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June 2004

IHES/P/04/30

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## Abstract

We present a method to construct matrix models on arbitrary simply connected oriented real two dimensional Riemannian manifolds. The actions and the path integral measure are invariant under holomorphic transformations of matrix coordinates.

# 1 Introduction

One of the most fascinating aspects of string theory is that it is a quantum theory that contains gravity. The discovery of Dirichlet branes [1] has revealed the importance of the view-point that gravitational theories which are perturbatively described by closed strings may have dual descriptions in terms of open strings. A peculiar feature of the open string description of D-branes is that the positions of multiple D-branes are described by matrices [2]. When those matrices are simultaneously diagonalizable, The eigenvalues have a natural interpretation as the positions of the D-branes. However, such an interpretation becomes obscure when those coordinate matrices do not commute. This non-abelian nature of matrix coordinates introduces non-locality which is typically exhibited in the formation of higher dimensional branes [3, 4]. This in turn may resolve problems arising from classical singularities in general relativity. Since open string degrees of freedom become important at the sub-stringy scale [5], we expect it to capture fundamental structures of space-time in the microscopic scale. This led to matrix model proposals where matrix coordinates are the fundamental variables [6, 7]. On the other hand, it does not seem straightforward to incorporate the non-locality with local symmetries of the target space, in particular diffeomorphisms, the classical symmetries of general relativity. But at least at sufficiently weak string coupling and low energy, diffeomorphisms should be realized in actions describing multiple D-branes on general gravitational backgrounds. Diffeomorphism symmetry will also be a key to understand how curved space-time will be realized in matrix models.<sup>1</sup>

Construction of multiple D-brane actions on Kähler manifolds was studied earlier based on Douglas' geodesic distance criterion, a requirement that the mass of fluctuations around a diagonal configuration be equal to the shortest geodesic distance between the diagonal elements [12, 13, 14]. Recently, approaches based on covariance under general coordinate transformations have been developed [15, 16, 17],<sup>2</sup> where actions that satisfy the geodesic distance criterion should in principle be obtained as special cases. However, in these works the complete form of the matrix coordinate transformation, or even its existence in a rigorous sense, were not shown. This remains as a challenging issue. Under these circumstances, it will be important to show that there exists a simple well-defined quantum model where the matrix coordinate transformation law is completely specified.

In this article, we construct matrix models on arbitrary simply connected oriented real two dimensional Riemannian manifolds. The generalization of holomorphic coordinate transformations to matrix coordinates is straightforward in this case. It is still non-trivial to construct covariant actions which in general contain holomorphic and anti-holomorphic matrix coordinates, in particular they will contain commutators of holomorphic and anti-holomorphic matrix coordinates. Further, we construct an invariant path integral measure which is used for defining the quantum models.

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<sup>1</sup>Our motivation for studying this issue was affected by communications with researchers in type IIB matrix model. See [8, 9, 10, 11] for other approaches on this issue.

<sup>2</sup>See also [18] for a case of local gauge symmetry in flat space. Some ideas and techniques appeared there were precursors of those in [17] and this article.

## 2 Holomorphically covariant matrix models

### 2.1 Holomorphic transformations for matrix coordinates

#### *Holomorphic transformations*

Let us consider an oriented real two dimensional Riemannian manifold  $\mathcal{M}$ . Note that such manifolds are always Kähler. Suppose we have a Kähler potential  $K(z, \bar{z})$  on  $\mathcal{M}$ , where  $z$  and  $\bar{z}$  are holomorphic and anti-holomorphic coordinates, respectively. In this article, we restrict ourselves to the case where  $\mathcal{M}$  is simply connected and we assume that the coordinate  $z$  covers the whole  $\mathcal{M}$ . We'd like to promote the holomorphic coordinate to  $N \times N$  complex matrix coordinate  $Z$ . In analogy with D-brane theories, we call  $N$  the number of D-instantons. We may need to take  $N$  to infinity in some appropriate scaling limit in order to define a matrix model relevant to nature, but here we are interested in the mathematical structure of matrix coordinate transformations which are common to multiple D-brane actions. For a holomorphic coordinate transformation

$$z \rightarrow f(z) \tag{1}$$

we define a corresponding holomorphic matrix coordinate transformation from  $Z$  to  $F$  through the Taylor series

$$Z \rightarrow F(Z) = f(Z) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \partial_z^n f(z_0) (Z - z_0)^n. \tag{2}$$

