Nonlinear Higher Spin Theories in Various Dimensions

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Abstract

In this article, an introduction to the nonlinear equations for completely symmetric bosonic higher spin gauge fields in anti de Sitter space of any dimension is provided. To make the presentation self-contained we explain in detail some related issues such as the MacDowell-Mansouri-Stelle-West formulation of gravity, unfolded formulation of dynamical systems in terms **of** free differential algebras and Young tableaux symmetry properties in terms of **Howe dual algebras**.

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1 Introduction

The Coleman-Mandula theorem [1] and its generalizations [2] strongly restrict the S-matrix symmetries of a nontrivial (*i.e.* interacting) relativistic field theory in flat space-time. More precisely, the extension of the space-time symmetry algebra is at most the (semi)direct sum of a (super)conformal algebra and an internal symmetry algebra spanned by elements that commute with the generators of the Poincaré algebra. Ruling out higher symmetries these theorems rule out higher spin (HS) gauge fields associated with them, allowing in practice only gauge fields of low spins (*i.e.* $s \leqslant 2$). However, as will be reviewed here, going beyond some assumptions of these no-go theorems allows to overcome both restrictions, on higher spins (*i.e.* $s > 2$) and on space-time symmetry extensions.

By now, HS gauge fields are pretty well understood at the free field level. Therefore, the main open problem in this topic is to find proper non-Abelian HS gauge symmetries which extend the space-time symmetries. These symmetries can possibly mix fields of different spins, as supersymmetry does. Even though one may never find HS particles in accelerators, non-Abelian HS symmetries might lead us to a better understanding of the true symmetries of unification models. From the supergravity perspective, the theories with HS fields may have more than 32 supercharges and may live in dimensions higher than 11. From the superstring perspective, several arguments support the conjecture [3] that the Stueckelberg symmetries of massive HS string excitations result from a spontaneous breaking of some HS gauge symmetries. In this picture, tensile string theory appears as a spontaneously broken phase of a more symmetric phase with only massless fields. In that case, superstring should exhibit higher symmetries in the high-energy limit as was argued long ago by Gross [4]. A more recent argument came from the AdS/CFT side after it was realized [5] that HS symmetries should be unbroken in the Sundborg–Witten limit

$$\lambda = g_{YM}^2 N = \left(\frac{R_{AdS}^2}{\alpha'}\right)^2 \longrightarrow 0\,,$$

because the boundary conformal theory becomes free. A dual string theory in the highly curved AdS space-time is therefore expected to be a HS theory (see also [6] and refs therein for recent developments).

One way to provide an explicit solution of the non-Abelian HS gauge symmetry problem is by constructing a consistent non-linear theory of massless HS fields. For several decades, a lot of efforts has been put into this direction although, from the very beginning, this line of research faced several difficulties. The first explicit attempts to introduce interactions between HS gauge fields and gravity encountered severe problems [7]. However, some positive results [8] were later obtained in flat space on the existence of consistent vertices for HS gauge fields interacting with each other, but not with gravity.

Fifteen years ago, the problem of consistent HS gravitational interactions was partially solved in four dimensions [9]. In order to achieve this result, the following conditions of the no-go theorems [1, 2] were relaxed:

(1) the theory is formulated around a flat background.

(2) the spectrum contains a finite number of HS fields.

The non-linear HS theory in four dimensions was shown to be consistent up to cubic order at the action level [9] and, later, at all orders at the level of equations of motion [10, 11]. The second part of these results was recently extended to arbitrary space-time dimensions [12]. The non-linear HS theory exhibits some rather unusual properties of HS gauge fields:

(1') the theory is perturbed around an $(A)dS$ background and does not admit a flat limit as long as HS symmetries are unbroken.

(2') the allowed spectra contain infinite towers of HS fields and do not admit a consistent finite truncation.

(3') the vertices have higher-derivative terms (that is to say, the higher derivatives appear in HS interactions - not at the free field level).

The properties (2') and (3') were also observed by the authors of [8] for HS gauge fields. Though unusual, these properties are familiar to high-energy theorists. The property (1') is verified by gauged supergravities with charged gravitinos [13, 14]. The property (2') plays an important role in the consistency of string theory. The property (3') is also shared by Witten's string field theory [15].

An argument in favor of an AdS background is that the S-matrix theorems [1, 2] do not apply since there is no well-defined S-matrix in AdS space-time. The AdS geometry plays a key role in the non-linear theory because cubic higher derivative terms are added to the free Lagrangian, requiring a non-vanishing cosmological constant Λ. These cubic vertices are schematically of the form

$$\mathcal{L}^{int} = \sum_{n,p} \Lambda^{-\frac{1}{2}(n+p)} D^n(\varphi_{\dots}) D^p(\varphi_{\dots}) \mathcal{R}^{\dots} ,$$

where $\varphi_{\dots}$ denotes some spin-s gauge field, and $\mathcal{R}$ stands for the fluctuation of the Riemann curvature tensor around the AdS background. Such vertices do not admit a $\Lambda \to 0$ limit. The order of derivatives which appear in the cubic vertex increases linearly with the spin [8, 9]: $n + p \sim s$. Since all spins $s > 2$ must be included in the non-Abelian HS algebra, the number of derivatives is not bounded from above. In other words, the HS gauge theory is *non-local*[1].

1.1 Plan of the paper

The purpose of these lecture notes is to present the bosonic nonlinear HS equations. Their structure is as follows.

[1] Non-local theories do not automatically suffer from the higher-derivative problem. Indeed, in some cases like string field theory, the problem is somehow cured [15, 16] if the free theory is well-behaved and if non-locality is treated perturbatively (see [17] for a comprehensive review on this point).

In the appendices, some elementary material is reviewed[2].

In Section 2, the MacDowell-Mansouri-Stelle-West formulation of gravity is recalled. In Section 3, some basics about Young tableaux and irreducible representations are introduced. In Section 4, the approach of Section 2 is generalized to HS fields, *i.e.* the free HS theory is formulated as a theory of 1-form connections. In Section 5, a non-Abelian HS algebra is constructed. The general definitions of free differential algebras are given in Section 6 and a strategy is explained on how to formulate non-linear HS in these terms. The sections 7 and 8 present the unfolded form of the massless scalar field and gravity, and free HS equations, respectively. In Section 9 is explained how the cohomologies of some operator σ_- describe the dynamical content of a theory. These cohomologies are calculated in the HS case in Section 10. The sections 11 and 12 introduce some tools (the star product and the twisted adjoint representation) useful for writing the nonlinear equations, which is done in Section 13. The nonlinear equations are analyzed perturbatively in Section 14. They are further discussed in Section 15. A brief conclusion inviting to further readings completes these lecture notes.

1.2 Conventions

Our conventions are as follows: A generic space-time **is denoted by $\mathcal{M}^d$ and is a (pseudo)-Riemannian manifold of dimension d, where the metric is taken to be "mostly minus".** Greek letters $\mu, \nu, \dots$ denote curved (*i.e.,* base) indices, while Latin letters $a, b, \dots$ denote fiber indices often referred to as tangent space indices. Both types of indices run from 0 to $d-1$. Capital Latin letters $A, B, \dots$ denote ambient space indices and their range of values is $0, 1, \dots, d-1, \hat{d}$, where the $(d+1)$-th direction is denoted by $\hat{d}$ to distinguish the tangent space index d from the value $\hat{d}$ that it can take. The tensor η_{ab} is the mostly minus Minkowski metric. The bracket $[\dots]$ denotes complete antisymmetrization with strength one (*e.g.* $A_{[a}B_{b]} = \frac{1}{2}(A_aB_b - A_bB_a)$) while the bracket $\{\dots\}$ denotes complete symmetrization with strength one (*e.g.* $A_{\{a}B_{b\}} = \frac{1}{2}(A_aB_b + A_bB_a)$). The de Rham complex $\Omega^*(\mathcal{M}^d)$ is the graded commutative algebra of differential forms that is endowed with the wedge product (the wedge symbol will always be omitted in this paper) and the exterior differential d. The basis elements dx^μ of the exterior algebra are Grassmann odd, *i.e.* anticommuting. The exterior differential is defined as $d = dx^\mu \partial_\mu$.

2 Gravity à la MacDowell - Mansouri - Stelle - West

Einstein's theory of gravity is a non-Abelian gauge theory of a spin-two particle, in the same way as Yang-Mills theories[3] are non-Abelian gauge theories of spin-one particles. Local symmetries of Yang-Mills theories originate from internal global symmetries. Similarly,

[2]For these lecture notes, the reader is only assumed to have basic knowledge of Yang-Mills theory, general relativity and group theory. The reader is also supposed to be familiar with the notions of differential forms and cohomology groups.

[3]See Appendix B for a brief review of Yang-Mills theories.

the gauge symmetries of Einstein gravity in the vielbein formulation[4] originate from global space-time symmetries of its most symmetric vacua. These symmetries are manifest in the formulation of MacDowell, Mansouri, Stelle and West [19, 20].

This section is devoted to the presentation of the latter formulation. In the first subsection 2.1, the Einstein-Cartan formulation of gravity is reviewed and the link with the Einstein-Hilbert action without cosmological constant is explained. A cosmological constant can be introduced into the formalism, which is done in Subsection 2.2. This subsection also contains an elegant action for gravity, written by MacDowell and Mansouri. In Subsection 2.3, the improved version of this action introduced by Stelle and West is presented, the covariance under all symmetries being made manifest.

2.1 Gravity as a Poincaré gauge theory

In this subsection, the frame formulation of gravity with zero cosmological constant is reviewed. We first introduce the dynamical fields and sketch the link to the Einstein-Hilbert metric. Then the action is written.

The basic idea is as follows: instead of considering the metric $g_{\mu\nu}$ as the dynamical field, two new dynamical fields are introduced: the vielbein or frame field e^a_μ and the Lorentz connection $\omega^{L\,ab}_\mu$.

The relevant fields appear through the 1-forms e^a and $\omega^{L\,ab} = -\omega^{L\,ba}$ defined by $e^a = e^a_\mu\, dx^\mu$ and $\omega^{L\,ab} = \omega^{L\,ab}_\mu\, dx^\mu$. The number of 1-forms is equal to $d + \frac{d(d-1)}{2} = \frac{(d+1)d}{2}$, which is the dimension of the Poincaré group $ISO(d-1,1)$. So they can be collected into a single 1-form taking values in the Poincaré algebra as $\omega = e^a P_a + \frac{1}{2}\,\omega^{L\,ab} M_{ab}$, where P_a and M_{ab} generate $iso(d-1,1)$ (see Appendix A). The corresponding curvature is the 2-form (see Appendix B):

$$R = d\omega + \omega^2 \equiv T^a P_a + \frac{1}{2}\, R^{L\,ab} M_{ab}\,,$$

where T^a is the torsion, given by

$$T^a = D^L e^a = de^a + \omega^{L\,a}{}_b e^b\ ,$$

and $R^{L\,ab}$ is the Lorentz curvature

$$R^{L\,ab} = D^L \omega^{L\,ab} = d\omega^{L\,ab} + \omega^{L\,a}{}_c \omega^{L\,cb}\,,$$

as follows from the Poincaré algebra (A.1)-(A.3).

To make contact with the metric formulation of gravity, one must assume that the frame e^a_μ has maximal rank d so that it gives rise to the non-degenerate metric tensor $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. One can also require the absence of torsion, $T_a = 0$. Then one solves this constraint and expresses the Lorentz connection in terms of the frame field, $\omega^L = \omega^L(e, \partial e)$. It can be checked that the tensor $R_{\mu\nu,\,\rho\sigma} = e^a_\mu e^b_\nu R^L_{ab\;\rho\sigma}$ expressed solely in terms of the metric is the Riemann tensor.

[4]See *e.g.* [18] for a pedagogical review on the gauge theory formulation of gravity and some of its extensions, like supergravity.

The first order action of the frame formulation of gravity is due to Weyl [21] and reads in any dimension $d > 1$

$$S[\, e^a_\mu, \omega^{L\,ab}_\mu] = \frac{1}{2\kappa^2} \int_{\mathcal{M}^d} R^{L\;bc} e^{a_1} \dots e^{a_{d-2}} \epsilon_{a_1 \dots a_{d-2} bc} \;, \tag{2.1}$$

where $\epsilon_{a_1 \dots a_d}$ is the invariant tensor of the special linear group $SL(d)$ and κ^2 is the gravitational constant, so that κ has dimension $(length)^{\frac{d}{2}-1}$. The Euler-Lagrange equations of the Lorentz connection

$$\frac{\delta S}{\delta \omega^{L\,bc}} \propto \epsilon_{a_1 \dots a_{d-2} bc}\, e^{a_1} \dots e^{a_{d-3}} T^{a_{d-2}} = 0$$

imply that the torsion vanishes. The Lorentz connection is then an auxiliary field, which can be removed from the action by solving its own (algebraic) equations of motion. The action $S = S[\, e\,,\, \omega^L(e, \partial e)\,]$ is now expressed only in terms of the vielbein [5]. Actually, only combinations of vielbeins corresponding to the metric appear and the action $S = S[\, g_{\mu\nu}]$ is indeed the second order Einstein-Hilbert action.

The Minkowski space-time solves $R^{L\;ab} = 0$ and $T^a = 0$. It is the most symmetrical solution of the Euler-Lagrange equations, whose global symmetries form the Poincaré group. The gauge symmetries of the action (2.1) are the diffeomorphisms and the local Lorentz transformations. Together, these gauge symmetries correspond to the gauging of the Poincaré group (see Appendix C for more comments).

2.2 Gravity as an $o(d-1,2)$ gauge theory

In the previous section, the Einstein-Cartan formulation of gravity with vanishing cosmological constant has been presented. We will now show how a non-vanishing cosmological constant can be added to this formalism. In these lectures, we will restrict ourselves to the *AdS* case but for the bosonic case we focus on everything can be rephrased for *dS*. One is mostly interested in the *AdS* case for the reason that it is more suitable for supersymmetric extensions. Furthermore, *dS* and *AdS* have rather different unitary representations (for *dS* there are unitary irreducible representations the energy of which is not bounded from below).

It is rather natural to reinterpret P_a and M_{ab} as the generators of the AdS_d isometry algebra $o(d-1,2)$. The curvature $R = d\omega + \omega^2$ then decomposes as $R = T^a P_a + \frac{1}{2} R^{ab} M_{ab}$, where the Lorentz curvature $R^{L\;ab}$ is deformed to

$$R^{ab} \equiv R^{L\;ab} + R^{cosm\;ab} \equiv R^{L\;ab} + \Lambda\, e^a e^b \,, \tag{2.2}$$

since (A.3) is deformed to (A.4).

MacDowell and Mansouri proposed an action [19], the Lagrangian of which is the product of two curvatures (2.2) in $d = 4$

$$S^{MM}[\, e, \omega] = \frac{1}{4\kappa^2 \Lambda} \int_{\mathcal{M}^4} R^{a_1 a_2} R^{a_3 a_4} \epsilon_{a_1 a_2 a_3 a_4} \,. \tag{2.3}$$

[5] In the context of supergravity, this action principle [14, 22] is sometimes called the "1.5 order formalism" [23] because it combines in some sense the virtues of first and second order formalism.

Expressing R^{ab} in terms of $R^{L\,ab}$ and $R^{cosm\,ab}$ by (2.2), the Lagrangian is the sum of three terms: a term $R^L R^{cosm}$, which is the previous Lagrangian (2.1) without cosmological constant, a cosmological term $R^{cosm}\,R^{cosm}$ and a Gauss-Bonnet term $R^L R^L$. The latter term contains higher-derivatives but it does not contribute to the equations of motion because it is a topological invariant.

The MacDowell-Mansouri action admits a higher dimensional generalization [24]

$$S^{MM}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{a_1a_2}R^{a_3a_4}e^{a_5}\dots e^{a_d}\epsilon_{a_1\dots a_d}\,. \tag{2.4}$$

The AdS_d space-time is defined as the most symmetrical solution of the Euler-Lagrange equations. As explained in more detail in subsection 2.3 it is a solution of the system $R^{ab}=0$, $T^a=0$ and the constraint: $\mathrm{rank}(e^a_\mu)=d$. Because the Gauss-Bonnet term

$$S^{GB}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{L\,a_1a_2}R^{L\,a_3a_4}e^{a_5}\dots e^{a_d}\epsilon_{a_1\dots a_d}\,.$$

is not topological beyond $d=4$, the field equations resulting from the action (2.4) are different from the Einstein equations in d dimensions. However the difference is by nonlinear terms that do not contribute to the free spin 2 equations [24] apart from replacing the cosmological constant Λ by $\frac{2(d-2)}{d}\Lambda$ (in such a way that no correction appears in $d=4$, as expected). One way to see this is by considering the action

$$S^{nonlin}[\,e,\omega] = S^{GB}[\,e,\omega] + \frac{d-4}{4\kappa^2}\int_{\mathcal{M}^d}\Big(\frac{2}{d-2}\,R^{L\,a_1a_2}e^{a_3}\dots e^{a_d} + \frac{\Lambda}{d}\,e^{a_1}\dots e^{a_d}\Big)\epsilon_{a_1\dots a_d}\,, \tag{2.5}$$

which is the sum of the Gauss-Bonnet term plus terms of the same type as the Einstein-Hilbert and cosmological terms (note that the latter are absent when $d=4$). The variation of (2.5) is equal to

$$\delta S^{nonlin}[\,e,\omega] = \frac{1}{4\kappa^2\Lambda}\int_{\mathcal{M}^d} R^{a_1a_2}R^{a_3a_4}\delta(e^{a_5}\dots e^{a_d})\,\epsilon_{a_1\dots a_d}\,, \tag{2.6}$$

when the torsion is required to be zero (i.e., applying the 1.5 order formalism to see that the variation over the Lorentz connection does not contribute). Indeed, the variation of the action (2.5) vanishes when $d=4$, but when $d>4$ the variation (2.6) is bilinear in the AdS_d field strength R^{ab}. Since the AdS_d field strength is zero in the vacuum AdS solution, the action S^{nonlin} **only contributes to corrections of the field equations which are nonlinear in the fluctuations near the** AdS **background**, having no effect on the free spin 2 equations. As a consequence, at the linearized level the Gauss-Bonnet term does not affect the form of the free spin 2 equations of motion, it merely redefines **an overall factor in front of the action and the cosmological constant via** $\kappa^2\rightarrow(\frac{d}{2}-1)\kappa^2$ **and** $\Lambda\rightarrow\frac{2(d-2)}{d}\Lambda$**, respectively** (as can be seen by substituting S^{GB} in (2.4) with its expression in terms of S^{nonlin} from (2.5)). Beyond the free field approximation the corrections to Einstein's field equations resulting from the action (2.4) are nontrivial for $d>4$ and nonanalytic in

Λ (as can be seen from (2.6)), having no meaningful flat limit. As will be shown later, this is analogous to the structure of HS interactions which also contain terms with higher derivatives and negative powers of Λ. The important difference is that in the case of gravity one can subtract the term (2.5) without destroying the symmetries of the model, while this is not possible in the HS gauge theories. The flat limit $\Lambda \to 0$ is perfectly smooth at the level of the algebra (*e.g.* $o(d-1,2) \to iso(d-1,1)$ for gravity, see Appendix A) and at the level of the free equations of motion, but it may be singular at the level of the action and nonlinear field equations.

2.3 MacDowell-Mansouri-Stelle-West gravity

The gauge symmetries of the MacDowell-Mansouri action (2.3) are the diffeomorphisms and the local Lorentz transformations. It is however possible to make the $o(d-1,2)$ symmetry manifest by combining the vielbein and the Lorentz connection into a single field $\omega = \omega_\mu{}^{AB} dx^\mu M_{AB}$. The fiber indices A, B now run from 0 to d. They are raised and lowered by the invariant mostly minus metric η_{AB} of $o(d-1,2)$ (see Appendix A).

In this subsection, the MacDowell-Mansouri-Stelle-West (MMSW) action [20] is written and it is shown how to recover the action presented in the previous subsection. The particular vacuum solution which corresponds to AdS space-time is also introduced. Finally the symmetries of the MMSW action and of the vacuum solution are analyzed.

In order to promote local $o(d-1,2)$ transformations to gauge symmetries, an additional field has to be introduced: the time-like vector V^A called *compensator*[6]. The compensator vector is constrained to have a constant norm ρ,

$$V^A V^B \eta_{AB} = \rho^2 \,. \tag{2.7}$$

As one will see, the constant ρ is related to the cosmological constant by

$$\rho^2 = -\Lambda^{-1} \,. \tag{2.8}$$

The MMSW action is ([20] for $d=4$ and [24] for arbitrary d)

$$S^{MMSW}[\,\omega^{AB}, V^A] = -\frac{\rho}{4\kappa^2} \int_{\mathcal{M}^d} \epsilon_{A_1 \dots A_{d+1}} R^{A_1 A_2} R^{A_3 A_4} E^{A_5} \dots E^{A_d} V^{A_{d+1}} \,, \tag{2.9}$$

where the curvature or field strength R^{AB} is defined by

$$R^{AB} \equiv d\omega^{AB} + \omega^{AC} \omega_C{}^B$$

and the frame field E^A by

$$E^A \equiv DV^A = dV^A + \omega^A_{\ B} V^B \,.$$

[6]This compensator field compensates additional symmetries serving for them as a Higgs field. The terminology is borrowed from application of conformal supersymmetry for the analysis of Poincaré supermultiplets (see *e.g.* [25]). It should *not* be confused with the homonymous - but unrelated - gauge field introduced in another approach to free HS fields [26].

Furthermore, in order to make link with Einstein gravity, two constraints are imposed: (i) the norm of V^A is fixed, and (ii) the frame field E_μ^A is assumed to have maximal rank equal to d. As the norm of V^A is constant, the frame field satisfies

$$E^A V_A = 0\,. \tag{2.10}$$

If the condition (2.7) is relaxed, then the norm of V^A corresponds to an additional dilaton-like field [20].

Let us now analyze the symmetries of the MMSW action. The action is manifestly invariant under

- Local $o(d-1,2)$ transformations:

$$\delta\omega^{AB}(x) = D\epsilon^{AB}(x)\,, \qquad \delta V^A(x) = -\epsilon^{AB}(x)V_B(x)\,; \tag{2.11}$$

- Diffeomorphisms:

$$\delta\omega_\nu^{AB} = \partial_\nu(\xi^\mu)\omega_\mu^{AB} + \xi^\mu\partial_\mu\omega_\nu^{AB}\,, \qquad \delta V^A = \xi^\nu\partial_\nu V^A\,. \tag{2.12}$$

Let us define the covariantized diffeomorphism as the sum of a diffeomorphism with parameter ξ^μ and a $o(d-1,2)$ local transformation with parameter $\epsilon^{AB}(\xi^\mu) = -\xi^\mu\omega_\mu^{AB}$. The action of this transformation is thus

$$\delta^{cov}\omega_\mu^{AB} = \xi^\nu R_{\nu\mu}^{AB}\,, \qquad \delta^{cov}V^A = \xi^\nu E_\nu^A \tag{2.13}$$

by (2.11)-(2.12).

The compensator vector is pure gauge. Indeed, by local $O(d-1,2)$ rotations one can gauge fix $V^A(x)$ to any values with $V^A(x)V_A(x) = \rho^2$. In particular, one can reach the standard gauge

$$V^A = \rho\,\delta^A_{\hat{d}}\,. \tag{2.14}$$

Taking into account (2.10), one observes that the covariantized diffeomorphism also makes it possible to gauge fix fluctuations of the compensator $V^A(x)$ near any fixed value. Because the full list of symmetries can be represented as a combination of covariantized diffeomorphism, local Lorentz symmetry and diffeomorphisms, in the standard gauge (2.14) the algebra of gauge symmetries is broken to the local $o(d-1,1)$ algebra and diffeomorphisms. In the standard gauge, one therefore recovers the field content and the gauge symmetries of the MacDowell-Mansouri action. Let us note that covariantized diffeomorphisms (2.13) do not affect the connection ω_μ^{AB} if it is flat (*i.e.* has zero curvature $R_{\nu\mu}^{AB}$). In particular covariantized diffeomorphisms do not affect the background AdS geometry.

To show the equivalence of the action (2.9) with the action(2.3), it is useful to define a Lorentz connection by

$$\omega^{L\,AB} \equiv \omega^{AB} - \rho^{-2}(E^AV^B - E^BV^A)\,. \tag{2.15}$$

In the standard gauge, the curvature can be expressed in terms of the vielbein $e^a \equiv E^a = \rho\, \omega^a{}_{\hat{d}}$ and the non-vanishing components of the Lorentz connection $\omega^{L\, ab} = \omega^{ab}$ as

$$
\begin{aligned}
R^{ab} &= d\omega^{ab} + \omega^{aC}\omega_C{}^b = d\omega^{L\, ab} + \omega^{L\, a}{}_c\omega^{L\, cb} - \rho^{-2} e^a e^b = R^{L\, ab} + R^{cosm\; ab}\,, \\
R^{a\hat{d}} &= \rho^{-1} T^a\,.
\end{aligned}
$$

Inserting these gauge fixed expressions into the MMSW action yields the action (2.3), where $\Lambda = -\rho^{-2}$. The MMSW action is thus equivalent to (2.3) by partially fixing the gauge invariance.

