A LA RECHERCHE DE LA m-THÉORIE PERDUE

Z-THEORY: CHASING m/f THEORY

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Z-theory: chasing $\mathfrak{m}/f$ theory

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Abstract

We present the evidence for the existence of the topological string analogue of M-theory, which we call **Z**-theory. The corners of **Z**-theory moduli space correspond to the Donaldson-Thomas theory, Kodaira-Spencer theory, Gromov-Witten theory, and Donaldson-Witten theory. We discuss the relations of **Z**-theory with Hitchin's gravities in six and seven dimensions, and make our own proposal, involving spinor generalization of Chern-Simons theory of three-forms. *To cite this article: N. Nekrasov , C. R. Physique 4 (2004).*

Résumé

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1 Introduction

The past ten years of string theory development have taught us that string theory is a wrong name for the fundamental theory of quantum gravity. We know that the theory has a moduli space of vacua, that this moduli space has some singularities, and we know that the expansion near different singularities look like different string theories, or like eleven dimensional supergravity [32]. In the context of topological strings the situation used to be different, but recent advances in this field suggest the similar picture. In the past few months a few striking conjectures have been put forward concerning the strong-weak dualities, relating topological strings of **A** and **B** types on the same Calabi-Yau three-fold X. The conjectures relate the perturbative type **A** string calculations to the D-brane **B** type calculations, and vice versa. So far most of the known checks of this S-duality conjecture involved only **B**-type branes.

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Strong-weak coupling duality in the physical superstring follows from the existence of some higher-dimensional theory, such that its compactification on tori gives rise to the dual theories. The purpose of this lecture is to draw a similar picture of what we called at various occasions (M)athematical M-theory, $\mathfrak{m}$, or $\mathfrak{f}$-theory. Some people call it topological M-theory [7], [11]. Since for us the main object is a certain partition function, which we denote by Z, in this paper we shall call the missing theory the **Z**-theory.

The simplest idea would be that the physical M-theory, whatever it is, is related to **Z**-theory, just like physical strings are related to the topological strings [4], [1]. This may well be true, but two warning signs are in order: this relation would not explain the relation between the topological gauge theory on $\mathbf{R}^4$ and the topological string on local Calabi-Yau manifold within **Z**-theory; to actually engineer the relation between the $\mathcal{N} = 1$ theories in four dimensions (which is what one gets by compactifying **M**-theory on $\mathcal{Z}_7$) and the topological strings one has to use CY compactifications with fluxes (which could be in principle related to G_2-compactifications, but this makes the whole construction less pretty) [8].

2 Evidence for Z-theory

In this section we describe briefly the topological string theory and the topological gauge theory computations which correspond to various degenerations of **Z**-theory.

2.1 **A** *story*

Gromov-Witten corner. Consider closed **A** type topological string on Calabi-Yau threefold X. Let k denote the (complexified) Kahler form of X, and $\mathfrak{t} = [k] \in H^2(X, \mathbf{C})$. The partition function is defined as a formal series in the string coupling constant $\hbar$:

$$Z_A(X, \mathfrak{t}; \hbar)^{GW} = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(X; \mathfrak{t}) \tag{1}$$

where

$$\mathcal{F}_g(X; \mathfrak{t}) = \sum_{\beta \in H_2(X; \mathbf{Z})} \exp\left(- \int_{\beta} \mathfrak{t} \right) N_g(\beta) \tag{2}$$

and $N_g(\beta)$ is the "number" of genus g stable holomorphic maps to X which land in the homology class β. The word "number" here can be defined more precisely using the virtual fundamental cycles but we shall not do that.

Lagrangian branes. By definition (1) the partition function is perturbative in $\hbar$. The relation to physical superstring [4] suggests that there are non-perturbative corrections to the "correct" definition of Z_A. These corrections, presumably, come from D-branes. There are natural D-branes in the topological string context. Namely, for any Lagrangian submanifold $L \subset X$ (where X is viewed as symplectic manifold), one can define the relative analogue of Gromov-Witten theory, i.e. stable maps of Riemann surfaces with boundaries, which land on L. Moreover, L may have several components, each component may have multiplicities and so on. In the most naive approach one would combine the effects of closed strings and open strings as follows:

$$Z_A^?(X;\mathfrak{t},\mathfrak{s};\hbar|\Lambda) = Z_A(X;\mathfrak{t};\hbar)^{GW} \times \sum_{\ell\in\Lambda\subset H_3(X,\mathbf{Z})} \exp\left(-\frac{1}{\hbar}\int_\ell \mathfrak{s}\right)\mathcal{N}_\ell(\hbar) \quad (3)$$

where $\mathcal{N}_\ell(\hbar) = \sum_{h\in\mathbf{Z}} \mathcal{N}_{\ell,h}\hbar^{2h-2}$ counts stable maps of (possibly disconnected) Riemann surfaces (hence the total genus can be arbitrary), with boundaries, which land on the Lagrangian submanifolds L_i, $i = 1,\ldots,k_\ell$ which represent the homology cycle ℓ. The homology cycles must belong to a Lagrangian (with respect to the intersection pairing) sublattice in $H_3(X,\mathbf{Z})$ [1] If these Lagrangian submanifolds are not simply-connected, then one modifies the definition of the numbers $\mathcal{N}_{\ell,h}$ by considering the moduli spaces of the pairs $(L_i,\mathcal{L}_i)$, where $\mathcal{L}_i$ is the rank $\mathrm{mult}(L_i)$ vector bundle on L_i with flat unitary connection. The stable map $(\Sigma,\partial\Sigma)$ is weighted with the weight $\mathrm{Tr}P\exp\oint_{\partial\Sigma}A$ where A is the pullback of the flat connection. The result it then somehow averaged over the moduli space of the Lagrangian submanifolds with unitary flat bundles over them. The logic of this construction is largely motivated by the corresponding one on the **B** side.