For the anti-holomorphic coordinate  $\bar{z}$ , we associate a matrix  $\bar{Z}$  which is the hermitian conjugate matrix of  $Z$ . We restrict ourselves to holomorphic transformations for the holomorphic coordinate, and anti-holomorphic transformations for the anti-holomorphic coordinate. The transformation involves only one matrix coordinate so there is no matrix ordering ambiguity. Requiring that when the coordinate matrix is diagonal the transformation reduces to the usual coordinate transformation for each diagonal element, the simple rule (2) is the only possibility. This simplifies the problem drastically compared to the higher dimensional case. As a nature of Taylor series, the definition (2) does not depend on the expansion point  $z_0$ . It also satisfies the composition law

$$H(F(Z)) = H \circ F(Z) \equiv h \circ f(Z). \tag{3}$$

In one complex dimension both independence of the expansion point and the composition law are realized simply, whereas in higher dimensions it is a challenging issue to incorporate both of them simultaneously.

#### *Defined region*

When the coordinate  $z$  is defined in a region  $\Omega_z$ , we define defined region of the corresponding matrix coordinate  $Z$  by requiring its eigenvalues to be in the region  $\Omega_z$ :

$$\text{spec } Z \subset \Omega_z. \tag{4}$$

Suppose the region is mapped as  $\Omega_z \rightarrow \Omega_f$  under a holomorphic map  $z \rightarrow f(z)$ . If  $Z$  satisfies (4), then  $F$  satisfies

$$\text{spec } F \subset \Omega_f. \quad (5)$$

Therefore, the definition of the defined region of matrix coordinate is consistent with holomorphic coordinate transformations. The definition (4) is also invariant under  $U(N)$  transformation

$$Z \rightarrow UZU^\dagger, \quad \bar{Z} \rightarrow U\bar{Z}U^\dagger \quad (6)$$

where  $U$  is an  $N \times N$  unitary matrix:

$$\text{spec } UZU^\dagger = \text{spec } Z. \quad (7)$$

We will loosely denote  $\text{spec } Z \subset \mathcal{M}$  when all the points whose positions in coordinate  $z$  are eigenvalues of  $Z$  are on  $\mathcal{M}$ , by identifying points with their coordinates.

### *Kähler normal coordinates*

In our construction of generally covariant actions for multiple D-branes in [17], a crucial role was played by a matrix-valued object which transforms as a tangent vector under coordinate transformations. It can be regarded as a matrix generalization of Riemann normal coordinates. Since here we are considering holomorphic coordinate transformations instead, we need an object which transforms as a holomorphic tangent vector. Such a vector for ordinary Kähler manifolds was constructed in [19, 20] and called Kähler normal coordinates. Kähler normal coordinates differ from Riemann normal coordinates by the anti-holomorphic part of the Riemann normal coordinates. The Kähler normal coordinate  $v_{z_0}^z$  at point  $z_0$  is defined via the holomorphic coordinate transformation

$$\begin{aligned} v_{z_0}^z(z) &= z - z_0 + \sum_{n=2}^{\infty} \frac{1}{n!} g^{z\bar{z}}(z_0, \bar{z}_0) \partial_z^n \partial_{\bar{z}} K(z_0, \bar{z}_0) (z - z_0)^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} g^{z\bar{z}}(z_0, \bar{z}_0) \partial_z^n \partial_{\bar{z}} K(z_0, \bar{z}_0) (z - z_0)^n \end{aligned} \quad (8)$$