Let us now consider the vacuum equations $R^{AB}(\omega_0) = 0$. They are equivalent to $T^a = 0$ and $R^{ab} = 0$ and, under the condition that $\mathrm{rank}(E^A_\nu) = d$, they uniquely define the local geometry of AdS_d with parameter ρ, in a coordinate independent way. The solution ω_0 also obviously satisfies the equations of motion of the MMSW action. To find the symmetries of the vacuum solution ω_0, one first notes that vacuum solutions are sent onto vacuum solutions by diffeomorphisms and local AdS transformations, because they transform the curvature homogeneously. Since covariantized diffeomorphisms do not affect ω_0, to find symmetries of the chosen solution ω_0 it is enough to check its transformation law under local $o(d-1,2)$ transformation. Indeed, by adjusting an appropriate covariantized diffemorphism it is always possible to keep the compensator invariant.

The solution ω_0 is invariant under local $o(d-1,2)$ transformations iff the parameter $\epsilon^{AB}(x)$ satisfies

$$
0 = D_0\epsilon^{AB}(x) = d\epsilon^{AB}(x) + \omega_0{}^A{}_C(x)\,\epsilon^{CB}(x) - \omega_0{}^B{}_C(x)\,\epsilon^{CA}(x)\,. \qquad (2.16)
$$

This equation fixes the derivatives $\partial_\mu \epsilon^{AB}(x)$ in terms of $\epsilon^{AB}(x)$ itself. In other words, once $\epsilon^{AB}(x_0)$ is chosen for some x_0, $\epsilon^{AB}(x)$ can be reconstructed for all x in a neighborhood of x_0, since by consistency[7] *all* derivatives of the parameter can be expressed as functions of the parameter itself. The parameters $\epsilon^{AB}(x_0)$ remain arbitrary, being parameters of the global symmetry $o(d-1,2)$. This means that, as expected for AdS space-time, the symmetry of the vacuum solution ω_0 is the global $o(d-1,2)$.

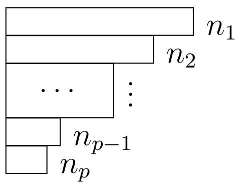
The lesson is that, to describe a gauge model that has a global symmetry h, it is useful to reformulate it in terms of the gauge connections ω and curvatures R of h in such a way that the zero curvature **condition** $R = 0$ solves the field equations and provides a solution with h as **its global symmetry**. If a symmetry h is not known, this observation can be used the other way around: by reformulating dynamics à la MacDowell-Mansouri one might guess the structure of an appropriate curvature R and thereby the non-Abelian algebra h.

[7] The identity $D_0^2 = R_0 = 0$ ensures consistency of the system (2.16), which is overdetermined because it contains $\frac{d^2(d+1)}{2}$ equations for $\frac{d(d+1)}{2}$ unknowns. Consistency in turn implies that higher space-time derivatives $\partial_{\nu_1}\ldots\partial_{\nu_n}\epsilon^{AB}(x)$ obtained by hitting (2.16) $n-1$ times with $D_{0\nu_k}$ are guaranteed to be symmetric in the indices $\nu_1\ldots\nu_n$.

3 Young tableaux and Howe duality

In this section, the Young tableaux are introduced. They characterize the irreducible representations of $gl(M)$ and $o(M)$. A representation of these algebras that will be useful in the sequel is built.

A *Young tableau* $\{n_i\}$ $(i = 1, \ldots, p)$ is a diagram which consists of a finite number $p > 0$ of rows of identical squares. The lengths of the rows are finite and do not increase: $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_p \geqslant 0$. The Young tableau $\{n_i\}$ is represented as follows:



Let us consider covariant tensors of $gl(M)$: $A_{a,\,b,\,c,\ldots}$ where $a, b, c, \ldots = 1, 2, \ldots, M$. Simple examples of these are the symmetric tensor $A^S_{a,\,b}$ such that $A^S_{a,\,b} - A^S_{b,\,a} = 0$, or the antisymmetric tensor $A^A_{a,\,b}$ such that $A^A_{a,\,b} + A^A_{b,\,a} = 0$.

A complete set of covariant tensors irreducible under $gl(M)$ is given by the tensors $A_{a^1_1 \ldots a^1_{n_1},\, \ldots\, ,\, a^p_1 \ldots a^p_{n_p}}$ $(n_i \geq n_{i+1})$ that are symmetric in each set of indices $\{a^i_1 \ldots a^i_{n_i}\}$ with fixed i and that vanish when one symmetrizes the indices of a set $\{a^i_1 \ldots a^i_{n_i}\}$ with any index a^j_l with $j > i$. The properties of these irreducible tensors can be conveniently encoded into Young tableaux. The Young tableau $\{n_i\}$ $(i = 1, \ldots, p)$ is associated with the tensor $A_{a^1_1 \ldots a^1_{n_1},\, \ldots\, ,\, a^p_1 \ldots a^p_{n_p}}$. Each box of the Young tableau is related to an index of the tensor, boxes of the same row corresponding to symmetric indices. Finally, the symmetrization of all the indices of a row with an index from any row below vanishes. In this way, the irreducible tensors A^S_{ab} and $A^A_{a,\,b}$ are associated with the Young tableaux $\square\square$ and $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$, respectively.

Let us introduce the polynomial algebra $\mathbb{R}[Y^a_i]$ generated by commuting generators Y^a_i where $i = 1, \ldots p$; $a = 1, \ldots M$. Elements of the algebra $\mathbb{R}[Y^a_i]$ are of the form

$$A(Y) = A_{a^1_1 \ldots a^1_{n_1},\, \ldots\, ,\, a^p_1 \ldots a^p_{n_p}} Y_1^{a^1_1} \ldots Y_1^{a^1_{n_1}} \ldots Y_p^{a^p_1} \ldots Y_p^{a^p_{n_p}} .$$

The condition that A is irreducible under $gl(M)$, *i.e.* is a Young tableau $\{n_i\}$, can be expressed as

$$\begin{aligned} Y^a_i \frac{\partial}{\partial Y^a_i} A(Y) &= n_i A(Y) \\ Y^b_i \frac{\partial}{\partial Y^b_j} A(Y) &= 0\,, \quad i < j \end{aligned} \tag{3.1}$$

where no sum on i is to be understood in the first equation. Let us first note that, as the generators Y^a_i commute, the tensor A is automatically symmetric in each set of indices $\{a^i_1 \ldots a^i_{n_i}\}$. The operator on the l.h.s. of the first equation of (3.1) then counts the number of Y^a_i's, the index i being fixed and the index a arbitrary. The first equation thus ensures that there are n_i indices contracted with Y_i, forming the set $a_1^i \dots a_{n_i}^i$. The second equation of (3.1) is equivalent to the vanishing of the symmetrization of a set of indices $\{a_1^i \dots a_{n_i}^i\}$ with an index b to the right. Indeed, the operator on the l.h.s. replaces a generator Y_j^b by a generator Y_i^b, $j > i$, thus projecting $A_{\dots, a_1^i \dots a_{n_i}^i, \dots b \dots}$ on its component symmetric in $\{a_1^i \dots a_{n_i}^i b\}$.

Two types of generators appear in the above equations: the generators

$$t_b^a = Y_i^a \frac{\partial}{\partial Y_i^b} \tag{3.2}$$

of $gl(M)$ and the generators

$$l_i^j = Y_i^a \frac{\partial}{\partial Y_j^a} \tag{3.3}$$

of $gl(p)$. These generators commute

$$[l_i^j, t_b^a] = 0 \tag{3.4}$$

and the algebras $gl(p)$ and $gl(M)$ are said to be *Howe dual* [27]. The important fact is that the irreducibility conditions (3.1) of $gl(M)$ are the highest weight conditions with respect to $gl(p)$.

When all the lengths n_i have the same value n, $A(Y)$ is invariant under $sl(p) \subset gl(p)$. Moreover, exchange of any two rows of this rectangular Young tableau only brings a sign factor $(-1)^n$, as is easy to prove combinatorically. The conditions (3.1) are then equivalent to:

$$\left(Y_i^a \frac{\partial}{\partial Y_j^a} - \frac{1}{p} \delta_i^j Y_k^a \frac{\partial}{\partial Y_k^a} \right) A(Y) = 0 \; . \tag{3.5}$$

Indeed, let us first consider the equation for $i = j$. The operator $Y_k^a \frac{\partial}{\partial Y_k^a}$ (where there is a sum over k and a) counts the total number m of Y's in $A(Y)$, while $Y_i^a \frac{\partial}{\partial Y_i^a}$ counts the number of Y_i's for some fixed i. The condition can only be satisfied if $\frac{m}{p}$ is an integer, *i.e.* $m = np$ for some integer n. As the condition is true for all i's, there are thus n Y_i's for every i. In other words, the tensorial coefficient of $A(Y)$ has p sets of n indices, *i.e.* is rectangular. The fact that it is a Young tableau is insured by the condition (3.5) for $i < j$, which is simply the second condition of (3.1). That the condition (3.5) is true both for $i < j$ and for $i > j$ is a consequence of the simple fact that any finite-dimensional $sl(p)$ module with zero $sl(p)$ weights (which are differences of lengths of the rows of the Young tableau) is $sl(p)$ invariant. Alternatively, this follows from the property that exchange of rows leaves a rectangular Young tableau invariant.

If there are p_1 rows of length n_1, p_2 rows of length n_2, etc. , then $A(Y)$ is invariant under $sl(p_1) \oplus sl(p_2) \oplus sl(p_3) \oplus \dots$, as well as under permutations within each set of p_i rows of length n_i.

To construct irreducible representations of $o(M-N, N)$, one needs to add the condition that A is traceless[8], which can be expressed as:

$$\frac{\partial^2}{\partial Y_i^a \partial Y_j^b}\,\eta^{ab}\,A(Y) = 0\,, \quad \forall\, i, j \tag{3.6}$$

where η_{ab} is the invariant metric of $o(M-N,N)$. The generators of $o(M-N,N)$ are given by

$$t_{ab} = \frac{1}{2}(\eta_{ac}t^c_b - \eta_{bc}t^c_a)\,. \tag{3.7}$$

They commute with the generators

$$k_{ij} = \eta_{ab}Y_i^a Y_j^b\,, \quad l_i^j = Y_i^a\frac{\partial}{\partial Y_j^a}\,, \quad m^{ij} = \eta^{ab}\frac{\partial^2}{\partial Y_i^a \partial Y_j^b}\,, \tag{3.8}$$

of $sp(2p)$. Note that the conditions (3.6) and (3.1) are highest weight conditions for the algebra $sp(2p)$ which is Howe dual to $o(M-N,N)$.

In the notation developed here, the irreducible tensors are manifestly symmetric in groups of indices. This is a convention: one could as well choose to have manifestly antisymmetric groups of indices corresponding to columns of the Young tableau. An equivalent implementation of the conditions for a tensor to be a Young tableau can be performed in the antisymmetric convention, by taking fermionic generators Y.

To end up with this introduction to Young diagrams, we give two "multiplication rules" of one box with an arbitrary Young tableau. More precisely, the tensor product of a vector (characterized by one box) with an irreducible tensor under $gl(M)$ characterized by a given Young tableau decomposes as the direct sum of irreducible tensors under $gl(M)$ corresponding to all possible Young tableaux obtained by adding one box to the initial Young tableau. For the (pseudo)orthogonal algebras, the tensor product of a vector (characterized by one box) with a traceless tensor characterized by a given Young tableau decomposes as the direct sum of traceless tensors under $o(M-N,N)$ corresponding to all possible Young tableaux obtained by adding or removing one box from the initial Young tableau (a box can be removed as a result of contraction of indices).

4 Free symmetric higher spin gauge fields as 1-forms

Properties of HS gauge theories are to a large extent determined by the HS global symmetries of their most symmetric vacua. The HS symmetry restricts interactions and fixes spectra of spins of massless fields in HS theories as ordinary supersymmetry does in supergravity. To elucidate the structure of a global HS algebra h it is useful to follow the approach in which fields, action and transformation laws are formulated in terms of the connection of h.

[8] For $M = 2N$ **modulo** 4, the irreducibility conditions also include the (anti)selfduality conditions on the tensors described by Young tableaux with $M/2$ rows. However, these conditions are not used in the analysis of HS dynamics in this paper.

The free HS gauge theories were originally formulated in terms of completely symmetric and double-traceless HS-fields $\varphi_{\nu_1\dots\nu_s}$ [28, 29], in a way analogous to the metric formulation of gravity (see [30] for a review on the metric-like formulation of HS theories). In Minkowski space-time $\mathbb{R}^{d-1,1}$, the spin-s Fronsdal action is

$$\begin{aligned} S_2^{(s)}[\varphi] \; = \; & -\frac{1}{2}\int d^dx \Big(\partial_\nu\varphi_{\mu_1\dots\mu_s}\partial^\nu\varphi^{\mu_1\dots\mu_s} \\ & -\frac{s(s-1)}{2}\,\partial_\nu\varphi^\lambda{}_{\lambda\mu_3\dots\mu_s}\partial^\nu\varphi_\rho{}^{\rho\mu_3\dots\mu_s} + s(s-1)\,\partial_\nu\varphi^\lambda{}_{\lambda\mu_3\dots\mu_s}\partial_\rho\varphi^{\nu\rho\mu_3\dots\mu_s} \\ & -s\,\partial_\nu\varphi^\nu{}_{\mu_2\dots\mu_s}\partial_\rho\varphi^{\rho\mu_2\dots\mu_s} - \frac{s(s-1)(s-2)}{4}\,\partial_\nu\varphi^{\nu\rho}{}_{\rho\mu_2\dots\mu_s}\partial_\lambda\varphi_\sigma{}^{\lambda\sigma\mu_2\dots\mu_s}\Big) \end{aligned} \tag{4.1}$$

where the metric-like field is double-traceless ($\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\varphi_{\mu_1\dots\mu_s}=0$) and has the dimension of $(length)^{1-d/2}$. This action is invariant under Abelian HS gauge transformations

$$\delta\varphi_{\mu_1\dots\mu_s} = \partial_{\{\mu_1}\epsilon_{\mu_2\dots\mu_s\}}\ , \tag{4.2}$$

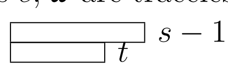
where the gauge parameter is a completely symmetric and traceless rank-$(s-1)$ tensor,

$$\eta^{\mu_1\mu_2}\epsilon_{\mu_1\dots\mu_{s-1}} = 0\ .$$

For spin $s=2$, (4.1) is the Pauli-Fierz action that is obtained from the linearization of the Einstein-Hilbert action via $g_{\mu\nu}=\eta_{\mu\nu}+\kappa\varphi_{\mu\nu}$ and the gauge transformations (4.2) correspond to linearized diffeomorphisms. The approach followed in this section is to generalize the MMSW construction of gravity to the case of HS gauge fields in AdS backgrounds [12]. The free HS dynamics will then be expressed in terms of 1-form connections taking values in certain representations of the AdS_d algebra $o(d-1,2)$.

The procedure is similar to that of Section 2 for gravity. The HS metric-like field is replaced by a frame-like field and a set of connections. These new fields are then united in a single connection. The action is given in terms of the latter connection and a compensator vector. It is constructed in such a way that it reproduces the dynamics of the Fronsdal formulation, once auxiliary fields are removed and part of the gauge invariance is fixed.

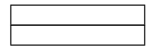
4.1 Frame-like formulation of higher spins

The double-traceless metric-like HS gauge field $\varphi_{\mu_1\dots\mu_s}$ is replaced by a frame-like field $e_\mu{}^{a_1\dots a_{s-1}}$, a Lorentz-like connection $\omega_\mu{}^{a_1\dots a_{s-1},\,b}$ [31] and an extra set of connections $\omega_\mu{}^{a_1\dots a_{s-1},\,b_1\dots b_t}$, where $t=2,\dots,s-1$ [32, 33]. All fields e, ω are traceless in the fiber indices a,b, which have the symmetry of the Young tableaux 

, where $t=0$ for the frame-like field and $t=1$ for the Lorentz-like connection. The metric-like field arises as the completely symmetric part of the frame field [31],

$$\varphi_{\mu_1\dots\mu_s} = e_{\{\mu_1,\,\mu_2\dots\mu_s\}}\ ,$$

where all fiber indices have been lowered using the AdS or flat frame field $e_{0\mu}^{\ a}$ defined in Section 2. From the fiber index tracelessness of the frame field follows automatically that the field $\varphi_{\mu_1\ldots\mu_s}$ is double traceless.

The frame-like field and other connections are then combined [24] into a connection 1-form $\omega^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ (where $A, B = 0, \ldots, d$) taking values in the irreducible $o(d-1,2)$-module characterized by the two-row traceless rectangular Young tableau  of length $s-1$, that is

$$\omega_\mu^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} = \omega_\mu^{\{A_1\ldots A_{s-1}\},B_1\ldots B_{s-1}} = \omega_\mu^{A_1\ldots A_{s-1},\{B_1\ldots B_{s-1}\}}\,,$$
$$\omega_\mu^{\{A_1\ldots A_{s-1},A_s\}\,B_2\ldots B_{s-1}} = 0\,,\quad \omega_\mu^{A_1\ldots A_{s-3}\,C}{}_{C,}{}^{B_1\ldots B_{s-1}} = 0\,. \tag{4.3}$$

One also introduces a time-like vector V^A of constant norm ρ^2. The component of the connection $\omega^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ that is most parallel to V^A is the frame-like field

$$E^{A_1\ldots A_{s-1}} = \omega^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}} V_{B_1}\ldots V_{B_{s-1}}\,,$$

while the less V-longitudinal components are the other connections. Note that the contraction of the connection with more than $s-1$ compensators V^A is zero by virtue of (4.3). Let us be more explicit in a specific gauge. As in the MMSW gravity reformulation, one can show that V^A is a pure gauge field and that one can reach the standard gauge $V^A = \delta^A_{\hat d}\rho$ (the argument will not be repeated here). In the standard gauge, the frame field and the connections are given by

$$\begin{aligned} e^{\ a_1\ldots a_{s-1}} &= \rho^{s-1}\omega^{\ a_1\ldots a_{s-1},\,\hat d\ldots\hat d} \\ \omega^{\ a_1\ldots a_{s-1},\,b_1\ldots b_t} &= \rho^{s-1-t}\omega^{\ a_1\ldots a_{s-1},\,b_1\ldots b_t\,\hat d\ldots\hat d} \end{aligned}$$

where the powers of ρ originate from a corresponding number of contractions with the compensator vector V^A. These normalization factors are consistent with the fact that the auxiliary fields $\omega_\mu^{\ a_1\ldots a_{s-1},\,b_1\ldots b_t}$ will be found to be expressed via t partial derivatives of the frame field $e_\mu^{\ a_1\ldots a_{s-1}}$ (ρ is a length scale) at the linearized level.

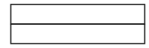
The linearized field strength or curvature is defined as the $o(d-1,2)$ covariant derivative of the connection $\omega^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$, *i.e.* by

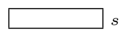
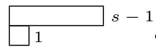
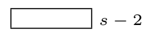
$$\begin{aligned} R_1^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} &= D_0\omega^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} \\ &= d\omega^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} + \omega_0{}^{A_1}{}_C\omega^{CA_2\ldots A_{s-1},\,B_1\ldots B_{s-1}} + \ldots \\ &\qquad + \omega_0{}^{B_1}{}_C\omega^{A_1\ldots A_{s-1},\,CB_2\ldots B_{s-1}} + \ldots\,, \end{aligned} \tag{4.4}$$

where the dots stand for the terms needed to get an expression symmetric in $A_1\ldots A_{s-1}$ and $B_1\ldots B_{s-1}$, and $\omega_0{}^A{}_B$ is the $o(d-1,2)$ connection associated to the AdS space solution, as defined in Section 2. The connection $\omega_\mu^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ has dimension $(length)^{2-s-d/2}$ in such a way that the field strength $R_{\mu\nu}^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ has proper dimension $(length)^{1-s-d/2}$.

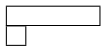
As $(D_0)^2 = R_0 = 0$, the linearized curvature R_1 is invariant under Abelian gauge transformations of the form

$$\delta\omega^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}} = D_0\epsilon^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}\,. \tag{4.5}$$

The gauge parameter $\epsilon^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}$ has the symmetry  and is traceless.

Before writing the action, let us analyze the frame field and its gauge transformations, in the standard gauge. According to the multiplication rule formulated in the end of section 3, the frame field $e_\mu{}^{a_1\ldots a_{s-1}}$ contains three irreducible (traceless) Lorentz components characterized by the symmetry of their indices:  ,  and  , where the last tableau describes the trace component of the frame field $e_\mu{}^{a_1\ldots a_{s-1}}$. Its gauge transformations are given by (4.5) and read

$$\delta e^{a_1\ldots a_{s-1}} = D_0^L \epsilon^{a_1\ldots a_{s-1}} - e_{0\,c}\epsilon^{a_1\ldots a_{s-1},\,c}\,.$$

The parameter $\epsilon^{a_1\ldots a_{s-1},\,c}$ is a generalized local Lorentz parameter. It allows us to gauge away the component  of the frame field. The other two components of the latter just correspond to a completely symmetric double traceless Fronsdal field $\varphi_{\mu_1\ldots\mu_s}$. The remaining invariance is then the Fronsdal gauge invariance (4.2) with a traceless completely symmetric parameter $\epsilon^{a_1\ldots a_{s-1}}$.

4.2 Action of higher spin gauge fields

For a given spin s, the most general $o(d-1,2)$-invariant action that is quadratic in the linearized curvatures (4.4) and, for the rest, built only from the compensator V^C and the background frame field $E_0^B = D_0 V^B$ is

$$S_2^{(s)}[\,\omega_\mu^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}\,,\omega_0^{AB}\,,V^C\,] \;=\; \frac{1}{2}\sum_{p=0}^{s-2} a(s,p)\,S^{(s,p)}[\,\omega_\mu^{A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}\,,\omega_0^{AB}\,,\,V^C\,] \tag{4.6}$$

where $a(s,p)$ is the *a priori* arbitrary coefficient of the term

$$\begin{aligned} S^{(s,p)}[\,\omega, V\,] \;&=\; \epsilon_{A_1\ldots A_{d+1}}\int_{M^d} E_0^{A_5}\ldots E_0^{A_d}V^{A_{d+1}}V_{C_1}\ldots V_{C_{2(s-2-p)}}\,\times \\ &\qquad \times R_1^{A_1B_1\ldots B_{s-2},\,A_2C_1\ldots C_{s-2-p}D_1\ldots D_p}\,R_1{}^{A_3}{}_{B_1\ldots B_{s-2},}{}^{A_4C_{s-1-p}\ldots C_{2(s-2-p)}}{}_{D_1\ldots D_p}\,. \end{aligned}$$

This action is manifestly invariant under diffeomorphisms, local $o(d-1,2)$ transformations (2.11) and Abelian HS gauge transformations (4.5) that leave invariant the linearized HS curvatures (4.4). Having fixed the AdS_d background gravitational field ω_0^{AB} and compensator V^A, diffeomorphisms and local $o(d-1,2)$ transformations break down to the AdS_d global symmetry $o(d-1,2)$.

As will be explained in Section 6, the connections $\omega_\mu{}^{a_1\ldots a_{s-1},\,b_1\ldots b_t}$ can be expressed as t derivatives of the frame-like field, via analogues of the torsion constraint. Therefore coefficients $a(s,p)$ must be chosen in such a way that the Euler-Lagrange derivatives are nonvanishing only for the frame field and the first connection ($t=1$). All other fields, *i.e.* the connections $\omega_\mu{}^{a_1\ldots a_{s-1},\,b_1\ldots b_t}$ with $t>1$, appear only through total derivatives. They are called extra fields[9]. This requirement makes sure that higher-derivative terms are absent from the free theory and fixes uniquely the spin-s free action up to a coefficient $b(s)$ in front of the action. More precisely, the coefficient $a(s,p)$ is essentially a relative coefficient given by [24]

$$a(s,p) = b(s)(-\Lambda)^{-(s-p-1)}\frac{(d-5+2(s-p-2))!!\,(s-p-1)}{(s-p-2)!}$$

where $b(s)$ is the arbitrary spin-dependent factor.

The equations of motion for $\omega_\mu{}^{a_1\ldots a_{s-1},\,b}$ are equivalent to the "zero-torsion condition"

$$R_{1\,A_1\ldots A_{s-1},\,B_1\ldots B_{s-1}}V^{B_1}\ldots V^{B_{s-1}} = 0\,.$$

They imply that $\omega_\mu{}^{a_1\ldots a_{s-1},\,b}$ is an auxiliary field that can be expressed in terms of the first derivative of the frame field. Substituting the found expression for $\omega_\mu{}^{a_1\ldots a_{s-1},\,b}$ into the HS action yields an action only expressed in terms of the frame field and its first derivative, modulo total derivatives. Actually, as gauge symmetries told us, the action actually depends only on the completely symmetric part of the frame field, *i.e.* the Fronsdal field. Moreover, the action (4.6) has the same gauge invariance as Fronsdal's one, thus it must be proportional to the Fronsdal action (4.1) because the latter is fixed up to a front factor by the requirements of being gauge invariant and of being of second order in the derivatives of the field [34].

