On the Hochschild-Kostant-Rosenberg map for graded manifolds

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ON THE HOCHSCHILD–KOSTANT–ROSENBERG MAP FOR GRADED MANIFOLDS

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Abstract. We show that the Hochschild–Kostant–Rosenberg map from the space of multivector fields on a graded manifold N (endowed with a Berezinian volume) to the cohomology of the algebra of multidifferential operators on N (as a subalgebra of the Hochschild complex of $C^\infty(N)$) is an isomorphism of Batalin–Vilkovisky algebras. Moreover, with an example inspired by string topology, we prove that in general the inclusion of multidifferential operators in the Hochschild complex is not a quasi-isomorphism.

1. Introduction

The multivector fields on a smooth manifold M can be seen as multidifferential operators on the algebra $\mathcal{C}^\infty(M)$ of smooth functions on M. This assignment is a particular case of the following general construction: given a graded associative algebra A, one defines the Hochschild–Kostant–Rosenberg map

$$\mathrm{HKR}\colon \mathcal{V}^\bullet(A) \to \mathsf{Hoch}^\bullet(A)$$

from the space of multivector fields $\mathcal{V}^\bullet(A) := S^\bullet(\mathrm{Der}(A)[-1])[1]$ to the Hochschild complex $\mathsf{Hoch}^\bullet(A)$, as the map which regards a multiderivation of A as a multilinear operator. Actually the image of HKR is contained in the subcomplex $\mathcal{D}^\bullet(A) \subset \mathsf{Hoch}^\bullet(A)$ of multidifferential operators.

If one considers $\mathcal{V}^\bullet(A)$ as a complex with trivial differential, then the HKR map is a morphism of complexes, and the classical Hochschild–Kostant–Rosenberg Theorem [9] states that when A is a smooth algebra, e.g., a polynomial algebra, the HKR map induces isomorphisms in cohomology $\mathcal{V}^\bullet(A) \simeq \mathrm{H}^\bullet(\mathcal{D}^\bullet(A)) \simeq \mathsf{HHoch}^\bullet(A)$.

In this paper we are primarily concerned with the case in which A is the algebra of smooth functions on a graded manifold N. In this case it is known that HKR still induces an isomorphism $\mathcal{V}^\bullet(N) \simeq \mathrm{H}^\bullet(\mathcal{D}^\bullet(N))$, where we used the short-hand notations $\mathcal{V}^\bullet(N)$ for $\mathcal{V}^\bullet(\mathcal{C}^\infty(N))$ and $\mathcal{D}^\bullet(N)$ for $\mathcal{D}^\bullet(\mathcal{C}^\infty(N))$; for a proof, see [22] in case N is an ordinary manifold and [1] for the general case.

The first result of the paper is that in general the inclusion $\mathcal{D}^\bullet(N) \subset \mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N))$ is not a quasi-isomorphism. In particular we consider the graded manifold $N = T[1]M$, where M is a smooth manifold, so that $\mathcal{C}^\infty(T[1]M)$ is the de Rham algebra $\Omega^\bullet(M)$ of M, and we prove the following

Theorem 5.3.b. *If M is a simply connected closed oriented smooth manifold of dimension greater than zero, then $\mathcal{V}^\bullet(\Omega^\bullet(M))$ is not isomorphic to $\mathsf{HHoch}^\bullet(\Omega^\bullet(M))$.*

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The key ingredient of the proof is the isomorphism [3] between the (shifted) homology $H_\bullet(\mathcal{L}M)[\dim M]$ of the free loop space $\mathcal{L}M$ of M and the Hochschild cohomology of the differential graded algebra $\Omega^\bullet(M)$.

We remark that when only ordinary smooth manifolds are considered, it is not known whether the space of multivector fields is quasi-isomorphic to the Hochschild cohomology. Up to our knowledge, only a partial result in this direction is known [15], namely, when M is a smooth manifold, $\mathcal{D}^\bullet(M)$ is quasi-isomorphic to the topological Hochschild complex $\mathsf{HHoch}_{\mathsf{top}}(\mathcal{C}^\infty(M))$ consisting of continuous multilinear homomorphisms (with respect to the Fréchet topology).

Many interesting algebraic structures can be defined on the objects introduced above. It is well known that $\mathcal{V}^\bullet(A)$ and $\mathsf{HHoch}^\bullet(A)$ are Gerstenhaber algebras [7], that $\mathrm{H}^\bullet(\mathcal{D}^\bullet(A))$ is a sub-Gerstenhaber algebra of $\mathsf{HHoch}^\bullet(A)$, and that HKR preserves these structures. Moreover, when A is a finite dimensional algebra endowed with a non-degenerate symmetric inner product compatible with the multiplication of A, then $\mathcal{V}^\bullet(A)$, $\mathrm{H}^\bullet(\mathcal{D}^\bullet(A))$ and $\mathsf{HHoch}^\bullet(A)$ become BV algebras [21].

The second purpose of this paper is to extend this construction to the case in which A is the algebra of smooth functions on a graded manifold N. In this case the algebra is not finite dimensional but we can remedy when N has a Berezinian volume. We prove in fact the following

Theorem 5.3.a. *Let N be a graded manifold endowed with a fixed Berezinian volume v and whose body is a closed smooth manifold. Then $\mathcal{V}^\bullet(N)$ and $\mathrm{H}^\bullet(\mathcal{D}^\bullet(N))$ can be endowed with BV algebra structures compatible with their classical Gerstenhaber structures. Moreover* HKR *is a map of BV algebras.*

The BV algebra structure on multidifferential operators was defined in [21], and corresponds to the BV structure found in [2] on the homology of the free loop space [4, 14]. The BV structure on $\mathcal{V}^\bullet(N)$ is the standard one on the space of multivector fields of a graded manifold N. Both structures depend on the choice of a Berezinian volume on N [11].