where the Kähler metric is given by  $g_{z\bar{z}}(z, \bar{z}) = \partial_z \partial_{\bar{z}} K(z, \bar{z})$ . We define matrix Kähler normal coordinate  $V_{z_0}^z$  at point  $z_0$  by simply replacing the ordinary coordinate in (8) with the matrix coordinate:

$$\begin{aligned} V_{z_0}^z(Z) &= Z - z_0 + \sum_{N=2}^{\infty} \frac{1}{n!} g^{z\bar{z}}(z_0, \bar{z}_0) \partial_z^n \partial_{\bar{z}} K(z_0, \bar{z}_0) (Z - z_0)^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} g^{z\bar{z}}(z_0, \bar{z}_0) \partial_z^n \partial_{\bar{z}} K(z_0, \bar{z}_0) (Z - z_0)^n. \end{aligned} \quad (9)$$

When  $Z$  is diagonal, so is  $V_{z_0}^z$  and each diagonal element is given by an ordinary Kähler normal coordinate. Under the holomorphic coordinate transformation (1), the matrix Kähler normal coordinate  $V_{z_0}^z$  transforms as a holomorphic tangent vector at point  $z_0$ :

$$V_{z_0}^z(Z) \rightarrow V_{f_0}^f(F) = \frac{\partial f}{\partial z}(z_0) V_{z_0}^z(Z). \quad (10)$$

The proof is essentially the same to the ordinary case [20], see the appendix.

## 2.2 Holomorphically covariant actions

The powerful result of [17] is that once we have a matrix-valued object which is built from coordinate matrices and transforms as a tangent vector at each point on the manifold, we can construct covariant actions for multiple D-branes. We use this result to write down our matrix model actions in the form of multiple D-instanton actions.

A key object in our construction was a matrix distribution function  $\delta(V_{z_0})$  which is defined by

$$\delta(V_{z_0}) = \int d\mu d\bar{\mu} e^{i(\mu_\alpha V_{z_0}^\alpha + \bar{\mu}_{\bar{\alpha}} \bar{V}_{z_0}^{\bar{\alpha}})} \quad . \quad (11)$$

Here,  $d\mu d\bar{\mu} \equiv \frac{d\mu_1 d\mu_2}{(2\pi)^2}$  where  $\mu = (\mu_1 + i\mu_2)/2$ ,  $\mu_1, \mu_2 \in \mathbf{R}$  and  $V_{z_0}^\alpha = e_z^\alpha(z_0, \bar{z}_0) V_{z_0}^z$  where  $e_z^\alpha(z_0, \bar{z}_0)$  being zweibein and  $\alpha$  and  $\bar{\alpha}$  are tangent space indices.  $V_{z_0}^\alpha$  and  $\mu_\alpha$  are scalars under holomorphic coordinate transformations and (co- and contra-variant) vectors under local Lorentz transformations. Thus  $\delta(V_{z_0})$  is a scalar under holomorphic coordinate transformations.

In addition to the invariance under the holomorphic matrix coordinate transformation (2), we also require invariance under the  $U(N)$  transformation (6) which is supposed to be a fundamental symmetry of matrix models. This is insured if the actions consist of traces of matrices. In the following we will only consider single-trace actions, but the extension to multi-trace case will be similar.

Using the matrix distribution function (11), a holomorphically covariant matrix model can be constructed with the action

$$S[Z, \bar{Z}] = \int dz_0 d\bar{z}_0 g_{z\bar{z}}(z_0, \bar{z}_0) \text{Tr} \left( \mathcal{L}(V_{z_0}^z(Z), \bar{V}_{\bar{z}_0}^{\bar{z}}(\bar{Z}), g_{z\bar{z}}(z_0, \bar{z}_0)) \delta(V_{z_0}) \right) \quad (12)$$

