AN INVARIANT FOR NON SIMPLY CONNECTED MANIFOLDS

Dan BURGHELEA and Stefan HALLER

Institut des Hautes Études Scientifiques

35, route de Chartres

91440 – Bures-sur-Yvette (France)

Février 2005

IHES/M/05/07

AN INVARIANT FOR NON SIMPLY CONNECTED MANIFOLDS

DAN BURGHELEA AND STEFAN HALLER

Abstract. For a closed manifold M we introduce the set of co-Euler structures and we define the modified Ray-Singer torsion, a positive real number associated to M, a co-Euler structure and an acyclic representation ρ of the fundamental group of M with $H^*(M;\rho)=0$. If the co-Euler structure is integral we show that the modified Ray–Singer torsion, regarded as a positive (real valued) function on the variety of some complex representations, is the absolute value of a (complex valued) rational function which carries interesting topological information about the manifold. This rational function is the invariant in the title. If the co-Euler structure is arbitrary one obtains a more general object, a holomorphic 1-cocycle. Interesting rational functions in topology appear in this way. The argument of this rational function when defined, is an interesting and apparently unexplored invariant which reminds the Atiyah–Patodi–Singer eta invariant.

Contents

1. Introduction 1
2. A few characteristic forms 4
3. Euler and co-Euler structures 6
4. Complex representations and Cochain complexes 8
5. Proof of the main results, Examples 11
References 12

1. Introduction

For a finitely presented group Γ denote by $\mathrm{Rep}(\Gamma;V)$ the algebraic set of all complex representations of Γ on the complex vector space V, cf. section 4.1. For a closed base pointed manifold (M,x_0) with $\Gamma=\pi_1(M,x_0)$ denote by $\mathrm{Rep}^M(\Gamma;V)$ the algebraic closure of $\mathrm{Rep}^M_0(\Gamma;V)$, the Zariski open set of representation $\rho\in\mathrm{Rep}(\Gamma;V)$ so that $H^*(M;\rho)=0$. The manifold M is called V-acyclic iff $\mathrm{Rep}^M(\Gamma;V)$, or equivalently $\mathrm{Rep}^M_0(\Gamma;V)$, is nonempty. If M is V-acyclic then the Euler–Poincaré

2000 *Mathematics Subject Classification.* 57R20, 58J52.

Key words and phrases. Euler structure; co-Euler structure; combinatorial torsion; analytic torsion; theorem of Bismut–Zhang; Chern–Simons theory; geometric regularization; mapping torus; rational function.

Part of this work was done while both authors enjoyed the hospitality of the Max Planck Institute for Mathematics in Bonn. A previous version was written while the second author enjoyed the hospitality of the Ohio State University. The second author is supported by the *Fonds zur Förderung der wissenschaftlichen Forschung* (Austrian Science Fund), project number P17108-N04.

characteristic $\chi(M)$ vanishes. There are plenty of $V-$ acyclic manifolds. All have vanishing Euler-Poincaré characteristic.

Recall that every representation $\rho \in \operatorname{Rep}(\Gamma; V)$ induces a canonical vector bundle F_ρ equipped with a canonical flat connection ∇_ρ, cf. section 4.1. The pair (F_ρ, ∇_ρ) will be denoted by $\mathbb{F}_\rho$.

For a closed smooth manifold with vanishing Euler–Poincaré characteristic one defines the set of co-Euler structures $\mathfrak{Eul}^*(M;\mathbb{R})$ as the set of equivalence classes of pairs (g,α), where g is a Riemannian metric on M, and $\alpha \in \Omega^{n-1}(M;\mathcal{O}_M)$ with $d\alpha = E(g)$. Here $E(g) \in \Omega^n(M;\mathcal{O}_M)$ denotes the Euler form of g, cf. section 2.1, and $\mathcal{O}_M$ denotes the orientation bundle. Of course such pairs exist only when $\chi(M)=0$. Two such pairs (g_1,α_1) and (g_2,α_2) are equivalent if $\alpha_2-\alpha_1 = \operatorname{cs}(g_1,g_2)$ where $\operatorname{cs}(g_1,g_2) \in \Omega^{n-1}(M;\mathcal{O}_M)/d\Omega^{n-2}(M;\mathcal{O}_M)$ denotes the Chern–Simons class, cf. section 2.1. The cohomology vector space $H^{n-1}(M;\mathcal{O}_M)$ acts freely and transitively on $\mathfrak{Eul}^*(M;\mathbb{R})$, cf. section 3.2. In particular given two co-Euler structures $\mathfrak{e}_1^*$ and $\mathfrak{e}_2^*$ there exists a unique element $(\mathfrak{e}_2^*-\mathfrak{e}_1^*) \in H^{n-1}(M;\mathcal{O}_M)$ which acting upon $\mathfrak{e}_1^*$ provides $\mathfrak{e}_2^*$. The set of co-Euler structures which is an affine vector space contains inside “the lattice” of **integral co-Euler structures**, cf. section 3.2.

The modified Ray-Singer torsion. Let us recall first the definition of the Ray–Singer torsion. Suppose $\mathbb{F}=(F,\nabla)$ is a real or complex vector bundle F equipped with a flat connection ∇. Let g be a Riemannian metric on M and let μ be a Hermitian (fiber) metric on F. The Ray–Singer torsion $T_{\mathrm{an}}(\nabla,g,\mu)$ is the positive real number given by:

$$\log T_{\mathrm{an}}(\nabla,g,\mu) := \frac{1}{2}\sum_k (-1)^{k+1} k \log \det{}' \Delta^k$$

Here Δ^k denotes the Laplacian in degree k of the elliptic complex $(\Omega^*(M;F), d_\nabla)$ when equipped with the scalar product induced from the Riemannian metric g and the Hermitian metric μ, and $\det{}'\Delta^k$ denotes its zeta regularized determinant, see [RS71].

