point for the three point function $R(\mu^*; \mu) = 0$ can be deduced from the existence of a fixed point for the running of beta function for the gauge coupling. The difference between the two fixed points amounts to a change of the renormalization prescriptions for the two-point functions.

3.3 Kugo–Ojima criterion on confinement using BFM

A thorough discussion of the Kugo–Ojima (KO) criterion on confinement is beyond the scope of this note and we refer the reader to the original literature [22]. However, before analysing the criterion within the Landau BFM, some critical remarks are in order. First of all, one should mention that the KO criterion is established within the asymptotic Fock space and is therefore intimately related with perturbation theory. Moreover, in contrast to the gauge-invariant Wilson-loop criterium of confinement, the KO criterion is established in the gauge-fixed theory and has to face the serious and well-known problem of Gribov copies.

Keeping these problems in mind, we now translate the KO criterion for the confinement into the context of the Landau background gauge. The advantage of this framework is that all composite operators needed in the present analysis are already present in the Lagrangian (2) coupled to the sources Ω_μ^a and $A_\mu^{*,a}$.

First we compute the Noether current for the $SU(N)$ invariance of the action (2). Simple algebraic manipulations lead to

$$-iJ_\mu = [A^\nu, F_{\mu\nu}] + [b, (A - \hat{A})_\mu] + [\bar{c}, \Omega_\mu] - [\bar{c}, \nabla_\mu c] - [\hat{\nabla}_\mu \bar{c}, c] \,, \tag{43}$$

which is conserved (using the equations of motions for the classical fields).

If we compute the equation of motion for the gauge field A_μ, we obtain a

18

related expression

$$
\begin{aligned}
\frac{\delta S}{\delta A_\mu} &= \nabla^\nu F_{\nu\mu} - \partial_\mu b - i[b, \hat{A}_\mu] + i[\nabla \bar{c}, c] + i[\bar{c}, \Omega] \\
&= \partial^\nu F_{\nu\mu} - J_\mu - s\left[\nabla_\mu \bar{c}\right] \\
&= \partial^\nu F_{\nu\mu} - J_\mu + \frac{\delta S}{\delta \hat{A}_\mu},
\end{aligned}
\tag{44}
$$

where we used in the last line the property that, in terms of A_μ and $\hat{A}_\mu$, the Lagrangian (2) depends on the background only through the gauge fixing.

We neglect the contribution of the antifields since they drop out from the computation of the previous current on the Fock space of on-shell states. In addition, for on-shell states one knows that the L.H.S. of (44) vanishes. So the KO charges can be defined by

$$
G = \int d^3x\, \partial^i F_{i0}, \qquad N = \int d^3x\, \frac{\delta S}{\delta \hat{A}_0}, \qquad Q = \int d^3x\, J_0,
\tag{45}
$$

which are related by $Q = N + G$. The KO criterion asserts that if there is a mass gap in the gluon propagator $\lim_{p^2 \to 0} p^2 E^T(p) = 0$, the first charge G vanishes because it is a total derivative. The second condition imposes that also N should vanish, so that Q is well-defined and all coloured states vanish on physical states. Notice that the second condition can expressed by studying the correlation function between the current $\delta S/\delta \hat{A}_\mu$ and a one-particle state with a gluon field, namely it can be related to the two-point function $\delta^2 S/\delta \hat{A}_\mu(x) A_\nu(y)$ of the background and quantum gauge field.

3.4 The correlation function $\langle T \nabla^\mu c(x) \nabla^\nu \bar{c}(y) \rangle$

The KO criterion can be reformulated in terms of the IR behaviour of a special correlation function involving the ghost fields. More precisely, one defines the matrix $u^{ab}(p^2)$ as

$$
\int d^4x\, e^{ip(x-y)} \langle (\nabla_\mu c)^a(x) (f^{bcd} A_\nu^c \bar{c}^d)(y) \rangle =:_{p \to 0} T_{\mu\nu} u^{ab}(p^2),
\tag{46}
$$

where the equality is valid in the limit of small momenta (i.e. one discards possible contributions from massive states to the correlation function in the
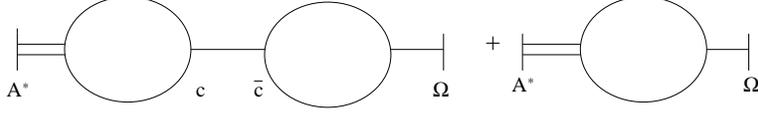
19