where  $\mathcal{L}(V_{z_0}^z, \bar{V}_{\bar{z}_0}^{\bar{z}}, g(z_0, \bar{z}_0))$  can be an arbitrary scalar built from holomorphic and anti-holomorphic vectors  $V_{z_0}^z$  and  $\bar{V}_{\bar{z}_0}^{\bar{z}}$  and tensors made from metric  $g_{z\bar{z}}(z_0, \bar{z}_0)$ . This action is manifestly covariant under holomorphic coordinate transformations. Integration over the expansion point  $z_0$  here is natural since configurations of D-instantons can form higher dimensional branes which should couple to the metric over some region of the space. However, one can expand the action at a single point and perform delta function integration explicitly to obtain an action on the point. Then the manifest covariance will be lost but it is useful when one studies around a configuration in which all D-instantons coincide. When  $V_{z_0}^z$  is diagonal,  $\delta(V_{z_0})$  reduces to a diagonal matrix of delta functions and the action reduces to a sum of terms corresponding to the individual D-instantons.

### 2.3 Invariant path integral measure

To define the quantum model by a path integral, we need to specify the integration measure. To keep the holomorphic coordinate transformation as a symmetry, the measure should also be invariant under the transformation. The path integral measure for a single D-instanton is  $\int dz d\bar{z} g_{z\bar{z}}(z, \bar{z}) = \int d\bar{d} K(z, \bar{z})$ . A natural generalization to  $N$  multiple D-instantons may be

$$\int \prod_{(ab)} dZ_{ab} d\bar{Z}_{ba} \det_{(cd,ef)} \frac{\partial}{\partial Z_{cd}} \frac{\partial}{\partial \bar{Z}_{ef}} \text{Tr} K(Z, \bar{Z}) \quad (13)$$

where  $a, b, \dots, f$  are matrix indices and the matrix-valued Kähler potential  $K(Z, \bar{Z})$  is given by

$$K(Z, \bar{Z}) \equiv \sum_{n,m=0}^{\infty} \frac{1}{n!} \frac{1}{m!} \partial_z^n \partial_{\bar{z}}^m K(z_0, \bar{z}_0) (Z - z_0)^n (\bar{Z} - \bar{z}_0)^m. \quad (14)$$

$K(Z, \bar{Z})$  transforms as a scalar under holomorphic matrix coordinate transformations:  $K(Z, \bar{Z}) = K(F, \bar{F})$ . This is explicitly checked in the appendix.

To be more precise, the integration measure for complex matrix  $Z$  can be more conveniently understood in terms of forms

$$\begin{aligned} dZ_{ab} \wedge d\bar{Z}_{ba} \wedge dZ_{ba} \wedge d\bar{Z}_{ab} &\propto d\text{Re}X_{ab}^1 \wedge d\text{Im}X_{ab}^1 \wedge d\text{Re}X_{ab}^2 \wedge d\text{Im}X_{ab}^2 \quad (a > b) \\ dZ_{aa} \wedge d\bar{Z}_{aa} &\propto dX_{aa}^1 \wedge dX_{aa}^2 \end{aligned} \quad (15)$$

where  $Z = X^1 + iX^2$ , and  $X^1, X^2$  are  $N \times N$  hermitian matrices. The overall numerical factor of the measure can be chosen appropriately.

The measure (13) is manifestly invariant under holomorphic matrix coordinate transformations (2) and  $U(N)$  transformations (6). It is also invariant under the shift of Kähler potential by holomorphic and anti-holomorphic functions:  $K(z, \bar{z}) \rightarrow K(z, \bar{z}) + \psi(z) + \bar{\psi}(\bar{z})$ . It also reduces to the usual matrix model measure when  $\mathcal{M}$  is flat  $\mathbf{R}^2$  with Cartesian coordinates. The measure (13) is physically natural, for suppose we have an action which makes the path integral strongly localized to a saddle point given by  $[Z, \bar{Z}] = 0$ . After the saddle point approximation for off-diagonal elements, the measure will reduce to  $\prod_a dZ_{aa} d\bar{Z}_{aa} g_{z\bar{z}}(Z_{aa}, \bar{Z}_{aa})$  which is a suitable measure for  $N$  non-interacting instantons on  $\mathcal{M}$ .