Let $\mathfrak{e}^*$ be a co-Euler structure. Represent $\mathfrak{e}^*$ by (g,α) with g the Riemannian metric chosen on M. Note that given a Riemannian metric g for any co-Euler structure $\mathfrak{e}^*$ one can find the form $\alpha \in \Omega^{n-1}(M;\mathcal{O}_M)$ so that (g,α) represents $\mathfrak{e}^*$, cf. section 3.2. Denote by $\omega(\nabla,\mu) \in \Omega^1(M;\mathbb{R})$ the Tondeur–Kamber form, cf. section 2.2 for definitions, and consider the quantity

$$T_{\mathrm{an}}(\nabla,g,\mu)e^{-\int_M \omega(\nabla,\mu)\wedge\alpha}.$$

This quantity will be called the *modified Ray–Singer torsion.*

Now given a representation $\rho \in \operatorname{Rep}^M(\Gamma;V)$ and an co-Euler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{R})$, consider the flat bundle $\mathbb{F}_\rho = (F_\rho,\nabla_\rho)$ induced by ρ. Choose a Riemannian metric g on M, a form α with (g,α) representing $\mathfrak{e}^*$ and a Hermitian metric μ on $\mathbb{F}_\rho$. It can be derived from the work of Bismut–Zhang [BZ92], cf. [BH04b] that:

Proposition 1. *If $H^*(M;\rho)=0$, then the quantity $T_{\mathrm{an}}(\nabla,g,\mu)e^{-\int_M \omega(\nabla_\rho,\mu)\wedge\alpha}$ is independent of g and μ i.e. depends only on the co-Euler structure $\mathfrak{e}^*$.*

Consequently it can be denoted by $T_{\mathrm{an}}^{\mathfrak{e}^*}(\rho)$. It defines a real valued function $T_{\mathrm{an}}^{\mathfrak{e}^*} : \operatorname{Rep}_0^M(\Gamma;V) \to \mathbb{R}_+$.

It is natural to ask:

Q1) Is the function $T^{\mathfrak{e}^*}_{\mathrm{an}}$ the absolute value of a holomorphic function?
And if so,
Q2) What is the topological information such holomorphic function carries?
Concerning (Q1) we show that:

Proposition 2. *a) For any $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{R})$ the function $T^{\mathfrak{e}^*}_{\mathrm{an}}$ is locally the absolute value of a holomorphic function.*

b) If $\mathfrak{e}^*$ is integral (cf. section 3.2 Definition 1) then $T^{\mathfrak{e}^*}_{\mathrm{an}}$ is the absolute value of a regular function on $\mathrm{Rep}^M_0(\Gamma;V)$ and therefore the restriction of a rational function on $\mathrm{Rep}^M(\Gamma;V)$.

In this paper both rational and regular functions are concepts in the sense of complex algebraic geometry.

In fact in view of the definitions in section 4.2, Proposition 2 can be strengthened to the following statement.

Proposition 3. *a) For a co-Euler structure $\mathfrak{e}^*$ the function $T^{\mathfrak{e}^*}_{\mathrm{an}}$ is the absolute value of a holomorphic 1-cocycle on $\mathrm{Rep}^M_0(\Gamma;V)$ with values in the multiplicative group of nonzero complex numbers. Moreover, this holomorphic 1-cocycle is the product of a holomorphic 1-cocycle on $\mathrm{Rep}^M(\Gamma;V)$ and of a rational function on $\mathrm{Rep}^M(\Gamma;V)$ which is a regular function on $\mathrm{Rep}^M_0(\Gamma;V)$ with values in $\mathbb{C}\setminus 0$.*

b) If the co-Euler structure $\mathfrak{e}^*$ is integral then the 1-cocycle claimed in a) is cohomologically trivial and is defined by a rational function on $\mathrm{Rep}^M(\Gamma;V)$ which is regular on $\mathrm{Rep}^M_0(\Gamma;V)$.

The invariant $A^{\mathfrak{e}^*}(\rho_1,\rho_2)$. Let us define now a real numbers valued invariant (which resembles the spectral flow) for two representations ρ_1,ρ_2 in the same component of $\mathrm{Rep}^M_0(\Gamma;V)$.We will consider it only mod $2\pi\mathbb{Z}$.

Given two representations $\rho_1,\rho_2 \in \mathrm{Rep}^M_0(\Gamma;V)$ a holomorphic path from ρ_1 to ρ_2 is given by a holomorphic map $\tilde\rho : U \to \mathrm{Rep}^M_0(\Gamma;V)$ where U is an open neighborhood of the segment of real numbers $[1,2]\times\{0\}\subset\mathbb{C}$ in the complex plane $\mathbb{C}$ so that $\tilde\rho(1)=\rho_1$ and $\tilde\rho(2)=\rho_1$. For $\mathfrak{e}^*$ a co-Euler structure $\rho_1,\rho_2\in\mathrm{Rep}^M_0(\Gamma;V)$ and $\tilde\rho(z)$ a holomorphic path in $\mathrm{Rep}^M_0(\Gamma;V)$ denote by

$$\arg^{\mathfrak{e}^*,\tilde\rho}(\rho_1,\rho_2) := \Re(2/i\int_1^2 (\partial T^{\mathfrak{e}^*}_{\mathrm{an}}(\tilde\rho(z)))/T^{\mathfrak{e}^*}_{\mathrm{an}}(\tilde\rho(z))dz).$$

Here for a smooth function φ of complex variable z, $\partial(\varphi)$ denotes the complex valued 1-form $(\partial\varphi/\partial z)dz$.

Proposition 4. *Let $\mathfrak{e}^*$ be an integral co-Euler structure .*

1. If $\tilde\rho'(z_)$ and $\tilde\rho''(z)$ are two holomorphic paths from ρ_1 to ρ_2 then

$$\arg^{\mathfrak{e}^*,\tilde\rho'}(\rho_1,\rho_2) = \arg^{\mathfrak{e}^*,\tilde\rho''}(\rho_1,\rho_2).$$

2. If $\tilde\rho'(z_),\tilde\rho''(z)$ and $\tilde\rho'''(z)$ are three holomorphic paths with $\tilde\rho'$ from ρ_1 to ρ_2, $\tilde\rho''$ from ρ_2 to ρ_3 and $\tilde\rho'''$ from ρ_1 to ρ_3 then

$$\arg^{\mathfrak{e}^*,\tilde\rho'''}(\rho_1,\rho_3) = \arg^{\mathfrak{e}^*,\tilde\rho'}(\rho_1,\rho_2) + \arg^{\mathfrak{e}^*,\tilde\rho''}(\rho_2,\rho_3).$$

Proposition 4 leads to a real valued numerical invariant associated to an integral co-Euler structure $\mathfrak{e}^*$ and two representations ρ_1,ρ_2 in the same connected component of $\mathrm{Rep}^M_0(\Gamma;V)$. The invariant denoted by $A^{\mathfrak{e}^*}(\rho_1,\rho_2)\in\mathbb{R}$ is defined by

$$A^{\mathfrak{e}^*}(\rho_1,\rho_2) := \arg^{\mathfrak{e}^*,\tilde\rho}(\rho_1,\rho_2)$$

with $\tilde{\rho}$ a holomorphic path from ρ_1 to ρ_2.