We should learn from this discussion that although perturbative **A** string only knows about the (complexified) symplectic structure of X, via $\mathfrak{t}$-dependence, the non-perturbative corrections bring in extra structure, the 3-form $\mathfrak{s}$, which turns out (upon complexification again) to be related to the $(3,0)$-form of the complex face of the Calabi-Yau manifold X; there is a corresponding term [2]

[1] The reason for taking only "half" of all possible 3-cycles is the electro-magnetic duality of the effective four dimensional sugra

[2] To arrive at the coupling (4) we note that in the presence of the boundary condition corresponding to L the scalar fermions ψ^μ in the worldsheet sigma model have zero modes corresponding to the motion along L. The zero-observable $\varnothing_{m_1\ldots m_p}\psi^{m_1}\ldots\psi^{m_p}$ saturates these zero modes if $p = \mathrm{dim}L$ (so, in particular, in the more general setup for the type **A** topological string one gets a similar coupling

in the target space theory action

$$S_{\mathrm{target}} = \int_L \mathfrak{s} + k\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) \tag{4}$$

2.2 **B** *story*

Mirror symmetry relates type **A** string on X to type **B** topological string on $X^\vee$ - another Calabi-Yau manifold. All the features of the type **A** string described above should be equally present for **B** string as well, order by order in $\hbar$. Indeed, mirror symmetry is the equivalence of sigma models before their coupling to the two dimensional gravity, and also holds in the presence of worldsheet boundaries.

Kodaira-Spencer corner. In particular, there exists a definition of the closed type B partition function, and [4] suggests that it is given by some field theory, i.e. instead of the integrals over the moduli spaces of Riemann surfaces one works with the integrals over the moduli spaces of Riemann graphs. The classical, i.e. genus zero, definition of the type **B** string free energy suggests an intimate connection to the symplectic geometry, via the special geometry. The full partition function involves $\hbar$-corrections, so the story involves some sort of quantization of the symplectic manifold, which gives rise to the special geometry, however no satisfactory proposal about it has been put forward so far. The *calibrated CY manifold* is a pair: $(X^\vee, \Omega)$, where $X^\vee$ is a complex threefold, $K_{X^\vee} \approx \mathcal{O}_{X^\vee}$ and Ω is nowhere vanishing holomorphic $(3,0)$-form. The moduli space $\widetilde{\mathcal{M}}$ of calibrated CY manifolds has complex dimension $1 + h^{2,1}_{X^\vee} = \frac{1}{2}\mathrm{dim}H^3(X^\vee, \mathbf{R})$. Moreover, one can choose local coordinates on $\widetilde{\mathcal{M}}$ to be the periods of Ω. These periods are not independent: choose some basis of A and B cycles in $H_3(X^\vee, \mathbf{Z})$: $A^i \circ A^j = 0$, $A^i \circ B_j = \delta^i_j$, $B_j \circ B_i = 0$, $i = 0, 1, \ldots, r$, $r = h^{2,1}(X^\vee)$, where $\circ$ stands for the intersection index, and define:

$$t_i = \oint_{A^i} \Omega, \qquad t^i_D = \oint_{B_i} \Omega \tag{5}$$

Then t^i are the local coordinates on $\widetilde{\mathcal{M}}$ and locally on $\widetilde{\mathcal{M}}$ there exists a holomorphic function $\mathcal{F}_0 = \mathcal{F}_0(X^\vee; t)$ such that :

$$t^i_D = \frac{\partial \mathcal{F}_0}{\partial t^i} \tag{6}$$

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for the $\frac{1}{2}$dim$X$-forms). The zero-observable inserted at the center of the disk breaks SL_2 down to the compact subgroup $U(1)$. The one-point function is non-vanishing, since the volume of $U(1)$ is finite.

This function, called prepotential, is the genus zero topological **B** string partition function [3]. The topological **B** string couples naively only to the complex structure deformations of $X^{\vee}$. However, it is well-known that the worldsheet theory is anomalous, and the choice of Ω enters the definition of the path integral measure. The moduli space $\widetilde{\mathcal{M}}$ is a cone over the moduli space $\mathcal{M}$ of complex structures of $X^{\vee}$. The rescaling of Ω does not change the complex structure of $X^{\vee}$, so that the quotient by this $\mathbf{C}^*$ action gives $\mathcal{M}$. This $\mathbf{C}^*$-action scales simultaneously t_i and t_D^i, which means that $\mathcal{F}_0$ should be a homogeneous function of degree 2. This 2 is related to the fact that the anomalous dependence on Ω we referred to earlier is Ω^{2-2g} on the genus g [4]

The full topological string partition function includes also the higher genus amplitudes:

$$Z_B(X^{\vee}; t, \hbar) = \exp \sum_{g=0}^{\infty} \hbar^{2-2g} \mathcal{F}_g(X^{\vee}; t) \tag{7}$$

For small $\hbar$ the partition function behaves as: $e^{\mathcal{F}_0/\hbar^2 + \cdots}$ which is a quasiclassical expression for a wave function, since $\mathcal{F}_0$, thanks to (6), is a generating function of a Lagrangian submanifold in $V = H^3(X^{\vee}, \mathbf{R})$. The wave function, then, corresponds to a state in the Hilbert space obtained by quantizing V. However the Planck constant in this "quantization" is $\hbar^2$, not $\hbar$. Moreover, [4] has shown, that Z_B cannot be viewed as a holomorphic function of t. Instead, the naive decoupling of $\bar{t}$ dependence is replaced by a certain linear partial differential equation on Z_B, called the holomorphic anomaly equation [4], which was interpreted in [34] as an equation, expressing the dependence of the wave function, obtained by the quantization of V, on the choice of holomorphic polarization [5].