Finally, using a covariant action constructed by (12) and the invariant measure (13), we define a holomorphically covariant quantum matrix model by the path integral

$$\int_{\text{spec } Z \subset \mathcal{M}} \prod_{(ab)} dZ_{ab} d\bar{Z}_{ba} \det_{(cd,ef)} \left( \frac{\partial}{\partial Z_{cd}} \frac{\partial}{\partial \bar{Z}_{ef}} \text{Tr} K(Z, \bar{Z}) \right) e^{-S[Z, \bar{Z}]} \quad (16)$$

where the path integral is performed over all configurations satisfying  $\text{spec } Z \subset \mathcal{M}$ .

### 3 Summary and future directions

In this article we presented a method to construct quantum matrix models on arbitrary simply connected oriented real two dimensional Riemannian manifolds. The actions are manifestly covariant and proposed path integral measure is invariant under holomorphic matrix coordinate transformations.

Apparently there are many interesting directions one can pursue using these models. One natural direction is to study the physics of non-commuting matrix coordinate configurations [21, 22, 23, 24] on non-trivial gravitational backgrounds. Now that we have concrete models on general curved backgrounds, it would be also possible to study the change of background under large  $N$  renormalization [13, 25].

In this article we restricted ourselves to simply connected manifolds covered with a single coordinate. In the case of manifold with non-trivial topology, we may need to understand how to describe the universal covering in terms of matrix coordinates. We hope to come back to this issue in the near future. There are few simple examples where the universal coverings by matrix coordinates are realized [26].

The extension to the higher dimensional case is a challenging issue. But it is encouraging that in our one complex dimensional models we could not only obtain consistent transformation law and covariant actions but also find the invariant path integral measure and thus define the quantum models. For Kähler manifolds, it may be the case that the invariant measure in higher dimensions takes a similar form to that in Eq. (13).

Although the motivation for studying matrix coordinate transformations came from string theory, we have not used string worldsheet techniques in this article. One obstacle is that it is hard to solve string theory and classify D-branes in general gravitational backgrounds. However, it would be interesting to study coordinate transformations in string theory with some particular target space and D-branes on it. Interesting clues in this direction can be found in [27, 28, 29, 30]. In particular, Ref. [29] studies non-trivial deformation of target space diffeomorphism symmetry in a topological open string theory on space-filling D-brane on Poisson manifolds. This would be more closely related to our model if one studies lower dimensional D-branes in that theory.

What we have done here is a construction of some kind of non-commutative geometry [31]. In particular, there is a close resemblance to deformation quantization [32], and string worldsheet calculations cited above were mostly done in backgrounds which give rise to deformation quantizations. On the other hand, there seems to be some differences between two. Here, associativity of the product is manifest, and there's no reference to a symplectic or Poisson structure. In deformation quantization the associative product is the thing to construct and this refers to the symplectic or Poisson structure. However, it is known that the deformation quantization of  $\mathbf{R}^{2d}$  appears in matrix models, and here although the matrix model actions do not explicitly contain the symplectic structure, it appears as a classical solution. It would be interesting to make contact with theories in mathematics.



## Acknowledgments

We are grateful to Dominic Brecher, Henry Ling and Mark Van Raamsdonk for the fruitful collaboration [17], reading the manuscript and helpful comments. We would also like to thank the IHES for hospitality where this work was completed. We'd like to thank Thibault Damour, Stefan Fredenhagen, Ofer Gabber, Anton Gerasimov, Pietro Grassi, Gyula Karolyi, Xiabo Liu, Nikita Nekrasov, Eric Novac, Emma Previato, Sanjaye Ramgoolam, Jeong-Hyuck Park, Zoran Scoda, Yuji Tachikawa and Satoshi Yamaguchi for useful discussions and stimulating conversations.