Let us recall that for a closed connected manifold with $\chi(M) = 0$ Turaev has introduced the set of Euler structures $\mathfrak{Eul}(M;\mathbb{Z})$ on which $H_1(M;\mathbb{Z})$ acts freely and transitively. In section 3.2 we construct a natural map $\mathrm{PD} : \mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}^*(M;\mathbb{R})$ whose image is a lattice in the affine space $\mathfrak{Eul}^*(M;\mathbb{R})$ and is affine over the Poincaré duality homomorphism $D : H_1(M;\mathbb{Z}) \to H^{n-1}(M;\mathcal{O}_M)$. The image of this map is the lattice mentioned above (and the integral co-Euler structure are exactly the elements of this lattice.)

Milnor–Turaev torsion. In section 4.4 we use Milnor's construction of torsion to assign to a V-acyclic closed manifold M with base point x_0, an Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$, and a cohomology orientation $\mathfrak{o}$ (of the finite dimensional vector space $\bigoplus_i H^i(M;\mathbb{R})$) a rational function $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ on $\mathrm{Rep}^M(\Gamma;V)$, $\Gamma = \pi_1(M;x_0)$) we call Milnor–Turaev torsion.

This rational function does, for $\mathfrak{e}^* = \mathrm{PD}(\mathfrak{e})$, the job claimed by Proposition 2 b). This function has the following properties, cf. section 4.4.

Theorem 1.

- (i) *The poles and zeros of $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ are contained in $\Sigma(M)$, the subvariety of representations ρ with $H^*(M;\rho) \neq 0$.*
- (ii) *The absolute value of $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho)$ calculated on $\rho \in \mathrm{Rep}^M_0(\Gamma;V)$ is the modified Ray–Singer torsion $T^{\mathfrak{e}^*}_{\mathrm{an}}(\rho)$, where $\mathfrak{e}^* = \mathrm{PD}(\mathfrak{e})$.*
- (iii) *If $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are two Euler structures then $\mathcal{T}^{\mathfrak{e}_2,\mathfrak{o}}_{\mathrm{comb}} = \mathcal{T}^{\mathfrak{e}_1,\mathfrak{o}}_{\mathrm{comb}} \cdot \det_{(\mathfrak{e}_2-\mathfrak{e}_1)}$ and $\mathcal{T}^{\mathfrak{e},-\mathfrak{o}}_{\mathrm{comb}} = (-1)^{\dim V} \cdot \mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ where $\det_{(\mathfrak{e}_2-\mathfrak{e}_1)}$ is the holomorphic function on $\mathrm{Rep}^M(\Gamma;V)$ defined in section 4.2. This function is actually regular.*
- (iv) *If ρ_1, ρ_2 belong to the same component of $\mathrm{Rep}^M_0(\Gamma;V)$ then*

$$\arg \mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho_2) - \arg \mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho_1) = A^{\mathrm{PD}(\mathfrak{e})}(\rho_1,\rho_2).$$

While the absolute value of Milnor–Turaev torsion is a more or less familiar quantity, its argument is a non-trivial topological invariant which to our knowledge has not yet been explored. In some special cases the Milnor–Turaev torsion agrees with familiar rational functions of one complex variable like the Lefschetz zeta function of a diffeomorphism, or the Alexander polynomial of the complement of a knot (divided by $(z-1)^2$,) cf. section 5.2. In these particular cases the relevant spaces of representations $\mathrm{Rep}^M(\Gamma;V)$ identify to $\mathbb{C} \setminus \{0\}$.

This paper describes the mathematics which makes precise the definitions and sketches the proof of the Propositions 1, 2, 4, and Theorem 1 stated above. Complete details are given in [BH04b]. The last section provides in addition to the sketch of the proof of the results some relevant examples. In Proposition 7 $\Gamma = \mathbb{Z}$, hence $\mathrm{Rep}(\mathbb{Z};V) = \mathrm{GL}(V)$, and in Example 1, $\dim V = 1$ and $\Gamma/[\Gamma,\Gamma]$ has rank one, hence $\mathrm{Rep}(\Gamma;\mathbb{C}) = \mathbb{C} \setminus 0$. Both, Proposition 7 and Example 1, show that the Milnor–Turaev torsion carries significant topological information. Question (Q2) in the case $\Gamma = \mathbb{Z}$ as well as in the case $\dim V = 1$ and for arbitrary V-acyclic manifolds will be address in forthcoming work.

2. A few characteristic forms

2.1. **Euler, Chern–Simons, and the global angular form.** Let M be smooth closed manifold of dimension n. Let $\pi : TM \to M$ denote the tangent bundle, and $\mathcal{O}_M$ the orientation bundle, a flat real line bundle over M. For a Riemannian metric g denote by

$$E(g) \in \Omega^n(M; \mathcal{O}_M)$$

its Euler form, and for two Riemannian metrics g_1 and g_2 by

$$\operatorname{cs}(g_1, g_2) \in \Omega^{n-1}(M; \mathcal{O}_M)/d(\Omega^{n-2}(M; \mathcal{O}_M))$$

their Chern–Simons class. The definition of both quantities is implicit in the formulae (4) and (5) below. They have the following properties which follow immediately from (4) and (5) below.

$$\begin{aligned} d\operatorname{cs}(g_1, g_2) &= E(g_2) - E(g_1) &(1)\\ \operatorname{cs}(g_2, g_1) &= -\operatorname{cs}(g_1, g_2) &(2)\\ \operatorname{cs}(g_1, g_3) &= \operatorname{cs}(g_1, g_2) + \operatorname{cs}(g_2, g_3) &(3) \end{aligned}$$

Let ξ denote the Euler vector field on TM which assigns to a point $x \in TM$ the vertical vector $-x \in T_xTM$.

A Riemannian metric g determines the Levi–Civita connection in the bundle $\pi : TM \to M$. There is a canonic $\operatorname{vol}(g) \in \Omega^n(TM; \pi^*\mathcal{O}_M)$, which vanishes when contracted with horizontal vectors and which assigns to an n-tuple of vertical vectors *their volume times their orientation.* The global angular form, see for instance [BT82], is the differential form

$$\Psi(g) := \frac{\Gamma(n/2)}{(2\pi)^{n/2}|\xi|^n} i_\xi \operatorname{vol}(g) \in \Omega^{n-1}(TM \setminus M; \pi^*\mathcal{O}_M).$$

Note that $\Psi(g)$ is the pull back of a form on $(TM \setminus M)/\mathbb{R}_+$. Moreover, we have the equalities

$$\begin{aligned} d\Psi(g) &= \pi^* E(g). &(4)\\ \Psi(g_2) - \Psi(g_1) &= \pi^* \operatorname{cs}(g_1, g_2) \mod \pi^* d\Omega^{n-2}(M; \mathcal{O}_M) &(5) \end{aligned}$$

For $M = \mathbb{R}^n$ equipped with the standard Riemannian metric g_0 we have

$$\Psi(g_0) = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \sum_{i=1}^{n} (-1)^i \frac{\xi_i}{(\sum \xi_i^2)^{n/2}} d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n.$$

in standard coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on TM.

