Rota-Baxter Algebras, Dendriform Algebras and Poincare-Birkhoff-Witt Theorem

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ROTA-BAXTER ALGEBRAS, DENDRIFORM ALGEBRAS AND POINCARÉ-BIRKHOFF-WITT THEOREM

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Abstract. Rota-Baxter algebras appeared in both the physics and mathematics literature. It is of great interest to have a simple construction of the free object of this algebraic structure. For example, free commutative Rota-Baxter algebras relate to double shuffle relations for multiple zeta values. The interest in the non-commutative setting arose in connection with the work of Connes and Kreimer on the Birkhoff decomposition in renormalization theory in perturbative quantum field theory. We construct free non-commutative Rota-Baxter algebras and apply the construction to obtain universal enveloping Rota-Baxter algebras of dendriform dialgebras and trialgebras. We also prove an analog of the Poincaré-Birkhoff-Witt theorem for universal enveloping algebra in the context of dendriform trialgebras. In particular, every dendriform dialgebra and trialgebra is a subalgebra of a Rota-Baxter algebra. We explicitly show that the free dendriform dialgebras and trialgebras, as represented by planar trees, are canonical subalgebras of free Rota-Baxter algebras.

Contents

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1. Introduction

It is well-known that the relationship between associative algebras and Lie algebras plays a fundamental role in the study of these algebraic structures and their applications. Central to this relation are the classical theorems of Cartier-Milnor-Moore-Quillen and of Poincaré-Birkhoff-Witt. Similar connections have been recently studied for some other algebraic structures [**Ch**, **Ron**, **L-R3**]

This paper explores the relationship between Rota-Baxter algebras and dendriform dialgebras and dendriform trialgebras.

1.1. **Rota-Baxter algebras.** A Rota-Baxter algebra is an algebra A with a linear endomorphism R satisfying the **Rota-Baxter relation** [1]:

$$R(x)R(y) = R\big(R(x)y + xR(y) + \lambda xy\big),\ \forall x, y \in A. \tag{1}$$

Here λ is a fix element in the base ring. It was introduced by the American mathematician Glen E. Baxter [**Ba**] in his probability study, and was popularized mainly by the work of Rota [**Ro1**, **Ro2**, **Ro3**] and his school.

A linear operator satisfying (1) in the context of Lie algebras was introduced independently by Belavin and Drinfeld [**B-D**], and Semenov-Tian-Shansky [**STS1**] in the 1980s related to solutions, called r-matrices, of the (modified) classical Yang-Baxter equation, named after the physicists Chen-ning Yang and Rodney Baxter. This curious coincidence of Baxter and Baxter just happens to indicate the connections of the Rota-Baxter operator with many areas of mathematics and physics. For example, a strikingly simple yet useful factorization theorem of the Rota-Baxter operator was discovered by Atkinson [**At**] in 1963, and then independently established for Lie algebras as a fundamental theorem of integrable systems by Reyman and Semenov-Tian-Shansky [**R-STS1**] in 1979, and according to the same authors "subsequently many times rediscovered".

Recently, there have been several interesting developments in Rota-Baxter algebras in relation to theoretical physics and mathematics, including quantum field theory [**C-K1**, **C-K2**, **Kr1**, **Kr2**], Yang-Baxter equations [**Ag1**, **Ag2**, **Ag3**], shuffle products [**E-G1**, **G-K1**, **G-K2**, **Ho**], operads [**EF1**, **A-L**, **Le1**, **Le2**, **Le3**, **E-G2**], Hopf algebras [**A-G-K-O**, **E-G1**], combinatorics [**Gu2**] and number theory [**B-B-B-L**, **E-G1**, **Gu5**, **Ho**]. The most prominent of these is the work of Connes and Kreimer in their Hopf algebraic approach to renormalization theory in perturbative quantum field theory [**C-K1**, **C-K2**], continued in [**E-G-K2**, **E-G-K3**].