5 Simplest higher spin algebras

In the previous section, the dynamics of free HS gauge fields has been expressed as a theory of 1-forms, the $o(d-1,2)$ fiber indices of which have symmetries characterized by two-row rectangular Young tableaux. This suggests that there exists a non-Abelian HS algebra $h\supset o(d-1,2)$ that admits a basis formed by a set of elements $T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$ in irreducible representations of $o(d-1,2)$ characterized by such Young tableaux. More precisely, the basis elements $T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$ satisfy the following properties $T_{\{A_1\ldots A_{s-1},A_s\}B_2\ldots B_{s-1}} = 0$, $T_{A_1\ldots A_{s-3}CC,}{}^{B_1\ldots B_{s-1}} = 0$, and the basis contains the $o(d-1,2)$ basis elements $T_{A,B} = -T_{B,A}$ such that all generators transform as $o(d-1,2)$ tensors

$$[T_{C,D}\,,\,T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}] = \eta_{DA_1}T_{CA_2\ldots A_{s-1},B_1\ldots B_{s-1}} + \ldots\,. \tag{5.1}$$

The question is whether a non-Abelian algebra h with these properties really exists. If yes, the Abelian curvatures R_1 can be understood as resulting from the linearization of the non-Abelian field curvatures $R = dW + W^2$ of h with the h gauge connection $W = \omega_0 + \omega$, where ω_0 is some fixed flat (*i.e.* vanishing curvature) zero-order connection of the subalgebra $o(d-1,2)\subset h$ and ω is the first-order dynamical part which describes massless fields of various spins.

[9]The extra fields show up in the non-linear theory and are responsible for the higher-derivatives as well as for the terms with negative powers of Λ in the interaction vertices.

The HS algebras with these properties were originally found for the case of AdS_4 [35, 36, 37] in terms of spinor algebras. Then this construction was extended to HS algebras in AdS_3 [38, 39, 40] and to $4d$ conformal HS algebras [41, 42] equivalent to the AdS_5 algebras [43]. $d = 7$ HS algebras [44] were also built in spinorial terms. Conformal HS algebra h in any dimension was first found by Eastwood [45]. Here we use the construction of the same algebra as given in [12], which is based on vector oscillator algebra (i.e., Weyl algebra).

5.1 Weyl algebras

The Weyl algebra A_{d+1} is generated by oscillators $\hat{Y}_i^A$, where $i = 1, 2$ and $A = 0, 1, \dots d$, satisfying the commutation relations

$$[\hat{Y}_i^A, \hat{Y}_j^B] = \epsilon_{ij}\, \eta^{AB} \,, \tag{5.2}$$

where $\epsilon_{ij} = -\epsilon_{j\,i}$ and $\epsilon_{12} = \epsilon^{12} = 1$. The invariant metrics $\eta_{AB} = \eta_{BA}$ and symplectic form ϵ^{ij} of $o(d-1, 2)$ and $sp(2)$, respectively, are used to raise and lower indices in the usual manner $A^A = \eta^{AB} A_B$, $a^i = \epsilon^{ij} a_j$, $a_i = a^j \epsilon_{j\,i}$. The Weyl algebra A_{d+1} is usually realized by taking as generators

$$\hat{Y}_1^A = \eta^{AB} \frac{\partial}{\partial X^B} \,, \quad \hat{Y}_2^A = X^A \,,$$

i.e. the real Weyl algebra is realized as the algebra $\mathbb{R} < X_A, \partial_B >$ of differential operators acting on formal power series $\Phi(X)$ in the variable X^A. One can also construct the Weyl algebra A_{d+1} starting from the associative algebra $\mathbb{R} < \hat{Y}_i^A >$ freely generated by the variables $\hat{Y}_i^A$, *i.e.* spanned by all (real) linear combinations of all possible products of the variables $\hat{Y}_i^A$. The Weyl algebra A_{d+1} is realized as the quotient of $\mathbb{R} < \hat{Y}_i^A >$ by the ideal made of all elements proportional to

$$\hat{Y}_i^A \hat{Y}_j^B - \hat{Y}_j^B \hat{Y}_i^A - \epsilon_{ij} \eta^{AB} \,.$$

In order to pick one representative of each equivalence class, we work with Weyl ordered operators. These are the operators completely symmetric under the exchange of $\hat{Y}_i^A$'s. The generic element of A_{d+1} is then of the form

$$f(\hat{Y}) = \sum_{p=0}^{\infty} \phi^{i_1 \dots i_p}_{A_1 \dots A_p} \hat{Y}_{i_1}^{A_1} \dots \hat{Y}_{i_p}^{A_p} \,, \tag{5.3}$$

where $\phi^{i_1 \dots i_p}_{A_1 \dots A_p}$ is symmetric under the exchange $(i_k, A_k) \leftrightarrow (i_l, A_l)$. Equivalently, one can define basis elements $S^{A_1 \dots A_m\,, B_1 \dots B_n}$ that are completely symmetrized products of m $\hat{Y}_1^A$'s and n $\hat{Y}_2^B$'s (*e.g.* $S^{A,B} = \{\hat{Y}_1^A \,, \hat{Y}_2^B\}$), and write the generic element as

$$f(\hat{Y}) = \sum_{m,n} f_{A_1 \dots A_m\,, B_1 \dots B_n} S^{A_1 \dots A_m\,, B_1 \dots B_n} \tag{5.4}$$

where the coefficients $f_{A_1 \dots A_m\,, B_1 \dots B_n}$ are symmetric in the indices A_i and B_j.

The elements

$$T^{AB} = -T^{BA} = \frac{1}{4}\,\{\hat{Y}^{Ai}\,, \hat{Y}^{B}_{i}\} \tag{5.5}$$

satisfy the $o(d-1,2)$ algebra

$$[T^{AB}, T^{CD}] = \frac{1}{2}\Big(\eta^{BC}T^{AD} - \eta^{AC}T^{BD} - \eta^{BD}T^{AC} + \eta^{AD}T^{BC}\Big)\,,$$

because of (5.2). When the Weyl algebra is realized as the algebra of differential operators $A_{d+1} \cong \mathbb{R} < X_A, \partial_B >$, then $T^{AB} = X^{[A}\partial^{B]}$ generates rotations of $\mathbb{R}^{d-1,2}$ acting on a scalar $\Phi(X^A)$.

The operators

$$t_{ij} = t_{ji} = \frac{1}{2}\{\hat{Y}^{A}_{i}\;, \hat{Y}^{B}_{j}\}\eta_{AB} \tag{5.6}$$

generate $sp(2)$. The various bilinears T^{AB} and t_{ij} commute

$$[T^{AB}, t_{ij}] = 0\,, \tag{5.7}$$

thus forming a Howe dual pair $o(d-1,2) \oplus sp(2)$.

5.2 Higher spin algebras

Let us consider the subalgebra $\mathcal{S}$ of elements $f(\hat{Y})$ of A_{d+1} that are invariant under $sp(2)$, *i.e.* $[f(\hat{Y}), t_{ij}] = 0$. Replacing $f(\hat{Y})$ by its Weyl symbol $f(Y)$, which is the ordinary function of commuting variables Y that has the same power series expansion as $f(\hat{Y})$ in the Weyl ordering, the $sp(2)$ invariance condition takes the form

$$\Big(\epsilon_{k\,j}Y^{A}_{i}\frac{\partial}{\partial Y^{A}_{k}} + \epsilon_{k\,i}Y^{A}_{j}\frac{\partial}{\partial Y^{A}_{k}}\Big)f(Y^{A}_{i}) = 0\,, \tag{5.8}$$

which is equivalent to (3.5) for $p = 2$. This condition implies that the coefficients $f_{A_1\ldots A_n,\,B_1\ldots B_m}$ vanish except when $n = m$, and the non-vanishing coefficients carry irreducible representations of $gl(d+1)$ corresponding to two-row rectangular Young tableaux. The $sp(2)$ invariance condition means in particular that (the symbol of) any element of $\mathcal{S}$ is an even function of Y^{A}_{i}. Let us note that the rôle of $sp(2)$ in our construction is reminiscent of that of $sp(2)$ in the conformal framework description of dynamical models (two-time physics) [46, 47].

However, the associative algebra $\mathcal{S}$ is not simple. It contains the ideal $\mathcal{I}$ spanned by the elements of $\mathcal{S}$ of the form $g = t_{ij}\,g^{ij}(\hat{Y})$. Due to the definition of t_{ij} (5.6), all traces of two-row Young tableaux are contained in $\mathcal{I}$. As a result, the associative algebra $\mathcal{A} = \mathcal{S}/\mathcal{I}$ contains only all traceless two-row rectangular tableaux. Let us choose a basis $\{T_s\}$ of $\mathcal{A}$ where the elements T_s carry an irreducible representation of $o(d-1,2)$ characterized by a two-row Young tableau with $s-1$ columns: $T_s \sim \begin{array}{|c|}\hline \qquad\quad \\ \hline \qquad\quad \\ \hline\end{array}\begin{smallmatrix} s-1 \\ s-1\end{smallmatrix}$.

Now consider the complex Lie algebra $h_{\mathbb{C}}$ obtained from the associative algebra $\mathcal{A}$ by complexification and taking the commutator as Lie bracket, the associativity property of

$\mathcal{A}$ thereby translating into the Jacobi identity of $h_{\mathbb{C}}$. It admits several inequivalent real forms $h_{\mathbb{R}}$ such that $h_{\mathbb{C}} = h_{\mathbb{R}} \oplus i\, h_{\mathbb{R}}$. The particular real form that corresponds to a unitary HS theory is denoted by $hu(1/sp(2)[d-1,2])$ (this notation refers to the Howe dual pair $sp(2) \oplus o(d-1,2)$ and to the fact that the related spin one Yang-Mills subalgebra is $u(1)$) [12]). It is spanned by the elements satisfying the following reality condition

$$(f(\hat{Y}))^\dagger = -f(\hat{Y}) \ , \tag{5.9}$$

at the condition that $\dagger$ is an involution[10] of the complex Weyl algebra defined by the relation

$$(\hat{Y}_i^A)^\dagger = i\hat{Y}_i^A \,. \tag{5.10}$$

Thanks to using the Weyl ordering prescription, reversing of the order of the oscillators has no effect so that $(f(\hat{Y}))^\dagger = \bar{f}(i\hat{Y})$ where the bar means complex conjugation of the coefficients in the expansion (5.3)

$$\bar{f}(\hat{Y}) = \sum_{p=0}^{\infty} \bar{\phi}^{\,i_1\ldots i_p}_{A_1\ldots A_p} \hat{Y}^{A_1}_{i_1} \ldots \hat{Y}^{A_p}_{i_p} \ . \tag{5.11}$$

As a result, the reality condition (5.9) implies that the coefficients in front of the generators T_s (*i.e.* the basis elements $S^{A_1\ldots A_{s-1}\,,B_1\ldots B_{s-1}}$) with even and odd s are, respectively, real and pure imaginary. In particular, the spin two generator T^{AB} enters with **a** real coefficient.

The real Lie algebra $hu(1/sp(2)[d-1,2])$ is infinite-dimensional. It contains the space-time isometry algebra $o(d-1,2)$ as the subalgebra generated by T^{AB}. The basis elements T_s ($s \geqslant 1$) will be associated with a spin-s gauge field. Taking two HS generators T_{s_1} and T_{s_2}, being polynomials of degrees $2(s_1-1)$ and $2(s_2-1)$ in $\hat{Y}$, respectively, one obtains (**modulo some coefficients**)

$$[T_{s_1}\,, T_{s_2}] = \sum_{m=1}^{min(s_1,s_2)-1} T_{s_1+s_2-2m} = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2}\,. \tag{5.12}$$

Let us notice that the formula (5.12) is indeed consistent with the requirement (5.1). Furthermore, once a gauge field of spin $s > 2$ appears, the HS symmetry algebra requires an infinite tower of HS gauge fields to be present, together with gravity. Indeed, the commutator $[T_s, T_s]$ of two spin-s generators gives rise to generators T_{2s-2}, corresponding to **a** gauge field of spin $s' = 2s-2 > s$, and also gives rise to generators T_2 of $o(d-1,2)$, corresponding to gravity fields. The spin-2 barrier separates theories with usual finite-dimensional lower-spin symmetries from those with infinite-dimensional HS symmetries. More precisely, the maximal finite-dimensional subalgebra of $hu(1/sp(2)[d-1,2])$ is the direct sum: $u(1)\oplus o(d-1,2)$, where $u(1)$ is the center associated with the elements proportional to the unit. Another consequence of the commutation relations (5.12) is that even spin generators T_{2p} ($p \geqslant 1$) span a proper subalgebra of the HS algebra.

[10] This means that $\dagger$ conjugates complex numbers, reverses the order of operators and squares to unity: $(\mu f)^\dagger = \bar{\mu} f^\dagger$, $(fg)^\dagger = g^\dagger f^\dagger$, $((f)^\dagger)^\dagger = f$. To be an involution of the Weyl algebra $\dagger$ is required to leave invariant its defining relation (5.2).

Note that the general structure of the commutation relations (5.12) follows from the fact that the associative Weyl algebra **possesses** an antiautomorphism $\rho(f(\hat{Y}))=f(i\hat{Y})$ in the Weyl ordering. (The difference between ρ and $\dagger$ is that the former does not conjugate complex numbers.) It induces an automorphism τ of the Lie algebra $hu(1/sp(2)[d-1,2])$ with

$$\tau(f(\hat{Y}))=-f(i\hat{Y})\,. \tag{5.13}$$

This automorphism is involutive in the HS algebra, *i.e.* $\tau^2=Id$, because $f(-\hat{Y})=f(\hat{Y})$. Therefore the algebra decomposes into subspaces of τ–odd and τ–even elements. Clearly, these are the subspaces of odd and even spins, respectively. This determines the general structure of the commutation relation (5.12), implying in particular that even spin subspace forms a proper subalgebra of $hu(1/sp(2)[d-1,2])$, which in notation of [12] is $ho(1/sp(2)[d-1,2])$.

Alternatively, the commutation relation (5.12) can be obtained from the following reasoning. The oscillator commutation relation (5.2) contracts two $\hat{Y}$ variables and produces a tensor ϵ_{ij}. Thus, because the commutator of two polynomials is antisymmetric, only odd numbers of contractions can survive. A HS generator T_s is a polynomial of degree $2(s-1)$ in $\hat{Y}$ with the symmetries associated with the two-row Young tableau of length $s-1$. Computing the commutator $[T_{s_1},T_{s_2}]$, only odd numbers $2m-1$ of contractions survive ($m\geqslant 1$) leading to polynomials of degree $2(s_1+s_2-2m-1)$ in $\hat{Y}$. They correspond to two-row rectangular Young tableaux[11] of length s_1+s_2-2m-1 that are associated to basis elements $T_{s_1+s_2-2m}$. The maximal number, say $2n-1$, of possible contractions is at most equal to the lowest polynomial degree in $\hat{Y}$ of the two generators. Actually, it must be one unit smaller since the numbers of surviving contractions are odd numbers while the polynomial degrees of the generators are even numbers. The lowest polynomial degree in Y of the two generators T_{s_1} and T_{s_2} is equal to $2\Big(min(s_1,s_2)-1\Big)$. Hence, $n=min(s_1,s_2)-1$. Consequently, the lowest possible polynomial degree of a basis element appearing on the right-hand-side of (5.12) is equal to $2(s_1+s_2-2n-1)=2(|s_1-s_2|+1)$. The corresponding generators are $T_{|s_1-s_2|+2}$.

The gauge fields of $hu(1/sp(2)[d-1,2])$ are the components of the connection 1-form

$$\omega(\hat{Y},x)=\sum_{s=1}^{\infty}dx^\mu\, i^{s-2}\omega_{\mu\ A_1\dots A_{s-1}\,,B_1\dots B_{s-1}}(x)\hat{Y}_1^{A_1}\dots\hat{Y}_1^{A_{s-1}}\hat{Y}_2^{B_1}\dots\hat{Y}_2^{B_{s-1}}\,.$$

They take values in the traceless two-row rectangular Young tableaux of $o(d-1,2)$. It is obvious from this formula why the basis elements T_s are associated to spin s fields. The curvature and gauge transformations have the standard Yang-Mills form:

$$R=d\omega+\omega^2\,,\qquad \delta\omega=D\epsilon\equiv d\epsilon+[\omega,\epsilon]\,. \tag{5.14}$$

[11]Note that the formal tensor product of two two-row rectangular Young tableaux contains various Young tableaux having up to four rows. The property that only two-row Young tableaux appear in the commutator of HS generators is a consequence of the $sp(2)$ invariance condition.

The formalism here presented is equivalent [48] to the spinor formalism developed previously for lower dimensions, where the HS algebra was realized in terms of commuting spinor oscillators $\hat{y}^\alpha, \hat{\bar{y}}^{\dot{\alpha}}$ (see, for example, [3, 49] for reviews) as

$$[\hat{y}^\alpha, \hat{y}^\beta] = i\epsilon^{\alpha\beta} \ , \quad [\hat{\bar{y}}^{\dot{\alpha}}, \hat{\bar{y}}^{\dot{\beta}}] = i\epsilon^{\dot{\alpha}\dot{\beta}} \ , \quad [\hat{y}^\alpha, \hat{\bar{y}}^{\dot{\beta}}] = 0 \ .$$

Though limited to $d = 3, 4$, the definition of the HS algebra with spinorial variables is simpler with respect to the case of vector oscillators Y_i^A, since generators are automatically traceless (because $\hat{y}^\alpha \hat{y}_\alpha = \hat{\bar{y}}^{\dot{\alpha}} \hat{\bar{y}}_{\dot{\alpha}} = const$), and there is no ideal to be factored out. However, spinorial realizations of $4d$ conformal HS algebras [41, 42] equivalent to the AdS_5 algebras [43] and $d = 7$ HS algebras [44] require the factorization of an ideal.

6 Free differential algebras and unfolded dynamics

The subsection 6.1 reviews some general definitions of the unfolded formulation of dynamical systems, a particular case of which are the HS field equations [50]. The strategy of the unfolded formalism is presented in Subsection 6.2. It makes use of free differential algebras in order to write consistent nonlinear dynamics.

6.1 Definition and examples of free differential algebras

Let us consider an arbitrary set of differential $p-$forms $W^\alpha \in \Omega^{p_\alpha}(\mathcal{M}^d)$ with $p_\alpha \geqslant 0$ (0-forms are included) and α is an index enumerating various forms, which, generically, may range in the infinite set $1 \leqslant \alpha < \infty$.

Let $R^\alpha \in \Omega^{p_\alpha+1}(\mathcal{M}^d)$ be the generalized curvatures defined by the relations

$$R^\alpha = dW^\alpha + G^\alpha(W^\beta)\,, \tag{6.1}$$

where $G^\alpha(W^\beta)$ are polynomial functions of W^β built with the aid of the exterior product of differential forms. The choice of a function $G^\alpha(W^\beta)$ satisfying the generalized Jacobi identity

$$G^\beta \frac{\delta^L G^\alpha}{\delta W^\beta} \equiv 0 \tag{6.2}$$

(the derivative with respect to W^β is left) defines a free differential algebra[12] [51]. We emphasize that the property (6.2) is a condition on the function $G^\alpha(W)$ to be satisfied identically for all W^β. The property (6.2) guarantees the generalized Bianchi identity

$$dR^\alpha = R^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}\,,$$

[12]We remind the reader that a differential d is a Grassmann odd nilpotent derivation of degree one, *i.e.* it satisfies the (graded) Leibniz rule and $d^2 = 0$. A differential algebra is a graded algebra endowed with a differential d. Actually, the "free differential algebras" (in physicist terminology) are more precisely christened "graded commutative free differential algebra" by mathematicians (this means that the algebra does not obey algebraic relations apart from graded commutativity). In the absence of 0-forms, the structure of these algebras is classified by Sullivan [52].

which tells us that the differential equations on W^β

$$R^\alpha = 0 \tag{6.3}$$

are consistent with $d^2 = 0$ and supercommutativity. Conversely, the property (6.2) is necessary for the consistency of the equation (6.3).

One defines the gauge transformations as

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}\,, \tag{6.4}$$

where $\varepsilon^\alpha(x)$ has form degree equal to $p_\alpha - 1$ (so that 0-forms W^α do not give rise to any gauge parameter). With respect to these gauge transformations the generalized curvatures transform as

$$\delta R^\alpha = -R^\gamma \frac{\delta^L}{\delta W^\gamma}\left(\varepsilon^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}\right)\,,$$

due to the property (6.2). This implies the gauge invariance of the equations (6.3). Also, since the equations (6.3) are formulated entirely in terms of differential forms, they are explicitly general coordinate invariant.

Unfolding means reformulation of the dynamics of one or another system in the form (6.3) which, as we explain below, is always possible by virtue of introducing enough auxiliary fields. Note that, according to (6.1), in this approach exterior differential of all fields is expressed in terms of the fields themselves.

Let h be a Lie (super)algebra, a basis of which is the set $\{T_\alpha\}$. Let $\omega = \omega^\alpha T_\alpha$ be a 1-form taking values in h. If one chooses $G(\omega) = \omega^2 \equiv \frac{1}{2}\omega^\alpha\omega^\beta[T_\alpha, T_\beta]$, then the equation (6.3) with $W = \omega$ is the zero-curvature equation $d\omega + \omega^2 = 0$. The relation (6.2) amounts to the usual Jacobi identity for the Lie algebra h (one can either check it explicitly in components or one can simply reinterpret the intrinsic formulas of Appendix B). In the same way, (6.4) is the usual gauge transformation of the connection ω.

If the set W^α also contains some p-forms denoted by $\mathcal{C}^i$ (*e.g.* 0-forms) and if the functions G^i are linear in ω and C,

$$G^i = \omega^\alpha (T_\alpha)^i{}_j \mathcal{C}^j\,, \tag{6.5}$$

then the relation (6.2) implies that the coefficients $(T_\alpha)^i{}_j$ define some matrices T_α forming a representation T of h, acting in a module V where the $\mathcal{C}^i$ take their values. The corresponding equation (6.3) is a covariant constancy condition $D_\omega \mathcal{C} = 0$ for $\mathcal{C}$ as a h-module, where $D_\omega \equiv d + \omega$.

6.2 Unfolding strategy

¿From the previous considerations, one knows that the system of equations

$$d\omega_0 + \omega_0^2 = 0 \tag{6.6}$$

$$D_{\omega_0}\mathcal{C} = 0 \tag{6.7}$$

forms a free differential algebra. The first equation usually describes a background (for example Minkowski or AdS) along with some pure gauge modes. The connection 1-form ω_0 takes value in some Lie algebra h. The second equation may describe nontrivial dynamics if $\mathcal{C}$ is a 0-form C that forms an infinite-dimensional h-module T appropriate to describe the space of all moduli of solutions (*i.e.* the initial data). One can wonder how the set of equations (6.6) and

$$D_{\omega_0} C = 0 \tag{6.8}$$

could describe any dynamics because it implies that (locally) the connection ω_0 is pure gauge and C is covariantly constant,

$$\begin{aligned} \omega_0(x) &= g^{-1}(x)\, dg(x)\,, & (6.9) \\ C(x) &= g(x)\, \mathrm{C}\,, & (6.10) \end{aligned}$$

where $g(x)$ is some function of the position x taking values in the Lie group associated with h (by exponentiation), and C is a constant vector of the h-module T. Since the gauge parameter $g(x)$ does not carry any physical degree of freedom, all physical information is contained in the value $C(x_0) = g(x_0)\mathrm{C}$ of the 0-form $C(x)$ in a fixed point x_0 of space-time. But as one will see in Section 7, if the 0-form $C(x)$ somehow parametrizes all derivatives of the original dynamical fields, then, supplemented with some algebraic constraints (that, in turn, single out an appropriate h-module), it can actually describe nontrivial dynamics. Indeed, the restrictions imposed on values of some 0-forms in a fixed point x_0 of space-time can lead to a nontrivial dynamics if the set of 0-forms is rich enough to describe all space-time derivatives of the dynamical fields in a fixed point of space-time, provided that the constraints just single out those values of the derivatives which are compatible with the original dynamical equations. By knowing a solution (6.10) one knows all the derivatives of the dynamical fields compatible with the field equations and can therefore reconstruct these fields by analyticity in some neighborhood of x_0.

The p-forms with $p > 0$ contained in $\mathcal{C}$ (if any) are still pure gauge in these equations. As will be clear from the examples below, the meaning of the 0-forms C contained in $\mathcal{C}$ is that they describe all gauge invariant degrees of freedom (*e.g.* the spin-0 scalar field, the spin-1 Maxwell field strength, the spin-2 Weyl tensor, etc., and all their on-mass-shell nontrivial derivatives). When the gauge invariant 0-forms are identified with derivatives of the gauge fields which are $p > 0$ forms, this is expressed by a deformation of the equation (6.7)

$$D_{\omega_0}\mathcal{C} = P(\omega_0)\mathcal{C}\,, \tag{6.11}$$

where $P(\omega_0)$ is a linear operator (depending on ω_0 at least quadratically) acting on $\mathcal{C}$. The equations (6.6) and (6.11) are of course required to be consistent, *i.e.* to describe some free differential algebra, which is a deformation of (6.6) and (6.7). If the deformation is trivial, one can get rid of the terms on the right-hand-side of (6.11) by a field redefinition. The interesting case therefore is when the deformation is nontrivial. A useful criterium of whether the deformation (6.11) is trivial or not is given in terms of the σ_- cohomology in Section 9.

The next step is to interpret the equations (6.6) and (6.11) as resulting from the linearization of some nonlinear system of equations with

$$W = \omega_0 + \mathcal{C} \tag{6.12}$$

in which ω_0 is some fixed zero-order background field chosen to satisfy (6.6) while $\mathcal{C}$ describes first order fluctuations. Consistency of this identification however requires nonlinear corrections to the original linearized equations because the full covariant derivatives built of $W = \omega_0 + \mathcal{C}$ develop nonzero curvature due to the right hand side of (6.11). Finding these nonlinear corrections is equivalent to finding interactions.

This suggests the following strategy for the analysis of HS gauge theories:

1. One starts from a space-time with some symmetry algebra s (*e.g.* Poincaré or Anti-de Sitter algebra) and a vacuum gravitational gauge field ω_0, which is a 1-form taking values in s and satisfying the zero curvature equations (6.6).