2.2. **Kamber–Tondeur one form.** Let E be a real or complex vector bundle over M. For a connection ∇ and a Hermitian structure μ on E we define a real valued one form $\omega(\nabla, \mu) := -\frac{1}{2}\operatorname{tr}_\mu(\nabla\mu) \in \Omega^1(M; \mathbb{R})$. More explicitly, for a tangent vector $X \in TM$

$$\omega(\nabla, \mu)(X) := -\frac{1}{2}\operatorname{tr}_\mu(\nabla_X \mu).$$

Note that if ∇ is flat then $\omega(\nabla, \mu)$ will be a closed one form. This can be easily seen by noticing that $\omega(\nabla^{\det E}, \mu^{\det E}) = \omega(\nabla, \mu)$ where $\nabla^{\det E}$ and $\mu^{\det E}$ denote the induced connection and Hermitian structure on the determinant line $\det E := \Lambda^{\operatorname{rank} E} E$.

3. Euler and co-Euler structures

3.1. **Euler structures.** Euler structures have been introduced by Turaev [T90] for manifolds with vanishing Euler–Poincaré characteristic. If one removes the hypothesis $\chi(M) = 0$ the concept of Euler structure can still be considered for any connected based pointed manifold (M, x_0) cf. [B99], [BH04b]. The set of Euler structures denoted here by $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ is equipped with a free and transitive action

$$m : H_1(M;\mathbb{Z}) \times \mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$$

which makes $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ an affine version of $H_1(M;\mathbb{Z})$.

If $\mathfrak{e}_1, \mathfrak{e}_2$ are two Euler structure we write $(\mathfrak{e}_2 - \mathfrak{e}_1)$ for the unique element in $H_1(M;\mathbb{Z})$ with $m((\mathfrak{e}_2 - \mathfrak{e}_1), \mathfrak{e}_1) = \mathfrak{e}_2$.

If M is a compact connected smooth manifold of dimension larger than 2 a convenient description of $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ and of the action m is the following:

$\mathfrak{Eul}_{x_0}(M;\mathbb{Z}) := \pi_0(\mathfrak{X}(M, x_0))$ where $\mathfrak{X}(M, x_0)$ denotes the space of vector fields of class C^r, $r \geq 0$ which vanish at x_0 and are non-zero elsewhere. We equip this space with the C^r–topology and note that the result $\pi_0(\mathfrak{X}(M, x_0))$ is the same for all r. Here $\pi_0(\mathfrak{X}(M, x_0))$ denotes the set of homotopy classes of such vector fields. If $X \in \mathfrak{X}(M, x_0)$ we will write $[X]$ for the corresponding class in $\pi_0(\mathfrak{X}(M, x_0))$.

The action m is described in the following way:

Represent $[\sigma] \in H_1(M;\mathbb{Z})$ by a simple closed curve σ. Choose a tubular neighborhood N of $\sigma(S^1)$ considered as vector bundle $N \to S^1$ and choose a fiber metric and a linear connection on N. Choose a representative of the Euler structure as a vector field X such that $X|_N = \frac{\partial}{\partial\theta}$, the horizontal lift of the canonic vector field on S^1. Choose a function $\lambda : [0, \infty) \to [-1, 1]$, which satisfies $\lambda(r) = -1$ for $r \leq \frac{1}{3}$ and $\lambda(r) = 1$ for $r \geq \frac{2}{3}$ and a function $\mu : [0, \infty) \to \mathbb{R}$ satisfying $\mu(r) = r$ for $r \leq \frac{1}{3}$, $\mu(r) = 0$ for $r \geq \frac{2}{3}$ and $\mu(r) > 0$ for all $r \in (\frac{1}{3}, \frac{2}{3})$. Now construct a new smooth vector field $\tilde{X}$ on M by setting

$$\tilde{X} := \begin{cases} X & \text{on } M \setminus N \\ \lambda(r)\frac{\partial}{\partial\theta} + \mu(r)\frac{\partial}{\partial r} & \text{on } N, \end{cases}$$

where $r : N \to [0, \infty)$ denotes the radius function determined by the fiber metric on N and $-r\frac{\partial}{\partial r}$ is the Euler vector field of N. The vector field $\tilde{X}$ represents the Euler structure $m([\sigma], [X])$. This construction is known as Reeb surgery, see e.g. [N03] or [BH04b] for more details.

Any smooth paths $\alpha : [0, 1] \to M$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ provides an identification of $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ to $\mathfrak{Eul}_{x_1}(M;\mathbb{Z})$. This is done in a obviously manner by choosing an isotopy $h_t, t \in [0, 1]$ of diffeomorphisms of M, with $h_0 = id$, $h_t(x_0) = \alpha(t)$ and assigning to the vector field X the vector field $(h_1)_*(X)$. If $\chi(M) = 0$ the identification is independent of the path making the base point x_0 irrelevant cf. [BH04b] and justifying the notation $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ for any of $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$.

Let τ be a smooth triangulation of M and consider the function $f_\tau : M \to \mathbb{R}$ linear on any simplex of the first baricentric subdivision and taking the value $\dim(\sigma)$ on the baricenter x_σ of σ. A continuous piecewise smooth vector field X on M with the property that at any point x in the open simplex σ is tangent to σ, satisfies $X(f_\tau)(x) < 0$ for any x which is not a baricenter and has the baricenters as non degenerate hyperbolic zeros of Morse index equal to the dimension of σ is called Euler vector field. Clearly any two Euler vector fields are homotopic by a homotopy of Euler vector fields.

Let $\pi := \{\pi_\sigma\}$ be a collection of simple paths π_σ with disjoint interior joining x_0 to the baricenter x_σ. As the set $\cup\pi_\sigma$ is contractible, a smooth regular neighborhood of it is the image by a smooth embedding $\varphi : (D^n, 0) \to (M, x_0)$. If X is an Euler vector field for τ then it is nonzero on $M \setminus \mathrm{Int}(D^n)$ and can be extended radially to a vector field $\tilde{X}$ on M with only one zero the point x_0, hence defines an Euler structure. The Euler structure is independent of the choice of the regular neighborhood and is the same for different π and π' provided that $\cup_\sigma \pi_\sigma * (\pi'_\sigma)^-1$ represents 0 in $H_1(M;\mathbb{Z})$. Here $*$ denote the juxtaposition of paths.