1.2. **Dendriform dialgebras and trialgebras.** A dendriform dialgebra is a module D with two binary operations $\prec$ and $\succ$ such that

$$\begin{aligned}&(x \prec y) \prec z = x \prec (y \prec z + y \succ z), (x \succ y) \prec z = x \succ (y \prec z),\\ &(x \prec y + x \succ y) \succ z = x \succ (y \succ z).\end{aligned} \tag{2}$$

Dendriform dialgebras were introduced by Loday [**Lo1**] in 1995 with motivation from algebraic K-theory, and have been further studied with connections to several areas in mathematics and physics, including operads [**Lo2**], homology [**Fra1**, **Fra2**], Hopf

[1] Sometimes the relation takes the form $R(x)R(y) + \theta R(xy) = R\big(R(x)y + xR(y)\big)$, so $\lambda = -\theta$.

algebras [**Ch**, **Hol2**, **Ron**, **L-R2**], Lie and Leibniz algebras, combinatorics [**Fo**, **L-R1**, **A-S1**, **A-S2**], arithmetic [**Lo3**] and quantum field theory [**Fo**].

The **dendriform trialgebra** of Loday and Ronco [**L-R2**] is a module T equipped with binary operations $\prec, \succ$ and $\cdot$ that satisfy the relations

$$\begin{aligned} &(x \prec y) \prec z = x \prec (y \star z), (x \succ y) \prec z = x \succ (y \prec z),\\ &(x \star y) \succ z = x \succ (y \succ z), (x \succ y) \cdot z = x \succ (y \cdot z),\\ &(x \prec y) \cdot z = x \cdot (y \succ z), (x \cdot y) \prec z = x \cdot (y \prec z), (x \cdot y) \cdot z = x \cdot (y \cdot z) \end{aligned} \tag{3}$$

for $x, y, z \in D$. Here

$$\star = \prec + \succ + \cdot . \tag{4}$$

The product $\star$ defined above, in terms of a linear combination of the dendriform trialgebra compositions $\prec$, $\succ$, and $\cdot$

$$x \star y = x \prec y + x \succ y + x \cdot y, \quad \forall x, y \in T \tag{5}$$

makes T into an associative algebra.

Dendriform algebras in general may be characterized by the so-called "splitting associativity", i.e. an associative product decomposes into a linear combination of several binary compositions. Since Loday first introduced the dendriform dialgebra, many more such structures have been found. Such as Leroux's ennea- and NS-algebra [**Le1**, **Le2**], or the quadri-algebra of Loday and Aguiar [**A-L**]. In [**E-G2**] we show how these more complex structures, equipped with large numbers of compositions and relations, may be derived from an operadic point of view in terms of products, see also [**Lo4**].

1.3. **The connection.** The first link between Rota-Baxter algebras and dendriform algebras was given by Aguiar [**Ag1**] who showed that a Rota-Baxter algebra of weight $\lambda = 0$ carries a dendriform dialgebra structure. This has been extended to further connections between linear operators and dendriform type algebras [**EF1**, **Le2**, **A-L**, **E-G2**], in particular to dendriform trialgebras by the first named author.

Theorem 1.1. (1) **(Aguiar [Ag2])** *A Rota-Baxter algebra of weight zero defines a dendriform dialgebra* $(A, \prec_R, \succ_R)$, *where*

$$x \prec_R y = xR(y),\ x \succ_R y = R(x)y.$$

(2) **(Ebrahimi-Fard [EF1])** *A Rota-Baxter algebra* (A, R) *of weight* λ *defines a dendriform trialgebra* $(A, \prec_R, \succ_R, \cdot_R)$, *where*

$$x \prec_R y = xR(y),\ x \succ_R y = R(x)y, x \cdot_R y = \lambda xy.$$

(3) **(Ebrahimi-Fard [EF1])** *A Rota-Baxter algebra* (A, R) *of weight* λ *defines a dendriform dialgebra* $(A, \prec'_R, \succ'_R)$, *where*

$$x \prec'_R y = xR(y) + \lambda xy,\ x \succ'_R y = R(x)y.$$

Thus there are natural functors from the category of associative Rota-Baxter algebras of weight λ to the categories of dendriform dialgebras and trialgebras. Our main goal in this paper is to study the adjoint functors. For this purpose we first study free Rota-Baxter algebras which play a central role in the study of the adjoint functor. This is in analogy to the role played by the free associative algebras in the study of the adjoint functor from the category of Lie algebras to the category of associative algebras.