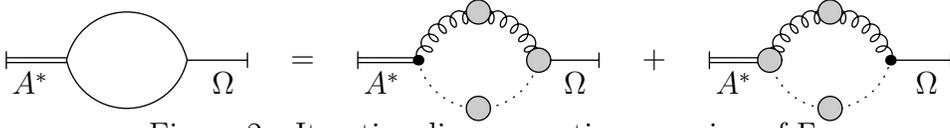Figure 1: Contribution to $\mathcal{G}_{\mu\nu}$ from its connected components

Figure 2: Iterative diagrammatic expansion of $\Gamma_{A^*\Omega}$.

L.H.S. of equation (46)). Then, according to Kugo and Ojima, the charge N is well defined provided that $u^{ab}(p^2)$ fulfills the condition [20]

$$u^{ab}(0) = -\delta^{ab} . \tag{47}$$

In the Landau background gauge, the Green function $\langle T\nabla^\mu c(x)\nabla^\nu \bar{c}(y)\rangle$ can be generated by functional differentiation with respect to the antifield A^*_μ and the background ghost Ω_ν:

$$\mathcal{G}_{\mu\nu}(x-y) = \langle T\nabla^\mu c(x)\nabla^\nu \bar{c}(y)\rangle^C = \left.\frac{\delta^2 W}{\delta A^*_\mu(x)\delta\Omega_\nu(y)}\right|_{J=\beta=0} . \tag{48}$$

In the Landau background field gauge many simplifications arise for the computation of $\mathcal{G}_{\mu\nu}$. First we observe that there are two contributions to $\mathcal{G}_{\mu\nu}$ coming from the connected graphs, which are shown in Figure 1.

Analytically we obtain, in the momentum space, using the parameterization in equation (18):

$$
\begin{aligned}
-\mathcal{G}_{\mu\nu}(p) &= -p_\mu A(p)\frac{G(p)}{p^2}p_\nu B(p) + L_{\mu\nu}C^L(p) + T_{\mu\nu}C^T(p) \\
&= -G^{-1}(p)L_{\mu\nu} + (G^{-1}(p) - 1)L_{\mu\nu} + C^T(p)T_{\mu\nu} \\
&= -L_{\mu\nu} + C^T(p)T_{\mu\nu} .
\end{aligned}
\tag{49}
$$

In the above equation we have used the simplifications coming from equation (19). This means that in the Landau background gauge the longitudinal

20

part of $\mathcal{G}_{\mu\nu}$ does not get renormalized and the only dynamical information is contained in $C^T(p)$. Moreover, this parameter can be extracted by looking at the 1-PI Green function $\Gamma_{\Omega_\mu A_\nu^*}$:

$$C^T(p) = -T^{\mu\nu}\Gamma_{\Omega_\mu A_\nu^*}. \tag{50}$$

By comparing equation (46) with equation (48) we arrive at the following identification:

$$C^T(p^2) = -u(p^2). \tag{51}$$

Therefore the full information contained in the matrix u can actually be derived from the knowledge of $\Gamma_{\Omega_\mu A_\nu^*}$ via equation (50).

In order to compute the 1-PI Green function $\Gamma_{A^*\Omega}$, we propose an iterative diagrammatic prescription motivated by the perturbative treatment and by some analogy with the SDE. Graphically we can represent it as in Figure 2. Notice that $\Gamma_{A^*\Omega}^{(0)} = 0$. Then we can write

$$\Gamma_{A_\mu^{a*}(-p)\Omega_\nu^b(p)} = \int d^4q \Gamma_{A_\mu^{*a}(-p)A_\nu^j(q)c^c(p+q)}^{(0)} D_{ji}^{\nu\rho}(q) \Gamma_{A_\rho^i(q)\Omega_\nu^b(p)\bar{c}^k(p+q)} D_{c\bar{c}}^{kc}(p+q) +$$

$$\int d^4q \Gamma_{A_\mu^{*a}(-p)A_\nu^j(q)c^c(p+q)} D_{ji}^{\nu\rho}(q) \Gamma_{A_\rho^i(q)\Omega_\nu^b(p)\bar{c}^k(p+q)}^{(0)} D_{c\bar{c}}^{kc}(p+q). \tag{52}$$

Here, $D_{ji}^{\nu\rho}$ and $D_{c\bar{c}}^{kc}$ stand for the full propagators of the gluons and the ghosts, $\Gamma_{A_\mu^{*a}A_\nu^j c^c}^{(0)}$ is the antifield-gluon-ghost tree-level vertex and $\Gamma_{A_\mu^{*a}A_\nu^j c^c}$ the full renormalized one, and analogously for $\Gamma_{A_\rho^i\Omega_\nu^b\bar{c}^k}^{(0)}$ and $\Gamma_{A_\rho^i\Omega_\nu^b\bar{c}^k}$.