Observation 1. *Given a smooth triangulation τ any Euler structure $\mathfrak{e}$ can be obtained from a collection of paths $\{\pi_\sigma\}$.*

A proof is implicit in [BH04b].

3.2. **Co-Euler structures.** Consider pairs (g, α) where g is a Riemannian metric on M and $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ with $d\alpha = E(g)$ where $E(g) \in \Omega^n(M; \mathcal{O}_M)$ denotes the Euler class of g, see section 2.1. We call two pairs (g_1, α_1) and (g_2, α_2) equivalent if

$$\mathrm{cs}(g_1, g_2) = \alpha_2 - \alpha_1 \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)/d\Omega^{n-2}(M \setminus x_0; \mathcal{O}_M).$$

We will write $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ for the set of equivalence classes and $[g, \alpha]$ for the equivalence class represented by the pair (g, α). Elements of $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ are called *co-Euler structures based at x_0.*

Observation 2. *Given a Riemannian metric g on M any co-Euler structure can be represented as a pair (g, α) for some α. If moreover $\chi(M) = 0$ one can choose $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$.*

There is a natural action

$$m^* : H^{n-1}(M; \mathcal{O}_M) \times \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$$

given by

$$m^*([\beta], [g, \alpha]) := [g, \alpha - \beta]$$

for $[\beta] \in H^{n-1}(M; \mathcal{O}_M)$. Since $H^{n-1}(M; \mathcal{O}_M) = H^{n-1}(M \setminus x_0; \mathcal{O}_M)$ this action is obviously free and transitive. In this sense $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ is an affine space over $H^{n-1}(M; \mathcal{O}_M)$.

If $\mathfrak{e}^*_1$, $\mathfrak{e}^*_2$ are two co-Euler structure we write $(\mathfrak{e}^*_2 - \mathfrak{e}^*_1)$ for the unique element in $H^{n-1}(M; \mathcal{O}_M)$ with $m^*((\mathfrak{e}^*_2 - \mathfrak{e}^*_1), \mathfrak{e}^*_1) = \mathfrak{e}^*_2$.

There is a map $PD : \mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \to \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ which combined with the Poincaré duality map $D : H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{R}) \to H^{n-1}(M; \mathcal{O}_M)$, the composition of the coefficient homomorphism for $\mathbb{Z} \to \mathbb{R}$ with the Poincaré duality isomorphism, makes the diagram below commutative.

$$\begin{tikzcd}
H_1(M;\mathbb{Z}) \times \mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \arrow[r, "m"] \arrow[d, "D\times\mathrm{PD}"'] & \mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \arrow[d, "\mathrm{PD}"] \\
H^{n-1}(M;\mathcal{O}_M) \times \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \arrow[r, "m^*"] & \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})
\end{tikzcd}$$

The map PD is defined in the following way. Suppose a vector field X is representing the Euler structure $\mathfrak{e}$. Choose a Riemannian metric g, regard X as mapping $X : M \setminus \{x_0\} \to TM \setminus M$, set $\alpha := X^*\Psi(g)$, and define $\mathrm{PD}(\mathfrak{e}) := [g, \alpha]$.

Observation 3. *Observe that if* $\chi(M) = 0$ *the obvious map* $\mathfrak{Eul}^*(M;\mathbb{R}) \to \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ *induced by inclusion of pairs* (g, α) *is bijective.*

Definition 1. A co-Euler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ is called *integral* if it belongs to the image of PD.

Clearly the subset of integral co-Euler structures define a lattice in the affine space $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$.

4. Complex representations and Cochain complexes

4.1. **Complex representations.** Let Γ be a finitely presented group with generators $g_1, \dots, g_r$ and relations $R_i(g_1, g_2, \dots, g_r) = e$, $i = 1, \dots, p$, and V be a complex vector space of dimension N. Let $\mathrm{Rep}(\Gamma; V)$ be the set of linear representations of Γ on V, i.e. group homomorphisms $\rho : \Gamma \to \mathrm{GL}_{\mathbb{C}}(V)$. By identifying V to $\mathbb{C}^N$ this set is in a natural way an algebraic set inside the space $\mathbb{C}^{rN^2+1}$ given by $pN^2 + 1$ equations. Precisely if $A_1, \dots, A_r, z$ represent the coordinates in $\mathbb{C}^{rN^2+1}$ with $A := (a^{ij})$, $a^{ij} \in \mathbb{C}$, so $A \in \mathbb{C}^{N^2}$ and $z \in \mathbb{C}$, then the equations defining $\mathrm{Rep}(\Gamma; V)$ are

$$\begin{aligned} z \cdot \det(A_1) \cdot \det(A_2) \cdots \det(A_r) &= 1 \\ R_i(A_1, \dots, A_r) &= \mathrm{id}, \qquad i = 1, \dots, p \end{aligned}$$

with each of the equalities R_i representing N^2 polynomial equations.

Suppose $\Gamma = \pi_1(M, x_0)$, M a closed manifold. Denote by $\mathrm{Rep}^M_0(\Gamma; V)$ the set of representations ρ with $H^*(M; \rho) = 0$ and notice that they form a Zariski open set in $\mathrm{Rep}(\Gamma; V)$ and denote the closure of this set by $\mathrm{Rep}^M(\Gamma; V)$. This is again an algebraic set which depends only on the homotopy type of M, can be empty (for example if $\chi(M) \neq 0$) but in some interesting cases can be the full space $\mathrm{Rep}(\Gamma; V)$, cf. section 5.2.

Recall that every representation $\rho \in \mathrm{Rep}(\Gamma; V)$ induces a canonical vector bundle F_ρ equipped with a canonical flat connection ∇_ρ. They are obtained from the trivial bundle $\tilde{M} \times V \to \tilde{M}$ and the trivial connection by passing to the Γ quotient spaces. Here $\tilde{M}$ is the canonical universal covering provided by the base point x_0. The Γ–action is the diagonal action of deck transformations on $\tilde{M}$ and the action ρ on V. The fiber of F_ρ over x_0 identifies canonically with V. The holonomy representation determines a right Γ–action on the fiber of F_ρ over x_0, i.e. an anti homomorphism $\Gamma \to \mathrm{GL}(V)$. When composed with the inversion in $\mathrm{GL}(V)$ we get back the representation ρ. The pair (F_ρ, ∇_ρ) will be denoted by $\mathbb{F}_\rho$.