Note that one has a lot of freedom in parameterizing $\widetilde{\mathcal{M}}$. The A-periods of Ω provide local holomorphic coordinates, but they may be not the most useful ones. The definition of these coordinates required a choice of the basis in $H_3(X^{\vee})$ but, since $H_3(X^{\vee}, \mathbf{R}) = H_3(X^{\vee}, \mathbf{Z}) \otimes \mathbf{R}$ this choice, made for some particular $X^{\vee}$, can be uniquely extended to all nearby CY's, and also globally up to monodromy in $Sp(2r+2, \mathbf{Z})$. In the holomorphic coordinates t the

[3] More precisely, its third derivative (in the special coordinates t^i) is the three-point function on a sphere of the zero-observables $\mu_{\bar{i}}^{j}(x, \bar{x}) \eta^{\bar{i}} \theta_j$ of **B** model, corresponding to the Beltrami differentials

[4] The fields of the sigma model part of the **B** string are: $x^i, \bar{x}^{\bar{i}}, \eta^{\bar{i}}, \theta_i, \psi_{\alpha}^i$, so the unbalanced are $2g$ zero modes of the 1-form ψ^i, one zero mode of x^i and one of θ_i. Since θ and ψ are fermions, their measure transforms as Ω^{1-2g}, while the one of the bosons x^i gives another factor of Ω

[5] There is some confusion, however, as to whether it is Z_B which has such an intepretation, or its square

prepotential $\mathcal{F}_0$ makes its most natural appearence, but it could be that it is not the most natural object to look at, especially in view of the holomorphic anomaly. Take, for example, the real part of Ω, $\Phi = \mathrm{Re}\Omega$, and parameterize $\widetilde{\mathcal{M}}$ by the cohomology class φ of Φ. In other words, let us pass from t_i parameterization to (p_i, q^i) parameterization: $p_i = \oint_{A^i} \Phi, q^i = \oint_{B_i} \Phi$. Clearly, from (5) we can express (p,q) via $t, \bar{t}$:

$$p_i = \tfrac{1}{2}\left(t_i + \bar{t}_i\right), \qquad q^i = \tfrac{1}{2}\left(\frac{\partial \mathcal{F}_0}{\partial t_i} + \frac{\partial \bar{\mathcal{F}}_0}{\partial \bar{t}_i}\right) \tag{8}$$

We claim that the transformation $(t, \bar{t}) \mapsto (p, q)$ is generated by the generating function, which turns out to be quite natural from the point of view of six dimensional topological gravity. In order to see that, introduce one more notation: $t_i = p_i + \sqrt{-1}\xi_i, i = 0, 1, \ldots, r$ and consider the following function on $\widetilde{\mathcal{M}}$:

$$\mathcal{V} = \frac{1}{2\sqrt{-1}} \int_{X^\vee} \Omega \wedge \bar{\Omega} \tag{9}$$

In the effective four dimensional supegravity obtained by compactifying Type **II** string on CY $X^\vee$ $\mathcal{V}$ gives the exponential of the Kahler potential. We note in passing that the (p,q) coordinates on $\widetilde{\mathcal{M}}$ are analogous to the Penner coordinates ℓ_i on the combinatorial moduli space of Riemann surfaces, which are not holomorphic, but are quite useful[18]. Thus their six dimensional analogues are also natural to consider. We can easily relate $\mathcal{V}$ to $\mathcal{F}_0$:

$$2\sqrt{-1}\mathcal{V} = \sum_j \left(\oint_{A^j} \Omega \oint_{B_j} \bar{\Omega} - \oint_{A^j} \bar{\Omega} \oint_{B_j} \Omega \right) = \sum_j \left(t_j \frac{\partial \bar{\mathcal{F}}_0}{\partial \bar{t}^j} - \bar{t}_j \frac{\partial \mathcal{F}_0}{\partial t^j} \right) =$$

$$2\sqrt{-1}(2\sum_j q^j \xi_j - H),$$

$$H = \frac{1}{\sqrt{-1}}\left(\mathcal{F}_0 - \bar{\mathcal{F}}_0\right) \qquad 2q^j = \frac{\partial H}{\partial \xi_j} \tag{10}$$

Thus, $\mathcal{V}$ is the Legendre transform of H with respect to ξ and it is more natural to view $\mathcal{V}$ as a function of p and q, not as a function of p and ξ. Note that the similar Legendre transform (and its quantum analogue) arise in the black hole entropy counting of [30].

Hitchin's approach. There exists an interesting variational problem, which produces $\mathcal{V}$ directly as a function of p and q [13]. We present a slightly reformulated version of Hitchin's construction (it was found independently in

[9]). In what follows we shall call this theory **H**6, since we shall also have to look at the higher dimensional versions of that theory. The fields of **H**6 theory are the closed three-form Φ on $X^\vee$ and a traceless vector-valued one-form J, i.e. a section of $End(TX^\vee, TX^\vee)$, in other words, a Higgs field acting on the (real!) tangent bundle to $X^\vee$. The one-form J is further constrained, which is implemented by yet another field, a six-form ε, which enters the functional linearly. Here is the functional:

$$S_{\mathbf{H}6} = \int_{X^\vee} \Phi \wedge \iota_J \Phi + i\varepsilon \left(\mathrm{Tr} J^2 + 6 \right) = \tag{11}$$