The HKR map lifts to an L_∞ map [10, 1] and, at least in the non graded case, to a G_∞ map [19] between complexes. One may conjecture that it also lifts to a BV_∞ map [20]. This would be the analogue, for a graded manifold, of Kontsevich's cyclic formality conjecture [17].

The plan of the paper is as follows. We begin by constructing the BV structure on the space of multivector fields in Section 2. Next we recall some facts on Hochschild cohomology in Section 3. Then we discuss BV structures on the space of multidifferential operators in Section 4. Finally in Section 5 we define the HKR map, describe its main properties, and prove Theorem 5.3.

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2. BV structure on multivector fields

Let A be a graded algebra and let $\mathsf{Der}(A) = \oplus_{j\in\mathbb{Z}}\mathsf{Der}^j(A)$ be the graded Lie algebra of derivations of A, namely $\mathsf{Der}^j(A)$ consists of linear maps $\phi\colon A \to A$

of degree j such that $\phi(ab) = \phi(a)b + (-1)^{j\,|a|}a\phi(b)$ and the bracket is $\{\phi, \psi\} = \phi \circ \psi - (-1)^{|\phi||\psi|}\psi \circ \phi$.

The space of multiderivations $\mathcal{V}^\bullet(A) := S^\bullet(\mathrm{Der}(A)[-1])[1]$ can be endowed with a Gerstenhaber structure, with the wedge product and the bracket which is the extension of the graded commutator $\{\cdot, \cdot\}$ on $\mathsf{Der}(A)$ to $\mathcal{V}^\bullet(A)$ by the Leibnitz rule.

We want to construct an operator Δ on $\mathcal{V}^\bullet(A)$ which makes this Gerstenhaber algebra into a BV algebra. We will use as an auxiliary tool the complex $\mathcal{I}^\bullet(A)$ of integral forms of A, closely following [6]; a different approach to the BV algebra structures on $\mathcal{V}^\bullet(A)$ can be found in [11]. Denote by $\Omega^1(A)$ the space of 1-forms of A, namely, the space $\mathrm{Hom}(\mathcal{V}^1(A), A)$, and assume that the Berezinian $\mathrm{Ber}(\Omega^1(A))$ is free and generated by one element v. To a *divergence operator* div, viz. an even linear map $\mathrm{div}\colon \mathsf{Der}(A) \to A$ satisfying

$$\mathrm{div}(fX) = f\mathrm{div}(X) + (-1)^{|f||X|}X(f),$$

we associate a linear operator $L\colon \mathcal{V}^1(A) \otimes_A \mathrm{Ber}(\Omega^1(A)) \to \mathrm{Ber}(\Omega^1(A))$ by the rule

$$L(X \otimes v) = \mathrm{div}(X)\, v.$$

Observe that for every $f \in A$ and every $X \in \mathsf{Der}(A)$, we have $L_X(fv) = X(f)\, v + (-1)^{|f||X|} f\, L_X(v)$ where we are using the notation $L_X(v) := L(X \otimes v)$.

We now introduce the space $\mathcal{I}^\bullet(A)$ of integral forms [6] as the A-module generated by the elements of $\mathrm{Ber}(\Omega^1(A))$ and by the operations ι_X with $X \in \mathcal{V}^1(A)$, acting on the left and subject to the rules $[\iota_X, \iota_Y] = 0$ and $\iota_{fX} = f\iota_X$. The action of L_X is extended to $\mathcal{I}^\bullet(A)$ by the rule $[L_X, \iota_Y] = \iota_{\{X,Y\}}$ One can define an exterior derivative d on $\mathcal{I}^\bullet(A)$ by imposing $\mathrm{d}v = 0$ and forcing Cartan's identity $\mathrm{d}\iota_X + \iota_X\mathrm{d} = L_X$. Indeed, a consequence of Cartan's formula is that $\mathrm{d}(\iota_{X_1} \cdots \iota_{X_k} v) = L_{X_1}(\iota_{X_2} \cdots \iota_{X_k} v) - \iota_{X_1}\mathrm{d}(\iota_{X_2} \cdots \iota_{X_k} v)$, and the action of d on elements of $\mathcal{I}^\bullet(A)$ can be computed inductively. The exterior derivative d defined by this procedure is a differential precisely when $[L_X, L_Y] = L_{\{X,Y\}}$. This is equivalent to the vanishing of the curvature of div; namely,

$$\mathrm{div}(\{X, Y\}) - X(\mathrm{div}(Y)) + (-1)^{|X|\,|Y|}Y(\mathrm{div}(X)) = 0.$$

Once the generator v of $\mathrm{Ber}(\Omega^1(A))$ is fixed, iterated "contractions" ι_X induce an isomorphism

$$\mathcal{V}^\bullet(A) \xrightarrow{\sim} \mathcal{I}^\bullet(A)$$

and the differential d induces on the space of multivector fields an operator Δ of degree -1 such that $\Delta^2 = 0$. An easy computation shows that $\Delta(X) = \mathrm{div}(X)$ for any $X \in \mathsf{Der}(A)$, and that Δ satisfies the seven term relation

(2.1)

$$\begin{aligned}\Delta(a \wedge b \wedge c) + \Delta(a) \wedge b \wedge c + (-1)^{|a|}a \wedge \Delta(b) \wedge c + (-1)^{|a|+|b|}a \wedge b \wedge \Delta(c) = \\ = \Delta(a \wedge b) \wedge c + (-1)^{|a|}a \wedge \Delta(b \wedge c) + (-1)^{(|a|+1)|b|}b \wedge \Delta(a \wedge c)\end{aligned}$$

and the compatibility with the bracket

$$\{a, b\} := (-1)^{|a|}\Big(\Delta(a \wedge b) - \Delta(a) \wedge b - (-1)^{|a|}a \wedge \Delta(b)\Big). \tag{2.2}$$