2. One reformulates the field equations of a given free dynamical system in the "unfolded form" (6.11). This can always be done in principle (the general procedure is explained in Section 7). The only questions are: "how simple is the explicit formulation?" and "what are the modules T of s for which the unfolded equation (6.8) can be interpreted as a covariant constancy condition?".

3. One looks for a nonlinear free differential algebra such that (6.3) reproduces correctly the free field equations (6.6) and (6.11) at the linearized level. More precisely, one looks for some function $G(W)$ verifying (6.2) and the Taylor expansion of which around ω_0 is given by

$$G(W) = \omega_0^2 + \Big(\omega_0 - P(\omega_0)\Big)\mathcal{C} + O(\mathcal{C}^2)\,,$$

where $\omega_0\mathcal{C}$ denotes the action of ω_0 in the h-module $\mathcal{C}$ and the terms denoted by $O(\mathcal{C}^2)$ are at least quadratic in the fluctuation.

It is not *a priori* guaranteed that some nonlinear deformation exists at all. If not, this would mean that no consistent nonlinear equations exist. But if the deformation is found, then the problem is solved because the resulting equations are formally consistent, gauge invariant and generally coordinate invariant as a consequence of the general properties of free differential algebras, and by construction, they describe the correct dynamics at the free field level.

To find some nonlinear deformation, one has to address two related questions. The first one is "what is a relevant s-module T in which 0-forms that describe physical degrees of freedom in the model can take values?" and the second is "which infinite-dimensional (HS) extension h of s, in which 1-form connections take their values, can act on the s-module T?". A natural candidate is a Lie algebra h constructed via commutators from the (complex) associative algebra $\mathcal{A}$

$$\mathcal{A} = \mathcal{U}(s)/Ann(T)\,,$$

where $\mathcal{U}(s)$ is the universal enveloping algebra[13] of s while $Ann(T)$ is the annihilator, *i.e.* the ideal of $\mathcal{U}(s)$ spanned by the elements which trivialize on the module T. Of course, this strategy may be too naive in general because not all algebras can be symmetries of a consistent field-theoretical model and only some subalgebras of h resulting from this construction may allow a consistent nonlinear deformation at the free differential level. A useful criterium is the *admissibility condition* [54] which requires that there should be a unitary h-module which describes a list of quantum single-particle states corresponding to all HS gauge fields described in terms of the connections of h. If no such representation exists, there is no chance to find a nontrivial consistent (in particular, free of ghosts) theory that admits h as a symmetry of its most symmetric vacuum. In any case, $\mathcal{U}(s)$ is the reasonable starting point to look for a HS algebra[14]. Indeed, the associative algebra $\mathcal{A}$ introduced in Section 5 is a quotient of $\mathcal{U}(o(d-1,2))$ [45]. The related real Lie algebra h is $hu(1/sp(2)[d-1,2])$. The space of single-particle quantum states of free massless HS fields of Section 8 provides a unitary module of $hu(1/sp(2)[d-1,2])$ in which all massless completely symmetric representations of $o(d-1,2)$ appear just once [48].

7 Unfolding lower spins

The dynamics of any consistent system can in principle be rewritten in the unfolded form $D_0 C = 0$ by adding enough auxiliary variables [55]. This technique is explained in the subsection 7.1. Two particular examples of the general procedure are presented: the unfolding of the Klein-Gordon equation and the unfolding of gravity, in Subsections 7.2 and 7.3, respectively.

7.1 Unfolded dynamics

Let $\omega_0 = e_0^a\, P_a + \frac{1}{2}\omega_0^{ab} M_{ab}$ be a vacuum gravitational gauge field taking values in some space-time symmetry algebra s. Let $C^{(0)}(x)$ be a given space-time field satisfying some dynamical equations to be unfolded. Consider for simplicity the case where $C^{(0)}(x)$ is a 0-form. The general procedure of unfolding goes schematically as follows:

For a start, one writes the equation

$$D_0^L C^{(0)} = e_0^a\ C_a^{(1)} \tag{7.1}$$

where D_0^L is the covariant Lorentz derivative and the field $C_a^{(1)}$ is auxiliary. Next, one checks whether the original field equations for $C^{(0)}$ impose any restrictions on the first derivatives of $C^{(0)}$. More precisely, some part of $\partial_\mu C^{(0)}$ might vanish on-mass-shell (*e.g.* for Dirac spinors). These restrictions in turn impose some restrictions on the auxiliary fields $C_a^{(1)}$.

[13] A universal enveloping algebra is defined as follows. Let $\mathcal{S}$ be the associative algebra that is freely generated by the elements of s. Let $\mathcal{I}$ be the ideal of $\mathcal{S}$ generated by elements of the form $xy - yx - [x,y]$ $(x, y \in s)$. The quotient $\mathcal{U}(s)$= $\mathcal{S}/\mathcal{I}$ is called the universal enveloping algebra of the Lie algebra s.

[14] Based on somewhat different arguments, this idea was put forward by Fradkin and Linetsky in [53].

If these constraints are satisfied by $C^{(1)}_a$, then these fields parametrize all on-mass-shell nontrivial components of first derivatives.

Then, one writes for these first level auxiliary fields an equation similar to (7.1)

$$D^L_0 C^{(1)}_a = e^b_0 \; C^{(2)}_{a,b} \,, \tag{7.2}$$

where the new fields $C^{(2)}_{a,b}$ parametrize the second derivatives of $C^{(0)}$. Once again one checks (taking into account the Bianchi identities) which components of the second level fields $C^{(2)}_{a,b}$ are non-vanishing provided that the original equations of motion are satisfied.

This process continues indefinitely, leading to a chain of equations having the form of some covariant constancy condition for the chain of fields $C^{(m)}_{a_1,a_2,\ldots,a_m}$ ($m \in \mathbb{N}$) parametrizing all on-mass-shell nontrivial derivatives of the original dynamical field. By construction, this leads to a particular unfolded equation **(6.3) with G^i given by** (6.5). As explained in subsection 6.1 this means that the set of fields realizes some module T of the space-time symmetry algebra s. In other words, the fields $C^{(m)}_{a_1,a_2,\ldots,a_m}$ are the components of a single field C living in some infinite-dimensional module of s. Then the infinite chain of equations can be rewritten as a single covariant constancy condition $D_0 C = 0$ where D_0 is the s-covariant derivative in the s-module where C takes its values.

7.2 The example of the scalar field

For simplicity, for the remaining of this section, we will consider a flat space-time background. The Minkowski solution can be written as

$$\omega_0 = dx^\mu \delta^a_\mu P_a \tag{7.3}$$

i.e. the flat frame is $(e_0)^a_\mu = \delta^a_\mu$ and the Lorentz connection vanishes. The equation (7.3) corresponds to the pure gauge solution (6.9) with

$$g(x) = \exp(x^\mu \, \delta^a_\mu \, P_a) \,. \tag{7.4}$$

where the space-time Lie algebra s is identified with the Poincaré algebra $iso(d-1,1)$.

As a preliminary to the gravity example considered in the next subsection, the simplest field-theoretical case of unfolding is reviewed, *i.e.* the unfolding of a massless scalar field $\phi(x)$, which was first described in [55]. The "unfolding" of the massless Klein-Gordon equation

$$\Box C(x) = 0 \tag{7.5}$$

is relatively easy to work out, so we give directly the final result and we comment about how it is obtained afterwards.

To describe dynamics of the spin zero massless field $C(x)$, let us introduce the infinite collection of 0-forms $C_{a_1\ldots a_n}(x)$ ($n = 0, 1, 2, \ldots$) which are completely symmetric traceless tensors

$$C_{a_1\ldots a_n} = C_{\{a_1\ldots a_n\}} \,, \quad \eta^{bc} C_{bca_3\ldots a_n} = 0 \,. \tag{7.6}$$

The "unfolded" version of the Klein-Gordon equation (7.5) has the form of the following infinite chain of equations

$$dC_{a_1\dots a_n} = e_0^b C_{a_1\dots a_n b}\,, \quad (n = 0, 1, \dots) \tag{7.7}$$

where we have used the opportunity to replace the Lorentz covariant derivative D_0^L by the ordinary exterior derivative d. It is easy to see that this system is formally consistent because applying d on both sides of (7.7) does not lead to any new condition,

$$d^2 C_{a_1\dots a_n} = -e_0^b dC_{a_1\dots a_n b} = e_0^b e_0^c dC_{a_1\dots a_n bc} = 0 \quad (n = 0, 1, \dots)$$

since $e_0^b e_0^c = -e_0^c e_0^b$ because e_0^b is a 1-form. As we know from Section 6.1, this property implies that the space T of 0-forms $C_{a_1\dots a_n}$ spans some representation of the Poincaré algebra $iso(d-1,1)$. In other words, T is an infinite-dimensional $iso(d-1,1)$-module[15].

To show that this system of equations is indeed equivalent to the free massless field equation (7.5), let us identify the scalar field $C(x)$ with the member of the family of 0-forms $C_{a_1\dots a_n}(x)$ at $n = 0$. Then the first two equations of the system (7.7) read

$$\partial_\nu C = C_\nu\,,$$

$$\partial_\nu C_\mu = C_{\mu\nu}\,,$$

where we have identified the world and tangent indices via $(e_0)_\mu^a = \delta_\mu^a$. The first of these equations just tells us that C_ν is the first derivative of C. The second one tells us that $C_{\nu\mu}$ is the second derivative of C. However, because of the tracelessness condition (7.6) it imposes the Klein-Gordon equation (7.5). It is easy to see that all other equations in (7.7) express highest tensors in terms of the higher-order derivatives

$$C_{\nu_1\dots\nu_n} = \partial_{\nu_1}\dots\partial_{\nu_n} C \tag{7.8}$$

and impose no new conditions on C. The tracelessness conditions (7.6) are all satisfied once the Klein-Gordon equation is true. From this formula it is clear that the meaning of the 0-forms $C_{\nu_1\dots\nu_n}$ is that they form a basis in the space of all on-mass-shell nontrivial derivatives of the dynamical field $C(x)$ (including the derivative of order zero which is the field $C(x)$ itself).

Let us note that the system (7.7) without the constraints (7.6) remains formally consistent but is dynamically empty just expressing all highest tensors in terms of derivatives of C according to (10.15). This simple example illustrates how algebraic constraints like tracelessness of a tensor can be equivalent to dynamical equations.

The above consideration can be simplified further by means of introducing the auxiliary coordinate u^a and the generating function

$$C(x,u) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{a_1\dots a_n}(x) u^{a_1}\dots u^{a_n}$$

[15]Strictly speaking, to apply the general argument of Section 6.1 one has to check that the equation remains consistent for any flat connection in $iso(d-1,1)$. It is not hard to see that this is true indeed.

with the convention that

$$C(x,0) = C(x)\,.$$

This generating function accounts for all tensors $C_{a_1 \ldots a_n}$ provided that the tracelessness condition is imposed, which in these terms implies that

$$\Box_u C(x,u) \equiv \frac{\partial}{\partial u^a}\frac{\partial}{\partial u_a} C = 0\,.$$

In other words, the $iso(d-1,1)$-module T is realized as the space of harmonic formal power series in u^a. The equations (7.7) then acquire the simple form

$$\frac{\partial}{\partial x^\mu} C(x,u) = \delta^a_\mu \frac{\partial}{\partial u^a} C(x,u)\,. \tag{7.9}$$

From this realization one concludes that the translation generators in the infinite-dimensional module T of the Poincaré algebra are realized as translations in the u–space, *i.e.*

$$P_a = -\frac{\partial}{\partial u^a}\,,$$

for which the equation (7.9) reads as a covariant constancy condition (6.7)

$$dC(x,u) + e_0^a P_a C(x,u) = 0\,. \tag{7.10}$$

One can find a general solution of the equation (7.10) in the form

$$C(x,u) = C(x+u,0) = C(0,x+u)$$

from which it follows in particular that

$$C(x) \equiv C(x,0) = C(0,x) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{\nu_1 \ldots \nu_n}(0) x^{\nu_1} \ldots x^{\nu_n}\,. \tag{7.11}$$

From (7.6) and (10.15) one can see that this is indeed the Taylor expansion for any solution of the Klein-Gordon equation which is analytic in $x_0 = 0$. Moreover one can recognize the equation (7.11) as a particular realization of the pure gauge solution (6.10) with the gauge function $g(x)$ of the form (7.4).

7.3 The example of gravity

The set of fields in Einstein-Cartan's formulation of gravity is composed of the frame field e^a_μ and the Lorentz connection ω^{ab}_μ. One supposes that the torsion constraint $T_a = 0$ is satisfied, in order to express the Lorentz connection in terms of the frame field. The Lorentz curvature can be expressed as $R^{ab} = e_c e_d\, R^{[ab]\,;\,[cd]}$, where $R^{ab\,;\,cd}$ is a rank four tensor with indices in the tangent space and which is antisymmetric in the pair of indices ab and cd, namely it has the symmetries of the tensor product $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array} \otimes \begin{array}{|c|}\hline c\\\hline d\\\hline\end{array}$. The algebraic Bianchi identity $e_b R^{ab} = 0$, which follows from the zero torsion constraint imposes that the tensor $R^{ab\,;\,cd}$ possesses the symmetries of the Riemann tensor, *i.e.* $R^{[ab\,;\,c]d} = 0$. More precisely, it carries an irreducible representation of $GL(d)$ characterized by the Young tableau $\begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array}$ in the antisymmetric basis. The vacuum Einstein equations state that this tensor is traceless, so that it is actually irreducible under the pseudo-orthogonal group $O(d-1,1)$ on-mass-shell. In other words, the Riemann tensor is equal on-mass-shell to the Weyl tensor.

For HS generalization, it is more convenient to use the symmetric basis. In this convention, the Einstein equations can be written as

$$T^a = 0\,, \qquad R^{ab} \;=\; e_c\, e_d\, C^{ac,\,bd} \tag{7.12}$$

where the 0-form $C^{ac,\,bd}$ is the Weyl tensor in the symmetric basis. More precisely, the tensor $C^{ac,bd}$ is symmetric in the pairs ac and bd and it satisfies the algebraic identities

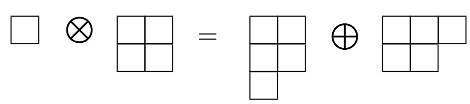
$$C^{\{ac,\,b\}d} = 0\,, \qquad \eta_{ac} C^{ac,\,bd} = 0\,.$$

Let us now start the unfolding of linearized gravity around the Minkowski background. The linearization of the second equation of (7.12) is

$$R_1^{ab} \;=\; e_{0\;c}\, e_{0\;d}\, C^{ac,\,bd} \tag{7.13}$$

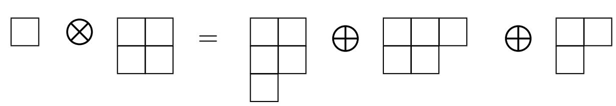
This equation is a particular case of the equation (6.11). What is lacking at this stage is the equations containing the differential of the Weyl 0-form $C^{ac,\,bd}$. Since we do not want to impose any additional dynamical restrictions on the system, the only restrictions on the derivatives of the Weyl 0-form $C^{ac,\,bd}$ may result from the Bianchi identities applied to (7.13).

A priori, the first Lorentz covariant derivative of the Weyl tensor is a rank five tensor in the following representation



(7.14)

decomposed according to irreducible representations of $gl(d)$. Since the Weyl tensor is traceless, the right hand side of (7.14) contains only one nontrivial trace, that is for traceless tensors we have the $o(d-1,1)$ Young decomposition by adding a three cell hook tableau, *i.e.*



The linearized Bianchi identity $dR_1^{ab} = 0$ leads to

$$e_{0\;c} e_{0\;d}\, dC^{ac,\,bd} = 0\,. \tag{7.15}$$

The components of the left-hand-side written in the basis $dx^\mu dx^\nu dx^\rho$ have the symmetry property corresponding to the tableau

$$\begin{array}{|c|c|}\hline \mu & a \\ \hline \nu & b \\ \hline \rho \\ \cline{1-1}\end{array} \;\sim\; \partial_{[\rho} C^{a}{}_{\mu}{}^{,b}{}_{\nu]}\,,$$

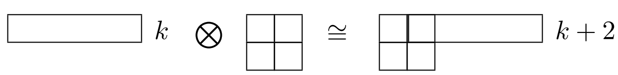
which also contains the single trace part with the symmetry properties of the three-cell hook tableau.

Therefore the consistency condition (7.15) says that in the decomposition (7.14) of the Lorentz covariant derivative of the Weyl tensor, the first term vanishes and the second term is traceless and otherwise arbitrary. Let $C^{abf,\,cd}$ be the traceless tensor corresponding to the second term in the decomposition (7.14) of the Lorentz covariant derivative of the Weyl tensor. This is equivalent to say that

$$dC^{ac,\,bd} \;=\; e_{0\,f}\,(2C^{acf,\;bd}+C^{acb,\;df}+C^{acd,\;bf})\,,$$

where the right hand side is fixed by the Young symmetry properties of the left hand side modulo an overall normalization coefficient. This equation looks like the first step (7.1) of the unfolding procedure. $C^{acf,\;bd}$ is irreducible under $o(d-1,1)$.

One should now perform the second step of the unfolding general scheme and write the analogue of (7.2). This process goes on indefinitely. To summarize the procedure, one can analyze the decomposition of the k-th Lorentz covariant derivatives (with respect to the Minkowski vacuum background, so they commute) of the Weyl tensor $C^{ac,\,bd}$. Taking into account the Bianchi identity, the decomposition goes as follows



(7.16)

The generic solution is

$$dC^{a_1\dots a_{k+2},\,b_1b_2} \;=\; e_{0\;c}\Big(\,(k+2)\,C^{a_1\dots a_{k+2}c,\,b_1b_2}\;+\;C^{a_1\dots a_{k+2}b_1,\,b_2c}\;+\;C^{a_1\dots a_{k+2}b_2,\,b_1c}\Big)\,,$$
$$(0\leqslant k\leqslant\infty) \qquad (7.17)$$

where the fields $C^{a_1\dots a_{k+2},\,b_1b_2}$ are in the irreducible representation of $o(d-1,1)$ characterized by the traceless two-row Young tableau on the right hand side of (7.16), *i.e.*

$$C^{\{a_1\dots a_{k+2},\,b_1\}b_2}=0\,,\qquad \eta_{a_1a_2}C^{a_1a_2\dots a_{k+2},\,b_1b_2}=0\,.$$

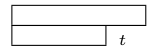
Note that, as expected, the system (7.17) is consistent with $d^2C^{a_1\dots a_{k+2},\,b_1b_2}=0$.

Analogously to the spin 0 case, the meaning of the 0-forms $C^{a_1\dots a_{k+2},\,b_1b_2}$ is that they form a basis in the space of all on-mass-shell nontrivial gauge invariant combinations of the derivatives of the spin 2 gauge field.

8 Free massless equations

In order to follow the strategy exposed in subsection 6.2 and generalize the example of gravity treated along these lines in subsection 7.3, we shall start by writing unfolded HS field equations in terms of the linearized HS curvatures (4.4). This result is christened the "central on-mass-shell theorem". It was originally obtained in [32, 50] for the case of $d=4$

and then extended to any d in [33, 24]. That these HS equations of motion indeed reproduce the correct physical degrees of freedom will be shown later in Section 10 via a cohomological approach explained in Section 9.

The linearized curvatures $R_1^{A_1\dots A_{s-1},\,B_1\dots B_{s-1}}$ were defined by (4.4). They decompose into the linearized curvatures with only V^A transverse, *i.e.* Lorentz fibre indices which have the symmetry properties associated with the two-row traceless Young tableau  $s-1$, t. It is convenient to use the standard gauge $V^A = \delta^A_{\hat{d}}$ (from now on we normalize V to unity). In the Lorentz basis, the linearized HS curvatures have the form

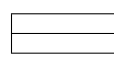
$$R_1^{a_1\dots a_{s-1},\,b_1\dots b_t} = D_0^L \omega^{a_1\dots a_{s-1},\,b_1\dots b_t} + e_{0\,c}\,\omega^{a_1\dots a_{s-1},\,b_1\dots b_t c} + O(\Lambda) \qquad (8.1)$$

For simplicity, in this section we discard the complicated Λ–dependent terms which do not affect the general analysis, i.e. we present explicitly the flat-space part of the linearized HS curvatures. It is important to note however that the Λ–dependent terms in (8.1) contain the field $\omega^{a_1\dots a_{s-1},\,b_1\dots b_{t-1}}$ which carries one index less than the linearized HS curvatures.

For $t=0$, these curvatures generalize the torsion of gravity, while for $t>0$ the curvature corresponds to the Riemann tensor. In particular, as we will demonstrate in Section 10, the analogues of the Ricci tensor and scalar curvature are contained in the curvatures with $t=1$ while the HS analog of the Weyl tensor is contained in the curvatures with $t=s-1$. (For the case of $s=2$ they combine into the level $t=1$ traceful Riemann tensor.)

The first on-mass-shell theorem states that the following free field equations in Minkowski or $(A)dS$ space

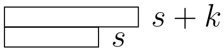
$$R_1^{a_1\dots a_{s-1},\,b_1\dots b_t} = \delta_{t,\,s-1}\;\, e_{0\,c}\,e_{0\,d}\, C^{a_1\dots a_{s-1}c,\,b_1\dots b_{s-1}d}\,, \qquad (0\leqslant t\leqslant s-1) \qquad (8.2)$$

properly describe completely symmetric gauge fields of generic spin $s\geq 2$. This means that they are equivalent to the proper generalization of the $d=4$ Fronsdal's equations of motion to any dimension, supplemented with certain algebraic constraints on the auxiliary HS connections which express the latter via derivatives of the dynamical HS fields. The 0-form $C^{a_1\dots a_s,\,b_1\dots b_s}$ is the spin-s Weyl-like tensor. It is irreducible under $o(d-1,1)$ and is characterized by a rectangular two-row Young tableau  s, s. The field equations generalize (7.13) of linearized gravity. The equations of motion put to zero all curvatures with $t\neq s-1$ and require $C^{a_1\dots a_s,\,b_1\dots b_s}$ to be traceless (that is the generalized Ricci-like tensors to be equal to zero).

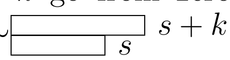
Note that the equations (8.2) result from the unfolding of the Fronsdal equations.[16] The analysis of the Bianchi identities of (8.2) works for any spin $s\geqslant 2$ in a way analogous to gravity. The final result is the following equation which presents itself like a covariant constancy condition

$$\begin{aligned} 0 = \widetilde{D}_0 C^{a_1\dots a_{s+k},\,b_1\dots b_s} \;\equiv\;& D_0^L C^{a_1\dots a_{s+k},\,b_1\dots b_s} \\ & - e_{0\,c}\Big((s+k)C^{a_1\dots a_{s+k}c,\,b_1\dots b_s} + s\,C^{a_1\dots a_{s+k}\{b_1,\,b_2\dots b_s\}c}\Big) + O(\Lambda)\,, \\ & (0\leqslant k\leqslant\infty) \end{aligned} \qquad (8.3)$$

[16]In fact, the action and equations of motion for totally symmetric massless HS fields in AdS_d with $d>4$ were originally obtained in [33] in the frame-like formalism.

where $C^{a_1\dots a_{s+k},\, b_1\dots b_s}$ are $o(d-1,1)$ irreducible tensors characterized by the Young tableaux 

. They describe on-mass-shell nontrivial k-th derivatives of the spin-s Weyl-like tensor, thus forming a basis in the space of gauge invariant combinations of $(s+k)$**-th** derivatives of a spin s HS gauge field. The system (8.3) is the generalization of the spin 0 system (7.7) and **the** spin 2 system (7.17) to arbitrary spin and to AdS background. Let us stress that for $s \geqslant 2$ the infinite system of equations (8.3) is a consequence of (8.2) by the Bianchi identity. For $s=0$ and $s=1$, the system (8.3) contains the dynamical Klein-Gordon and Maxwell equations, respectively. Note that (8.2) makes no sense for $s=0$ because there is no spin 0 gauge potential while (8.3) with $s=0$ reproduces the unfolded spin 0 equation (7.7) and its AdS generalization. For the spin 1 case, (8.2) only gives a definition of the spin 1 Maxwell field strength $C^{a,b} = -C^{b,a}$ in terms of the potential ω_μ. The dynamical equations for spin 1, *i.e.* Maxwell equations, are contained in (8.3). The fields $C^{a_1\dots a_{k+1},\, b}$, characterized by the Lorentz irreducible (*i.e.* traceless) two-row Young tableaux with one cell in the second row, form a basis in the space of on-mass-shell nontrivial derivatives of the Maxwell tensor $C^{a,\, b}$.

Since s and k go from zero to infinity, it is clear that the complete set of 0-forms $C^{a_1\dots a_{s+k},\, b_1\dots b_s} \sim$ 

covers the set of all two-row Young tableaux. This suggests that the Weyl-like 0-forms take values in the linear space of $hu(1/sp(2)[d-1,2])$ that obviously forms an $o(d-1,1)$- (*i.e.* Lorentz) module. Following the strategy sketched in Subsection 6.2, one can expect that they belong to an $o(d-1,2)$-module. But the idea to use the adjoint representation of $hu(1/sp(2)[d-1,2])$ does not work because, according to the commutation relation (5.1), the commutator of the background gravity connection $\omega_0 = \omega_0^{AB} T_{AB}$ with a generator of $hu(1(2)[d-1,2])$ preserves the rank of the generator, while the covariant derivative $\widetilde{D}$ in (8.3) acts on the infinite set of Lorentz tensors of infinitely increasing ranks. Fortunately, the appropriate representation only requires a slight modification compared to the adjoint representation. As will be explained in Section 12, the 0-forms C belong to the so-called "twisted adjoint representation".