$\int_{X^\vee} \Phi_{[abc} J^m_d \Phi_{ef]m} (\mathrm{d}^6 x)^{abcdef}$, provided that the constraints

$$J^b_a J^a_b = -6, \quad J^a_a = 0 \tag{12}$$

are imposed. Finally, the real variable is not Φ, but a two-form B, which enters as follows: fix $\varphi = [\Phi] \in H^3(X^\vee, \mathbf{R})$. We shall denote by the same letter the de Rham representative of φ, i.e. a closed 3-form. Then we write:

$$\Phi = \varphi + \mathrm{d}B, \qquad B \in \Lambda^2 T^* X^\vee \tag{13}$$

The claim of Hitchin's is that by minimizing S with respect to B one gets $\mathcal{V}$, where $\Omega = \Phi_* + \sqrt{-1}\iota_{J_*}\Phi_*$, $J_*^2 = -1$, and $*$ means that we take the values of Φ and J at the critical point of S. We note in passing that the Lagrangian, analogous to (11), can be written in two dimensions, where one replaces Φ by a closed 1-form. In this case one gets Polyakov's formulation of the sigma model coupled to two dimensional gravity, where, however, the two dimensional metric appears only through the complex structure J: $J^\beta_\alpha = \sqrt{g} g^{\beta\gamma}\epsilon_{\alpha\gamma}$. Minimizing with respect to J is not very sensible unless one considers not one, but at least two closed one-forms Φ^i (target space should be at least two dimensional, for the classical Polyakov string to be equivalent to the Nambu string). Such a generalization is also possible in six dimensional context [9]. Note, however, that unlike two dimensional case, one can only consider "flat tensorial target spaces with constant $\mathfrak{B}$-field": $S_{\mathfrak{G},B} = \int \mathfrak{G}_{\alpha\beta}\Phi^\alpha \wedge \iota_J \Phi^\beta + \mathfrak{B}_{\alpha\beta}\Phi^\alpha \wedge \Phi^\beta$ where the index α runs from 1 to d, the dimension of the tensorial target space, and $\mathfrak{G}$ and $\mathfrak{B}$ are the constant metric and the antisymmetric tensor respectively [6].

In a sense, **H**6-theory exhibits T-duality, just like two dimensional sigma model on a circle [7]. However, it maps quadratic constraint $\mathrm{Tr} J^2 = -6$ into the non-

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[6] For $d = 2$ such theories are closely related to the theories on (p, q) fivebranes in **IIB** string theory

[7] The implementation of the T-duality is quite interesting in view of applications to the black hole entropy counting [30]. One relaxes the constraint that Φ is closed,

linear one: $\mathrm{Tr} J^{-2} = -6$. At any rate, $\mathbf{H}6$ provides an interesting off-shell extension of the prepotential.

Kodaira-Spencer theory. We don't know whether this is a "correct" off-shell extension, i.e. if it reproduces the higher genus amplitudes $\mathcal{F}_g$, $g \geq 1$. It has been argued in [4] that these ampltidues can be calculated using the Feynmann rules of the so-called Kodaira-Spencer (KS) theory of gravity, which is a cubic field theory, whose propagating field is a $(1,1)$ two-form on $X^\vee$. In this sense KS theory is similar to $\mathbf{H}6$ theory, although in the latter the propagating field B is just a two-form. Both theories are background dependent: in KS case one has to fix the reference complex structure and an element in $H^{2,1}(X^\vee)$; in $\mathbf{H}6$ theory one has to fix φ. We refer to [9] for the suggestion for the construction of the map between the KS and $\mathbf{H}6$ theories, although neither [9] nor we suggest that any of these formulations are the suitable nonperturbative formulations of the type $\mathbf{B}$ topological string. We stress that $\mathbf{H}6$ theory, if at all, is related to the "square" of the topological $\mathbf{B}$ string, since it is $\mathcal{F}_0 - \bar{\mathcal{F}}_0$ which is the Legendre transform of H, not the prepotential itself. The same Legendre transform occured in the recent discussion of black hole entropy [30].

Nonperturbative corrections. Neither KS nor Hitchin's functionals know about the non-perturbative corrections coming from D-branes in the type $\mathbf{B}$ string. The latter correspond to the coherent sheaves on $X^\vee$, and the simplest ones are the ideal sheaves of points and holomorphic curves on $X^\vee$. The actual counting problem which their consideration leads to will be described in the Donaldson-Thomas section. Here we shall simply mention that their enumeration brings extra parameters the partition function of the type $\mathbf{B}$ string must depend on: the two-form $s \in H^2(X^\vee, \mathbf{C})$, which couples to the worldvolumes of the D-strings, the Poincare duals of ch_2 of the corresponding sheaves:

$$Z_B^{?}(X^\vee; t, s, \hbar | \Lambda) = Z_B(X^\vee; t) \times \sum_{l \in \Lambda \subset H^{even}(X^\vee, \mathbf{Z})} \exp\left(\frac{1}{\hbar}\int_{X^\vee} l \wedge s\right) \int_{[\mathbf{M}_l]^{vir}} 1 \tag{14}$$

where $\mathbf{M}_l$ is the moduli space of (stable?) coherent sheaves $\mathcal{I}$ on $X^\vee$, with the Chern character $\mathrm{ch}(\mathcal{I}) = l$, and Λ is some Lagrangian sublattice[8] in H^{even}, and $[\mathbf{M}_l]$ is the virtual fundamental cycle.