Therefore we have proven

Lemma 2.1. *If the Berezinian* $\mathrm{Ber}(\Omega^1(A))$ *is a free A-module of rank one and* div *is a curvature-free divergence operator, then the operator Δ defined as above endows $\mathcal{V}^\bullet(A)$ with a BV structure compatible with the usual Gerstenhaber structure.*

The main example of this construction is when $A = \mathcal{C}^\infty(N)$, N being a graded manifold endowed with a Berezinian volume v. In this case the operators L_X and ι_X are just the classical Lie derivatives and contraction operators, and the complex $\mathcal{I}^\bullet(N)$ is the complex of integral forms of the graded manifold. Since the Berezinian is a line bundle and v is a nowhere zero section, there exists an operator div defined uniquely by the equation $L_Y(v) = \mathrm{div}(Y)\, v$, which is indeed a divergence operator whose curvature vanishes. Observe that in the case when N is a oriented smooth manifold, this amounts to choosing an ordinary volume form v. In the case when $N = T[1]M$, with M an oriented smooth manifold, there is a canonical Berezinian volume v characterized by

$$\int_N \alpha\, v = \int_M \alpha, \qquad \forall \alpha \in C^\infty(N) = \Omega^\bullet(M).$$

Another example when there is canonical Berezinian volume is the manifold T^*M with its Liouville volume form.

Remark 2.2. The two examples are actually related. Namely, let $A = \Gamma(S^\bullet TM)$ be the subalgebra of smooth functions on T^*M that are polynomial along the fibers. Observe that $\mathcal{V}^\bullet(A)$ is a BV subalgebra of $\mathcal{V}(T^*M)$. As a consequence of the "Fourier transform" [1, 16], the Gerstenhaber algebras $\mathcal{V}^\bullet(T[1]M)$ and $\mathcal{V}^\bullet(A)$ are isomorphic. But it can be easily verified that they are also isomorphic as BV algebras.

Remark 2.3. For a smooth manifold M, integral forms are just ordinary differential forms and $\mathcal{I}^\bullet(M)$ is naturally identified with $\Omega^\bullet(M)$. On the other hand, for a graded manifold N which is non trivial in odd degree, the complex $\mathcal{I}^\bullet(N)$ of integral forms is not isomorphic to the de Rham complex of N (see [6] for details).

3. BV structure on Hochschild cohomology

The aim of this Section is to recall some standard facts about Hochschild cohomology and fix notations for the rest of the paper. We address the reader to [12] and [21] for a comprehensive treatment.

3.1. **Hochschild cohomology.** Let $A = \oplus_{j\in\mathbb{Z}} A_j$ be a differential graded algebra over $\mathbb{R}$, with an associative product μ, a unit $\mathbf{1}$ and a differential d of degree $+1$. We also suppose that A is endowed with a non degenerate symmetric inner product compatible with the algebra multiplication, namely such that $\langle a, b\rangle = (-1)^{|a|\,|b|}\langle b, a\rangle$ and $\langle \mu(a \otimes b)\,|\, c\rangle = \langle a\,|\,\mu(b\otimes c)\rangle$. Finally, a bimodule B over the algebra A is given.

Let us set $T(A) := \bigoplus_{k\geq 0} A^{\otimes k}$ and $T^B(A) := \mathbb{R} \oplus \bigoplus_{k,l\geq 0} A^{\otimes k}\otimes B \otimes A^{\otimes l}$. It is well known that $T(A)$ is a coalgebra and $T^B(A)$ a bi-comodule over $T(A)$ with the coproducts

$$T(A) \to T(A) \otimes T(A)$$
$$(a_1, \ldots, a_n) \mapsto \sum_{i=0}^{n} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n)$$

and

$$T^B(A) \to (T(A)\otimes T^B(A)) \oplus (T^B(A)\otimes T(A))$$

$$(a_1,\dots,a_k,b,a_{k+1},\dots,a_n) \mapsto \sum_{i=0}^{k}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,b,\dots,a_n)+ \\ + \sum_{i=k}^{n}(a_1,\dots,b,\dots,a_i)\otimes(a_{i+1},\dots,a_n).$$

Hence we can define the space $\mathsf{Coder}(T(A),T(A))$, resp. $\mathsf{Coder}(T(A),T^B(A))$, of coderivations from $T(A)$ to $T(A)$, resp. from $T(A)$ to $T^B(A)$, with respects to the above coproducts.

The Hochschild cochain complex of A with values in B is defined as

$$\mathsf{Hoch}^\bullet(A,B) := \mathsf{Coder}(T(A[1]),T^{B[1]}(A[1]))[-1]$$

where by $A[1]$ we mean the graded algebra obtained by shifting the degrees of A by 1; namely, $A[1]=\oplus_{j\in\mathbb{Z}}(A[1])_j$ with $(A[1])_j := A_{j+1}$. As usual one can make the identification

$$\mathsf{Hoch}^\bullet(A,B) = \prod_{n\geq 0}\mathsf{Hom}(A^{\otimes n},B)[-n].$$

Let us denote by $\widetilde{\mu^B}$, $\widetilde{\mu}$ and $\widetilde{\mathrm{d}}$ the lifts of the bimodule structure $\mu^B\colon A\otimes B\otimes A\to B$, of the multiplication $\mu\colon A\otimes A\to A$ and of the differential $\mathrm{d}\colon A\to A$, to coderivations of $T^{B[1]}(A[1])$ and $T(A[1])$. Then, on the Hochschild cochain complex we can define a differential $\delta^B\colon \mathsf{Hoch}^\bullet(A,B)\to\mathsf{Hoch}^\bullet(A,B)$, by setting $\delta^B(f) := \delta^B_0(f)+\delta^B_1(f)$ where $\delta^B_0(f) := \widetilde{\mathrm{d}}\circ f-(-1)^{|f|}f\circ\widetilde{\mathrm{d}}$ and $\delta^B_1(f) := \widetilde{\mu^B}\circ f-(-1)^{|f|}f\circ\widetilde{\mu}$. It is easy to check that $(\delta^B)^2=(\delta^B_0)^2=(\delta^B_1)^2=0$. The cohomology with respect to the total differential δ^B is called Hochschild cohomology of A with values in B and it is denoted by $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A,B)$, where the subscript DG means that we are working in the category of differential graded algebras. We could also forget the differential of A and define the Hochschild cohomology of A as a graded associative algebra. This simply amounts to consider the cohomology with respect to the differential δ^B_1; the relevant cohomology is denoted by $\mathsf{HHoch}^\bullet(A,B)$.