As pointed out by Cartier [**Ca**] thirty years ago, "The existence of free (Rota-)Baxter algebras follows from well-known arguments in universal algebra but remains quite immaterial as long as the corresponding word problem is not solved in an explicit way as Rota was the first to do." Both Rota's aforementioned construction [**Ro1**] and the construction of Cartier himself in the above cited paper dealt with free commutative Rota-Baxter algebras. A third construction was obtained by the second named author and Keigher [**G-K1**] later as a generalization of shuffle product algebras. We will construct free non-commutative Rota-Baxter algebras using a non-commutative shuffle product. This is given in Section 2.

Then in Section 3 we apply this construction to obtain adjoint functors of the functors in Theorem 1.1 and to prove a Poicaré-Birkhoff-Witt theorem for the universal enveloping Rota-Baxter algebras of a dendriform dialgebra and trialgebra, identifying a basis of the universal enveloping algebras in terms of the dendriform algebras. In particular we show that every dendriform trialgebra is a subalgebra of a Rota-Baxter algebra.

The special case for free algebras is considered in Section 4 where we realize the free dendriform dialgebra and trialgebra of Loday and Loday-Ronco in terms of decorated planar rooted trees as canonical subalgebras of free Rota-Baxter algebras.

1.4. **Notation.** In this paper, $\mathbf{k}$ is a commutative unitary ring which will be further assumed to be a field in sections 3 and 4. Let $\mathbf{Alg}$ be the category of unitary $\mathbf{k}$-algebras A whose unit is identified with the unit $\mathbf{1}$ of $\mathbf{k}$ by the structure homomorphism $\mathbf{k} \to A$. Let $\mathbf{Alg}^0$ be the category of nonunitary $\mathbf{k}$-algebras. Similarly let $\mathbf{RB}_\lambda$ (resp. $\mathbf{RB}^0_\lambda$) be the category of unitary (resp. nonunitary) Rota-Baxter $\mathbf{k}$-algebras of weight λ. The subscript λ will be suppressed if there is no danger of confusion. Let $\mathbf{DD}$ and $\mathbf{DT}$ be the category of dendriform dialgebras over $\mathbf{k}$ and dendriform trialgebras over $\mathbf{k}$.

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2. Free Rota-Baxter algebras

We first construct free unitary Rota-Baxter algebras over an algebra with an augmented ideal, in the category of unitary Rota-Baxter algebras. This will be generalized to free unitary Rota-Baxter algebras over an arbitrary unitary algebra in §2.4. Then in §2.5, we modify the construction to give free nonunitary Rota-Baxter algebras. The later free Rota-Baxter algebras will be applied in sections 3 and 4 to study universal enveloping Rota-Baxter algebras of dendriform dialgebras and trialgebras, and free dendriform dialgebras and trialgebras.

2.1. **The basis.**

2.1.1. *Rota-Baxter parenthesized words.* Let B be a $\mathbf{k}$-algebra with the following conditions.

Condition 2.1. (1) The structure homomorphism $\mathbf{k} \to B$ is injective.
(2) B has an augmentation ideal I (i.e., $B/I \cong \mathbf{k}$) such that I is a free $\mathbf{k}$-module.

Let X be a basis of the free $\mathbf{k}$-module I. Let $\tilde{X} = X \cup \{1\}$. Then B is a free $\mathbf{k}$-module with basis $\tilde{X}$. Let $X' = \tilde{X} \cup \{\lfloor, \rfloor\}$. Let $M(X')$ be the free semigroup (not the free monoid) generated by X'. We will use concatenation to denote the product in $M(X')$ and use $\cdot$ to denote the product in B.

Definition 2.2. A word in $M(X')$ is called a **Rota-Baxter (parenthesized) word (RBW)** if the following conditions are satisfied.