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and adds a term $\int \Phi \wedge \tilde{\Phi}$ to the action, where $\tilde{\Phi}$ is the closed three-form, with fixed cohomology class. Field theoretically it is more natural (and, as we explain below, necessary after coupling to the gauge fields) to assume that $[\tilde{\Phi}] \in H^3(X^\vee, \mathbf{Z})$

[8] The most studied, so far, example, corresponds to

$$\Lambda = (1 \in H^0(X^\vee, \mathbf{Z})) \oplus 0 \oplus H^4(X^\vee, \mathbf{Z}) \oplus H^6(X^\vee, \mathbf{Z})$$

Donaldson-Thomas corner. The Donaldson-Thomas (DT) theory is the mathematical version of the physical "integrating out the D-branes" procedure. The theory is not yet constructed, but some partial results are already available, especially in the rank one case. We consider Calabi-Yau threefold Y [9]. Just as topological string of type **A** allows generalizations to non-CY spaces, the DT theory also has a non-CY version, however we shall not discuss it here. Type **B** open strings couple to $(0,1)$-connections $\bar{A}$, which correspond to the Q-closed boundary operators iff $F^{0,2}=0$ [33], the naive guess would be that we should look for the solutions of the equations $F^{0,2}=0$ modulo (complexified) gauge transformations. This is the same thing as solving holomorphic Chern-Simons (hCS) equations of motion. However, deformation-theoretically this is not a very well-posed problem. Indeed, $\bar{A}$ has three functional degrees of freedom, $F^{0,2}=0$ imposes three equations, and we have one gauge invariance. Therefore, the virtual dimension of the space of solutions is minus infinity. This problem can be, however, cured, by introducing the adjoint Higgs field ϖ which is a $(0,3)$-form with values in the endomorphisms of the bundle where $\bar{A}$ acts. The equations $F^{0,2}=0$ are replaced by the so-called Donaldson-Uhlenbeck-Yau equations:

$$F^{0,2}_{\bar{A}} = \bar{\partial}^{\dagger}_{\bar{A}}\varpi, \qquad F^{1,1}\wedge k\wedge k = [\varpi,\bar{\varpi}] \tag{15}$$

where we have also partly fixed the gauge, leaving only the unitary gauge transformations. This partial gauge fixing is the physical implementation of the stability condition. It explicitly depends on the Kahler form k. The equations (15) no longer follow from the hCS action. Instead, they describe the localization locus of the (partially) topologically twisted $\mathcal{N}=2$ six dimensional gauge theory, which lives on the Euclidean D5-brane wrapping a six-fold inside a CY fourfold. If sixfold itself is a CY threefold, then ϖ is a scalar, and moreover on the solutions of (15) ϖ vanishes, formally reducing us to the original hCS problem. However, the presence of ϖ is important in evaluating the determinants of the fluctuations around the solutions to (15) and, ultimately, in construction of the virtual fundamental cycle of $\mathbf{M}_l$.

The hCS theory has the Lagrangian, derived in the **B** string context in [33]:

$$\int_{X^\vee} \Omega\wedge \mathrm{Tr}(AdA+\frac{2}{3}A^3) \tag{16}$$

If we follow the previous philosophy and couple the **B** model to $\bar{\mathbf{B}}$ model, then

[9] We choose here another letter for the CY space, since depending on the context Y may stand for X or for $X^\vee$ in the discussions above

we should replace (16) by:

$$\int_{X^\vee} \Phi \wedge \mathrm{Tr}(AdA + \frac{2}{3}A^3) = \int_{X^\vee} \varphi \wedge CS(A) + B \wedge \mathrm{Tr}(F \wedge F) \tag{17}$$

The theory with the action (16) makes almost no sense, since the exponential of the action (16) is not gauge invariant. Indeed, its gauge invariance requires integrality of $[\Omega] \in H^3(X, \mathbf{C})$ but it is impossible to achieve for compact non-singular CY. The action (17), on the other hand, is perfectly sensible, since the requirement of integrality of $\varphi = [\Phi]$ is quite a reasonable one. Moreover, this requirement is *precisely* the condition [20] on the complex moduli of CY to be the solution of *attractor equations*. We see therefore that the topological strings know something about the black holes, constructed by wrapping D-branes on various cycles in the CY. Perhaps this remark would clarify some of the mysteries uncovered in [30].

The last term in (17) couples B to the second Chern class of the bundle. In general we should consider not just bundles and connections, but the complexes of bundles, with connections and maps (bi-fundamental matter) between them. The notion of connection on the object in derived category gets complicated but the Chern character and its component are still simple.

In fact, even if the complex of bundles corresponds to the ideal sheaf of a curve or of a collection of curves and points, which are the objects of study in the DT theory, the second Chern class has a simple meaning. It is the Poincare dual to the cycles, represented by the curves. The coupling (17) describes, then, the coupling of the B-field to these curves. We thus learn that the B-field of **H**6 theory plays the role of the Kahler form for the D-strings of the corresponding type **B** topological string!

DT theory and quantum space-time foam. The discovery of [15] was the realization that the counting of ideal sheaves [19], which is performed by the DT theory can be viewed as the Kahler gravity path integral, where one sums over fluctuating topologies of the six dimensional space-time. In this interpretation, the "curvature" F which is used to represent the Chern classes of the sheaves, is viewed as the (discrete) deformation of the Kahler form: $k = k_0 + \hbar F$. We should stress that the results of [15] do not imply the discreteness of the fundamental description of Kahler gravity. We do not know what is the correct Lagrangian of that theory, nor what are its fundamental degrees of freedom. However, the localization technique can be applied to this theory, and the fixed points of the symmetry group action (the symmetry in question was the torus action, which was the isometry of the background toric CY, and acted on the space of Kahler metrics which asymptote the background one) corresponded to the blowups of the original CY along the ideals of the torus invariant curves. The visible part of the Kahler gravity action is the volume of the space:

$$S_{\text{kahler}} = \frac{1}{6\hbar^2}\int k \wedge k \wedge k = S_0 + \int k_0 \wedge \mathrm{ch}_2 + \hbar \int \mathrm{ch}_3 \tag{18}$$

As explained in [23] the equality of the DT and GW partition functions is the particular case of more general phenomenon – S-duality which relates **A** and **B** topological strings on the same CY manifold, while inverting the string coupling $\hbar$. In the case of toric CY's the KS contribution to the **B**-partition function is trivial, as the D-brane contribution to the **A**-partition function. The S-duality can be derived from the S-duality of physical **IIB** string [23], [17].