When $B=A$ with the canonical bimodule structure we write $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A)$ and $\mathsf{HHoch}^\bullet(A)$ for $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A,A)$ and $\mathsf{HHoch}^\bullet(A,A)$ respectively. Moreover δ^A_0, δ^A_1 and δ^A are denoted by δ_0, δ_1 and δ.

3.2. **Operations on the Hochschild cochain complex.** On the Hochschild cochain complex $\mathsf{Hoch}^\bullet(A)$ one can define various operations. First, there is a composition $f\circ g$ whose graded antisymmetrization $\{f,g\} := f\circ g-(-1)^{|f|\,|g|}g\circ f$ gives rise to a graded odd Lie bracket of degree $+1$, also known as the Gerstenhaber bracket. Notice that the associativity of the product μ of A is equivalent to $\{\widetilde{\mu},\widetilde{\mu}\}=0$, the fact that d is a derivation for μ is equivalent to $\{\widetilde{\mathrm{d}},\widetilde{\mu}\}=0$, and $\mathrm{d}^2=0$ is equivalent to $\{\widetilde{\mathrm{d}},\widetilde{\mathrm{d}}\}=0$. These three properties immediately imply that the Hochschild differential $\delta(f)=\{\widetilde{\mu}+\widetilde{\mathrm{d}},f\}$ indeed squares to zero. Similar relations holds for $\widetilde{\mu^B}$ and δ^B.

Next, using the identification of $\mathsf{Hoch}^\bullet(A)$ with $\prod_{n\geq 0}\mathsf{Hom}(A^{\otimes n},A)[-n]$ we define a product between $\phi\in\mathsf{Hom}(A^{\otimes k},A)[-k]$ and $\psi\in\mathsf{Hom}(A^{\otimes l},A)[-l]$ as

$$(\phi\cup\psi)(a_1\otimes\cdots\otimes a_{k+l}) := (-1)^{kl}\mu(\phi(a_1\otimes\cdots\otimes a_k)\otimes\psi(a_{k+1}\otimes\cdots\otimes a_{k+l})).$$

This associative product is non-commutative but it gives rise to a graded commutative product in cohomology. The cup product and the Gerstenhaber bracket satisfy in cohomology the graded Leibnitz rule

$$\{a, b \cup c\} = \{a, b\} \cup c + (-1)^{(|a|+1)|b|} b \cup \{a, c\}.$$

Therefore $(\mathsf{HHoch}^\bullet(A), \cup, \{\cdot,\cdot\})$ is a Gerstenhaber algebra [7]. In addition, on the complex $\mathsf{Hoch}^\bullet(A, A^*)$ one has an operator β given by the dual to Connes' B-operator [5]. More explicitly, one defines $\beta\colon \mathsf{Hoch}^\bullet(A, A^*) \to \mathsf{Hoch}^{\bullet-1}(A, A^*)$ as

$$(\beta(f)(a_1, \dots, a_n))(a_{n+1}) := \sum_{i=1}^{n+1} (-1)^\epsilon (f(a_i, \dots, a_{n+1}, a_1, \dots, a_{i-1}))(\mathbf{1})$$

where $\mathbf{1}$ is the unit of A and $\epsilon = |f| + |a_1| + \cdots + |a_{n+1}| + (|a_i| + \cdots + |a_n|)(|a_1| + \cdots + |a_{i-1}|)$.

The inner product on A gives rise to an injection $P\colon A \to A^*$ which is an A-bimodule map, and, by composing the Hochschild cochains with the injection P, one obtains an injective map $\wp\colon \mathsf{Hoch}^\bullet(A) \to \mathsf{Hoch}^\bullet(A, A^*)$. If moreover $\wp$ is a quasi-isomorphism, i.e., induces an isomorphism $H(\wp)$ in cohomology, then we can define an operator Δ_β of degree -1 on $\mathsf{HHoch}^\bullet(A)$ by setting $\Delta_\beta = H(\wp)^{-1} \circ \beta \circ H(\wp)$. As shown in [21] (see also [13]), the operator Δ_β squares to zero in cohomology and is compatible with the Gerstenhaber structure on $\mathsf{HHoch}^\bullet(A)$ in the sense that (cf. equation (2.1))

$$\begin{aligned}\Delta_\beta(a \cup b \cup c) + \Delta_\beta(a) \cup b \cup c + (-1)^{|a|} a \cup \Delta_\beta(b) \cup c + (-1)^{|a|+|b|} a \cup b \cup \Delta_\beta(c) = \\ = \Delta_\beta(a \cup b) \cup c + (-1)^{|a|} a \cup \Delta_\beta(b \cup c) + (-1)^{(|a|+1)|b|} b \cup \Delta_\beta(a \cup c)\end{aligned}$$

and (cf. equation (2.2))

$$\{a, b\} = (-1)^{|a|} \Big(\Delta_\beta(a \cup b) - \Delta_\beta(a) \cup b - (-1)^{|a|} a \cup \Delta_\beta(b)\Big).$$

In other words $(\mathsf{HHoch}^\bullet(A), \cup, \{\cdot,\cdot\}, \Delta_\beta)$ is a Batalin–Vilkovisky (BV) algebra. Summing up, we have

Proposition 3.1. *If the map $\wp\colon \mathsf{Hoch}^\bullet(A) \to \mathsf{Hoch}^\bullet(A, A^*)$ induced by the inner product of A is a quasi-isomorphism, then $\mathsf{HHoch}^\bullet(A)$ is endowed with a BV algebra structure, compatible with its Gerstenhaber structure.*

Notice that the same construction also works in the differential graded case, i.e., we have a BV structure on $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A)$ under the hypothesis that $\wp\colon \mathsf{Hoch}^\bullet_{\mathsf{DG}}(A) \to \mathsf{Hoch}^\bullet_{\mathsf{DG}}(A, A^*)$ is a quasi-isomorphism.