The equations (8.2), (8.3) provide the unfolded form of the free equations of motion for completely symmetric massless fields of all spins in any dimension. This fact is referred to as central on-mass-shell theorem because it plays a distinguished role in various respects. The idea of the proof will be explained in section 10. Let us note that the right-hand-side of the equation (8.2) is a particular realization of the deformation terms (6.11) in free differential algebras.

9 Dynamical content via σ_- cohomology

In this section, we perform a very general analysis of equations of motion of the form

$$\widehat{D}_0 \mathcal{C} = 0\,. \tag{9.1}$$

via a cohomological reformulation of the problem [57, 42, 58, 59]. It will be applied to the HS context in the next section.

In the unfolded HS equations the background covariant derivative $\widehat{D}_0$ decomposes as the sum

$$\widehat{D}_0 = \sigma_- + D_0^L + \sigma_+ \,, \tag{9.2}$$

where the operator $\sigma_\pm$ modifies the rank of a Lorentz tensor by ± 1 and the background Lorentz covariant derivative D_0^L does not change it. Let us introduce the number of Lorentz indices as a grading G of the space of tensors. In the present consideration we do not fix the module V on which $\widehat{D}$ acts. In the context of the HS theory the interesting cases are when $\widehat{D}$ acts either in the adjoint representation or in the twisted adjoint representation of the HS algebra. For example, the covariant constancy condition (8.3) takes the form

$$\widetilde{D}_0 C = (D_0^L + \sigma_- + \sigma_+)\, C = 0 \,,$$

where σ_+ denotes the Λ–dependent terms.

The cohomological classification exposed here only assumes the following abstract properties: the grading operator G is diagonalizable in a vector space V and it possesses a spectrum bounded from below. The grading properties of the Grassmann odd operators D_0^L and σ_- are summarized in the commutation relations

$$[G, D_0^L] = 0 \,, \qquad [G, \sigma_-] = -\sigma_- \,.$$

The operator σ_+ is a sum of operators of strictly positive grade. (In HS applications σ_+ has grade one, i.e. $[G, \sigma_+] = \sigma_+$, but this is not essential for the general analysis.) It is further assumed that the operator σ_- acts vertically in the fibre V, *i.e.* **it does not act on space-time coordinates. (In HS models, only the operator D_0^L acts non trivially on the space-time coordinates (differentiates).)** The background covariant derivative $\widehat{D}_0$ defined by (9.2) is nilpotent. The decomposition of the nilpotency equation $(\widehat{D}_0)^2 = 0$ gives the following identities

$$(\sigma_-)^2 = 0 \,, \quad D_0^L \sigma_- + \sigma_- D_0^L = 0 \,, \quad (D_0^L)^2 + \sigma_+\sigma_- + \sigma_-\sigma_+ + D_0^L\sigma_+ + \sigma_+ D_0^L + (\sigma_+)^2 = 0 \,. \tag{9.3}$$

(If σ_+ has definite grade $+1$ as is the case in the HS theories, the last relation is equivalent to the three conditions $(\sigma_+)^2 = 0$, $\quad D_0^L\sigma_+ + \sigma_+ D_0^L = 0$, $\quad (D_0^L)^2 + \sigma_+\sigma_- + \sigma_-\sigma_+ = 0$.)

An important property is the nilpotency of σ_-. The point is that the analysis of Bianchi identities (as was done in details in Subsection 7.3 for gravity) is, in fact, equivalent to the analysis of the cohomology

$$H(\sigma_-) \equiv \frac{Ker(\sigma_-)}{Im(\sigma_-)}$$

of the differential σ_-.

Let us suppose that the dynamical field $\mathcal{C}$ is a p-form taking values in the complex $V \otimes \Omega(\mathcal{M}^d)$. By "dynamical" field, one means here that it is not expressed as derivatives of something else (*e.g.* the metric-like gauge field of Fronsdal's approach). The fields that are expressed as derivatives of the dynamical fields by virtue of the field equations are referred to as "auxiliary". The field equation $\widehat{D}_0\mathcal{C} = 0$ is invariant under the gauge transformation $\delta\mathcal{C} =$

$\widehat{D}_0\varepsilon$, where the gauge parameter ε is a $(p-1)$-form. These gauge transformations contain both differential gauge transformations (like linearized diffeomorphisms) and Stueckelberg gauge symmetries (like linearized local Lorentz transformations[17]). A field that is neither auxiliary nor pure gauge by Stueckelberg gauge symmeries is said to be a "nontrivial" dynamical field. One can prove [57, 42] (see also [58, 59]) the following propositions:

A. Nontrivial dynamical fields are non-vanishing elements of $H^p(\sigma_-)$.

B. Differential gauge symmetry parameters ε are classified by $H^{p-1}(\sigma_-)$.

C. Inequivalent differential field equations on the nontrivial dynamical fields contained in $\widehat{D}_0\mathcal{C}=0$ are in one-to-one correspondence with representatives of $H^{p+1}(\sigma_-)$.

Proof of A: The first claim is almost obvious. Indeed, let us decompose the field $\mathcal{C}$ according to the degree G:

$$\mathcal{C}=\sum_{n=0}\mathcal{C}_n\,,\qquad G\mathcal{C}_n=n\,\mathcal{C}_n\,,\qquad (n=0,1,2,\ldots)\,.$$

The field equation (9.1) thus decomposes as

$$\widehat{D}_0\mathcal{C}|_{\,n-1}=\sigma_-\mathcal{C}_n+D_0^L\mathcal{C}_{n-1}+\Big(\sigma_+\sum_{m\leq n-2}\mathcal{C}_m\Big)\Big|_{n-1}=0\,. \tag{9.4}$$

By a straightforward induction on $n=1,2,\ldots$, one can convince oneself that all fields $\mathcal{C}_n$ that contribute to the first term of the right hand side of the equation (9.4) are thereby expressed in terms of derivatives of lower grade fields, hence they are auxiliary [18]. As a result only fields annihilated by σ_- are not auxiliary. Taking into account the gauge transformation

$$\delta\mathcal{C}_n=\sigma_-\varepsilon_{n+1}+D_0^L\varepsilon_n+\Big(\sigma_+\sum_{m\leq n-1}\varepsilon_m\Big)\Big|_n \tag{9.5}$$

one immediately realizes that all components $\mathcal{C}_n$ which are σ_- exact, *i.e.* which belong to the image of σ_-, are Stueckelberg, *i.e.* they can be gauged away. Therefore, a nontrivial dynamical p-form field in $\mathcal{C}$ should belong to the quotient $Ker(\sigma_-)/Im(\sigma_-)$. $\square$

For Einstein-Cartan's gravity, the Stueckelberg gauge symmetry is local Lorentz symmetry and indeed what distinguishes the frame field and the metric tensor is that the latter actually belongs to the cohomology $H^1(\sigma_-)$ while the former contains a σ_- exact part.

Proof of B: The proof follows the same lines as for A. The first step has been performed already in the sense that (9.5) already told us that the parameters such that $\sigma_-\varepsilon\neq 0$ are

[17] We remind the reader that the metric tensor corresponds to the symmetric part $e_{\{\mu\ a\}}$ of the frame field. The antisymmetric part of the frame field $e_{[\mu\ a]}$ can be gauged away by fixing local Lorentz symmetry, because it contains as many independent components as the Lorentz gauge parameter ε^{cd}.

[18] Here we use the fact that the operator σ_- acts vertically (that is, it does not differentiate space-time coordinates) thus giving rise to algebraic conditions which express auxliary fields via derivatives of the other fields.

Stueckelberg and can be used to completely gauge away trivial parts of the field $\mathcal{C}$. Thus differential parameters must be σ_- closed. The only subtlety is that one should make use of the fact that the gauge transformation $\delta_\varepsilon \mathcal{C} = \widehat{D}_0 \varepsilon$ are reducible. More precisely, gauge parameters obeying the reducibility identity $\varepsilon = \widehat{D}_0 \zeta$ are trivial in the sense that they do not perform any gauge transformation, $\delta_{\widehat{D}_0 \zeta} \mathcal{C} = 0$. The second step of the proof is a mere decomposition of the reducibility identity in order to see that σ_- exact parameters correspond to reducible gauge transformations[19]. $\square$

Proof of C: Suppose that one has already obtained and analyzed the system of equations (9.1) in degree G from $n = 1$ up to $n = n_0 - 1$. Applying the operator $\widehat{D}_0$ on the covariant derivative $\widehat{D}_0 \mathcal{C} = 0$ gives identically zero, $(\widehat{D}_0)^2 \mathcal{C} = 0$. Decomposing the latter Bianchi identity gives in grade equal to $n_0 - 1$

$$(\widehat{D}_0)^2 \mathcal{C}|_{n_0-1} = \sigma_- \Big(\widehat{D}_0 \mathcal{C}|_{n_0} \Big) + D_0^L \Big(\widehat{D}_0 \mathcal{C}|_{n_0-1} \Big) + \Big(\sigma_+ \widehat{D}_0 \mathcal{C} \Big) \Big|_{n_0-1} = 0 \,.$$

By the induction hypothesis, the equations $\widehat{D}_0 \mathcal{C}|_m = 0$ with $m \leqslant n_0 - 1$ have been imposed and analyzed already. Therefore

$$\sigma_- \Big(\widehat{D}_0 \mathcal{C}|_{n_0} \Big) = 0 \,.$$

In other words, $\widehat{D}_0 \mathcal{C}|_{n_0} \in Ker(\sigma_-)$. It can contain a σ_- exact part and a nontrivial part:

$$\widehat{D}_0 \mathcal{C}|_{n_0} = \sigma_- E_{n_0+1} + F_{n_0} \,, \qquad F_{n_0} \in H^{p+1}(\sigma_-) \,.$$

The trivial part can be compensated by a field redefinition of the component $\mathcal{C}_{n_0+1}$ which was not treated before (by the induction hypothesis). More precisely one performs

$$\mathcal{C}_{n_0+1} \rightarrow \mathcal{C}'_{n_0+1} := \mathcal{C}_{n_0+1} - E_{n_0+1} \,.$$

So we are left with $\widehat{D}_0 \mathcal{C}'|_{n_0} = F_{n_0}$. Thence, the field equation $\widehat{D}_0 \mathcal{C}'|_{n_0} = 0$ not only expresses $\mathcal{C}'_{n_0+1}$ in terms of derivatives of lower grade p-forms but also sets F_{n_0} to zero. This imposes some $\mathcal{C}_{n_0+1}$-independent conditions on the derivatives of the fields $\mathcal{C}_k$ with $k \leq n_0$, thus leading to differential restrictions on the nontrivial dynamical fields. Therefore, to each representative of $H^{p+1}(\sigma_-)$ there corresponds a differential field equation. $\square$

Note that if $H^{p+1}(\sigma_-) = 0$, the equation (9.1) contains only constraints which express auxilary fields via derivatives of the dynamical fields, imposing no restrictions on the latter. If D_0^L is a first order differential operator and σ_+ is at most a second order differential operator, **which** is true in HS applications, then, if $H^{p+1}(\sigma_-)$ is nonzero in the grade k sector, the associated differential equations on a grade l dynamical field are of order $k + 1 - l$.

[19] Note that factoring out the σ_- exact parameters accounts for algebraic reducibility of gauge symmetries. The gauge parameters in $H^{p-1}(\sigma_-)$ may still have differential reducibility analogous to differential gauge symmetries for nontrivial dynamical fields. For the examples of HS systems considered below the issue of reducibility of gauge symmetries is irrelevant however because there are no p-form gauge parameters with $p > 0$.

10 σ_- cohomology in higher spins

As was shown in the previous section, the analysis of unfolded dynamical equations amounts to the computation of the cohomology of σ_-. In this section we apply this technique to the analysis of HS field equations.

10.1 Computation of some σ_- cohomology groups

As explained in subsection 4.1, HS gauge fields are 1-forms taking values in various two-row traceless (*i.e.* Lorentz irreducible) Young tableaux. Following Section 3, two-row Young tableaux in the symmetric basis can be conveniently described as a subspace of the polynomial algebra $\mathbb{R}[Y^a, Z^b]$ generated by the $2d$ commuting generators Y^a and Z^b. Vectors of the space $\Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ are p-forms taking values in $\mathbb{R}[Y, Z]$. A generic element reads

$$\alpha = \alpha_{a_1 \dots a_{s-1},\, b_1 \dots b_t}(x, dx) Y^{a_1} \dots Y^{a_{s-1}} Z^{b_1} \dots Z^{b_t}\,,$$

where $\alpha_{a_1 \dots a_{s-1},\, b_1 \dots b_t}(x, dx)$ are differential forms.

By looking at the definition of linearized curvatures (8.1) and taking into account that Λ-dependent terms in this formula denote some operator σ_+ that increases **the** number of Lorentz indices, it should be clear that

$$\sigma_- \equiv e_0^a \frac{\partial}{\partial Z^a} \tag{10.1}$$

for the 1-form sector of HS theory. In other words, σ_- is the "de Rham differential" of the "manifold" parametrized by the Z-variables where the generators dZ^a of the exterior algebra are identified with the background vielbein e_0^a. This remark is very helpful because it already tells us that the cohomology of σ_- is zero in the space $\Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ with $p > 0$ because its topology is trivial in the Z-variable sector. The actual physical situation is less trivial because one has to take into account the Lorentz irreducibility properties of the HS fields. Firstly, there is the Young tableau condition (3.1), *i.e.* in this case

$$Y^a \frac{\partial}{\partial Z^a} \alpha = 0\,. \tag{10.2}$$

The condition singles out a module $W \subset \Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ for which the topology is no more trivial in Z. Secondly, the HS fields are furthermore irreducible under $o(d-1, 1)$, which is equivalent to the tracelessness condition (3.6), *i.e.* in our case it is sufficient to impose

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} \alpha = 0\,. \tag{10.3}$$

This condition further restricts to the module $\hat{W} \subset W$. Both conditions, (10.2) and (10.3), do commute with σ_-, so one can restrict the cohomology to the corresponding subspaces. Since the topology in Z space is not trivial any more, the cohomology groups are no more empty.

Note that from (10.2) and (10.3) follows that all traces are zero

$$\eta^{ab}\frac{\partial^2}{\partial Z^a \partial Y^b}\alpha = 0\,, \qquad \eta^{ab}\frac{\partial^2}{\partial Z^a \partial Z^b}\alpha = 0\,.$$

As explained in Section 9, for HS equations formulated in terms of the connection 1-form ($p=1$), the cohomology groups of interest are $H^p(\sigma_-)$ with $p=0,1$ and 2. The computation of the cohomology groups obviously increases in complexity as the form degree increases. In our analysis we consider simultaneously the cohomology $H^p(\sigma_-,W)$ of traceful two-row Young tableaux (*i.e.* relaxing the tracelessness condition (10.3)) and the cohomology $H^p(\sigma_-,\hat{W})$ of traceless two-row Young tableaux.

Form degree zero: This case corresponds to gauge parameters ε, the cocycle condition $\sigma_-\varepsilon = 0$ states that the gauge parameters do not depend on Z. In addition, they cannot be σ_--exact since they are on the bottom of the form degree. Therefore, the elements of $H^0(\sigma_-,W)$ are the completely symmetric tensors which correspond to the unconstrained 0-form gauge parameters $\varepsilon(Y)$ in the traceful case (like in [60]) while they are furthermore traceless in $H^0(\sigma_-,\hat{W})$ and correspond to Fronsdal's gauge parameters [28] $\eta^{ab}\frac{\partial^2}{\partial Y^a \partial Y^b}\varepsilon(Y) = 0$ in the traceless case.

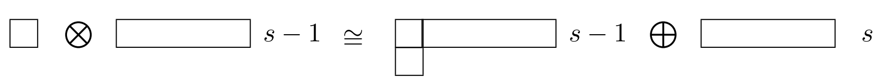
Form degree one: Because of the Poincaré Lemma, any σ_- closed 1-form $\alpha(Y,Z)$ admits a representation

$$\alpha(Y,Z) = e_0^a \frac{\partial}{\partial Z^a}\phi(Y,Z)\,. \tag{10.4}$$

The right hand side of this relation should satisfy the Young condition (10.2), *i.e.*, taking into account that it commutes with σ_-,

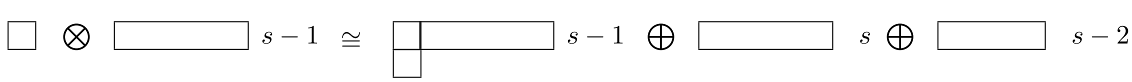
$$e_0^a \frac{\partial}{\partial Z^a} Y^b \frac{\partial}{\partial Z^b}\phi(Y,Z) = 0\,.$$

From here it follows that $\phi(Y,Z)$ is either linear in Z^a (Z-independent ϕ do not contribute to (10.4)) or satisfies the Young property itself. In the latter case the $\alpha(Y,Z)$ given by (10.4) is σ_- exact. Therefore, nontrivial cohomology can only appear in the sector of elements of the form $e_0^a\beta_a(Y)$ which are arbitrary in the traceful case of W and harmonic in Y in the traceless case of $\hat{W}$. Decomposing $\beta_a(Y)$ into irreps of $gl(d)$



one observes that the hook (*i.e.*, the two-row tableau) is the σ_- exact part, while the one-row part describes $H^1(\sigma_-,W)$. These are the rank s totally symmetric **tracefull** dynamical fields which appear in the unconstrained approach [29, 60].

In the traceless case, decomposing $\beta_a(Y)$ into irreps of $o(d-1,1)$ one obtains



(10.5)

where all tensors associated with the various Young tableaux are traceless. Again, the hook (*i.e.* two-row tableau) is the σ_- exact part, while the one-row traceless tensors in (10.5) describe $H^1(\sigma_-, \hat{W})$ which just matches the Fronsdal fields [28] because a rank s double traceless symmetric tensor is equivalent to a pair of rank s and rank $s-2$ traceless symmetric tensors.

Form degree two: The analysis of $H^2(\sigma_-, W)$ and $H^2(\sigma_-, \hat{W})$ is still elementary, but a little bit more complicated than that of $H^0(\sigma_-)$ and $H^1(\sigma_-)$. Skipping technical details we therefore give the final results.

By following a reasoning similar to the one in the previous proof, one can show that, in the traceful case, $H^2(\sigma_-, W)$ is spanned by 2-forms of the form

$$F \;=\; e_0^a\, e_0^b\, \frac{\partial^2}{\partial Y^a \partial Z^b}\, C(Y,Z)\,, \tag{10.6}$$

where the 0-form $C(Y,Z)$ satisfies the Howe dual $sp(2)$ invariance conditions

$$Z^b \frac{\partial}{\partial Y^b} C(Y,Z) = 0\,, \qquad Y^b \frac{\partial}{\partial Z^b} C(Y,Z) = 0\,,$$

and, therefore,

$$(Z^b \frac{\partial}{\partial Z^b} - Y^b \frac{\partial}{\partial Y^b}) C(Y,Z) = 0\,. \tag{10.7}$$

In accordance with the analysis of Section 3, this means that

$$C(Y,Z) = C_{a_1 \dots a_s,\, b_1 \dots b_s} Y^{a_1} \dots Y^{a_s} Z^{b_1} \dots Z^{b_s}\,. \tag{10.8}$$

where the component 0-forms $C_{a_1 \dots a_s,\, b_1 \dots b_s}$ have the symmetry properties corresponding to the rectangular two-row Young tableau of length s

$$C_{a_1 \dots a_s,\, b_1 \dots b_s} \sim \begin{array}{|c|}\hline \qquad\qquad \\ \hline \qquad\qquad \\ \hline \end{array}\begin{array}{l} s \\ s \end{array} \;. \tag{10.9}$$

From (10.6) it is clear that F is σ_- closed and $F \in W$. It is also clear that it is not σ_- exact in the space W. Indeed, suppose that $F = \sigma_- G$, $G \in W$. For any polynomial $G \in W$ its power in Z cannot be higher than the power in Y (because of the Young property, the second row of a Young tableau is not longer than the first row). Since σ_- decreases the power in Z, the degree in Z of σ_- exact elements $\sigma_- G$ is strictly less than the degree in Y. This is not true for the elements (10.6) because of the condition (10.7). The tensors (10.8) correspond to the linearized curvature tensors introduced by de Wit and Freedman [29].

Let us now consider the traceless case of $H^2(\sigma_-, \hat{W})$. The formula (10.6) still gives cohomology but now $C(Y,Z)$ must be traceless,

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} C(Y,Z) = 0\,, \qquad \eta^{ab} \frac{\partial^2}{\partial Z^a \partial Z^b} C(Y,Z) = 0\,, \qquad \eta^{ab} \frac{\partial^2}{\partial Y^a \partial Z^b} C(Y,Z) = 0\,.$$

In this case they correspond to Weyl-like tensors, *i.e.* on-shell curvatures. They form the so-called "Weyl cohomology". But this is not the end of the story because there are other elements in $H^2(\sigma_-, \hat{W})$. They span **the** "Einstein cohomology" and contain two different types of elements:

$$r_1 = e_0^a e_0^b \Big((Z_b\, Y^c - Y_b\, Z^c)\, \frac{\partial^2}{\partial Y^a \partial Y^c}\, \rho_1(Y) \Big)\,, \tag{10.10}$$

$$\begin{aligned} r_2 \;\; &= \;\; e_0^a e_0^b \Big((d-1)(d + Y^c \frac{\partial}{\partial Y^c} - 2) Y_a Z_b + d Y_c Y^c Z_a \frac{\partial}{\partial Y^b} \\ &- \;\; (d + Y^c \frac{\partial}{\partial Y^c} - 2) Y_e Z^e Y_a \frac{\partial}{\partial Y^b} + Y_e Y^e Y_a Z^c \Big) \frac{\partial}{\partial Y^c} \frac{\partial}{\partial Y^b}\, \rho_2(Y)\,, \end{aligned} \tag{10.11}$$

where $\rho_{1,2}(Y)$ are arbitrary harmonic polynomials

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} \rho_{1,2}(Y) = 0 \;,$$

thus describing completely symmetric traceless tensors.

One can directly see that r_1 and r_2 belong to $\hat{W}$ (*i.e.*, satisfy (10.2) and (10.3)) and are σ_- closed, $\sigma_- r_{1,2} = 0$ (the check is particularly simple for r_1). It is also easy to see that $r_{1,2}$ are in the nontrivial cohomology class. Indeed, the appropriate trivial class is described in tensor notations by the 2-form $e_0^c \omega_{a_1 \ldots a_{s-1}, bc}$, where

$$\omega_{a_1 \ldots a_{s-1}, bc} = e_0^f \omega_{f; a_1 \ldots a_{s-1}, bc} \tag{10.12}$$

is a 1-form that has the properties of traceless two-row Young tableau with $s-1$ cells in the first row and two cells in the second row. It is obvious that the trivial cohomology class neither contains a rank $s-2$ tensor like ρ_2, that needs a double contraction in $\omega_{f;a_1 \ldots a_{s-1}, bc}$ in (10.12), nor a rank s symmetric tensor like ρ_2 because symmetrization of a contraction of the tensor $\omega_{f;a_1 \ldots a_{s-1}, bc}$ over any s indices gives zero. A slightly less trivial (although still elementary) fact is that the Einstein cohomology r_1 and r_2 together with the Weyl cohomology F span $H^2(\sigma_-, \hat{W})$.

10.2 Higher-spin field equations

These cohomological results tell us that there are several possible choices for invariant differential HS field equations. The form of $r_{1,2}$ (10.10) and (10.11) indicates that Einstein cohomology is responsible for the field equations of completely symmetric double traceless fields. Indeed, carrying one power of Z^a, they are parts of the HS curvatures $R_{a_1, \ldots a_{s-1}, b}$ with one cell in the second row of the corresponding Young tableau. For spin s, $\rho_1(Y)$ is a degree s harmonic polynomial while $\rho_2(Y)$ is a degree $s-2$ harmonic polynomial. As a result, the field equations which follow from $r_1 = 0$ and $r_2 = 0$ are of second order in derivatives of the dynamical Fronsdal fields taking values in $H^1(\sigma_-, \hat{W})$ and, as expected for Lagrangian equations in general, there are as many equations as dynamical fields.

For example, spin 2 equations result from the conditions $r_1 = 0$ and $r_2 = 0$ imposed on the dynamical fields associated with the elements **of** $H^0(\sigma_-, \hat{W})$ of the form $\rho_2(Y) = r$, $\rho_1(Y) = r_{ab} Y^a Y^b$ with arbitrary r and traceless r_{ab}. These are, respectively, the trace and traceless parts of the linearized Einstein equations on the scalar trace and second rank traceless tensor which together constitute a **tracefull** metric tensor. Analogously, higher order $r_1 = 0$ and $r_2 = 0$ correspond to the trace and traceless parts of the Fronsdal equations (which are, of course, double traceless). Note that spin 0 and spin 1 equations are not described by this cohomology. Klein-Gordon and Maxwell equations result from the cohomology of the twisted adjoint representation, that encodes equations that can be imposed in terms of the generalized Weyl tensors which contain the spin 1 Maxwell tensor and the spin 0 scalar as the lower spin particular cases.