2.3 Donaldson-Witten corner

Z-theory also knows about four dimensional gauge theory. Fix a gauge group G. Then one can define a partition function:

$$Z(\mathfrak{a}, \varepsilon_1, \varepsilon_2, q) = Z^{pert}(\mathfrak{a}, \varepsilon_1, \varepsilon_2, q) \times \sum_{n=0}^{\infty} q^{2h^{\vee}_G n}\, \mathrm{Tr}_{\mathbf{C}[\mathbf{M}_n(G)]}\left(\exp(\mathfrak{a}), e^{\varepsilon_1}, e^{\varepsilon_2}\right) \tag{19}$$

where $\mathbf{M}_n(G)$ denotes the moduli space of framed holomorphic $G_{\mathbf{C}}$-bundles $\mathcal{P}$ over $\mathbf{CP}^2 = \mathbf{C}^2 \cup \mathbf{CP}^1_{\infty}$, which are trivial on $\mathbf{CP}^1_{\infty}$, together with the choice of the trivialization:

$$\mathcal{P}_{\mathbf{CP}^1_{\infty}} \approx \mathbf{CP}^1_{\infty} \times G_{\mathbf{C}}$$

The number n is the second Chern class $c_2(\mathcal{P})$ of the bundles, $\exp\mathfrak{a} \in \mathbf{T}$ – the maximal torus of $G_{\mathbf{C}}$, $(e^{\varepsilon_1}, e^{\varepsilon_2}) \in \mathbf{C}^* \times \mathbf{C}^*$ – the maximal torus of $SO_4(\mathbf{C})$ – the complexification of the of group of isometries of $\mathbf{R}^4 \approx \mathbf{C}^2$, and finally $\mathbf{C}[X]$ denotes the space of holomorphic functions on X (for the classical gauge groups from the A,B,C,D seria one can think of the polynomials in the ADHM data). The group $G_{\mathbf{C}} \times SO_4$ acts on $\mathbf{M}_n(G)$ by changing the trivialization at infinity and by automorphisms of $\mathbf{CP}^2$, preserving infinity. Finally, the Z^{pert} is the perturbative (from the gauge theory point of view) piece, which can be found in [22]. The trace in (19) arises in the counting of the BPS states in the five dimensional gauge theory compactified on a circle [25]. This gauge theory can be engineered [16] by compactifying **M**-theory on local Calabi-Yau manifold X_G, which is the A, D, E singularity fibered over $\mathbf{P}^1$ (for non-simply laced G one also twists by the automorphisms of the Dynkin diagrams of the corresponding A, D, E "parent" groups). The correspondence between physical **M**-theory compactification and the low-energy physics of the resulting four dimensional gauge theory implies that the same Z function should be equal to the Gromov-Witten partition function of X_G, where $\mathfrak{a}$ and

$\log(q)$ correspond to the Kahler moduli of X_G [24]. The parameters $\varepsilon_1, \varepsilon_2$ are trickier to identify. In the simplest specialization, when $\varepsilon_1 = -\varepsilon_2 = \hbar$, the latter is identified with the string coupling constant. However, general case is harder to intepret. Presumably it corresponds to the equivariant GW (= equivariant DT) theory of X_G.

3 Towards Z-theory

- Who's Zed?

- Zed's dead, baby, Zed's dead.

In our discussions of GW, KS, DT corners of **Z**-theory we arrived at the picture where the complete(d) topological string partition function depends on both $\mathfrak{s}$ and $\mathfrak{t}$ variables, or t and s variables. It is plain to identify them with the *full moduli* of CY metric on Y. Nonperturbative topological string cares about both the calibrated complex and symplectic aspects of CY geometry. We are not saying that the exact CY metric is what the string couples to. Rather, it is the *homological CY geometry* that the topological string cares about. One way to make the unification between the Kahler and calibrated complex moduli of Y is to consider the manifold $\mathcal{Z} = Y \times \mathbf{S}^1$. Its third cohomology splits as $H^2(Y) \oplus H^3(Y)$. In this way our moduli are nothing but the moduli of three-forms in seven dimensional theory. Moreover, the Lagrangian branes and topological **A** strings are nothing but the associative cycles in the G_2-manifold $\mathcal{Z}$ [10].

3.1 Hitchin theory in seven dimensions: Polyakov formulation

Agent 007 reports to M.

Hitchin has proposed [13] a seven dimensional theory (**H**7), analogous to **H**6, which classifies G_2-holonomy metrics on $\mathcal{Z}$ in terms of the closed three-forms on $\mathcal{Z}$. We present here its Polyakov-like formulation. The fields of the **H**7 theory are: the metric h_{ij} and a closed three-form $C = C_{ijk} \mathrm{d}y^i \mathrm{d}y^j \mathrm{d}y^k$. The Lagrangian is:

$$S_{\mathbf{H}7} = \int_{\mathcal{Z}} \left(h^{ij} C_i \wedge C_j \wedge C + \sqrt{h} \right) \tag{20}$$