(1) The number of $\lfloor$ in the word equals the number of $\rfloor$ in the word;
(2) Counting from the left to the right, the accumulative number of $\lfloor$ at each location is always greater or equal to that of $\rfloor$;
(3) There is no occurrence of x_1x_2 in the word, with $x_1, x_2 \in X$;
(4) There is no occurrence of $\rfloor\lfloor$, $\lfloor\rfloor$ in the word;
(5) $\mathbf{1}$ only occurs as the word $\mathbf{1}$ or as a part of $\lfloor\mathbf{1}\rfloor$ in the word. (So there is no $\lfloor\mathbf{1}\lfloor$, $\rfloor\mathbf{1}\lfloor$, $\mathbf{1}x$ with $x \in X$, etc., and $\mathbf{1}$ can not occur at the beginning or the end of a multi-letter word.)

Let $\mathfrak{M}(X)$ be the set of RBWs on X and let $Ш^{NC}(B)$ be the free $\mathbf{k}$-module generated by $\mathfrak{M}(X)$.

We remark that $\mathbf{1}$ is the identity of B. It is not the identity of the free monoid generated by X' which will play no role in this paper.

The concept of parenthesized words has appeared in the work of Kreimer [**Kr1**] to represent Hopf algebra structure on Feynman diagrams in pQFT, with a different set of restrictions on the words. We use $\lfloor$ and $\rfloor$ instead of (and) to avoid confusion with the usually meaning of parentheses.

For a word $y := y_1 \cdots y_n \in M(X')$, $y_i \in X', 1 \leq i \leq n$, we let $d_i(y)$ be the number of $\lfloor$ in $y_1, \cdots, y_i$ minus the number of $\rfloor$ in $y_1, \cdots, y_i$. Then the first two conditions for $\mathfrak{M}(X)$ are equivalent to $d_i(y) \geq 0, 1 \leq i \leq n$ and $d_n(y) = 0$; Define $d(y) := \max\{d_i(y),\ 1 \leq i \leq n\}$ to be the **depth** of y.

2.1.2. *An alternative description.* Let Y, Z be two subsets of $M(X')$.

$$\begin{aligned} A(Y,Z) \;&=\; \Big(\bigcup_{r\geq 1} (Y\lfloor Z\rfloor)^r\Big) \bigcup \Big(\bigcup_{r\geq 0} (Y\lfloor Z\rfloor)^r Y\Big) \\ &\bigcup \Big(\bigcup_{r\geq 0} \lfloor Z\rfloor (Y\lfloor Z\rfloor)^r\Big) \bigcup \Big(\bigcup_{r\geq 0} \lfloor Z\rfloor (Y\lfloor Z\rfloor)^r Y\Big). \end{aligned} \tag{6}$$

We construct a sequence of subsets of $M(X')$ by the following recursion. Let $X_0 = X$ and $\tilde{X}_0 = X \cup \{\mathbf{1}\}$. Define $X_1 = A(X, \tilde{X}_0)$, $\tilde{X}_1 = X_1 \cup \{\mathbf{1}\}$. In general, for $n \geq 0$, define

$$X_{n+1} \;=\; A(X, \tilde{X}_n), \tag{7}$$

$$\tilde{X}_{n+1} = X_{n+1} \cup \{\mathbf{1}\}, \quad (8)$$

$$X_\infty = \bigcup_{n\geq 0} X_n = \varinjlim X_n, \quad (9)$$

$$\tilde{X}_\infty = \bigcup_{n\geq 0} \tilde{X}_n = \varinjlim \tilde{X}_n. \quad (10)$$

Here the second equations in Eq. (9) and (10) follows since $X_1 \supseteq X_0$, $\tilde{X}_1 \supseteq \tilde{X}_0$ and, assuming $X_n \supseteq X_{n-1}$, $\tilde{X}_n \supseteq \tilde{X}_{n-1}$, we get $X_{n+1} = A(X, \tilde{X}_n) \supseteq A(X, \tilde{X}_{n-1}) = A_n$ and thus $\tilde{X}_{n+1} \supseteq \tilde{X}_n$.