Thus, in the traceless case, the proper choice to reproduce dynamical field equations equivalent to the equations resulting from the Fronsdal Lagrangian is to keep only the Weyl cohomology nonzero. Setting elements of the Einstein cohomology to zero, that imposes the second-order field equations on the dynamical fields, leads to

$$R_1 \;=\; e_0^a\, e_0^b\, \epsilon_{ij}\, \frac{\partial^2}{\partial Y_i^a \partial Y_j^b}\, C(Y_k^c)\,, \tag{10.13}$$

which is exactly (8.2) where one makes contact with the HS algebra convention via the identification of variables (Y, Z) with (Y_1, Y_2). Thus, the generalized Weyl tensors $C(Y_k^c)$ on the right hand side of (8.2) parametrize the Weyl cohomology in the HS curvatures just to make the equations (8.2) equivalent to the HS field equations that follow from the Fronsdal's action. In the HS 0-form sector, the spin 0 and spin 1 field equations are contained in (8.3). Let us stress that our analysis works both in the flat space and in the $(A)dS_d$ case originally considered in [33]. Indeed, although the nonzero curvature affects the explicit form of the background frame and the Lorentz covariant derivative D_0^L and also requires a non-zero operator σ_+ denoted by $O(\Lambda)$ in (8.1), all this does not change the analysis of σ_- cohomology because the operator σ_- remains of the form (10.1) with a nondegenerate frame e_ν^a.

Alternatively, one can set the Weyl cohomology to zero, keeping the Einstein cohomology arbitrary. It is well known that in the spin 2 case of gravity the generic solution of the condition that the Weyl tensor is zero leads to conformally flat metrics. It is tempting to conjecture that the analogous condition for an arbitrary spin $s > 2$ singles out the "conformally flat" single trace HS fields of the form

$$\varphi_{\nu_1,\ldots\nu_s}(x) = g_{\{\nu_1,\nu_2}(x)\psi_{\nu_3\ldots\nu_s\}}(x) \tag{10.14}$$

with traceless symmetric $\psi_{\nu_3\ldots\nu_s}(x)$. Indeed, it is easy to see that the conformally flat free HS fields (10.14) have zero generalized Weyl tensor simply because it impossible to build a traceless tensor (10.9) from derivatives of $\psi_{\nu_1\ldots\nu_{s-2}}(x)$.

In the traceful case the equation (10.13) with traceful $C(Y_i^a)$ does not impose any differential restrictions on the fields in $H^1(\sigma_-, W)$ because $e_0^a\, e_0^b\, \epsilon_{ij}\, \frac{\partial^2}{\partial Y_i^a \partial Y_j^b}\, C(Y_k^c)$ span full $H^2(\sigma_-, W)$. This means that the equation (10.13) describes an infinite set of constraints which express all fields in terms of derivatives of the dynamical fields in $H^1(\sigma_-, W)$. In this sense, the equation (10.13) for a traceful field describes off-mass-shell constraints identifying the components of $C(Y)$ with the deWit-Freedman curvature tensors[20].

The cohomology analysis outlined here can be extended to the space $\hat{W}_n$ of tensors required to have their n-th trace equal zero, *i.e.* with (10.3) replaced by

$$\left(\eta^{ab}\frac{\partial^2}{\partial Y^a \partial Y^b}\right)^n \alpha = 0\,,$$

It is tempting to conjecture that the resulting gauge invariant field equations will contain $2n$ derivatives.

Let us make the following comment. The analysis of the dynamical content of the covariant constancy equations $\widehat{D}_0 C = 0$ may depend on the choice of the grading operator G and related graded decomposition (9.2). This may lead to different definitions of σ_- and, therefore, different interpretations of the same system of equations. For example, one can choose a different definition of σ_- in the space W of traceful tensors simply by decomposing W into a sum of irreducible Lorentz tensors (*i.e.*, traceless tensors) and then defining σ_- within any of these subspaces as in $\hat{W}$. In this basis, the equations (8.2) will be interpreted as dynamical equations for an infinite set of traceless dynamical fields. This phenomenon is not so surprising, taking into account the well-known analogous fact that, say, an off-mass-shell scalar can be represented as an integral over the parameter of mass of an infinite set of on-mass-shell scalar fields. More generally, to avoid paradoxical conclusions one has to take into account that σ_- may or may not have a meaning in terms of the elements of the Lie algebra Lie h that gives rise to the covariant derivative (9.2).

Nonlinear equations should replace the linearized covariant derivative $\widetilde{D}_0$ with the full one, $\widetilde{D}$, containing the h-valued connection ω. They should also promote the linearized curvature R_1 to R. Indeed, (8.3) and (10.13) cannot be correct at the nonlinear level because the consistency of $\widetilde{D}C = 0$ implies arbitrarily high powers of C in the r.h.s. of the modified equations since:

$$\widetilde{D}\widetilde{D}C \sim RC \sim O(C^2) + \text{higher order terms}\ ,$$

the last relation being motivated by (10.13).

Apart from dynamical field equations, the unfolded HS field equations contain constraints on the auxiliary components of the HS connections, expressing the latter via derivatives of the nontrivial dynamical variables, (*i.e.* Fronsdal fields) modulo pure gauge ambiguity. Originally, all HS gauge connections have **dimension** *length*$^{2-d/2}$ so that the HS field strength (5.14) needs no dimensionful parameter to have **dimension** *length*$^{1-d/2}$**.** However, this means that when some of the gauge connections are expressed via derivatives of the others, these expressions must involve space-time derivatives in the dimensionless combination $\rho\frac{\partial}{\partial x^\nu}$ where ρ is some parameter of dimension *length*. The only dimensionful parameter available in the analysis of free dynamics is the radius ρ of the AdS space-time related to the cosmological constant by (2.8). Recall that it appears through the definition of the frame

[20]However, if one imposes $C(Y_i^a)$ to be harmonic, then the corresponding field equations (10.13) imposes the deWit-Freedman curvature to be traceless. In this sense, it is possible to remove the tracelessness requirement in the frame-like formulation (see Subsection 4.1) without changing the physical content of the free field equations.

field (2.10) with $V^A \sim \rho$ adapted to make the frame E^A (and, therefore, the metric tensor) dimensionless. As a result, the HS gauge connections are expressed by the unfolded field equations through the derivatives of the dynamical fields as

$$\omega^{\,a_1\dots a_{s-1},\,b_1\dots b_t\hat{d}\dots\hat{d}} = \Pi\left(\rho^t \frac{\partial}{\partial x^{b_1}}\dots\frac{\partial}{\partial x^{b_t}}\omega^{\,a_1\dots a_{s-1},\,\hat{d}\dots\hat{d}}\right) + \text{lower derivative terms}\,, \qquad (10.15)$$

where Π is some projector that permutes indices (including the indices of the forms) and projects out traces. Plugging these expressions back into the HS field strength (5.14) one finds that HS connections with $t > 1$ (*i.e.* extra fields that appear for $s > 2$) contribute to **the** terms with higher derivatives which blow up in the flat limit $\rho \to \infty$. This mechanism brings higher derivatives and negative powers of the cosmological constant into HS interactions (but not into the free field dynamics because the free action is required to be independent of the extra fields). Note that a similar phenomenon takes place in the sector of the generalized Weyl 0-forms $C(Y)$ in the twisted adjoint representation.

11 Star product

We shall formulate consistent nonlinear equations using the star product. In other words we shall deal with ordinary commuting variables Y_i^A instead of operators $\hat{Y}_i^A$. In order to avoid ordering ambiguities, we choose the Weyl prescription. An operator is said to be Weyl ordered if it is completely symmetric under the exchange of operators $\hat{Y}_i^A$. One establishes a one to one correspondence between each Weyl ordered polynomial $f(\hat{Y})$ (5.4) and its symbol $f(Y)$, defined by substituting each operator $\hat{Y}_i^A$ with the commuting variable Y_i^A. Thus $f(Y)$ admits a formal expansion in power series of Y_i^A identical to that of $f(\hat{Y})$, *i.e.* with the same coefficients,

$$f(Y) = \sum_{m,n} f_{A_1\dots A_m\,,B_1\dots B_n} Y_1^{A_1}\dots Y_1^{A_m} Y_2^{B_1}\dots Y_2^{B_n}\,. \qquad (11.1)$$

To reproduce the algebra A_{d+1}, one defines the *star product* in such a way that, given any couple of functions f_1, f_2, which are symbols of operators $\hat{f}_1, \hat{f}_2$ respectively, $f_1 * f_2$ is the symbol of the operator $\hat{f}_1\hat{f}_2$. The result is nontrivial because the operator $\hat{f}_1\hat{f}_2$ should be Weyl ordered. It can be shown that this leads to the definition

$$(f_1 * f_2)(Y) = f_1(Y)\, e^{\frac{1}{2}\overleftarrow{\partial}^j_A \overrightarrow{\partial}^i_B \eta^{AB}\epsilon_{ji}}\, f_2(Y)\ , \qquad (11.2)$$

where $\partial^j_A \equiv \frac{\partial}{\partial Y_j^A}$ and $\overleftarrow{\partial}$, as usual, means that the partial derivative acts to the left while $\overrightarrow{\partial}$ acts to the right. The rationale behind this definition is simply that higher and higher powers of the differential operator in the exponent produce more and more contractions. One can show that the star product is an associative product law, and that it is regular, which means that the star product of two polynomials in Y is still a polynomial. From (11.2) it follows that the star product reproduces the proper commutation relation of oscillators,

$$[Y_i^A, Y_j^B]_* \equiv Y_i^A * Y_j^B - Y_j^B * Y_i^A = \epsilon_{ij}\eta^{AB}\ .$$

The star product has also an integral definition, equivalent to the differential one given by (11.2), which is

$$(f_1 * f_2)(Y) = \frac{1}{\pi^{2(d+1)}} \int dS dT\, f_1(Y+S)\, f_2(Y+T) \exp(-2 S_i^A T_A^i) \; . \tag{11.3}$$

The whole discussion of Section 5 can be repeated here, with the prescription of substituting operators with their symbols and operator products with star products. For example, the $o(d-1,2)$ generators (5.5) and the $sp(2)$ generators (5.6) are realized as

$$T^{AB} = -T^{BA} = \frac{1}{2} Y^{iA} Y_i^B \; , \quad t_{ij} = t_{ji} = Y_i^A Y_{jA} \; , \tag{11.4}$$

respectively. Note that

$$Y_i^A * = Y_i^A + \frac{1}{2} \frac{\overrightarrow{\partial}}{\partial Y_A^i}$$

and

$$* Y_i^A = Y_i^A - \frac{1}{2} \frac{\overleftarrow{\partial}}{\partial Y_A^i} \; .$$

From here it follows that

$$[Y_i^A \, , f(Y)]_* = \frac{\partial}{\partial Y_A^i} f(Y) \tag{11.5}$$

and

$$\{Y_i^A \, , f(Y)\}_* = 2 Y_i^A f(Y) \, . \tag{11.6}$$

With the help of these relations it is easy to see that the $sp(2)$ invariance condition $[t_{ij}, f(Y)]_* = 0$ indeed has the form (5.8) and singles out two-row rectangular Young tableaux, *i.e.* it implies that the coefficients $f_{A_1 \dots A_m \, , B_1 \dots B_n}$ are nonzero only if $n = m$, and symmetrization of any $m+1$ indices of $f_{A_1 \dots A_m \, , B_1 \dots B_m}$ gives zero.

One can then introduce the gauge fields taking values in this star algebra as functions $\omega(Y)$ of oscillators,

$$\omega(Y) = \sum_{s \geq 1} \omega^{A_1 \dots A_{s-1} \, , B_1 \dots B_{s-1}}(x) Y_{1A_1} \dots Y_{1A_{s-1}} Y_{2B_1} \dots Y_{2B_{s-1}} \; , \tag{11.7}$$

with their field strength defined by

$$R(Y) = d\omega(Y) + (\omega * \omega)(Y) \tag{11.8}$$

and gauge transformations

$$\delta\omega(Y) = d\epsilon(Y) + [\omega \, , \epsilon]_*(Y) \, . \tag{11.9}$$

For the subalgebra of $sp(2)$ singlets we have

$$D(t_{ij}) = 0 \, , \qquad [t_{ij} \, , \epsilon]_* = 0 \, , \qquad [t_{ij} \, , R]_* = 0 \, . \tag{11.10}$$

Note that $d(t_{ij}) = 0$ and, therefore from the first of these relations it follows that $[t_{ij} \, , \omega]_* = 0$, which is the $sp(2)$ invariance relation. Furthermore, one can get rid of traces by factoring out the ideal $\mathcal{I}$ spanned by the elements of the form $t_{ij} * g^{ij}$.

12 Twisted adjoint representation

As announced in Section 8, we give here a precise definition of the module in which the Weyl-like 0-forms take values, in such a way that (8.3) is reproduced at the linearized level.

To warm up, let us start with the adjoint representation. Let $\mathcal{A}$ be an associative algebra endowed with a product denoted by $*$. The $*$-commutator is defined as $[a,b]_* = a*b - b*a\ ,\ a,b\in\mathcal{A}$. As usual for an associative algebra, one constructs a Lie algebra g from $\mathcal{A}$, the Lie bracket of which is the $*$-commutator. Then g has an adjoint representation the module of which coincides with the algebra itself and such that the action of an element $a\in g$ is given by

$$[a,X]_*\ ,\forall X\in\mathcal{A}\ .$$

Let τ be an automorphism of the algebra $\mathcal{A}$, that is to say

$$\tau(a*b)=\tau(a)*\tau(b)\,,\qquad \tau(\lambda a+\mu b)=\lambda\tau(a)+\mu\tau(b),\qquad \forall a,b\in\mathcal{A}\,,$$

where λ and μ are any elements of the ground field $\mathbb{R}$ or $\mathbb{C}$. The τ-twisted adjoint representation of g has the same definition as the adjoint representation, except that the action of g on its elements is modified by the automorphism τ:

$$a(X)\ \rightarrow\ a*X-X*\tau(a)\ .$$

It is easy to see that this gives a representation of g.

The appropriate choice of τ, giving rise to the infinite bunch of fields contained in the 0-form C (matter fields, generalized Weyl tensors and their derivatives) is the following:

$$\tau(f(Y))=f(\widetilde{Y})\ ,\tag{12.1}$$

where

$$\widetilde{Y}_i^A=Y_i^A-2V^AV^BY_{Bi}$$

is the oscillator Y_i^A reflected with respect to the compensator (recall that we use the normalization $V_AV^A=1$). So one can say that the automorphism τ is some sort of parity transformation in the V-direction, leaving unaltered the Lorentz components of the oscillators. More explicitly, in terms of the transverse and longitudinal components

$${}^{\perp}Y_i^A=Y_i^A-V^AV_BY_i^B\,,\qquad {}^{\parallel}Y_i^A=V^AV^BY_{Bi}\,,$$

the automorphism τ is the transformation

$${}^{\perp}Y_i^A\rightarrow{}^{\perp}Y_i^A\,,\qquad {}^{\parallel}Y_i^A\rightarrow-{}^{\parallel}Y_i^A\,,$$

or, in the standard gauge $Y_i^a\rightarrow Y_i^a$, $Y_i^{\hat d}\rightarrow -Y_i^{\hat d}$. From (11.2) it is obvious that τ is indeed an automorphism of the star product algebra.

Thus, at the linearized level one can define a covariant derivative for the 0-form C in the twisted adjoint representation as

$$\widetilde{D}_0C=dC+\omega_0*C-C*\widetilde{\omega}_0\ .\tag{12.2}$$

One decomposes ω_0 into its Lorentz and translational part, $\omega_0 = \omega_0^L + \omega_0^{transl}$, via (2.15), which gives

$$\begin{aligned}\omega_0^L &\equiv \frac{1}{2}\,\omega_0^{AB}\,{}^\perp Y_A^i\,{}^\perp Y_{Bi} = \frac{1}{2}\,\omega_0^{ab}\,Y_a^i\,Y_{bi}\ ,\\ \omega_0^{transl} &\equiv \omega_0^{AB}\,{}^\perp Y_A^i\,{}^\parallel Y_{Bi} = e_0^a Y_a^i Y_{Bi} V^B\ .\end{aligned}$$

Taking into account the definition (12.1), it is clear that τ changes the sign of ω_0^{transl} while leaving ω_0^L untouched. This is tantamount to say that $\widetilde{D}_0$ contains an anticommutator with the translational part of the connection instead of a commutator,

$$\widetilde{D}_0 C = D_0^L C + \{\omega_0^{transl}, C\}_*\ ,$$

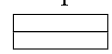
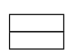
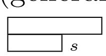
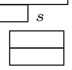
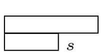
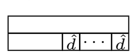
where D_0^L is the usual Lorentz covariant derivative, acting on Lorentz indices. Expanding the star products, we have

$$\widetilde{D}_0 = D_0^L + 2E_0^A V^B({}^\perp Y_A^i\,{}^\parallel Y_{Bi} - \frac{1}{4}\epsilon^{ij}\frac{\partial^2}{\partial\,{}^\perp Y^{Ai}\partial\,{}^\parallel Y^{Bj}})\ , \tag{12.3}$$

the last term being due to the noncommutative structure of the star algebra.

The equation (12.3) suggests that there exists a grading operator

$$N^{tw} = N_\perp - N_\parallel =^\perp Y_i^A\frac{\partial}{\partial^\perp Y_i^A} - {}^\parallel Y_i^A\frac{\partial}{\partial^\parallel Y_i^A} \tag{12.4}$$

commuting with $\widetilde{D}_0$, and whose eigenvalues $N_\perp - N_\parallel = 2s$, where s is the spin, classify the various irreducible submodules into which the twisted adjoint module decomposes as $o(d-1,2)$-module. In other words, the system of equations $\widetilde{D}_0 C = 0$ decomposes into an infinite number of independent subsystems, the fields of each subset satisfying $N^{tw}C = 2sC$, for some nonnegative integer s. Let us give some more detail about this fact. Recall that requiring $sp(2)$ invariance restricts us to the rectangular two row AdS_d Young tableaux  . By means of the compensator V^A we then distinguish between transverse (Lorentz) and longitudinal indices. Clearly $N^{tw} \geq 0$, since having more than half of vector indices in the extra direction V would imply symmetrization over more than half of all indices, thus giving zero because of the symmetry properties of Young tableaux. Then, each independent sector $N^{tw} = 2s$ of the twisted adjoint module starts from the rectangular Lorentz-Young tableau corresponding to the (generalized) Weyl tensor  ^s_s , and admits as further components all its "descendants"  $^{s+k}_{s}$, which the equations themselves set equal to k Lorentz covariant derivatives of  ^s_s. From the AdS_d-Young tableaux point of view, the set of fields forming an irreducible submodule of the twisted adjoint module with some fixed s is nothing but the components of the fields $C^{A_1\dots A_u,B_1\dots B_u}$ ($u = s,\dots\infty$) that have $k = u - s$ indices parallel to V^A:  $^{s+k=u}_{s}$ $\sim$  ^u_u

Quotienting by the ideal $\mathcal{I}$ admits an infinite number of possible ways of choosing representatives. The condition (3.6), which in our case amounts to

$$\frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}}\omega(Y) = 0\ ,$$

is not convenient for the twisted adjoint representation because it does not preserve the grading (12.4), *i.e.* it does not commute with N^{tw}. A version of the factorization condition that commutes with N^{tw} is

$$\Big(\frac{\partial^2}{\partial^{\perp}Y_i^A\partial^{\perp}Y_{Aj}} - 4^{\|}Y^{Ai\|}Y_A^j\Big)C(Y) = 0\ .$$

In practical computations it is sometimes even more convenient to require Lorentz tracelessness, *i.e.* tracelessness with respect to transversal indices. Recall that the final result is insensitive to the particular choice of the factorization condition, that is, choosing one or another condition is a matter of convenience. Upon application of one or another procedure for factorizing out the traces, the spin s submodule of the twisted adjoint module forms an irreducible $o(d-1,2)$-module. It is straightforward to compute a value of the Casimir operator in this irreducible $o(d-1,2)$-module

$$2T^{AB}T_{AB} = (s-1)(s+d-3)\,,$$

where $2s$ is the eigenvalue of N^{tw}. This value coincides with that for the unitary massless representations of any spin in AdS_d [56]. This fact is in agreement with the general observation [61, 42] that the representations carried by the 0-form sector in the unfolded dynamics are dual by a nonunitary Bogolyubov transform to the Hilbert space of single-particle states in the quantized sector of the theory.

Thus the subset of the fields C in the twisted adjoint representation with some fixed s matches the set of spin s generalized Weyl 0-forms of section 8. Not surprisingly, they form equivalent $o(d-1,2)$-modules. In particular, it can be checked that, upon an appropriate rescaling of fields, (12.3) reproduces (8.3) in the standard gauge. The precise form of the Λ–dependent terms in (8.3) follows from this construction.

Note that, in agreement with the general arguments of section 6, each irreducible spin-s submodule of the twisted adjoint representation is infinite-dimensional. This means that, in the unfolded formulation, the dynamics of any fixed spin s field is described in terms of an infinite set of fields related by the first-order unfolded differential equations. These equations contain both the physical equations of motions and some generalized torsion constraints, which separate dynamical from auxiliary fields. By virtue of these constraints, one is able to relate the latter fields to derivatives of the physical field of spin s, just as in gravity where the null torsion constraint leads to the identification of the Lorentz connection with derivatives of the vielbein. Thus auxiliary fields parametrize all on-mass-shell nontrivial combinations of derivatives of dynamical fields. Of course, to make it possible to describe a field-theoretical dynamical system with infinite number of degrees of freedom, the set of auxiliary 0-forms associated with all gauge invariant combinations of derivatives of dynamical fields should be infinite-dimensional.

13 Nonlinear field equations

We are now ready to search for nonlinear corrections to the free field dynamics. We will see that it is indeed possible to find a unique form for interactions, modulo field redefinitions, if one demands that the $sp(2)$ invariance of Section 5.2 is maintained at the nonlinear level. This condition is of crucial importance because, if the $sp(2)$ invariance was broken, then the resulting nonlinear equations might involve new tensor fields, different from the two-row rectangular Young tableaux one started with, and this might have no sense (for example, the new fields may contain ghosts). Thus, to have only usual HS fields as independent degrees of freedom, one has to require that $sp(2)$ invariance survives at the nonlinear level, or, in other words, that there should be a modified $sp(2)$ generator,

$$t_{ij}^{int} = t_{ij} + O(C) \ ,$$

that still satisfies $D(t_{ij}^{int}) = 0$ which is a deformation of the free field condition (11.10).

The construction of nonlinear corrections to the free field dynamics and the check of consistency order by order is quite cumbersome. These have been performed explicitly up to second order in the Weyl 0-forms [50, 62] in terms of the spinorial formulation of the $4d$ HS theory. More refined methods have been developed to formulate the full dynamics of HS gauge fields in a closed form first in four dimensions [10, 11] and more recently in any dimension [12]. The latter is presented now.

13.1 Doubling of oscillators

A trick that simplifies the formulation is to introduce additional noncommutative variables Z. This allows one to describe complicate nonlinear corrections as solutions of some simple differential equations with respect to such variables. **The** form of these equations is fixed by formal consistency **and by the** existence of nonlinear $sp(2)$ generators that guarantee correct spectrum of fields and the gauge invariance of all nonlinear terms they encode.

More precisely, this step amounts to the doubling of the oscillators $Y_i^A \rightarrow (Z_i^A, Y_i^A)$, and correspondingly one needs to enlarge the star product law. It turns out that a sensible definition is the following,

$$(f * g)(Z,Y) = \frac{1}{\pi^{2(d+1)}} \int dS dT \, e^{-2S_i^A T_A^i} f(Z+S, Y+S) g(Z-T, Y+T) \ , \tag{13.1}$$

which is an associative and regular product law in the space of polynomial functions $f(Z,Y)$, and gives rise to the commutation relations

$$[Z_i^A, Z_j^B]_* = -\epsilon_{ij}\eta^{AB} \, , \qquad [Y_i^A, Y_j^B]_* = \epsilon_{ij}\eta^{AB} \, , \qquad [Y_i^A, Z_j^B]_* = 0 \ .$$

The definition (13.1) has the meaning of a normal ordering with respect to the "creation" and "annihilation" operators $Z - Y$ and $Z + Y$, respectively. Actually, from (13.1) follows that the left star multiplication by $Z - Y$ and the right star multiplication by $Z + Y$ are equivalent to usual multiplications by $Z-Y$ and $Z+Y$, respectively. Note that Z independent functions $f(Y)$ form a proper subalgebra of the star product algebra (13.1) with the Moyal star product (11.3).

One can also check that the following formulae are true:

$$Y_i^A * = Y_i^A + \frac{1}{2}\left(\frac{\overrightarrow{\partial}}{\partial Y_A^i} - \frac{\overrightarrow{\partial}}{\partial Z_A^i}\right) \ , \qquad * Y_i^A = Y_i^A - \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial Y_A^i} + \frac{\overleftarrow{\partial}}{\partial Z_A^i}\right) \ , \tag{13.2}$$

$$Z_i^A * = Z_i^A + \frac{1}{2}\left(\frac{\overrightarrow{\partial}}{\partial Y_A^i} - \frac{\overrightarrow{\partial}}{\partial Z_A^i}\right) , \qquad * Z_i^A = Z_i^A + \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial Y_A^i} + \frac{\overleftarrow{\partial}}{\partial Z_A^i}\right) . \tag{13.3}$$

Furthermore, the appropriate reality conditions for the Lie algebra built from this associative star product algebra via commutators are

$$\bar{f}(Z,Y) = -f(iZ,-iY) , \tag{13.4}$$

where the bar denotes complex conjugation of the coefficients of the expansion of $f(Z,Y)$ in powers of Z and Y. This condition results from (5.9) with the involution † defined by the relations

$$(Y_i^A)^\dagger = -iY_i^A , \qquad (Z_i^A)^\dagger = iZ_i^A . \tag{13.5}$$

The distinguishing property of the extended definition of the star product is that it admits the inner Klein operator

$$\mathcal{K} = \exp(-2z_i y^i) , \tag{13.6}$$

where

$$y_i \equiv V_A Y_i^A , \qquad z_i \equiv V_A Z_i^A$$

are the projections of the oscillators along V^A. Using the definitions (13.1)-(13.6), one can show that $\mathcal{K}$ (i) generates the automorphism τ as an inner automorphism of the extended star algebra,

$$\mathcal{K} * f(Z,Y) = f(\widetilde{Z},\widetilde{Y}) * \mathcal{K} , \tag{13.7}$$

and (ii) is involutive

$$\mathcal{K} * \mathcal{K} = 1 \tag{13.8}$$

(see Appendix D for a proof of these properties).