[10] In his lecture at Les Houches School in the summer of 2001 S. Shatashvili proposed the unification of the topological **A** and **B** models as the main motivation for the study of the analogue of the topological twist in the Type **II** superstring compactifications on G_2 and $Spin(7)$ holonomy manifolds[31]

where $C_i = C_{ijk}\mathrm{d}y^j \wedge \mathrm{d}y^k$. Again, the dynamical field is not C but the two-form B, such that $C = \sigma + \mathrm{d}B$, and $\sigma = [C] \in H^3(\mathcal{Z}, \mathbf{R})$ is fixed. Minimizing (20) with respect to h one arrives at the Lagrangian, proposed by N. Hitchin in [13]. Note that there exists a T-dual version of (20) which is non-polynomial in the propagating three-form. As in the six dimensional case, one can treat C as an independent field, and add a term $\int C \wedge \tilde{G}$ to the action, where $\tilde{G}$ is a closed 4-form, with fixed cohomology class $[\tilde{G}]$. Then classically one can eliminate C and arrive at the action involving $\tilde{G}$ and h. Further eliminating h, one arrives at the dual formulation of the theory, also present in [13].

What is the meaning of the expression (20)? We found the following simple way of repackaging it. Introduce Dirac matrices γ^i, which obey

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2h^{ij} \cdot \mathbf{1}$$

Then (20) can be rewritten as:

$$S_{\mathbf{H}7} = \int_{\mathcal{Z}} \sqrt{h} \left(1 + \frac{2}{3} \mathrm{Tr} \hat{C}_{(3)} \hat{C}_{(3)} \hat{C}_{(3)} \right) \tag{21}$$

where $\hat{C}_{(3)} = C_{ijk}\gamma^{[i}\gamma^j\gamma^{k]}$ and the trace is taken in the spin representation of $Spin(7)$. We note that on the solutions of (20) the three-form C is harmonic with respect to the metric h (which in turn depends on σ). This condition can be neatly expressed as:

$$\{ \not{\mathcal{D}} \, , \hat{C}_{(3)} \} = 0 \tag{22}$$

where $\not{\mathcal{D}}$ is the Dirac operator.

The theory (20) on $\mathcal{Z} = \mathbf{S}^1 \times Y$ on $\mathbf{S}^1$-invariant fields reduces to the sum of the Kahler gravity and the $\mathbf{H}6$ theory (although the constraint $\mathrm{Tr} J^2 = -6$ is replaced by $\mathrm{Det} J = -1$, and one has to integrate out the dilaton and the KK vector field). This makes the theory (20) a suggestive candidate for the correct theory. However, this on-shell verification is not sufficient for our purposes.

3.2 Speculations on unification

The action (21) involves the metric h_{ij} and integrating it out seems as difficult, as it is in four dimensions. We better use the lesson of [15] and replace the physical metric by the gauge fields. We know in the case $\mathcal{Z} = \mathbf{S}^1 \times Y$ that the quantum foam picture is defined (and completed) quantum mechanically by summing over holomorhic curves, which are the worldsheets of D-strings with D(-1)-instantons bound to them. Lifted to seven dimensions this summation becomes the summation over associative cycles. These are cycles for which the volume form, obtained from the induced metric, coincides with the restriction of the three form C, which should solve the equations of motion of (20).

We also should not forget about the Chern-Simons theory $\int CdC$ of 3-forms in seven dimensions. After all, it is this theory which explains in a most natural way the holomorphic anomaly equation [34], [9]. However this theory does not include the D-brane effects in the topological string. So we must supplement it with the contribution of membranes, which are charged under the C-field:

$$S_{?} = \frac{1}{2}\int_{\mathcal{Z}} C\mathrm{d}C + \sum_{l \in H_3(\mathcal{Z},\mathbf{Z})} \mathcal{N}_l e^{-\int_l C} \tag{23}$$

where $\mathcal{N}_l$ is the number of associative cycles in the homology class l. Indeed, the associative cycles would project to the holomorphic curves upon reduction to six dimensions. However, the notion of associativity requires the G_2 structure, which is only avalable (if at all) for a special gauge representative of C. Also, the expression (23) cannot possibly qualify for the definition of fundamental theory – it is at best some sort of effective action. The analogy we have in mind is the three dimensional compact electrodynamics of Polyakov [29], where one writes the effective $U(1)$ theory contains instanton corrections, which reflect the presence of some more fundamental degrees of freedom (for example, non-abelian gauge fields).

If we invoke the instanton interpretation of the membranes in (23), we could hope to reproduce the sum over l from some sort of gauge theory on $\mathcal{Z}$, with gauge field A interacting with C via the coupling:

$$\mathcal{S}_{??} = \frac{1}{2}\int C\mathrm{d}C + \int C \wedge \mathrm{Tr} F \wedge F + \int CS_7(A) \tag{24}$$

The couplings in (24) can be shown to reproduce the on-shell values of the action of Kahler gravity (18). However, (24) does not contain terms which would ensure localization of the gauge theory path integral. For special gauge groups, like $U(1)$ or $SO(8)$ or E_8 the variations of the gauge field $A \mapsto A + \delta A$ can be compensated by the Green-Schwarz-like variation of the three-form: $C \mapsto C + \mathrm{Tr}(\delta A F)$, so the theory has an enormous gauge invariance as far as A is concerned[11]. The gauge-fixed theory could be defined along the lines of [2],[5] using the three-form C, which does not have to solve (20), thus suggesting an off-shell extension. Perhaps it is worth observing that (24) can also be interpreted as a Chern-Simons action for the superconnection $\mathcal{A} =$

[11] In fact, for these gauge groups by redefining $C \mapsto C + CS_3(A)$ one can eliminate A dependence in the trivial instanton sector

$A + C$:

$$\mathrm{d}^{-1}\mathrm{Tr}e^{\mathcal{F}} \tag{25}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}^2$.