Note that If ρ_0 is a representation in the connected component $\mathrm{Rep}_\alpha(\Gamma; V)$ one can identify $\mathrm{Rep}_\alpha(\Gamma; V)$ to the connected component of ∇_{ρ_0} in the complex analytic space of flat connections of the bundle F_{ρ_0} modulo the group of bundle isomorphisms of F_{ρ_0} which restrict to the identity on the fiber above x_0.

4.2. **1-Cocycle.** Let $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ denote the multiplicative group of nonzero complex numbers. Let X be a complex analytic space. A holomorphic 1-cocycle on X is an equivalence class of systems $(U_\alpha, h_\alpha : U_\alpha \to \mathbb{C}_*)$, α in a set of indices with $\{U_\alpha\}$ an open cover of X and h_α a holomorphic map so that $h^{\alpha_2}_{\alpha_1}/h^{\alpha_1}_{\alpha_2}$ is constant where $h^\beta_\alpha := h_\alpha|_{U_\alpha \cap U_\beta}$. Two such systems (U_α, h_α) and (U'_β, h'_β) are equivalent if together, $(U_\alpha, U'_\beta, h_\alpha, h'_\beta)$ form a system as above. A 1-cocycle is cohomologically trivial iff is equivalent to a system whose covering consists of one open set, the space X, only. In particular, it can be represented by holomorphic functions which on each connected component of X is unique up to multiplication by a constant.

i. An element $a \in H_1(M;\mathbb{Z})$ defines a holomorphic function $\det_a : \mathrm{Rep}^M(\Gamma; V) \to \mathbb{C}_*$ and in particular a cohomologically trivial holomorphic 1-cocycle. The complex number $\det_a(\rho)$ is the evaluation on $a \in H_1(M;\mathbb{Z})$ of $\det(\rho) : \Gamma \to \mathbb{C}_*$ which factors through $H_1(M;\mathbb{Z})$. Note that for $a, b \in H_1(M;\mathbb{Z})$ we have $\det_{a+b} = \det_a \det_b$.

If a is a torsion element then $\det_a$ is constant and equal to a root of unity of order equal to the order of a on that component. Therefore any $\hat{a} \in H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))$ defines a cohomologically trivial holomorphic 1-cocycle.

ii. An element $\hat{b} \in H_1(M;\mathbb{R})$ defines a holomorphic 1-cocycle. When $\hat{b}$ is in $H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z})) \subseteq H_1(M;\mathbb{R})$ this cocycle is cohomologically trivial and representable by the holomorphic function described above. Here is how this 1-cocycle is constructed:

1: Choose a representation of $\hat{b} = \mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_r a_r$ where μ_k are real numbers and $a_k \in H_1(M;\mathbb{Z})$, with $k = 1, \ldots, r$.

2: For each $\rho \in \mathrm{Rep}^M(\Gamma; V)$ construct the open set $U_\rho^{a_1,\ldots,a_r}$ and the holomorphic map $h_\rho^{a_1,a_2,\ldots,a_r} : U_\rho^{a_1,\ldots,a_r} \to \mathbb{C}_*$ as follows:

- represent each of the complex numbers $\det_{a_k}(\rho)$ as $\det_{a_k}(\rho) = \lambda_k e^{i\theta_k}$ with $\lambda_k \in \mathbb{R}_+$ and $\theta_k \in [0, 2\pi)$.

- define the open set $U_\rho^{a_1,\ldots,a_r} \subseteq \mathrm{Rep}^M(\Gamma; V)$ by

$$U_\rho^{a_1,\ldots,a_r} := \left\{\rho' \in \mathrm{Rep}^M(\Gamma; V) \;\middle|\; \theta_k - \pi < \arg(\det{}_{a_k}(\rho')) < \theta_k + \pi, \text{for all } 1 \le k \le r\right\}.$$

Note that for ρ' in $U_\rho^{a_1,\ldots,a_r}$ there is a unique writing $\det_{a_i}(\rho') = \lambda(\rho') e^{i\theta(\rho')}$ with $\lambda(\rho') \in \mathbb{R}_+$ and $\theta_k - \pi < \theta(\rho') < \theta_k + \pi$, and therefore for any real number μ the complex number $(\det_{a_k}(\rho'))^\mu$ is well defined. Therefore we get a well defined complex number

$$h_\rho^{a_1,\ldots,a_r}(\rho') := \prod_{k=1,\ldots,r} (\det{}_{a_k}(\rho'))^{\mu_k}.$$

It is immediate to check that $h_\rho^{a_1,\ldots,a_r}$ is holomorphic and that $(U_\rho^{a_1,\ldots,a_r}, h_\rho^{a_1,\ldots,a_r})$ provides a holomorphic 1-cocycle. It is also easy to check that the 1-cocycle so defined is independent on the representation of $\hat{b}$.

4.3. **The space of Cochain complexes.** Let $(k_0, k_1, \ldots, k_n)$ be a string of non-negative integers which satisfy the following requirements:

$$k_0 - k_1 + k_2 \mp \cdots + (-1)^n k_n \quad = \quad 0 \tag{6}$$

$$k_i - k_{i-1} + k_{i-2} \mp \cdots + (-1)^i k_0 \quad \geq \quad 0 \quad \text{for any } i \le n-1. \tag{7}$$

Denote by $\mathbb{D}(k_0, \ldots, k_n)$ the collection of Cochain complexes of the form

$$C = (C^*, d^*) : 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \to 0$$

with $C^i := \mathbb{C}^{k_i}$, and by $\mathbb{D}_{\mathrm{ac}}(k_0, \ldots, k_n) \subseteq \mathbb{D}(k_0, \ldots, k_n)$ the subset of acyclic complexes. The Cochain complex C is determined by the collection $\{d^i\}$ of linear maps $d^i : \mathbb{C}^{k_i} \to \mathbb{C}^{k_{i+1}}$. If regarded as the subset of $\{d^i\} \in \mathbb{L}(k_0, \ldots, k_n) = \bigoplus_{i=0}^{n-1} L(\mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}})$[1] which satisfy the quadratic equations

$$d^{i+1} \cdot d^i = 0,$$

[1]We denote by $L(V, W)$ the space of linear maps from V to W.

the set $\mathbb{D}(k_0, \dots, k_n)$ is an affine algebraic set given by degree two homogeneous polynomials and $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ is a Zariski open set.