Equation (52) could be used as a starting point to derive, by iterative solutions, the information on $C^T(p)$ in equation (50), possibly also beyond standard perturbation theory.

4   Conclusion and Outlook

In the present paper, the complete algebraic structure of the BRST symmetry for Landau background gauge fixing is exploited. We recover all the results presented in the literature for the non-renormalization of the trilinear ghost-gluon vertex and different simplifications provided by the BFM. In addition, we show that all the information regarding the IR behaviour of the theory

21

is encoded in a new Green function $\Gamma_{\Omega_\mu A_\nu^*}(p)$ which appears naturally in the Landau background gauge. We provide a prescription to compute this new Green function to all orders in perturbation theory and beyond the perturbative analysis.

It may be hoped that the present formalism could enhance the distinctive simplifications of the Landau gauge, which is widely used in the study of the IR properties of Green functions for QCD and in the strong regime of gauge theories.

Acknowledgements

References

[1] H. Kluberg-Stern and J. B. Zuber, "Ward identities and some clues to the renormalization of gauge invariant operators," Phys. Rev. D **12**, 467 (1975);
H. Kluberg-Stern and J. B. Zuber, "Renormalization of nonabelian gauge theories in a background field gauge: 1. Green functions," Phys. Rev. D **12**, 482 (1975);
L. F. Abbott, "The background field method beyond one loop," Nucl. Phys. B **185**, 189 (1981);
S. Ichinose and M. Omote, "Renormalization using the background field method," Nucl. Phys. B **203**, 221 (1982);
D. M. Capper and A. MacLean, "The background field method at two loops: a general gauge Yang-Mills calculation," Nucl. Phys. B **203**, 413 (1982);
A. Denner, G. Weiglein and S. Dittmaier, "Application of the background field method to the electroweak standard model," Nucl. Phys. B **440**, 95 (1995) [arXiv:hep-ph/9410338];
P. A. Grassi, "Stability and renormalization of Yang-Mills theory with

background field method: a regularization independent proof," Nucl. Phys. B **462**, 524 (1996) [arXiv:hep-th/9505101];
P. A. Grassi, "Renormalization of non-semisimple gauge models with the background field method," Nucl. Phys. B **560**, 499 (1999) [arXiv:hep-th/9908188].

[2] C. R. Stephens, "Nonperturbative background field calculations," Annals Phys. **181**, 120 (1988);
M. Luscher and P. Weisz, "Background field technique and renormalization in lattice gauge theory," Nucl. Phys. B **452**, 213 (1995) [arXiv:hep-lat/9504006];
S. Capitani, "Lattice perturbation theory," Phys. Rept. **382**, 113 (2003) [arXiv:hep-lat/0211036];
J. P. Bornsen and A. E. M. van de Ven, "Three-loop Yang-Mills beta-function via the covariant background field method," Nucl. Phys. B **657**, 257 (2003) [arXiv:hep-th/0211246];
H. D. Trottier, "Higher-order perturbation theory for highly-improved actions," Nucl. Phys. Proc. Suppl. **129-130**, 142 (2004) [arXiv:hep-lat/0310044].

[3] G. 't Hooft, "Computation of the quantum effects due to a four-dimensional pseudoparticle," Phys. Rev. D **14**, 3432 (1976) [Erratum-ibid. D **18**, 2199 (1978)];
D. Amati and A. Rouet, "Renormalization Of Yang-Mills Theory Developed Around An Instanton," Nuovo Cim. A **50**, 265 (1979);
L. G. Yaffe, "Quantizing gauge theories: nonclassical field configurations, broken symmetries, and gauge copies," Nucl. Phys. B **151**, 247 (1979);
V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "Exact Gell-Mann-Low function of supersymmetric Yang-Mills theories from instanton calculus," Nucl. Phys. B **229**, 381 (1983);
T. R. Morris, D. A. Ross and C. T. Sachrajda, "Higher order quantum corrections in the presence of an instanton background field," Nucl. Phys. B **255**, 115 (1985);
A. S. Goldhaber, A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "Quantum corrections to mass and central charge of supersymmetric solitons," arXiv:hep-th/0401152.