As to the **H**7 theory, we believe that the $\mathrm{Tr}\hat{C}^3_{(3)}$ representation should be taken as hint to what the correct formulation of the theory should be. We suggest to study the spectral action [6], associated with the generalized Dirac operator:

$$\hat{D} = \not{\mathcal{D}} + \hat{C} \tag{26}$$

where $\hat{C} = \sum_{p=0}^{3} \hat{C}_{(2p+1)}, \hat{C}_{(2p+1)} = C_{i_1 i_2 \ldots i_{2p+1}} \gamma^{[i_1} \gamma^{i_2} \ldots \gamma^{i_{2p+1}]}$. Consider the following generalization of the trace of heat kernel

$$I(t) = \mathrm{Tr}_{\mathcal{H}} e^{-t\hat{D}^2} \tag{27}$$

where $\mathcal{H}$ is the space of sections of spin bundle over $\mathcal{Z}$ (we neglect here the subtleties about the existence of spin structure). The small t expansion of $I(t)$ gives the functionals of $C_{(p)}$ and the metric h, which are invariant under the following gauge transformations:

$$\left(\not{\mathcal{D}} + \hat{C}\right) \mapsto e^{-\hat{B}} \left(\not{\mathcal{D}} + \hat{C}\right) e^{\hat{B}} \tag{28}$$

where $\hat{B} = \sum_{p=0}^{3} \hat{B}_{(2p)}, \hat{B}_{(2p)} = B_{i_1 i_2 \ldots i_{2p+1}} \gamma^{[i_1} \gamma^{i_2} \ldots \gamma^{i_{2p}]}$, so that ultimately one should take $\hat{C}$ to contain all forms, while reversing the statistics of even ones, as they would correspond to the ghosts. One can also consider the eight-dimensional index density,

$$\int_0^1 dt \ \mathrm{Lim}_{\beta \to 0} \ \mathrm{Tr} \ \gamma^9 \ e^{-\beta(\gamma^8 \partial_t + \not{\mathcal{D}} + t\hat{C})^2};$$

and extract the corresponding Chern-Simons-like action:

$$\mathcal{S}_{???} = \int_{\mathcal{Z}} \sqrt{h} \left(\mathrm{Tr} \left(\hat{C} \{ \not{\mathcal{D}}, \hat{C} \} + \frac{2}{3} \hat{C}^3 \right) + 1 \right) \tag{29}$$

Varying (29) with respect to $\hat{C}_{(5)}$ we get equations on the gauge field $\hat{C}_{(1)}$, similar to (15). What is left is the sum of the CdC Chern-Simons action and **H**7-action (20).

It would be obviously interesting to derive the actions (24),(29) by:

i) integrating out free fields (like free fermions in 3d), which might be related to the gravitino of physical **M**-theory (see [21] for the recent discussions of the subtleties of the latter, also see [9] for alternative suggestions for the free field formulation);

ii) from some sort of topological open membrane theory [28].

The action (29) might be also related to the recent studies of flux compactifications and generalized CY manifolds [12], [14]. Perhaps the gauge field is nothing but the component $\hat{C}_{(1)}$, and we have to include higher Chern-Simons terms in the action (29). It is also possible that the components $\hat{C}_{(5)}, \hat{C}_{(7)}$ should be viewed as BV anti-fields, and should be gauge-fixed.

We also mention another connection to the noncommutative geometry. Obviously, the expression (20) reminds very much the star product of differential forms, viewed as functions on $\Pi T\mathcal{Z}$, where the non(anti)commutativity is introduced along the odd directions. It is plausible that one can write some sort of matrix model, where γ^i's will be independent variables, so that the Dirac anticommutation relations would correspond to the minima of the action. In this way the metric h^{ij} will be just a parameter of the classical solution, and not a fundamental field. The analogous phenomenon in the contenxt of non-commutativity is well-known [26].

Obviously, all this deserves further investigation [12].

As one of the indications of the naturalness of the seven dimensional theory we give here the formula for the partition function of this theory in the Ω-background on $\mathbf{S}^1 \times \mathbf{R}^6$, which is a generalization of the equivariant MacMahon function [19]:

$$Z_{7\mathrm{d}} = \prod_{a,b=1}^{\infty} \frac{(1 - q_+^a (q_1 q_2 q_3)^{b-1})}{(1 - q_-^a (q_1 q_2 q_3)^{b-1})} \prod_{\alpha=1}^{3} \left[\frac{(1 - q_-^a q_\alpha^b)}{(1 - q_+^a q_\alpha^b)} \right] \tag{30}$$

where q_1, q_2, q_3 are the equivariant parameters, and $q_\pm = q(q_1 q_2 q_3)^{\pm\frac{1}{2}}$, $q = -e^{i\hbar}$. The partition function is the (conjectural) answer for the sum over 3d partitions with the K-theoretic analogue of the equivariant vertex measure. The cohomological version of (30), which corresponds to the six dimensional DT theory, is known to be true [19].

Zed and two notes. The seven dimensional theory is not the final word. The Chern-Simons action (29) clearly suggests eight-dimensional Donaldson-like theory, whose boundary action would be (29). The equations (15), due to

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[12] As we were submitting this paper to the archive, an interesting paper [3] appeared, which might also prove useful in the construction of the **Z**-theory

the extra field ϖ, also are most naturally interpreted in the eight dimensional terms. Moreover, (30) exhibits most symmetries when written in terms of four equivariant parameters, $q_4 = (q_1 q_2 q_3)^{-1}$:

$$Z_{7\to 8} = \prod_{a,b=1}^{\infty} \prod_{\mu=1}^{4} \left[\frac{(1 - q_-^a q_\mu^b)}{(1 - q_+^a q_\mu^b)} \right] \tag{31}$$

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