The map

$$\pi_0 : \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n) \to \mathrm{Emb}(C^0, C^1)$$

which associates to $C \in \mathbb{D}(k_0, \dots, k_n)$ the linear map d^0, is a bundle whose fiber is isomorphic to $\mathbb{D}_{\mathrm{ac}}(k_1 - k_0, k_2, \dots, k_n)$.

As a consequence we have

Proposition 5.

(i) $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ *is a connected smooth quasi algebraic set of dimension*

$$k_0 \cdot k_1 + (k_1 - k_0) \cdot k_2 + \cdots + (k_{n-1} - k_{n-2} + \cdots \pm k_0) \cdot k_n.$$

(ii) *The closure* $\hat{\mathbb{D}}_{\mathrm{ac}}(k_0, \dots, k_n)$ *is an irreducible algebraic set, hence an affine algebraic variety.*

For any cochain complex in $C \in \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ denote by

$$B^i := \mathrm{img}(d^{i-1}) \subseteq C^i = \mathbb{C}^{k_i}$$

and consider the short exact sequence

$$0 \to B^i \xrightarrow{\mathrm{inc}} C^i \xrightarrow{d^i} B^{i+1} \to 0.$$

Choose a bases b_i for each B_i, and choose lifts $\bar{b}_{i+1}$ of b_{i+1} in C^i using d^i, i.e. $d^i(\bar{b}_{i+1}) = b_{i+1}$. Clearly $\{b_i, \bar{b}_{i+1}\}$ is a base of C^i.

Consider the base $\{b_i, \bar{b}_{i+1}\}$ as a collection of vectors in $C^i = \mathbb{C}^{k_i}$ and write them as a columns of a matrix $[b_i, \bar{b}_{i+1}]$. Define the torsion of the acyclic complex C, by

$$\tau(C) := (-1)^N \prod_{i=0}^{n} \det[b_i, \bar{b}_{i+1}]^{(-1)^i}$$

where $(-1)^N$ is Turaev's sign, see [FT00]. The result is independent of the choice of the bases b_i and of the lifts $\bar{b}_i$ cf. [M66] [FT00], and leads to the function

$$\tau : \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n) \to \mathbb{C} \setminus \{0\}.$$

Turaev provided a simple formula for this function, cf. [T01], which permits to recognize τ as the restriction of a rational function on $\hat{\mathbb{D}}_{\mathrm{ac}}(k_0, k_1, \dots, k_n)$, although this is rather obvious from general reasons.

4.4. **Milnor–Turaev torsion.** Consider a smooth triangulation τ of M whose set of simplexes of dimension q is denoted by $\mathcal{X}_q$ and the collection of integers $k_i = \sharp(\mathcal{X}_i) \cdot \dim V$ where V is a fixed complex vector space. Let $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ be an Euler structure, $\mathfrak{o}$ a cohomology orientation, and suppose that M is V-acyclic. Then the integers $(k_0, \dots, k_n)$ satisfy (6) and (7). Consider a collection of simple paths $\pi_{\mathfrak{e}} \equiv \{\pi_\sigma\}$ from x_0 to the barycenters x_σ as in cf. section 3.2 an ordering o of the barycenters (zeros of X_τ) which induces the cohomology orientation $\mathfrak{o}$, and a framing ϵ of V.

Consider the chain complex $(C^*_\tau(M; \rho), d_\tau(\rho))$ associated with the triangulation τ which computes the cohomology $H^*(M; \rho)$. Using $\pi_{\mathfrak{e}}$, o and ϵ one can identify $C^q_\tau(M; \rho)$ with $\mathbb{C}^{k_q}$. We obtain in this way a map

$$t_{\pi_{\mathfrak{e}}, o, \epsilon} : \mathrm{Rep}(\Gamma; V) \to \mathbb{D}(k_0, \dots, k_n)$$

which sends $\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)$ to $\mathbb{D}_{\mathrm{ac}}(k_0,\dots,k_n)$. A look at the explicit definition of $d_\tau(\rho)$ implies that $t_{\pi_{\mathfrak{e}},o,\epsilon}$ is actually a regular map between two algebraic sets. Change of ϵ, $\pi_{\mathfrak{e}}$ and o changes the map $t_{\pi_{\mathfrak{e}},o,\epsilon}$. However, in view of the arguments of [M66] or [T86] one can derive (cf. [BH04b])

Proposition 6. *If $\rho\in\mathrm{Rep}_0^M(\Gamma;V)$ different choices τ,π_e,o,ϵ provide the same composition $\tau\cdot t_{\pi_e,o,\epsilon}(\rho)$ provided $\tau and \pi_e$'s define the same Euler structure $\mathfrak{e}$ add o's define the same cohomology orientation $\mathfrak{o}$.*

We therefore obtain a well defined complex valued rational function on $\mathrm{Rep}^M(\Gamma;V)$ named Milnor–Turaev torsion and denoted from now on by $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$.

5. Proof of the main results, Examples

5.1. **Proof of the results.** The proof of Proposition 1 uses the formulae which describe the change of Ray–Singer torsion when one changes the Riemannian metric g and of the fiber metric (Hermitian) metric μ and are given in [BZ92]. The invariance of the modified Ray–Singer torsion follows then from the formulae (4) and (5) and simple manipulations with Stokes formula. For details see [BFK01] and [BH04b].

The proof of Proposition 2(b) follows from the equality of the modified Ray–Singer torsion associated with $\mathrm{PD}(\mathfrak{e})$ and the Milnor–Turaev torsion $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ for $\rho\in\mathrm{Rep}_0^M(\Gamma;V)$. This is implicit in [BZ92] and explicit in [BH04b].

Theorem 1(i) follows from the definition of the Milnor–Turaev torsion while (ii) from Proposition 2(b) the discussion about Euler and co-Euler structures and section 4.2 cf. [BH04b].

Theorem 1(iii) follows from Milnor's description of the change of torsion when one changes the ordered base [BH04b]. The reader should be aware of the discussion in section 4.3. Part (iv) is the formula for the difference between the arguments of the value of a holomorphic function in terms of its absolute value.

Proposition 2(a) follows from Theorem 1(iii). Proposition 4 is then a consequence of Theorem 1 and the discussion in section 4.3, by comparing an arbitrary co-Euler structure with an integral one.

Details of these derivation are contained in [BH04b].