3.3. **Examples.** A trivial example is when A is finite dimensional, and hence $\wp$ is an isomorphism. A more interesting case is the algebra of functions on a graded manifold N endowed with a Berezinian volume v. In this case the pairing is defined by

$$\langle f_1, f_2 \rangle = \int_N f_1 f_2\, v. \tag{3.1}$$

In the special case $N = T[1]M$, the algebra of functions is the de Rham algebra $\Omega^\bullet(M)$ and the pairing is just the Poincaré duality. If M is closed, the induced map $\wp\colon \mathsf{Hoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M)) \to \mathsf{Hoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M), \Omega^\bullet(M)^*)$ is a quasi-isomorphism [14], and, by Proposition 3.1, there exists a BV algebra structure on $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$. This BV algebra structure coincides, via Chen's isomorphism, with the Chas–Sullivan BV structure on the homology of the free loop space of M [2, 3, 4, 8, 14, 21].

Remark 3.2. In general, when N is a graded manifold, $\mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N))$ is not necessarily quasi-isomorphic to $\mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N), \mathcal{C}^\infty(N)^*)$, and hence we do not know whether we can define a BV structure on $\mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N))$. However we will see in Section 4 that a version of Proposition 3.1 can be applied to a certain subcomplex of the Hochschild complex.

4. BV structure on multidifferential operators

The Hochschild complex of A has a sub-Gerstenhaber algebra $\mathcal{D}^\bullet(A)$ consisting of multidifferential operators, namely sums of cochains of the form $(a_1, \dots, a_n) \mapsto \prod_{i=1}^n \phi_i(a_i)$ where ϕ_i are composition of derivations. We now want to discuss under which conditions the cohomology of $\mathcal{D}^\bullet(A)$ admits a natural BV structure. As above we are assuming that there exists a non degenerate symmetric inner product on A compatible with the multiplication, and hence an injective map $\wp\colon \mathsf{Hoch}^\bullet(A) \to \mathsf{Hoch}^\bullet(A, A^*)$. The point is to determine when the Connes cyclic β-operator $\beta\colon \mathsf{Hoch}^\bullet(A, A^*) \to \mathsf{Hoch}^{\bullet-1}(A, A^*)$ induces an operator $\Delta_\beta\colon \mathcal{D}^\bullet(A) \to \mathcal{D}^{\bullet-1}(A)$ making the diagram

$$\begin{array}{ccc} \mathcal{D}^\bullet(A) & \xrightarrow{\wp} & \mathsf{Hoch}^\bullet(A, A^*) \\ {\scriptstyle \Delta_\beta}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathcal{D}^{\bullet-1}(A) & \xrightarrow{\wp} & \mathsf{Hoch}^{\bullet-1}(A, A^*) \end{array}$$

commutative. To answer this question, we look at the problem from a more general perspective; namely, let $C^\bullet(A)$ be any sub-Gerstenhaber algebra of $\mathsf{Hoch}^\bullet(A)$ whose $\wp$-image in $\mathsf{Hoch}^\bullet(A, A^*)$ is closed under β. Since $\wp$ is injective, β induces a well-defined operator Δ_β on the complex $C^\bullet(A)$. Following [21] and [13], the operator Δ_β squares to zero in the cohomology of $C^\bullet(A)$, and endows $\mathrm{H}^\bullet(C^\bullet(A))$ with a BV algebra structure compatible with its Gerstenhaber structure. The same construction also works with $C^\bullet_{\mathsf{DG}}(A) \subseteq \mathsf{Hoch}^\bullet_{\mathsf{DG}}(A)$.

We now specialize to the case when $A = C^\infty(N)$, where N is a graded manifold endowed with a Berezinian volume v. In order to prove that the cohomology $\mathrm{H}^\bullet(\mathcal{D}^\bullet(N))$ of the algebra of multidifferential operator admits a natural BV structure, we only need to prove that $(\beta \circ \wp)(\mathcal{D}^\bullet(N)) \subseteq \wp(\mathcal{D}^\bullet(N))$ with $\wp$ induced by the pairing (3.1). We first need the following "integration-by-parts" Lemma.

Lemma 4.1. *Let D be a multidifferential operator. Then there exist a multidifferential operator $\tilde{D}$ such that*

$$\langle D(f_1, \dots, f_n), \mathbf{1} \rangle = \langle \tilde{D}(f_1, \dots, f_{n-1}), f_n \rangle$$

Then we observe that for every $D \in \mathcal{D}^n(N)$ and for every $i = 1, \dots, n$, the operator

$$D_i(f_1, \dots, f_n) := D(f_i, \dots, f_n, f_1, \dots, f_{i-1}), \qquad f_1, \dots, f_n \in A,$$

is still in $\mathcal{D}^n(N)$. Finally

$$(\beta \circ \wp(D))(f_1, \dots, f_{n-1}))(f_n) = \sum_{i=1}^{n} (-1)^{\epsilon} \langle D(f_i, \dots, f_n, f_1, \dots, f_{i-1}), \mathbf{1} \rangle =$$
$$= \sum_{i=1}^{n} (-1)^{\epsilon} \langle D_i(f_1, \dots, f_n), \mathbf{1} \rangle = \sum_{i=1}^{n} (-1)^{\epsilon} \langle \tilde{D}_i(f_1, \dots, f_{n-1}), f_n \rangle =$$
$$= \wp \left(\sum_{i=1}^{n} (-1)^{\epsilon} \tilde{D}_i \right) (f_1, \dots, f_{n-1})(f_n).$$