[4] L. F. Abbott, M. T. Grisaru and R. K. Schaefer, "The background field method and the S-matrix," Nucl. Phys. B **229**, 372 (1983);
C. Becchi and R. Collina, "Further comments on the background field method and gauge invariant effective actions," Nucl. Phys. B **562**, 412 (1999) [arXiv:hep-th/9907092];
R. Ferrari, M. Picariello and A. Quadri, "Algebraic aspects of the background field method," Annals Phys. **294**, 165 (2001) [arXiv:hep-th/0012090].

[5] P. A. Grassi, T. Hurth and M. Steinhauser, "Practical algebraic renormalization," Annals Phys. **288**, 197 (2001) [arXiv:hep-ph/9907426];
P. A. Grassi, T. Hurth and M. Steinhauser, "The algebraic method," Nucl. Phys. B **610**, 215 (2001) [arXiv:hep-ph/0102005];
P. A. Grassi, T. Hurth and M. Steinhauser, "Non-invariant two-loop counterterms for the three-gauge-boson vertices," JHEP **0011**, 037 (2000) [arXiv:hep-ph/0011067];
P. A. Grassi and T. Hurth, "On the two-loop electroweak amplitude of the muon decay," in *Proc. of the 5th International Symposium on Radiative Corrections (RADCOR 2000)* ed. Howard E. Haber, arXiv:hep-ph/0101183.

[6] J. Papavassiliou, "On the connection between the pinch technique and the background field method," Phys. Rev. D **51**, 856 (1995) [arXiv:hep-ph/9410385];
R. Haussling, E. Kraus and K. Sibold, "Gauge parameter dependence in the background field gauge and the construction of an invariant charge," Nucl. Phys. B **539**, 691 (1999) [arXiv:hep-th/9807088];
D. Binosi and J. Papavassiliou, "Pinch technique and the Batalin-Vilkovisky formalism," Phys. Rev. D **66**, 025024 (2002) [arXiv:hep-ph/0204128];
D. Binosi and J. Papavassiliou, "The pinch technique to all orders," Phys. Rev. D **66**, 111901 (2002) [arXiv:hep-ph/0208189].

[7] P. A. Grassi, T. Hurth and A. Quadri, "Super background field method for N = 2 SYM," JHEP **0307**, 008 (2003) [arXiv:hep-th/0305220].

[8] A. Blasi, O. Piguet and S. P. Sorella, "Landau gauge and finiteness," Nucl. Phys. B **356**, 154 (1991);

N. Maggiore and S. P. Sorella, "Finiteness of the topological models in the Landau gauge," Nucl. Phys. B **377**, 236 (1992).

[9] O. Piguet and S.P. Sorella, "Algebraic renormalization", Lectures Notes Monographs, Springer-Verlag, Berlin/Heidelberg,1995, p.28.

[10] P. A. Grassi, "The Abelian anti-ghost equation for the standard model in the 't Hooft background gauge," Nucl. Phys. B **537**, 527 (1999) [arXiv:hep-th/9804013].

[11] J. Polchinski, "Renormalization and effective Lagrangians," Nucl. Phys. B **231**, 269 (1984).

[12] C.M. Becchi, "On the construction of renormalized gauge theories using renormalization group techniques," Lecture given at the Parma Theoretical Physics Seminar, 1991, arXiv:hep-th/9607188.

[13] C. Wetterich, "Average action and the renormalization group equations," Nucl. Phys. B **352**, 529 (1991);
M. Reuter and C. Wetterich, "Effective average action for gauge theories and exact evolution equations," Nucl. Phys. B **417**, 181 (1994);
M. Reuter and C. Wetterich, "Exact evolution equation for scalar electrodynamics," Nucl. Phys. B **427**, 291 (1994);
M. Reuter and C. Wetterich, "Indications for gluon condensation for nonperturbative flow equations," arXiv:hep-th/9411227;
M. Reuter, "Effective Average Actions and Nonperturbative Evolution Equations," arXiv:hep-th/9602012.