Let us note that *a priori* the star product (13.1) is well-defined for the algebra of polynomials (which means that the star product of two polynomials is still a polynomial). Thus the star product admits an ordinary interpretation in terms of oscillators as long as we deal with polynomial functions. But $\mathcal{K}$ is not a polynomial because it contains an infinite number of terms with higher and higher powers of $z_i y^i$. So, *a priori* the star product with $\mathcal{K}$ may give rise to divergencies arising from the contraction of an infinite number of terms (for example, an infinite contribution may appear in the zeroth order like a sort of vacuum energy). What singles out the particular star product (13.1) is that this does not happen for the class of functions which extends the space of polynomials to include $\mathcal{K}$ and similar functions.

Indeed, the evaluation of the star product of two exponentials like $\mathcal{A} = \exp(A^{ij}_{AB} W_{1}{}^A_i W_{2}{}^A_j)$, where $W_{1,2}{}^A_j$ are some linear combinations of Y_i^A and Z_i^A, amounts to evaluating the Gaussian integral resulting from (13.1). The potential problem is that the bilinear form B of the integration variables in the Gaussian integral may be degenerate for some exponential factors A_1 and A_2 in $A_1 * A_2$, which leads to an infinite result because the Gaussian evaluates $det^{-1/2}|B|$. As was shown originally in [11] (see also [63]) for the analogous spinorial star product in four dimensions, the star product (13.1) is well-defined for the class of functions, which we call "regular", that can be expanded into a finite sum of functions f of the form

$$f(Z,Y) = P(Z,Y) \int_{M^n} d^n t \; \rho(t) \exp\Big(\phi(t) z_i y^i\Big) , \tag{13.9}$$

where the integration is over some compact domain $M^n \subset \mathbb{R}^n$ parametrized by the coordinates t_i $(i = 1, \ldots, n)$, the functions $P(Z,Y)$ and $\phi(t)$ are arbitrary polynomials of (Z,Y) and t_i, respectively, while $\rho(t)$ is integrable in M^n. The key point of the proof is that the star product (13.1) is such that the exponential in the Ansatz (13.9) never contributes to the quadratic form in the integration variables simply because $s_i s^i = t_i t^i \equiv 0$. As a result, a star product of two elements (13.9) never develops an infinity and the class (13.9) turns out to be closed under star multiplication pretty much as usual polynomials. The complete proof is given in Appendix E. The Klein operator $\mathcal{K}$ obviously belongs to the regular class (13.9): $n = 1$, $\rho(t) = \delta(t+2)$, $\phi(t) = t$, $P(Z,Y) = 1$, so our manipulations with $\mathcal{K}$ are safe. This property can be lost however if one uses a different star product realization of the same oscillator algebra. For example, usual Weyl ordering prescription is not helpful in that respect.

13.2 Field equations

The nonlinear equations are formulated in terms of the fields $W(Z,Y|x)$, $S(Z,Y|x)$ and $B(Z,Y|x)$, where B is a 0-form, while

$$W(Z,Y|x) = dx^\mu W_\mu(Z,Y|x)\,, \qquad S(Z,Y|x) = dZ_i^A S^i_A(Z,Y|x)$$

are connection 1-forms, in space-time and auxiliary Z_i^A directions, respectively. They satisfy the reality conditions analogous to (13.4)

$$\bar{W}(Z,Y|x) = -W(iZ,-iY|x)\ , \quad \bar{S}(Z,Y|x) = -S(iZ,-iY|x)\ ,$$

$$\bar{B}(Z,Y|x) = -\widetilde{B}(iZ,-iY|x)\ .$$

The fields ω and C are identified with the "initial data" for the evolution in Z variables as follows:

$$\omega(Y|x) = W(0,Y|x)\,, \qquad C(Y|x) = B(0,Y|x)\,.$$

The differentials satisfy the standard anticommutation relations $dx^\mu dx^\nu = -dx^\nu dx^\mu$, $dZ_i^A dZ_j^B = -dZ_j^B dZ_i^A$, $dx^\mu dZ_i^A = -dZ_i^A dx^\mu$, and commute with all other variables. The dependence on Z variables will be reconstructed by the imposed equations (modulo pure gauge ambiguities).

We require that all $sp(2)$ indices are contracted covariantly. This is achieved by imposing the conditions

$$[t^{tot}_{ij}, W]_* = 0\,, \qquad [t^{tot}_{ij}, B]_* = 0\,, \qquad [t^{tot}_{ij}, S^A_k]_* = \epsilon_{jk} S^A_i + \epsilon_{ik} S^A_j\ , \tag{13.10}$$

where the diagonal $sp(2)$ generator

$$t^{tot}_{ij} \equiv Y_i^A Y_{Aj} - Z_i^A Z_{Aj} \tag{13.11}$$

generates inner $sp(2)$ rotations of the star product algebra

$$[t^{tot}_{ij}, Y^A_k]_* = \epsilon_{jk} Y^A_i + \epsilon_{ik} Y^A_j\,, \qquad [t^{tot}_{ij}, Z^A_k]_* = \epsilon_{jk} Z^A_i + \epsilon_{ik} Z^A_j\,. \tag{13.12}$$

Note that the first of the relations (13.10) can be written covariantly as $D(t_{ij}) = 0$, taking into account that $d(t_{ij}) = 0$.

The full nonlinear system of equations for completely symmetric HS fields is

$$dW + W * W = 0 \ , \tag{13.13}$$

$$dB + W * B - B * \widetilde{W} = 0 \ , \tag{13.14}$$

$$dS + W * S + S * W = 0 \ , \tag{13.15}$$

$$S * B - B * \widetilde{S} = 0 \ , \tag{13.16}$$

$$S * S = -\frac{1}{2}(dZ^i_A dZ^A_i + 4\Lambda^{-1} dZ_{Ai} dZ^i_B V^A V^B B * \mathcal{K}) \ , \tag{13.17}$$

where we define

$$\widetilde{S}(dZ, Z, Y) = S(\widetilde{dZ}, \widetilde{Z}, \widetilde{Y}) \ . \tag{13.18}$$

Solutions of the system (13.13)-(13.17) admit factorization over the ideal generated by the nonlinear $sp(2)$ generators (13.22) defined in the next section as nonlinear deformations of the generators (11.4) used in the free field analysis. The system resulting from this factorization gives the nonlinear HS interactions to all orders.

The first three equations are the only ones containing space-time derivatives, via the de Rham differential $d = dx^\mu \frac{\partial}{\partial x^\mu}$. They have the form of zero-curvature equations for the space-time connection W (13.13) and the Z-space connection S (13.15) together with a covariant constancy condition for the 0-form B (13.14). These equations alone do not allow any nontrivial dynamics, so the contribution coming from (13.16) and (13.17) is essential. Note that the last two equations are constraints from the space-time point of view, not containing derivatives with respect to the x-variables, and that the nontrivial part only appears with the "source" term $B * \mathcal{K}$ in the V^A longitudinal sector of (13.17) (the first term on the right hand side of (13.17) is a constant). The inverse power of the cosmological constant Λ is present in (13.17) to obtain a Weyl tensor with Λ independent coefficients in the linearized equations in such a way that their flat limit also makes sense. In the following, however, we will again keep V normalized to 1, which means $\Lambda = 1$ (see Section 2.3).

The system is formally consistent, *i.e.* compatible with $d^2 = 0$ and with associativity. A detailed proof of this statement can be found in the appendix F. Let us however point out here the only tricky step. To prove the consistency, one has to show that the associativity relation $S * (S * S) = (S * S) * S$ is compatible with the equations. This is in fact the form of the Bianchi identity with respect to the Z variables, because S actually acts as a sort of exterior derivative in the noncommutative space (as will be shown in the next section). Associativity seems then to be broken by the source term $B * \mathcal{K}$ which anticommutes with dz_i as a consequence of (13.16) and the definition (13.18), and brings in a term proportional to $dz_i dz^i dz_j$ (where $dz_i = V_A dZ^A_i$). This is not a problem however: since i is an $sp(2)$ index, it can take only two values and, as a result, the antisymmetrized product of three indices vanishes identically: $dz_i dz^i dz_j = 0$.

In a more compact way, one can prove consistency by introducing the noncommutative extended covariant derivative $\mathcal{W} = d + W + S$ and assembling eqs. (13.13)-(13.17) into

$$\mathcal{W} * \mathcal{W} = -\frac{1}{2}(dZ_A^i dZ_i^A + 4dZ_{Ai} dZ_B^i V^A V^B B * \mathcal{K}) ,$$

$$\mathcal{W} * B = B * \widetilde{\mathcal{W}} .$$

In other words, $S*S$ is nothing but the ZZ component of an (x, Z)-space curvature, and it is actually the only component of the curvature allowed to be nonvanishing, xx and xZ being trivial according to (13.13) and (13.15), respectively. Consistency then amounts to the fact that the associativity relations $\mathcal{W}*(\mathcal{W}*\mathcal{W}) = (\mathcal{W}*\mathcal{W})*\mathcal{W}$ and $(\mathcal{W}*\mathcal{W})*B = \mathcal{W}*(\mathcal{W}*B)$ are respected by the nonlinear equations. Recall, however, that it was crucial for the consistency that the symplectic indices take only two values.

According to the general scheme of free differential algebras, the consistency of the nonlinear equations implies gauge symmetry under the local transformations

$$\delta\mathcal{W} = [\epsilon, \mathcal{W}]_* , \qquad \delta B = \epsilon * B - B * \widetilde{\epsilon} , \tag{13.19}$$

where $\epsilon = \epsilon(Z, Y|x)$ is $sp(2)$ invariant, *i.e.* $[t_{ij}^{tot}, \epsilon]_* = 0$, and otherwise arbitrary.

Consistency of the system, which means compatibility of the equations with the Bianchi identities, both in the x and in the Z sector (the latter case being verified by associativity of S), guarantees that the perturbative analysis works systematically at all orders.

13.3 sp(2) invariance

The $sp(2)$ symmetry is crucial for the consistency of the free system, to avoid unwanted ghost degrees of freedom. But as said before, survival of the $sp(2)$ invariance at the full nonlinear level is also very important, in the sense that it fixes the form of the nonlinear equations and prevents a mixture of unwanted degrees of freedom at the nonlinear level.

The rationale behind this is as follows. The conditions (13.10) guarantee the $sp(2)$ covariance of the whole framework. But this is not enough because one has to remove traces by factoring out terms which are themselves proportional to the $sp(2)$ generators. The third commutation relation in (13.10) makes this difficult. Indeed, it means that the operators S_i^A transform elements of the algebra proportional to t_{ij}^{tot} into t_{ij}^{tot} independent elements, *i.e.* the equations (13.13)-(13.17) do not allow a factorization with respect to the ideal generated by t_{ij}^{tot}.

To avoid this problem at the full nonlinear level one has to build proper generators

$$t_{ij}^{int} = t_{ij} + t_{ij}^1 + \dots ,$$

where t_{ij}^1 and higher terms denote the field-dependent corrections to the original $sp(2)$ generators (5.6), such that they satisfy the $sp(2)$ commutation relations

$$[t_{ij}^{int}, t_{kl}^{int}] = \epsilon_{ik} t_{jl}^{int} + \epsilon_{jk} t_{il}^{int} + \epsilon_{il} t_{jk}^{int} + \epsilon_{jl} t_{ik}^{int}$$

and

$$Dt^{int}_{ij} = 0 \ , \qquad [S, t^{int}_{ij}]_* = 0 \ , \qquad B * \widetilde{t}^{int}_{ij} - t^{int}_{ij} * B = 0 \, . \tag{13.20}$$

What fixes the form of the nontrivial equations (13.16) and (13.17) is just the requirement that such nonlinearly deformed $sp(2)$ generators t^{int}_{ij} do exist. Actually, getting rid of the dZ's in (13.16) and (13.17) in the longitudinal sector, these equations read

$$[s^i, s^j]_* = -\epsilon^{ij}(1 + 4B * \mathcal{K}) \ , \qquad s^i * B * \mathcal{K} = -B * \mathcal{K} * s^i$$

(where $s^i \equiv V^A S^i_A$). This is just a realization [64] of the so called deformed oscillator algebra found originally by Wigner [65] and discussed by many authors [66]

$$[\hat{y}^i, \hat{y}^j]_* = \epsilon^{ij}(1 + \nu \hat{k}) \ , \qquad \{\hat{y}^i, \hat{k}\}_* = 0 \ , \tag{13.21}$$

ν being a central element. The main property of this algebra is that, for any ν, the elements $\tau_{ij} = -\frac{1}{2}\{s_i, s_j\}_*$ form the $sp(2)$ algebra that rotates properly s_i

$$[\tau_{ij}, s_k]_* = \epsilon_{ik} s_j + \epsilon_{jk} s_i \, .$$

As a consequence, there exists another $sp(2)$ generator

$$\mathcal{T}_{ij} = -\frac{1}{2}\{S^A_i, S_{Aj}\}_* \, ,$$

which acts on S^A_i as

$$[\mathcal{T}_{ij}, S^A_k]_* = \epsilon_{ik} S^A_j + \epsilon_{jk} S^A_i \, .$$

As a result, the difference

$$t^{int}_{ij} \equiv t^{tot}_{ij} - \mathcal{T}_{ij} \tag{13.22}$$

satisfies the $sp(2)$ commutation relation and the conditions (13.20), taking into account (13.10) and (13.13) - (13.17). Moreover, at the linearized level, where $S^A_i = Z^A_i$ as will be shown in the next section, t^{int}_{ij} reduces to t_{ij}. This means that, if nonlinear equations have the form given above, interaction terms coming from the evolution along noncommutative directions do not spoil the $sp(2)$ invariance and allow the factorization of the elements proportional to t^{int}_{ij}. This, in turn, implies that the nonlinear equations admit an interpretation in terms of the tensor fields we started with in the free field analysis. Let us also note that by virtue of (13.22) and (13.13)-(13.17) the conditions (13.20) are equivalent to (13.10).

An interesting interpretation of the deformed oscillator algebra (13.21) is [64] that it describes a two-dimensional fuzzy sphere of a ν-dependent radius. Comparing this with the equations (13.16) and (13.17) we conclude that the nontrivial HS equations describe a two-dimensional fuzzy sphere embedded into a non-commutative space of variables Z^A and Y^A. Its radius varies from point to point of the usual (commutative) space-time with coordinates x, depending on the value of the HS curvatures collectively described by the Weyl 0-form $B(Z, Y|x)$.

14 Perturbative analysis

Let us now expand the equations around a vacuum solution, checking that the full system of HS equations reproduces the free field dynamics at the linearized level.

The vacuum solution (W_0, S_0, B_0) around which we will expand is defined by $B_0 = 0$, which is clearly a trivial solution of (13.14) and (13.16). Furthermore, it cancels the source term in (13.17), which is then solved by

$$S_0 = dZ_i^A Z_A^i \,. \tag{14.1}$$

The equation (13.15) at the zeroth order then becomes $\{W_0, S_0\}_* = 0$, and taking into account that

$$[Z_i^A, f]_*(Z, Y) = -\frac{\partial}{\partial Z_A^i} f(Z, Y) \tag{14.2}$$

(see (13.3)) one concludes that W_0 can only depend on Y and not on Z. One solution of (13.13) is the AdS connection, bilinear in Y,

$$W_0 = \omega_0^{AB}(x) T_{AB}(Y) \,, \tag{14.3}$$

which thus appears as a natural vacuum solution of HS nonlinear equations. The vacuum solution (14.1), (14.3) satisfies also the $sp(2)$ invariance condition (13.10).

Let us point out that from (14.1) and (14.2) it follows that the adjoint action of S_0 is equivalent to the action of the exterior differential in the Z-space. As a result, the space-time constraints (13.16) and (13.17) actually correspond to differential equations with respect to the noncommutative Z variables.

Our goal is now to see whether free HS equations emerge from the full system as first order correction to the vacuum solution. We thus set

$$W = W_0 + W_1 \,, \qquad S = S_0 + S_1 \,, \qquad B = B_0 + B_1 \,,$$

and keep terms up to the first order in W_1, S_1, B_1 in the nonlinear equations.

We begin by looking at (13.16). B is already first order, so we can substitute S by S_0 to obtain that B_1 is Z-independent

$$B_1(Z, Y) = C(Y|x) \,. \tag{14.4}$$

Inserting this solution into (13.14) just gives the twisted adjoint equation (12.2), one of the two we are looking for.

Next we attempt to find S_1 substituting (14.4) into (13.17), taking into account that

$$f(Z, Y) * \mathcal{K} = \exp(-2z_i y^i)\, f(Z_i^A - V^A(z_i + y_i),\, Y_i^A - V^A(z_i + y_i))$$

(see Appendix D), which one can write as

$$f * \mathcal{K} = \exp(-2z_i y^i)\, f({}^{\perp}Z - {}^{\parallel}\mathrm{Y}, {}^{\perp}Y - {}^{\parallel}Z) \,.$$

This means that $\mathcal{K}$ acts on functions of Z and Y by interchanging their respective longitudinal parts (taken with a minus sign) and multiplication by a factor of $\exp(-2z_i y^i)$.

Looking at the ZZ part of the curvature, one can see that the V^A transversal sector is trivial at first order and that the essential Z-dependence is concentrated only in the longitudinal components. One can then analyze the content of (13.17) with respect to the longitudinal direction only, getting

$$\partial^i s_1^j - \partial^j s_1^i = -4\varepsilon^{ij} C(-{}^{\parallel}Z, {}^{\perp}Y) \exp(-2z_i y^i) \,, \tag{14.5}$$

with $\partial^i = \frac{\partial}{\partial z_i}$. The general solution of the equation $\partial_i f^i(z) = g(z)$ is $f_i(z) = \partial_i \epsilon + \int_0^1 dt\, t\, z_i g(z)$. Applying this to (14.5) one has

$$s_1^i = \partial^i \epsilon_1 + 2z^i \int_0^1 dt\, t\, C(-t\,{}^{\parallel}Z, {}^{\perp}Y) \exp(-2tz_k y^k) \,,$$

Analogously, in the V transverse sector one obtains that ${}^{\perp}S_i^A$ is pure gauge so that

$$S^i_{1\,A} = \frac{\partial}{\partial Z_i^A}\epsilon_1 + 2V_A z^i \int_0^1 dt\, t\, C(-t\,{}^{\parallel}Z, {}^{\perp}Y) \exp(-2tz_k y^k) \,,$$

where the first term on the r.h.s. is the Z-exact part. This term is the pure gauge part with the gauge parameter $\epsilon_1 = \epsilon_1(Z;Y|x)$ belonging to the extended HS algebra. One can conveniently set $\frac{\partial}{\partial Z_i^A}\epsilon_1 = 0$ **by** using part of the gauge symmetry (13.19). This choice fixes the Z-dependence of the gauge parameters to be trivial and leaves exactly the gauge freedom one had at the free field level, $\epsilon_1 = \epsilon_1(Y|x)$. Moreover, let us stress that with this choice one has reconstructed S_1 entirely in terms of B_1. Note that s_1^i belongs to the regular class of functions (13.9) compatible with the star product.

We now turn our attention to the equation (13.15), which determines the dependence of W on z. In the first order, it gives

$$\partial^i W_1 = ds_1^i + W_0 * s_1^i - s_1^i * W_0 \,.$$

The general solution of the equation $\frac{\partial}{\partial z_i}\varphi(z) = \chi^i(z)$ is given by the line integral

$$\varphi(z) = \varphi(0) + \int_0^1 dt\, z_i \chi^i(tz) \,,$$

provided that $\frac{\partial}{\partial z_i}\chi^i(z) = 0$ $(i = 1, 2)$. Consequently,

$$W_1 = \omega(Y|x) - z^j \int_0^1 dt\, (1-t)\, e^{-2tz_i y^i} E_0^B \frac{\partial}{\partial^{\perp} Y^{jB}} C(-t\,{}^{\parallel}Z, {}^{\perp}Y) \,, \tag{14.6}$$

taking into account that the term $z_i ds_1^i$ vanishes because $z_i z^i = 0$. Again, W_1 is in the regular class. Note also that in (14.6) only the frame field appears, while the dependence on the Lorentz connection cancels out, because of the local Lorentz symmetry which forbids the presence of ω^{ab} if not inside Lorentz covariant derivatives.

One still has to analyze (13.13), which at first order reads

$$dW_1 + \{W_0, W_1\}_* = 0 \ .$$

Plugging in (14.6), one **gets**

$$R_1 = O(C) \ , \qquad R_1 = d\omega + \{\omega \, , W_0\}_* \, , \tag{14.7}$$

where corrections on the r.h.s. of the first equation in (14.7) come from the second term in (14.6), and prevent (14.7) from being trivial, that would imply ω to be a pure gauge solution[21]. The formal consistency of the system with respect to Z variables allows one to restrict the study of (13.13) to the physical space $Z = 0$ only (with the proviso that the star products are to be evaluated before sending Z to zero). This is due to the fact that the dependence on Z is reconstructed by the equations in such a way that if (13.13) is satisfied for $Z = 0$, it is true for all Z. By elementary algebraic manipulations one obtains the final result

$$R_1 = \frac{1}{2} E_0^A \, E_0^B \, \frac{\partial^2}{\partial \, {}^\perp Y_i^A \partial \, {}^\perp Y^{iB}} C(0, {}^\perp Y) \ ,$$

which, together with the equation for the twisted adjoint representation previously obtained, reproduces the free field dynamics for all spins (10.13) and (8.3).

Following the same lines one can now reconstruct order-by-order all nonlinear corrections to the free HS equations of motion. Note that all expressions that appear in this analysis belong the regular class (13.9), and therefore the computation as a whole is free from divergencies, being well defined. **At the same time, the substitution of expressions** like (10.15) for auxiliary fields will give rise to nonlinear corrections with higher derivatives, which are nonanalytic in the cosmological constant.

15 Discussion

The main message of these lectures is that nonlinear dynamics of HS gauge fields can be consistently formulated in all orders in interactions in Anti de Sitter space-time of any dimension $d \geqslant 4$.

A surprising issue related to the structure of the HS equations of motion is that, within this formulation of the dynamics, one can get rid of the space-time variables. Indeed, (13.13)-(13.15) are zero curvature and covariant constancy conditions, admitting pure gauge solu-

[21] In retrospective, one sees that this is the reason why it was necessary for the Z coordinates to be noncommutative, allowing nontrivial contractions, since corrections are obtained from perturbative analysis as coefficients of an expansion in powers of z obtained by solving for the z-dependence of the fields from the full system.

tions, *i.e.* locally

$$\begin{aligned} W(x) &= g^{-1}(x) * dg(x) \ , \\ B(x) &= g(x) * B_0 * \widetilde{g}^{-1}(x) \ , \\ S(x) &= g(x) * S_0 * g^{-1}(x) \ , \end{aligned} \tag{15.1}$$

the whole dependence on space-time points being absorbed into a gauge function $g(x)$, which is an arbitrary invertible element of the star product algebra, while $B_0 = B_0(Z, Y)$ and $S_0 = S_0(Z, Y)$ are arbitrary x-independent elements of the star product algebra. Because the system is gauge invariant, the gauge functions disappear from the remaining two equations, which then encode the whole dynamics though being independent of x. This turns out to be possible, in the unfolded formulation, just because of the presence of an infinite bunch of fields, supplemented by an infinite number of appropriate constraints, determined by consistency. As previously seen in the lower spin examples (see section 7), the 0-form B turns out to be the generating function of all on-mass-shell nontrivial derivatives of the dynamical fields. Thus it locally reconstructs their x-dependence through their Taylor expansion which in turn is just given by the formulas (15.1). So, within the unfolded formulation, the dynamical problem is well posed once all 0-forms assembled in B_0 are given at one space-time point x_0, because this is sufficient to obtain the whole evolution of fields in any neighborhood of x_0 by analyticity (note that S_0 is reconstructed in terms of B_0 up to a pure gauge part). This way of solving the nonlinear system, getting rid of x variables, is completely equivalent to the one used in the previous section, or, in other words, the unfolded formulation involves a trade between space-time variables and auxiliary noncommutative variables (Z, Y). Nevertheless, the way we see and perceive the world seems to require the definition of local events, and it is this need for locality that makes the reduction to the "physical" subspace $Z = 0$ (keeping the x-dependence instead of gauging it away) more appealing. On the other hand, as mentioned in subsection (13.3), the HS equations in the auxiliary noncommutative space has a clear geometrical meaning of describing embeddings of a two-dimensional noncommutative sphere into the Weyl algebra.