Proof of Lemma 4.1. The proof is by induction on the order of the multidifferential operator D. If D is homogeneous of order zero,

$$D(f_1, \dots, f_n) = \lambda f_1 \cdots f_n$$

for some constant λ, so that

$$\langle D(f_1, \dots, f_n), \mathbf{1} \rangle = \int_N \lambda f_1 \cdots f_n \, v = \langle \lambda f_1 \cdots f_{n-1}, f_n \rangle$$

and we are done. Now assume the claim proved for operators up to order k and prove it for order $k+1$ operators by the following argument. A homogeneous component of an order $k+1$ multidifferential operator can be written as

$$D(f_1, \dots, f_n) = D_0(f_1, \dots f_{i-1}, X(f_i), f_{i+1}, \dots, f_n)$$

for a suitable multidifferential operator D_0 of order k, some index i and some vector field X. We compute

$$\langle D(f_1, \dots, f_n), \mathbf{1} \rangle = \langle D_0(f_1, \dots, X(f_i), \dots, f_n), \mathbf{1} \rangle$$

Here we have to distinguish two cases. If $i \neq n$, by the induction hypothesis applied to D_0, we can write

$$\langle D_0(f_1, \dots, X(f_i), \dots, f_n), \mathbf{1} \rangle = \langle \tilde{D}_0(f_1, \dots, X(f_i), \dots, f_{n-1}), f_n \rangle$$

and we are done. If $i = n$ then the induction hypothesis gives

$$\langle D_0(f_1, \dots, f_{n-1}, X(f_n)), \mathbf{1} \rangle = \langle \tilde{D}_0(f_1, \dots, f_{n-1}), X(f_n) \rangle.$$

For any vector field Y, Cartan's formula gives $L_Y(v) = \mathrm{d}i_Y(v) + i_Y\mathrm{d}(v) = \mathrm{d}i_Y(v)$, since $\mathrm{d}(v) = 0$ [6]. Hence, by Stokes' Theorem we have that

$$0 = \int_N \mathrm{d}i_Y(f\, v) = \int_N Y(f)\, v + (-1)^{|f|\,|Y|} \int_N f\, L_Y(v).$$

Recall for Section 2 that there exists an operator div defined uniquely by the equation $L_Y(v) = \mathrm{div}(Y)\, v$. Therefore

$$\langle Y(f), \mathbf{1} \rangle = \int_N Y(f)\, v = -(-1)^{|f|\,|Y|} \int_N f\, \mathrm{div}(Y)\, v = -\langle \mathrm{div}(Y), f \rangle. \tag{4.1}$$

Going back to our problem with D_0, we apply the previous formula to the vector field $Y = \tilde{D}_0(f_1, \dots, f_{n-1})X$ and obtain

$$\langle \tilde{D}_0(f_1, \dots, f_{n-1}), X(f_n) \rangle = \int_N \tilde{D}_0(f_1, \dots, f_{n-1}) X(f_n)\, v$$
$$= \langle \mathrm{div}(\tilde{D}_0(f_1, \dots, f_{n-1})X), f_n \rangle.$$

The map $(f_1, \cdots, f_{n-1}) \mapsto \operatorname{div}(\tilde{D}_0(f_1, \ldots, f_{n-1})X)$ is a multidifferential operator, and the Lemma is proven by setting $\tilde{D}(f_1, \ldots, f_{n-1}) = \operatorname{div}(\tilde{D}_0(f_1, \ldots, f_{n-1})X)$. □

5. The Hochschild–Kostant–Rosenberg map

Recall from Section 4 that the Hochschild complex of A has a subalgebra $\mathcal{D}^\bullet(A)$ consisting of multidifferential operators; namely, sums of cochains of the form $(a_1, \ldots, a_n) \mapsto \prod_{i=1}^n \phi_i(a_i)$ where each ϕ_i is a composition of derivations. We define the Hochschild–Kostant–Rosenberg (HKR) map as follows:

$$\begin{array}{ccc} \mathcal{V}^\bullet(A) & \longrightarrow & \mathcal{D}^\bullet(A) \\ \phi_1 \wedge \cdots \wedge \phi_n & \mapsto & \dfrac{1}{n!} \displaystyle\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\ \phi_{\sigma(1)} \cup \cdots \cup \phi_{\sigma(n)}. \end{array} \tag{5.1}$$

We have already observed that both $\mathcal{V}^\bullet(A)$ and $\mathsf{HHoch}^\bullet(A)$ are Gerstenhaber algebras, and it is well known that the HKR map in fact preserves these structures. More explicitly

Theorem 5.1. *If $\mathcal{V}^\bullet(A)$ is endowed with the zero differential, then* HKR *is a morphism of complexes. Moreover the induced map in cohomology is a morphism of Gerstenhaber algebras.*

Proof. This is a standard result: the fact that HKR respects the product structures in cohomology follows directly from the fact that the cup product is commutative in cohomology [7]. An easy check shows that for $X, Y \in \mathsf{Der}(A)$ we have

$$\{\mathrm{HKR}(X), \mathrm{HKR}(Y)\} - \mathrm{HKR}(\{X, Y\}) = 0$$

and hence, by the compatibility between the bracket and the product, HKR induces in cohomology a map of Gerstenhaber algebras. □

The HKR map actually takes its values in the subcomplex $\mathcal{D}^\bullet(A)$ of multidifferential operator. The classical Theorem of Hochschild, Kostant and Rosenberg [9], combined with the above result, can then be stated as follows.

Theorem 5.2. *If A is a smooth algebra, then*

a) HKR: $\mathcal{V}^\bullet(A) \to \mathrm{H}^\bullet(\mathcal{D}^\bullet(A))$ *is an isomorphism of Gerstenhaber algebras;*
b) the inclusion $\mathcal{D}^\bullet(A) \hookrightarrow \mathsf{Hoch}^\bullet(A)$ is a quasi-isomorphism.