[14] U. Ellwanger, "Flow equations and BRS invariance for Yang-Mills theories," Phys. Lett. B **335**, 364 (1994) [arXiv:hep-th/9402077];
U. Ellwanger, M. Hirsch and A. Weber, "Flow equations for the relevant part of the pure Yang-Mills action," Z. Phys. C **69**, 687 (1996) [arXiv:hep-th/9506019];
U. Ellwanger, M. Hirsch and A. Weber, "The heavy quark potential from Wilson's exact renormalization group," Eur. Phys. J. C **1**, 563 (1998) [arXiv:hep-ph/9606468].

[15] D. Litim and N. Tetradis, "Approximate solutions of exact renormalization group equations," arXiv:hep-th/9501042.

[16] M. D'Attanasio and T. R. Morris, "Gauge invariance, the quantum action principle, and the renormalization group," Phys. Lett. B **378**, 213 (1996) [arXiv:hep-th/9602156].

[17] F. Freire, D. F. Litim and J. M. Pawlowski, "Gauge invariance and background field formalism in the exact renormalisation group," Phys. Lett. B **495**, 256 (2000) [arXiv:hep-th/0009110];
F. Freire, D. F. Litim and J. M. Pawlowski, "Gauge invariance, background fields and modified Ward identities," Int. J. Mod. Phys. A **16**, 2035 (2001) [arXiv:hep-th/0101108];
D. F. Litim and J. M. Pawlowski, "Wilsonian flows and background fields," Phys. Lett. B **546**,279 (2002) [arXiv:hep-th/0208216].

[18] J. M. Pawlowski, D. F. Litim, S. Nedelko and L. von Smekal, "Infrared behaviour and fixed points in Landau gauge QCD," arXiv:hep-th/0312324.

[19] M. Bonini and E. Tricarico, "Background field method in the Wilson formulation," Nucl. Phys. B **606**, 231 (2001) [arXive:hep-th/0104255].

[20] R. Alkofer and L. von Smekal, "The infrared behavior of QCD Green's functions: Confinement, dynamical symmetry breaking, and hadrons as relativistic bound states," Phys. Rept. **353**, 281 (2001) [arXiv:hep-ph/0007355].

[21] R. F. Sobreiro, S. P. Sorella, D. Dudal and H. Verschelde, "Gribov horizon in the presence of dynamical mass generation in Euclidean Yang-Mills theories in the Landau gauge," arXiv:hep-th/0403135.

[22] T. Kugo and I. Ojima, "Local covariant operator formalism of non-abelian gauge theories and quark confinement problem," Prog. Theor. Phys. Suppl. **66**, 1 (1979).

[23] A. C. Aguilar and A. A. Natale, arXiv:hep-ph/0405024.

[24] C. M. Becchi and C. Imbimbo, "Gribov horizon, contact terms and Čech-De Rham cohomology in 2D topological gravity," Nucl. Phys. B **462**, 571 (1996) [arXiv:hep-th/9510003].
C. Becchi, S. Giusto and C. Imbimbo, "The functional measure of gauge theories in the presence of Gribov horizons," arXiv:hep-th/9811018.

[25] L. Baulieu, P. A. Grassi and D. Zwanziger, "Gauge and topological symmetries in the bulk quantization of gauge theories," Nucl. Phys. B **597**, 583 (2001) [arXiv:hep-th/0006036].

[26] G. Barnich, F. Brandt and M. Henneaux, "Local BRST cohomology in gauge theories," Phys. Rept. **338**, 439 (2000) [arXiv:hep-th/0002245].

[27] F. Brandt, "Jet coordinates for local BRST cohomology," Lett. Math. Phys. **55**, 149 (2001) [arXiv:math-ph/0103006].

[28] F. Brandt, "Local BRST cohomology and covariance," Commun. Math. Phys. **190**, 459 (1997) [arXiv:hep-th/9604025].

[29] A. Quadri, "Algebraic properties of BRST coupled doublets," JHEP **0205**, 051 (2002) [arXiv:hep-th/0201122].

[30] A. Quadri, "Higher-order non-symmetric counterterms in pure Yang-Mills theory," J. Phys. G **30**, 677 (2004) [arXiv:hep-th/0309133].

[31] C. Lerche and L. von Smekal, "On the infrared exponent for gluon and ghost propagation in Landau gauge QCD," Phys. Rev. D **65**, 125006 (2002) [arXiv:hep-ph/0202194].