5.2. **Milnor–Turaev torsion for mapping tori.** Let N be a closed connected manifold and $\varphi: N\to N$ a diffeomorphism. Define the mapping torus $M=N_\varphi$ by gluing the boundaries of $N\times I$ with the help of φ, more precisely identifying $(x,1)$ with $(\varphi(x),0)$. The manifold M comes with a projection $p:M\to S^1$, and an inclusion $N\to M$, $x\mapsto(x,0)$. Set $\Gamma:=\pi_1(S^1,0)$. Let $\mathbb{K}$ be a field of characteristics zero, and suppose V is a finite dimensional vector space over $\mathbb{K}$. For $\rho\in\mathrm{Rep}(\Gamma;V)=\mathrm{GL}(V)$ define

$$P^k_\varphi(\rho):=\det\Big(\varphi^*\circ\rho_*-\mathrm{id}:H^k(N;V)\to H^k(N;V)\Big)$$

and the Lefschetz zeta function:

$$\zeta_\varphi(\rho):=\frac{\prod_{k\text{ even}}P^k_\varphi(\rho)}{\prod_{k\text{ odd}}P^k_\varphi(\rho)}$$

This is a rational function $\zeta_\varphi:\mathrm{GL}(V)\to\mathbb{K}$.

The mapping torus structure on M equips M with a canonical Euler structure $\mathfrak{e}$ and canonical cohomology orientation $\mathfrak{o}$.

Choose a base point $x_0 \in M$ with $p(x_0) = 0 \in S^1$. Every representation $\rho \in \operatorname{Rep}(\Gamma; V)$ gives rise to a representation $p^*\rho$ of $\pi_1(M, x_0)$ on V. It is not hard to see that $H^*(M; p^*\rho) = 0$ iff $P^k_\varphi(\rho) \neq 0$ for all k. Indeed, we have a long exact sequence, the Wang sequence:

$$\cdots \to H^*(M; p^*\rho) \to H^*(N; V) \xrightarrow{\varphi^* \circ \rho_* - \mathrm{id}} H^*(N; V) \to H^{*+1}(M; p^*\rho) \to \cdots \quad (8)$$

This shows that $\operatorname{Rep}^M(\Gamma; V) = \mathrm{GL}(V)$.

The Euler structure $\mathfrak{e}$ is defined by a vector field X with $\omega(X) < 0$ where $\omega := p^*dt \in \Omega^1(M)$. All are homotopic. Strictly speaking one needs vector fields with only one zero at x_0. Clearly from any nowhere zero vector field one can obtain one with such property by a standard modification. The cohomology orientation is derived from the Wang long exact sequence applied to the trivial one dimensional real representation. For details see [BH04b].

We have

Proposition 7. *With these notations* $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(p^*\rho) = (-1)^{\dim(V) \cdot z_\varphi} \zeta_\varphi(\rho)$ *where*

$$z_\varphi = \sum_q \dim H^q(N; \mathbb{R}) \cdot \dim \ker\big(\varphi^* - \mathrm{id} : H^q(N; \mathbb{R}) \to H^q(N; \mathbb{R})\big)$$

A proof of this proposition is contained in [BH04b]. We close this section with the following additional example.

Example 1. If M is obtained by surgery on a framed knot, and $\dim V = 1$, the function $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ coincides with the Alexander polynomial of the knot, see [T02]. We hope that passing to higher dimensional representations $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ captures more subtle knot invariants.

References

[APS75] M.F. Atiyah, V.K. Patodi and I.M. Singer, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78**(1975), 405–432.

[BZ92] J.–M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque. **205**, Société Mathématique de France, 1992.

[BT82] R. Bott and L.W. Tu, *Differential forms in algebraic topology.* Graduate texts in Mathematics **82**. Springer Verlag, New York–Berlin, 1982.

[B99] D. Burghelea, *Removing Metric Anomalies from Ray–Singer Torsion*, Lett. Math. Phys. **47**(1999), 149–158.

[BFK01] D. Burghelea, L. Friedlander and T. Kappeler, *Relative Torsion*, Commun. Contemp. Math. **3**(2001), 15–85.

[BH04a] D. Burghelea and S. Haller, *Laplace transform, dynamics and spectral geometry*, preprint `math.DG/0405037`

[BH04b] D. Burghelea and S. Haller, *Euler structures, the variety of representations and the Milnor–Turaev torsion*, preprint Max-Planck-Institut für Mathematik, Bonn. **121**(2004).

[F02] M. Farber, *Singularities of the analytic torsion*, J. Differential Geom. **41**(1995), 528–572.

[FT00] M. Farber and V. Turaev, *Poincaré–Reidemeister metric, Euler structures and torsion*, J. Reine Angew. Math. **520**(2000), 195–225.

[H77] R. Hartshorne, *Algebraic Geometry.* Graduate Texts in Mathematics **52**. Springer Verlag, New York–Heidelberg, 1977.

[M98] J. Marcsik, ..., Thesis OSU **8**(1998).

[MQ86] V. Mathai and D. Quillen, *Superconnections, Thom Classes, and Equivariant differential forms*, Topology **25**(1986), 85–110.

[M66] J. Milnor, *Whitehead trosion*, Bull. Amer. Math. Soc. **72**(1966), 358–426.

[N03] L. Nicolaescu, *The Reidemeister Torsion of* 3*–manifolds*, de Gruyter Studies in Mathematics **30**, Walter de Gruyter & Co., Berlin, 2003.

[Q85] D. Quillen, *Determinants of Cauchy–Riemann operators on Riemann surfaces*, Functional Anal. Appl. **19**(1985), 31–34. Translation of Funktsional. Anal. i Prilozhen. **19**(1985), 37–41.

[RS71] D.B. Ray and I.M. Singer, *R–torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. **7**(1971), 145–210.

[S99] N. Steenrod, *The topology of fiber bundles*. Reprint of the 1957 edition. Princeton University Press, Princeton, NJ, 1999.

[T86] V. Turaev, *Reidemeister torsion in Knot theory*, Uspekhi Mat. Nauk **41**(1986), 119–182.

[T90] V. Turaev, *Euler structures, nonsingular vector fields, and Reidemeister-type torsions*, Math. USSR–Izv. **34**(1990), 627–662.

[T01] V. Turaev, *Introduction to combinatorial torsions*. Notes taken by Felix Schlenk. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001.

[T02] V. Turaev, *Torsion of 3-dimensional manifolds*. Progress in Mathematics **208**. Birkhäuser Verlag, Bassel–Boston–Berlin, 2002.

[W78] G.W. Whitehead, *Elements of homotopy theory*. Graduate Texts in Mathematics **61**. Springer Verlag, New York–Berlin, 1978.

Dan Burghelea, Dept. of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, USA.

E-mail address: `burghele@mps.ohio-state.edu`

Stefan Haller, Department of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090, Vienna, Austria.

E-mail address: `stefan.haller@univie.ac.at`