Our main result is a version of Theorem 5.2 for graded manifolds, namely, we prove

Theorem 5.3. *Let N be a graded manifold endowed with a fixed Berezinian volume v and whose body is a smooth closed manifold. Then $\mathcal{V}^\bullet(N)$ and $\mathrm{H}^\bullet(\mathcal{D}^\bullet(N))$ can be endowed with BV algebra structures compatible with their classical Gerstenhaber structures. Moreover*

a) HKR: $\mathcal{V}^\bullet(N) \to \mathrm{H}^\bullet(\mathcal{D}^\bullet(N))$ *is an isomorphism of BV algebras;*
b) in general the inclusion $\mathcal{D}^\bullet(N) \hookrightarrow \mathsf{Hoch}^\bullet(\mathcal{C}^\infty(N))$ is not a quasi-isomorphism, and in particular it is never a quasi-isomorphism when $N = T[1]M$ with M a simply connected closed oriented smooth manifold of dimension greater than zero.

Proof of part a). We have seen in Sections 4 and 2 that, in case $A = \mathcal{C}^\infty(N)$ is the algebra of smooth functions of a graded manifold N endowed with a Berezinian volume form, then both $\mathcal{V}^\bullet(N)$ and $\mathrm{H}^\bullet(\mathcal{D}^\bullet(N))$ are BV algebras in a way compatible with their classical Gerstenhaber structures.

We know from Theorems 5.1 that HKR induces in cohomology a morphism of Gerstenhaber algebras. Moreover we know from [1] that $\mathrm{HKR}\colon \mathcal{V}^\bullet(N) \to \mathcal{D}^\bullet(N)$ is a quasi-isomorphism. Therefore, by the compatibility between the BV Laplacian and the Gerstenhaber bracket, we only need to prove that for every vector field $X \in \mathcal{V}^1(N)$ on a graded manifold N, we have

$$\mathrm{HKR}(\Delta(X)) = \Delta_\beta(\mathrm{HKR}(X)).$$

To see this, consider the diagram

$$\begin{array}{ccccc}
\mathcal{V}^1(N) & \xrightarrow{\mathrm{HKR}} & \mathcal{D}^1(N) & \xrightarrow{\wp} & \mathsf{Hoch}^1(\mathcal{C}^\infty(N), \mathcal{C}^\infty(N)^*) \\
\Delta \downarrow & & \Delta_\beta \downarrow & & \beta \downarrow \\
\mathcal{V}^0(N) & \xrightarrow{\mathrm{HKR}} & \mathcal{D}^0(N) & \xrightarrow{\wp} & \mathsf{Hoch}^0(\mathcal{C}^\infty(N), \mathcal{C}^\infty(N)^*)
\end{array}$$

Since the diagram on the right commutes and $\wp$ is injective, commutativity of the diagram on the left follows from the commutativity of the external diagram. This is indeed the case since on the one side, for $X \in \mathcal{V}^1(N)$ and $f \in \mathcal{C}^\infty(N)$, we have that

$$(\beta(\wp(\mathrm{HKR}(X))))\,(f) = -\langle X(f), \mathbf{1}\rangle, \tag{5.2}$$

on the other side

$$(\wp(\mathrm{HKR}(\Delta(X))))\,(f) = \langle \Delta(X), f\rangle. \tag{5.3}$$

By Section 2, $\Delta(X) = \mathrm{div}(X)$, and the right-hand sides of equations (5.2) and (5.3) coincide by equation 4.1.

Proof of part b). Thanks to part *a)* of the Theorem, part *b)* is equivalent to saying that for a general graded manifold N, the HKR map $\mathcal{V}^\bullet(N) \hookrightarrow \mathsf{Hoch}^\bullet(C^\infty(N))$ does not induce an isomorphism in cohomology. We need a short digression on differential graded algebras. In case the graded algebra A has a differential $\mathrm{d}\colon A \to A$, one can see d as a multivector field and consider the operator $\{\mathrm{d}, \cdot\}\colon \mathcal{V}^\bullet(A) \to \mathcal{V}^\bullet(A)$. Clearly it squares to zero since $\mathrm{d}^2 = 0$, and we denote by $\mathrm{H}^\bullet(\mathcal{V}^\bullet(A), \{\mathrm{d}, \cdot\})$ the relevant cohomology.

We say that a differential graded algebra A satisfies the HKR Theorem if the following equation holds

$$\mathrm{HKR}\colon \mathcal{V}^\bullet(A) \xrightarrow{\sim} \mathsf{HHoch}^\bullet(A). \tag{5.4}$$

If we replace $\mathsf{HHoch}^\bullet(A)$ by $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(A)$, the natural object that replaces $\mathcal{V}^\bullet(A)$ on the left-hand side of equation (5.4) is $\mathrm{H}^\bullet(\mathcal{V}^\bullet(A), \{\mathrm{d}, \cdot\})$, and we say that A satisfies the differential graded version of the HKR Theorem ($\mathrm{HKR}_{\mathsf{DG}}$) if we have

$$\mathrm{HKR}\colon \mathrm{H}^\bullet(\mathcal{V}^\bullet(A), \{\mathrm{d}, \cdot\}) \xrightarrow{\sim} \mathsf{HHoch}^\bullet_{\mathsf{DG}}(A). \tag{5.5}$$

We now consider the graded manifold $N = T[1]M$.

Lemma 5.4. $\mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) \cong \mathrm{H}^\bullet_{\mathrm{deRham}}(M)$.