## Appendix

To show the transformation laws of the holomorphic matrix vector  $V_{z_0}^z$  and the matrix-valued Kähler potential  $K(Z, \bar{Z})$  under holomorphic matrix coordinate transformations, we first prove the following formula. It is presented in a form applicable to higher dimensional case. In the following, indices after a comma denote partial differentiations. For simplicity, let us consider a holomorphic coordinate transformation  $z^i \rightarrow z'^i = z'^i(z)$  which keeps the origin. Extension to the case that doesn't keep the origin is obvious. We shall show that the transformation law of  $K_{,i_1 \dots i_{n'} \bar{i}_1 \dots \bar{i}_{m'}}(z, \bar{z})$  ( $n', m' \geq 1$ ) under the holomorphic coordinate transformation is given by

$$\begin{aligned} & K_{,i_1 \dots i_{n'} \bar{i}_1 \dots \bar{i}_{m'}}(z, \bar{z}) \\ \rightarrow & K_{,i'_1 \dots i'_{n'} \bar{i}'_1 \dots \bar{i}'_{m'}}(z', \bar{z}') \\ &= \sum_{n=1}^{n'} \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} K_{,k_1 \dots k_n \bar{k}_1 \dots \bar{k}_m}(z, \bar{z}) \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_n})}{\partial z'^{i'_1} \dots \partial z'^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}'^{\bar{i}'_1} \dots \partial \bar{z}'^{\bar{i}'_{m'}}} \right]_* \end{aligned} \quad (17)$$

where  $[\dots]_*$  means that terms including  $z$  ( $\bar{z}$ ) that are differentiated by no  $z'$  ( $\bar{z}'$ ) are omitted. The proof is essentially the same to the one in [20] and presented here for readers' convenience. We show Eq. (17) by induction.

i) In the case  $n' = m' = 1$ , Eq. (17) means

$$K_{,i' \bar{i}'}(z', \bar{z}') = K_{,i \bar{i}}(z, \bar{z}) \frac{\partial z^i}{\partial z'^{i'}} \frac{\partial \bar{z}^{\bar{i}}}{\partial \bar{z}'^{\bar{i}'}} \quad (18)$$

which follows from the chain rule for differentiation. (18) is the transformation law for the Kähler metric.

ii) We assume that Eq. (17) holds for  $n', m'$ . Differentiation of Eq. (17) by  $z'^{i'_{n'+1}}$  gives

$$\begin{aligned} & K_{,i'_1 \dots i'_{n'+1} \bar{i}'_1 \dots \bar{i}'_{m'}} \\ = & \sum_{n=1}^{n'} \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} K_{,k_1 \dots k_{n+1} \bar{k}_1 \dots \bar{k}_m} \frac{\partial z^{k_{n+1}}}{\partial z'^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_n})}{\partial z'^{i'_1} \dots \partial z'^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}'^{\bar{i}'_1} \dots \partial \bar{z}'^{\bar{i}'_{m'}}} \right]_* \\ & + \sum_{n=1}^{n'} \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} K_{,k_1 \dots k_n \bar{k}_1 \dots \bar{k}_m} \frac{\partial}{\partial z'^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_n})}{\partial z'^{i'_1} \dots \partial z'^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}'^{\bar{i}'_1} \dots \partial \bar{z}'^{\bar{i}'_{m'}}} \right]_* . \end{aligned} \quad (19)$$

The first term can be rewritten as

$$\sum_{n=2}^{n'+1} \sum_{m=1}^{m'} \frac{1}{(n-1)!} \frac{1}{m!} K_{,k_1 \dots k_n \bar{k}_1 \dots \bar{k}_m} \frac{\partial z^{k_n}}{\partial z^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_{n-1}})}{\partial z^{i'_1} \dots \partial z^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}^{\bar{i}'_1} \dots \partial \bar{z}^{\bar{i}'_{m'}}} \right]_* . \quad (20)$$