The system of gauge invariant nonlinear equations for all spins in AdS_d here presented can be generalized [12] to matrix-valued fields, $W \to W_\alpha{}^\beta$, $S \to S_\alpha{}^\beta$ and $B \to B_\alpha{}^\beta$, $\alpha, \beta = 1, ..., n$, giving rise to Yang-Mills groups $U(n)$ in the $s = 1$ sector while remaining consistent. It is also possible to truncate to smaller inner symmetry groups $USp(n)$ and $O(n)$ by imposing further conditions based on certain antiautomorphisms of the star product algebra [37, 12]. Apart from the possibility of extending the symmetry group with matrix-valued fields and modulo field redefinitions, it seems that there is no ambiguity in the form of nonlinear equations. As previously noted, this is due to the fact that the $sp(2)$ invariance requires (13.16) and (13.17) to have the form of a deformed oscillator algebra.

HS models have just one dimensionless coupling constant

$$g^2 = |\Lambda|^{\frac{d}{2}-1} \kappa^2 \, .$$

To introduce the coupling constant, one has to rescale the fluctuations ω_1 of the gauge fields (*i.e.* additions to the vacuum field) as well as the Weyl 0-forms B by a coupling constant

g so that it cancels out in the free field equations. In particular, g is identified with the Yang-Mills coupling constant in the spin one sector. Its particular value is artificial however because it can be rescaled away in the classical theory (although it is supposed to be a true coupling constant in the quantum theory where it is a constant in front of the whole action in the exponential inside the path integral). Moreover, there is no dimensionful constant allowing us to discuss a low-energy expansion, *i.e.* an expansion in powers of a dimensionless combination of this constant and the covariant derivative. The only dimensionful constant present here is Λ. The dimensionless combinations $\bar{D}_\mu \equiv \Lambda^{-1/2} D_\mu$ are not good expansion entities, since the commutator of two of them is of order 1, as a consequence of the fact that the AdS curvature is roughly $R_0 \sim D^2 \sim \Lambda gg$. For this reason also it would be important to find solutions of HS field equations different from AdS_d, thus introducing in the theory a massive parameter different from the cosmological constant.

Finally, let us note that a variational principle giving the nonlinear equations (13.13)-(13.17) is still unknown in all orders. Indeed, at the action level, gauge invariant interactions have been constructed only up to the cubic order [9, 24, 67].

16 Conclusion

Since it was impossible to cover in these lectures all the interesting and important directions of research in the modern HS gauge theory, an invitation to further readings is provided as a conclusion.

The level of generality of the analyzes covered in these lectures has been restricted in the following points: only completely symmetric bosonic HS gauge fields have been considered, and only at the level of the equations of motion. For general topics in HS gauge theories, the reader is referred to the review papers [3, 6, 30, 49, 68, 69, 70, 71]. Among the specific topics that have not been addressed here, one can mention:

(i) spinor realizations of HS superalgebras in $d = 3, 4$ [35, 36, 37, 63], $d = 5$ [41, 43], $d = 7$ [44] and the recent developments in any dimension [48],

(ii) cubic action interactions [9, 24, 67],

(iii) application of unfolded dynamics to go to larger (super)space, *e.g.* free HS theories in tensorial superspaces [72, 42, 59] or HS theories in usual superspace [73],

(iv) group-theoretical classification of invariant equations via unfolded formalism [74],

(v) HS gauge fields different from the completely symmetric Fronsdal fields: *e.g.* mixed symmetry fields [75] and partially massless fields [76].

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A Isometry algebras

By "space-time" symmetries one means symmetries of the corresponding space-time manifold $\mathcal{M}^d$ of dimension d, which may be isometries or conformal symmetries. The most symmetrical solutions of vacuum Einstein equations, with or without cosmological constant Λ, are (locally) Minkowski space ($\Lambda = 0$), de Sitter ($\Lambda > 0$) and Anti de Sitter ($\Lambda < 0$) spaces. In this paper, we only consider $\mathbb{R}^{d-1,1}$ and AdS_d spaces though the results generalize easily to dS_d space.

The Poincaré group $ISO(d-1,1) = \mathbb{R}^{d-1,1} \rtimes SO(d-1,1)$ has translation generators P_a and Lorentz generators M_{ab} $(a, b = 0, 1, \dots, d-1)$ satisfying the algebra

$$[\, M_{ab}\, ,\, M_{cd}\,] = \eta_{ac}\, M_{db} - \eta_{bc}\, M_{da} - \eta_{ad}\, M_{cb} + \eta_{bd}\, M_{ca}\, , \tag{A.1}$$

$$[\, P_a\, ,\, M_{bc}] = \eta_{ab}\, P_c - \eta_{ac}\, P_b\, , \tag{A.2}$$

$$[P_a, P_b] = 0\, . \tag{A.3}$$

"Internal" symmetries are defined as transformations that commute with the translations generated by P_a and the Lorentz transformations generated by M_{ab} [1]. The relation (A.1) is the Lorentz algebra $o(d-1,1)$ while the relations (A.2)-(A.3) state that the Poincaré algebra is a semi-direct product $iso(d-1,1) = \mathbb{R}^d \ni o(d-1,1)$.

The algebra of isometries of the AdS_d space-time is given by the commutation relations (A.1)-(A.2) and

$$[P_a, P_b] = -\frac{1}{\rho^2}\, M_{ab}\, . \tag{A.4}$$

where ρ is proportional to the radius of curvature of AdS_d and is related to the cosmological constant via $|\Lambda| = \rho^{-2}$. The (non-commuting) transformations generated by P_a are called transvections in AdS_d to distinguish them from the (commuting) translations. By defining $M_{\hat{d}\, a} = \rho\, P_a$, it is possible to collect all generators into the generators M_{AB} where $A =$

$0, 1, \ldots, \hat{d}$. These generators M_{AB} span $o(d-1,2)$ algebra since they satisfy the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{DB} - \eta_{BC} M_{DA} - \eta_{AD} M_{CB} + \eta_{BD} M_{CA} \,,$$

where η_{AB} is the mostly minus invariant metric of $o(d-1,2)$. This is easily understood from the geometrical construction of AdS_d as the hyperboloid defined by $X^A X_A = \frac{(d-1)(d-2)}{2}\rho^2$ which is obviously invariant under the isometry group $O(d-1,2)$. Since transvections are actually rotations in ambient space, it is normal that they do not commute. It is possible to derive the Poincaré algebra from the AdS_d isometry algebra by taking the infinite-radius limit $\rho \to \infty$. This limiting procedure is called Inönü-Wigner contraction [77].

B Gauging internal symmetries

In this subsection, a series of tools used in any gauge theory is briefly introduced. One considers the most illustrative example of Yang-Mills theory, which corresponds to gauging an internal symmetry group.

Let g be a (finite-dimensional) Lie algebra of basis $\{T_\alpha\}$ and Lie bracket $[\ ,\ ]$. The structure constants are defined by the following products of basis elements T_α as $[T_\alpha, T_\beta] = T_\gamma f^\gamma{}_{\alpha\beta}$. The Yang-Mills theory is conveniently formulated only in purely algebraic terms by using differential forms taking values in the Lie algebra g.

- The connection $A = dx^\mu A^\alpha_\mu T_\alpha$ is defined in terms the vector gauge field A^α_μ.
- The curvature $F = dA + A^2 = \frac{1}{2}\, dx^\mu dx^\nu F^\alpha_{\mu\nu} T_\alpha$ is associated with the field strength tensor $F^\alpha_{\mu\nu} = \partial_{[\mu} A^\alpha_{\nu]} + f^\alpha{}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu$.
- The Bianchi identity $dF + AF - FA = 0$ is a consequence of $d^2 = 0$ and the Jacobi identity in the Lie algebra.
- The gauge parameter $\epsilon = \epsilon^\alpha T_\alpha$ is associated with the infinitesimal gauge transformation $\delta_\epsilon A = d\epsilon + A\epsilon - \epsilon A$ which transforms the curvature as $\delta_\epsilon F = F\epsilon - \epsilon F$. In components, this reads as $\delta_{\epsilon^\alpha} A^\alpha_\mu = \partial_\mu \epsilon^\alpha + f^\alpha_{\beta\gamma} A^\beta_\mu \epsilon^\gamma$ and $\delta F^\alpha_{\mu\nu} = f^\alpha_{\beta\gamma} F^\beta_{\mu\nu} \epsilon^\gamma$.

The algebra $\Omega(\mathcal{M}^d) \otimes g$ is a Lie superalgebra, the product of which is the graded Lie bracket denoted by $[\ ,\ ]$ [22]. The elements of $\Omega^p(\mathcal{M}^d) \otimes g$ are p-forms taking values in g. The interest of the algebra $\Omega(\mathcal{M}^d) \otimes g$ is that it contains the gauge parameter $\epsilon \in \Omega^0(\mathcal{M}^d) \otimes g$, the connection $A \in \Omega^1(\mathcal{M}^d) \otimes g$, the curvature $F \in \Omega^2(\mathcal{M}^d) \otimes g$ and the Bianchi identity takes place in $\Omega^3(\mathcal{M}^d) \otimes g$.

To summarize, the Yang-Mills theory is a fibre bundle construction where the Lie algebra g is the fiber, A the connection and F the curvature. The Yang-Mills action takes the form

$$S^{YM}[A^a_\mu] \propto \int_{\mathcal{M}^d} Tr[F^* F] \,,$$

[22]Here, the grading is identified with that in the exterior algebra so that the graded commutator is evaluated in terms of the original Lie bracket $[\ ,\ ]$.

in which case g is taken to be semisimple in such a way that the Killing form is non-degenerate. Here * is the Hodge star producing a dual form. Note that the action contains the metric tensor which is needed to make the Yang-Mills action invariant under diffeomorphisms. Furthermore, because of the cyclicity of the trace, the Yang-Mills Lagrangian $Tr[F^* F]$ is also manifestly invariant under the gauge transformations.

An operatorial formulation is also useful for its compactness. Let us now consider some matter fields Φ living in some space V on which acts the Lie algebra g, via a representation T_α. In other words, the elements T_α are reinterpreted as operators acting on some representation space (also called module) V. The connection A becomes thereby an operator. For instance, if T is the adjoint representation then the module V is identified with the Lie algebra g and A acts as $A \cdot \Phi = [A, \Phi]$. The connection A defines the covariant derivative $D \equiv d + A$. For any representation of g, the transformation law of the matter field is $\delta\Phi = -\epsilon\phi$, where ϵ is a constant or a function of x according to whether g is a global or a local symmetry. The gauge transformation law of the connection can also be written as $\delta A = \delta D = [D, \epsilon]$ and is such that $\delta(D\Phi) = D(\delta\Phi)$. The curvature is economically defined as an operator $F = \frac{1}{2}[D, D] = D^2$, which holds because of the identity $[d, A] = dA$. In space-time components, the latter equation reads as usual $[D_\mu, D_\nu] = F_{\mu\nu}$. The Bianchi identity is a direct consequence of the associativity of the differential algebra and Jacobi identities of the Lie algebra and reads in space-time components $[D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] = 0$. The graded Jacobi identity leads to $\delta F = \frac{1}{2}\Big(\,[\,[D, \epsilon]\,, D\,] + [\,D, [D, \epsilon]\,]\,\Big) = [D^2\,,\, \epsilon] = [F\,,\, \epsilon]$.

The present notes make an extension of the previous compact notations and synthetic identities. Indeed, they generalize it straightforwardly to other gauge theories formulated via a non-Abelian connection, *e.g.* HS gauge theories[23].

C Gauging space-time symmetries

The usual Einstein-Hilbert action $S[g]$ is invariant under diffeomorphisms. The same is true for $S[e, \omega]$, defined by (2.1), since everything is written in terms of differential forms. The action (2.1) is also manifestly invariant under local Lorentz transformations $\delta\omega = d\epsilon + [\omega, \epsilon]$ with gauge parameter $\epsilon = \epsilon^{ab} M_{ab}$, because $\epsilon_{a_1 \ldots a_d}$ is an invariant tensor of $SO(d-1, 1)$. The gauge formulation of gravity shares many common features with a Yang-Mills theory formulated in terms of a connection ω taking values in the Poincaré algebra.

However, gravity is actually *not* a Yang-Mills theory with Poincaré as (internal) gauge group. The aim of this section is to express precisely the distinction between internal and space-time gauge symmetries.

To warm up, let us mention several obvious differences between Einstein-Cartan's gravity and Yang-Mills theory. First of all, the Poincaré algebra $iso(d-1, 1)$ is not semisimple (since it is **not a *direct* sum** of simple Lie algebras). Secondly, the action (2.1) cannot be written

[23] More precisely, one can take g as an infinite-dimensional Lie algebra that arises from an associative algebra with product law $*$ and is endowed with the (sometimes twisted) commutator as bracket. Up to these subtleties and some changes of notation, all previous relations hold for HS gauge theories considered here, and they might simplify some explicit checks by the reader.

in a Yang-Mills form $\int Tr[F^* F]$. Thirdly, the action (2.1) is not invariant under the gauge transformations $\delta\omega = d\epsilon + [\omega, \epsilon]$ generated by *all* Poincaré algebra generators, *i.e.* with gauge parameter $\epsilon(x) = \epsilon^a(x)P_a + \epsilon^{ab}(x)M_{ab}$. For $d > 3$, the action (2.1) is invariant only when $\epsilon^a = 0$. (For $d = 3$, the action (2.1) describes a genuine Chern-Simons theory with local $ISO(2,1)$ symmetries.)

This latter fact is not in contradiction with the fact that one actually gauges the Poincaré group in gravity. Indeed, the torsion constraint allows one to relate the local translation parameter ϵ^a to the infinitesimal change of coordinates parameter ξ^μ. Indeed, the infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ acts as the Lie derivative

$$\delta_\xi = \mathcal{L}_\xi \equiv i_\xi d + d i_\xi$$

where the inner product i is defined by

$$i_\xi \equiv \xi^\mu \frac{\partial}{\partial(dx^\mu)}\,,$$

where the derivative is understood to act from the left. Any coordinate transformation of the frame field can be written as

$$\delta_\xi e^a = i_\xi(de^a) + d(i_\xi e^a) = i_\xi T^a + \underbrace{\epsilon^a{}_b e^b + D^L \epsilon^a}_{=\delta_\epsilon\, e^a}\,,$$

where the Poincaré gauge parameter is given by $\epsilon = i_\xi \omega$. Therefore, when T^a vanishes any coordinate transformation of the frame field can be interpreted as a local Poincaré transformation of the frame field, and reciprocally.

To summarize, the Einstein-Cartan formulation of gravity is indeed a fibre bundle construction where the Poincaré algebra $iso(d-1,1)$ is the fiber, ω the connection and R the curvature, but, unlike for Yang-Mills theories, the equations of motion imposes some constraints on the curvature ($T^a = 0$), and some fields are auxiliary (ω^{ab}). A fully covariant formulation is achieved in the AdS case with the aid of compensator formalism as explained in Section 2.3.

D Two properties of the inner Klein operator

In this appendix, we shall give a proof of the properties (13.7) and (13.8). One can check the second property with the help of (13.1), which in this case amounts to

$$\mathcal{K} * \mathcal{K} = \frac{1}{\pi^{2(d+1)}} \int dS dT\, e^{-2S_i^A T_A^i} e^{-2(s_i+z_i)(y^i+s^i)} e^{-2(z_i-t_i)(y^i+t^i)}\,,$$

with

$$s_i \equiv V_A S_i^A\,, \qquad t_i \equiv V_A T_i^A. \tag{D.1}$$

Using the fact that $s_i s^i = t_i t^i = 0$ and rearranging the terms yields

$$\begin{aligned}\mathcal{K} * \mathcal{K} &= \frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dS dT\, e^{-2S_i^A T_A^i} e^{-2s^i(z_i - y_i)} e^{-2t^i(z_i+y_i)} \\ &= \frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dS dT\, e^{-2S_A^i [V^A(z_i - y_i) - T_i^A]} e^{-2t^i(z_i+y_i)} \ .\end{aligned}$$

Since

$$\frac{1}{\pi^{2(d+1)}} \int dS e^{-2S_A^i (Z_i^A - Y_i^A)} = \delta(Z_i^A - Y_i^A) \ , \tag{D.2}$$

one gets

$$\mathcal{K} * \mathcal{K} = e^{-4z_i y^i} \int dT \delta(V^A(z_i - y_i) - T_i^A) e^{-2T_B^i V^B (z_i + y_i)}$$

which, using $V_B V^B = 1$, gives $\mathcal{K} * \mathcal{K} = e^{-4z_i y^i} e^{-2(-y^i z_i + z^i y_i)} = 1$. The formula (D.2) might seem unusual because of the absence of an i in the exponent. It is consistent however, since one can assume that all oscillator variables, including integration variables, are genuine real variables times (some fixed) square root of i, *i.e.* that the integration is along appropriate lines in the complex plane. One is allowed to do so without coming into conflict with the definition of the star algebra, because its elements are analytic functions of the oscillators, and can then always be continued to real values of Y.

The proof of (13.7) is quite similar. Let us note that, with the help of (13.8), it amounts to check that $\mathcal{K} * f(Z, Y) * \mathcal{K} = f(\tilde{Z}, \tilde{Y})$. One can prove, by going through almost the same steps shown above, that

$$\mathcal{K} * f(Z, Y) = \mathcal{K}\, f(Z_A^i + V_A(y^i - z^i), Y_A^i - V_A(y^i - z^i)) \ .$$

Explicitly, one has

$$\begin{aligned}\mathcal{K} * f(Z, Y) &= \frac{\mathcal{K}}{\pi^{2(d+1)}} \int dS dT e^{-2S_i^A (T_A^i + V_A(y^i - z^i))} f(Z - T, Y + T) \\ &= \mathcal{K} \int dT\, \delta(T_A^i + V_A(y^i - z^i)) f(Z - T, Y + T) \\ &= \mathcal{K}\, f(Z_A^i + V_A(y^i - z^i), Y_A^i - V_A(y^i - z^i)) \ ,\end{aligned}$$

where we have made use of (D.2). Another star product with $\mathcal{K}$ leads to

$$\mathcal{K} * f(Z, Y) * \mathcal{K} =$$

$$\frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dS dT e^{-2T_A^i [V^A(z_i + y_i) + S_i^A]} e^{-2s_i(y^i - z^i)} f(Z_A^i + V_A(y^i - z^i) + S_A^i, Y_A^i - V_A(y^i - z^i) + S_A^i) \ ,$$

which, performing the integral over T and using (D.2), gives in the end

$$e^{-4z_i y^i} e^{2(z_i + y_i)(y^i - z^i)} f(Z_A^i + V_A(y^i - z^i) - V_A(z^i + y^i), Y_A^i - V_A(y^i - z^i) - V_A(z^i + y^i)) = f(\tilde{Z}, \tilde{Y}) \ .$$

E Regularity

We will prove that the star product (13.1) is well-defined for the regular class of functions (13.9). This extends the analogous result for the spinorial star product in three and four dimension obtained in [11, 63].

Theorem E.1. *Given two regular functions $f_1(Z,Y)$ and $f_2(Z,Y)$, their star product (13.1) $(f_1 * f_2)(Z,Y)$ is a regular function.*

Proof:

$$f_1 * f_2 = P_1(Z,Y)\int_{M_1} d\tau_1\rho_1(\tau_1)\exp[2\phi_1(\tau_1)z_iy^i] \, * \, P_2(Z,Y)\int_{M_2} d\tau_2\rho_2(\tau_2)\exp[2\phi_2(\tau_2)z_iy^i]$$

$$= \int_{M_1} d\tau_1\rho_1(\tau_1)\exp[2\phi_1(\tau_1)z_iy^i]\int_{M_2} d\tau_2\rho_2(\tau_2)\exp[2\phi_2(\tau_2)z_iy^i]\frac{1}{\pi^{2(d+1)}}\int dSdT\times$$

$$\times\exp\{-2S_i^AT_A^i+2\phi_1(\tau_1)[s^i(z-y)_i]+2\phi_2(\tau_2)[t^i(z+y)_i]\}P_1(Z+S,Y+S)P_2(Z-T,Y+T)\ ,$$

with s_i and t_i defined in (D.1). Inserting

$$P(Z+U,Y+U)=\exp\left[U_i^A\left(\frac{\partial}{\partial Z_{1\,i}^A}+\frac{\partial}{\partial Y_{1\,i}^A}\right)\right]P(Z_1,Y_1)\Big|_{\substack{Z_1=Z\\Y_1=Y}}\ , \tag{E.1}$$

one gets

$$f_1 * f_2 = \int_{M_1\times M_2} d\tau_1 d\tau_2\rho_1(\tau_1)\rho_2(\tau_2)\exp\{2[\phi_1(\tau_1)+\phi_2(\tau_2)]z_iy^i\}\times$$

$$\times\int dSdT\exp\left\{-2S_A^i\left[-\phi_1(\tau_1)V^A(z-y)_i+\frac{1}{2}\left(\frac{\partial}{\partial Z_{1\,A}^i}+\frac{\partial}{\partial Y_{1\,A}^i}\right)\right]\right\}P_1(Z_1,Y_1)\Big|_{\substack{Z_1=Z\\Y_1=Y}}\times$$

$$\times\exp\left\{-2T_A^i\left[S_i^A-\phi_2(\tau_2)V^A(z+y)_i+\frac{1}{2}\left(-\frac{\partial}{\partial Z_{2\,A}^i}+\frac{\partial}{\partial Y_{2\,A}^i}\right)\right]\right\}P_2(Z_2,Y_2)\Big|_{\substack{Z_2=Z\\Y_2=Y}}$$

$$= \int_{M_1\times M_2} d\tau_1 d\tau_2\rho_1(\tau_1)\rho_2(\tau_2)\exp\{2[\phi_1(\tau_1)+\phi_2(\tau_2)+2\phi_1(\tau_1)\phi_2(\tau_2)]z_iy^i\}\times$$

$$\times\exp\left\{-\frac{1}{2}\left(\frac{\partial}{\partial Z_{1\,A}^i}+\frac{\partial}{\partial Y_{1\,A}^i}\right)\left(\frac{\partial}{\partial Z_{2\,A}^i}-\frac{\partial}{\partial Y_{2\,A}^i}\right)\right\}P_1[Z_1-\phi_2(\tau_2)\,{}^{\parallel}(Z+Y),Z_2-\phi_2(\tau_2)\,{}^{\parallel}(Z+Y)]\times$$

$$P_2(Z_2+\phi_1(\tau_1)\,{}^{\parallel}(Z-Y),Y_2-\phi_1(\tau_1)\,{}^{\parallel}(Z-Y))\Big|_{\substack{Z_1=Z_2=Z\\Y_1=Y_2=Y}}\ ,$$

as one can check by using (D.2) (or, equivalently, Gaussian integration, the rationale for the equivalence being given in Appendix D). The product of two compact domains $M_1\subset R^n$ and $M_2\subset R^m$ is a compact domain in R^{n+m} and P_1,P_2 are polynomials, so one concludes that the latter expression is a finite sum of regular functions of the form (13.9). □

As explained in Subsection 13.1, it is important for this proof that the exponential in the Ansatz (13.9) never contributes to the quadratic form in the integration variables, which is thus independent of the particular choice of the functions f_1 and f_2. This property makes sure that the class of functions (13.9) is closed under star multiplication, and it is a crucial consequence of the definition (13.1).

F Consistency of the nonlinear equations

We want to show explicitly that the system of equations (13.13)-(13.17) is consistent with respect to both x and Z variables.

We can start by acting on (13.13) with the x-differential d. Imposing $d^2 = 0$, one has

$$dW * W - W * dW = 0 ,$$

which is indeed satisfied by associativity, as can be checked by using (13.13) itself. So (13.13) represents its own consistency condition.

Differentiating (13.14), one gets

$$dW * B - W * dB - dB * \widetilde{W} - B * d\widetilde{W} = 0 .$$

Using (13.14), one can substitute each dB with $-W * B + B * \widetilde{W}$, obtaining

$$dW * B + W * W * B - B * \widetilde{W} * \widetilde{W} - B * d\widetilde{W} = 0 .$$

This is identically satisfied by virtue of (13.13), which is thus the consistency condition of (13.14).

The same procedure works in the case of (13.15), taking into account that, although S is a space-time 0-form, $dx^\mu dZ_i^A = -dZ_i^A dx^\mu$. Again one gets a condition which amounts to an identity because of (13.13).

Hitting (13.16) with d and using (13.14) and (13.15), one obtains

$$-W * S * B - S * B * \widetilde{W} + W * B * \widetilde{S} + B * \widetilde{S} * \widetilde{W} = 0 ,$$

which is identically solved taking into account (13.16).

Finally, (13.15) turns the differentiated l.h.s. of (13.17) into $-W * S * S + S * S * W$, while using (13.14) the differentiated r.h.s. becomes $-2\Lambda^{-1} dz_i dz^i(-W * B * \mathcal{K} + B * \widetilde{W} * \mathcal{K})$. Using (13.7), one is then able to show that the two sides of the equation obtained are indeed equal if (13.17) holds.

S being an exterior derivative in the noncommutative directions, in the Z sector the consistency check is more easily carried on by making sure that a covariant derivative of each equation does not lead to any new condition, *i.e.* leads to identities. This amounts to implementing $d_Z^2 = 0$.

So commuting S with (13.13) gives identically 0 by virtue of (13.13) itself. The same is true for (13.15), (13.14) and (13.16), with the proviso that in these latter two cases one has to take an anticommutator of the equations with S because one is dealing with odd-degree-forms (1-forms). The only nontrivial case then turns out to be (13.17), which is treated in Subsection 13.2.

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