Proof. First, we show that for every $m \geq 1$ we have $\mathrm{H}^m(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) = 0$. We fix local coordinates $\{x^i, \theta^j\}$ on $T[1]M$, where x^i are (even) coordinates on M and θ^j (odd) coordinates on the fibers. Consider the globally well-defined derivation ι_E which on the local generators of multivector fields acts as

$$\iota_E\left(x^i\right) = 0\,; \quad \iota_E\left(\theta^i\right) = 0\,; \quad \iota_E\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial \theta^i}\,; \quad \iota_E\left(\frac{\partial}{\partial \theta^i}\right) = 0\,.$$

Recall that the derivation $\{\mathrm{d}, \cdot\}$ acts as

$$\left\{\mathrm{d}, x^i\right\} = \theta^i\,; \quad \left\{\mathrm{d}, \theta^i\right\} = 0\,; \quad \left\{\mathrm{d}, \frac{\partial}{\partial x^i}\right\} = 0\,; \quad \left\{\mathrm{d}, \frac{\partial}{\partial \theta^i}\right\} = \frac{\partial}{\partial x^i}\,.$$

It follows that $L_E = \{\mathrm{d}, \cdot\} \circ \iota_E + \iota_E \circ \{\mathrm{d}, \cdot\}$ is a derivation on $\mathcal{V}(T[1]M)$ which, when restricted to the fields of degree m, is the multiplication by m; namely

$$L_E\left(x^i\right) = 0\,; \quad L_E\left(\theta^i\right) = 0\,; \quad L_E\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\,; \quad L_E\left(\frac{\partial}{\partial \theta^i}\right) = \frac{\partial}{\partial \theta^i}\,.$$

Now, suppose that Ψ is a $\{\mathrm{d}, \cdot\}$-closed multivector field of degree $m \geq 1$. Then it is also $\{\mathrm{d}, \cdot\}$-exact:

$$\begin{aligned} \Psi &= \frac{1}{m} L_E(\Psi) = \frac{1}{m}\{\mathrm{d}, \iota_E \Psi\} + \frac{1}{m}\iota_E(\{\mathrm{d}, \Psi\}) \\ &= \{\mathrm{d}, \frac{1}{m}\iota_E \Psi\} \end{aligned}$$

This shows that higher cohomology groups vanish, and we are left to prove that $\mathrm{H}^0(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) = \mathrm{H}^\bullet_{\mathrm{deRham}}(M)$. To see this, just notice that the 0-vector fields on $T[1]M$ are the differential forms on M and the action of $\{\mathrm{d}, \cdot\}$ on $\mathcal{V}^0(T[1]M)$ is precisely the action of the de Rham differential on $\Omega^\bullet(M)$. □

Observe that another way of proving Lemma 5.4 goes through the Gerstenhaber isomorphism described in Remark 2.2. In fact, it is not difficult to see that the image of the multivector field d under this isomorphism is the restriction to $A = \Gamma(S^\bullet TM)$ of the canonical Poisson bivector field on the symplectic manifold T^*M. Thus, $\mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\})$ is isomorphic to the Poisson cohomology of T^*M (restricted to functions polynomial along the fibers) which in turn (by nondegeneracy of the Poisson structure) is isomorphic to the de Rham cohomology of the total space and hence of the base.

Lemma 5.5. *If* $\Omega^\bullet(M)$ *satisfies the* HKR *Theorem, then it also satisfies the* $\mathrm{HKR}_{\mathsf{DG}}$ *Theorem.*

Proof. We have to show that if equation (5.4) holds, then also equation (5.5) is true. Since the differentials δ_0 and δ_1 commute, we can compute $\mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$ using a spectral sequence. By the hypothesis that the HKR Theorem holds, we have

$$E_1 = \mathsf{HHoch}^*(\Omega^\bullet(M)) \cong \mathcal{V}^\bullet(T[1]M).$$

The E_2 term is given by the δ_0-cohomology of the E_1, where δ_0 is the commutator with d. Using the fact that HKR preserves the Gerstenhaber bracket in cohomology, we have an isomorphism of complexes

$$(\mathsf{HHoch}^\bullet(\Omega^\bullet(M)), \delta_0) \cong (\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}).$$

and hence

$$E_2 = \mathrm{H}^\bullet(\mathsf{HHoch}^\bullet(\Omega^\bullet(M)), \delta_0) \cong \mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}).$$

From the previous Lemma we know that $\mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) \cong \mathrm{H}^\bullet_{\mathrm{deRham}}(M)$ concentrated in (vertical) degree zero. Hence the spectral sequence collapses at the E_2 term, namely

$$\mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) = E_2 = E_\infty = \mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M)).$$

□

We are now able to exhibit a family of examples of graded manifolds N for which the inclusion $\mathcal{D}^\bullet(N) \hookrightarrow \mathsf{Hoch}(\mathcal{C}^\infty(N))$ is not a quasi-isomorphism. Let $N = T[1]M$ with M a simply connected closed oriented smooth manifold of dimension greater than zero. Assume by contradiction that $\mathcal{V}^\bullet(\Omega^\bullet(M))$ is isomorphic to $\mathsf{HHoch}^\bullet(\Omega^\bullet(M))$. Then by Lemma 5.5 $\Omega^\bullet(M)$ satisfies the $\mathrm{HKR}_{\mathsf{DG}}$ Theorem. By combining Chen's isomorphism [3, 8]

$$H_\bullet(LM) \cong \mathsf{HHoch}^\bullet_{\mathsf{DG}}(\Omega^\bullet(M))$$

with the $\mathrm{HKR}_{\mathsf{DG}}$ isomorphism, we have

$$H_\bullet(LM) \cong \mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}).$$

On the other hand, by Lemma 5.4 there is an isomorphism

$$\mathrm{H}^\bullet(\mathcal{V}^\bullet(T[1]M), \{\mathrm{d}, \cdot\}) \cong \mathrm{H}^\bullet_{\mathrm{deRham}}(M),$$

and we obtain the isomorphism

$$\mathrm{H}_\bullet(LM) \cong \mathrm{H}^\bullet_{\mathrm{deRham}}(M),$$

which is false for any manifold as above [18].

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