Therefore, we have

$$\begin{aligned} & K_{,i'_1 \dots i'_{n'+1} \bar{i}'_1 \dots \bar{i}'_{m'}} \\ = & \sum_{m=1}^{m'} \frac{1}{m!} K_{,k_1 \bar{k}_1 \dots \bar{k}_m} \frac{\partial^{n'+1} z^{k_1}}{\partial z^{i'_1} \dots \partial z^{i'_{n'+1}}} \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}^{\bar{i}'_1} \dots \partial \bar{z}^{\bar{i}'_{m'}}} \right]_* \\ & + \sum_{n=2}^{n'} \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} K_{,k_1 \dots k_n \bar{k}_1 \dots \bar{k}_m} \left\{ n \frac{\partial z^{k_n}}{\partial z^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_{n-1}})}{\partial z^{i'_1} \dots \partial z^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}^{\bar{i}'_1} \dots \partial \bar{z}^{\bar{i}'_{m'}}} \right]_* \right. \\ & \quad \left. + \frac{\partial}{\partial z^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_n})}{\partial z^{i'_1} \dots \partial z^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}^{\bar{i}'_1} \dots \partial \bar{z}^{\bar{i}'_{m'}}} \right]_* \right\} \\ & + \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} K_{,k_1 \dots k_{n'+1} \bar{k}_1 \dots \bar{k}_m} \frac{\partial z^{k_{n'+1}}}{\partial z^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_{n'}})}{\partial z^{i'_1} \dots \partial z^{i'_{n'}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}^{\bar{i}'_1} \dots \partial \bar{z}^{\bar{i}'_{m'}}} \right]_* . \quad (21) \end{aligned}$$

For the terms in the curly brackets, we apply the relation

$$\begin{aligned} & n \frac{\partial z^{k_n}}{\partial z^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_{n-1}})}{\partial z^{i'_1} \dots \partial z^{i'_{n'}}} \right]_* + \frac{\partial}{\partial z^{i'_{n'+1}}} \left[ \frac{\partial^{n'}(z^{k_1} \dots z^{k_n})}{\partial z^{i'_1} \dots \partial z^{i'_{n'}}} \right]_* \\ = & \left[ \frac{\partial^{n'+1}(z^{k_1} \dots z^{k_n})}{\partial z^{i'_1} \dots \partial z^{i'_{n'+1}}} \right]_* \end{aligned} \quad (22)$$

where symmetrization of the first term on the left-hand side is implied. Then,

$$\begin{aligned} & K_{,i'_1 \dots i'_{n'+1} \bar{i}'_1 \dots \bar{i}'_{m'}} \\ = & \sum_{n=1}^{n'+1} \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} K_{,k_1 \dots k_n \bar{k}_1 \dots \bar{k}_m} \left[ \frac{\partial^{n'+1}(z^{k_1} \dots z^{k_n})}{\partial z^{i'_1} \dots \partial z^{i'_{n'+1}}} \right]_* \left[ \frac{\partial^{m'}(\bar{z}^{\bar{k}_1} \dots \bar{z}^{\bar{k}_m})}{\partial \bar{z}^{\bar{i}'_1} \dots \partial \bar{z}^{\bar{i}'_{m'}}} \right]_* . \quad (23) \end{aligned}$$

Thus Eq. (17) holds for  $n'+1, m'$ . That Eq. (17) also holds for  $n', m'+1$  can be shown similarly. From i) and ii) Eq. (17) is proved.

Now let us go back to our one complex dimensional case. From Eq. (17) we can immediately see that under holomorphic coordinate transformations,  $\partial_z^{n'} \partial_{\bar{z}}^{m'} K$  transforms according to

$$\begin{aligned} & \partial_z^{n'} \partial_{\bar{z}}^{m'} K|_0 \\ \rightarrow & \partial_f^{n'} \partial_{\bar{f}}^{m'} K|_0 = \sum_{n=1}^{n'} \sum_{m=1}^{m'} \frac{1}{n!} \frac{1}{m!} \partial_z^n \partial_{\bar{z}}^m K|_0 \left[ \frac{\partial^{n'}(z^n)}{\partial^{n'} f} \right]_0 \left[ \frac{\partial^{m'}(\bar{z}^m)}{\partial^{m'} \bar{f}} \right]_0 \\ = & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n!} \frac{1}{m!} \partial_z^n \partial_{\bar{z}}^m K|_0 \left[ \frac{\partial^{n'}(z^n)}{\partial^{n'} f} \right]_0 \left[ \frac{\partial^{m'}(\bar{z}^m)}{\partial^{m'} \bar{f}} \right]_0 . \quad (24) \end{aligned}$$

Here, the subscript 0 denotes that it is a value at the origin. The second equality holds because the terms in  $[\cdot \cdot]_0$  vanish when  $n > n', m > m'$ .

Using Eq. (24) (the third line for the holomorphic part and the second line for the anti-holomorphic part), the left-hand side of Eq. (10) can be explicitly calculated from the definition (8):

$$\begin{aligned}
V_0^f(F) &= \sum_{n=1}^{\infty} \frac{1}{n!} g^{f\bar{f}} \partial_f^n \partial_{\bar{f}} K|_0 F^n \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} g^{f\bar{f}}|_0 \left( \sum_{m=1}^{\infty} \frac{1}{m!} \partial_z^m \partial_{\bar{z}} K|_0 \left[ \frac{\partial \bar{z}}{\partial \bar{f}} \right]_0 \left[ \frac{\partial^n (z^m)}{\partial^n f} \right]_0 \right) F^n \\
&= \frac{\partial f}{\partial z}(0) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n!} \frac{1}{m!} g^{z\bar{z}} \partial_z^m \partial_{\bar{z}} K|_0 \left[ \frac{\partial^n (z^m)}{\partial^n f} \right]_0 F^n \\
&= \frac{\partial f}{\partial z}(0) \sum_{m=1}^{\infty} \frac{1}{m!} g^{z\bar{z}} \partial_z^m \partial_{\bar{z}} K|_0 Z^m = \frac{\partial f}{\partial z}(0) V_0^z(Z). \tag{25}
\end{aligned}$$

Thus, under holomorphic coordinate transformations the matrix Kähler normal coordinate  $V_{z_0}^z$  transforms as a holomorphic tangent vector at point  $z_0$ .

It is also straightforward to show that the matrix-valued Kähler potential  $K(Z, \bar{Z})$  transforms as a scalar:

$$\begin{aligned}
K(F, \bar{F}) &= \sum_{n', m'=0}^{\infty} \frac{1}{n'!} \frac{1}{m'!} \partial_f^{n'} \partial_{\bar{f}}^{m'} K|_0 F^{n'} \bar{F}^{m'} \\
&= K_0 + \partial_z K|_0 \frac{\partial z}{\partial f}(0) F + \partial_{\bar{z}} K|_0 \frac{\partial \bar{z}}{\partial \bar{f}}(0) \bar{F} \\
&\quad + \sum_{n', m'=1}^{\infty} \frac{1}{n'!} \frac{1}{m'!} \sum_{n, m=1}^{\infty} \frac{1}{n!} \frac{1}{m!} \partial_z^n \partial_{\bar{z}}^m K|_0 \left[ \frac{\partial^{n'} (z^n)}{\partial^{n'} f} \right]_0 \left[ \frac{\partial^{m'} (\bar{z}^m)}{\partial^{m'} \bar{f}} \right]_0 F^{n'} \bar{F}^{m'} \\
&= \sum_{n, m=0}^{\infty} \frac{1}{n!} \frac{1}{m!} \partial_z^{n'} \partial_{\bar{z}}^{m'} K|_0 Z^n \bar{Z}^m \\
&= K(Z, \bar{Z}). \tag{26}
\end{aligned}$$

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