We note that, for each $n \geq 1$, the union of $X_n = A(X, \tilde{X}_{n-1})$ expressed in Eq.(6) is disjoint:

$$\begin{aligned} A(X,\tilde{X}_{n-1}) &= \Big(\dot{\bigcup}_{r\geq 1} (X\lfloor \tilde{X}_{n-1}\rfloor)^r\Big) \dot{\bigcup} \Big(\dot{\bigcup}_{r\geq 0} (X\lfloor \tilde{X}_{n-1}\rfloor)^r X\Big) \\ &\dot{\bigcup} \Big(\dot{\bigcup}_{r\geq 0} \lfloor \tilde{X}_{n-1}\rfloor (X\lfloor \tilde{X}_{n-1}\rfloor)^r\Big) \dot{\bigcup} \Big(\dot{\bigcup}_{r\geq 0} \lfloor \tilde{X}_{n-1}\rfloor (X\lfloor \tilde{X}_{n-1}\rfloor)^r X\Big). \end{aligned} \quad (11)$$

For an element $\mathfrak{X}$ in $(X\lfloor \tilde{X}_{n-1}\rfloor)^r$ (resp. $(X\lfloor \tilde{X}_{n-1}\rfloor)^r X$, resp. $\lfloor \tilde{X}_{n-1}\rfloor (X\lfloor \tilde{X}_{n-1}\rfloor)^r$, resp. $\lfloor \tilde{X}_{n-1}\rfloor (X\lfloor \tilde{X}_{n-1}\rfloor)^r X$), define its **length** to be $\ell(\mathfrak{X}) = 2r$ (resp. $2r+1$, resp. $2r+1$, resp. $2r+2$); its **head** to be $h(\mathfrak{X}) = 0$ (resp. 0, resp. 1, resp. 1); its **tail** to be $t(\mathfrak{X}) = 1$ (resp. 0, resp. 1, resp. 0); its **signature** to be $s(\mathfrak{X}) = (h(\mathfrak{X}), t(\mathfrak{X}))$. Shortly speaking, the head (resp. tail) of $\mathfrak{X}$ is 0 or 1 if $\mathfrak{X}$ starts (resp. ends) with an element in X or $\lfloor \mathfrak{M}(X)\rfloor$. Also define $\ell(1) = 0$. To summarize, we have the table

$\mathfrak{X}$	$(X\lfloor \tilde{X}_{n-1}\rfloor)^r$	$(X\lfloor \tilde{X}_{n-1}\rfloor)^r X$	$\lfloor \tilde{X}_{n-1}\rfloor (X\lfloor \tilde{X}_{n-1}\rfloor)^r$	$\lfloor \tilde{X}_{n-1}\rfloor (X\lfloor \tilde{X}_{n-1}\rfloor)^r X$
$\ell(\mathfrak{X})$	$2r$	$2r+1$	$2r+1$	$2r+2$
$h(\mathfrak{X})$	0	0	1	1
$t(\mathfrak{X})$	1	0	1	0
$s(\mathfrak{X})$	(0,1)	(0,0)	(1,1)	(1,0)

We will use the following more convenient description of $\mathfrak{M}(X)$.

Lemma 2.3. *$\mathfrak{M}(X) = \tilde{X}_\infty$. In fact, $\tilde{X}_n$ consists of words in $\mathfrak{M}(X)$ of depth less or equal n.*

Proof. We only need to prove the second statement for which we use the induction on n. Let $\mathfrak{M}_n$ be the RBWs of depth n.

When $n = 0$, we have $\tilde{X}_0 = X \cup \{\mathbf{1}\}$. On the other hand, RBWs of depth 0 do not have the occurrence of $\lfloor$ or $\rfloor$. So by condition 3, they are precisely $X \cup \{\mathbf{1}\}$. So $\tilde{X}_0 = \mathfrak{M}_0$.

Assume $\tilde{X}_n = \cup_{i\leq n} \mathfrak{M}_i$ for $0 \leq n \leq k$. By the description of $\tilde{X}_{k+1} = A(X, \tilde{X}_k)$ in Eq.(11) and the induction hypothesis, we have

$$\tilde{X}_{k+1} \subseteq \cup_{i\leq k+1} \mathfrak{M}_i.$$

On the other hand, let $\mathfrak{X}$ be a RBW of depth at most $k+1$. We explain how to express $\mathfrak{X}$ in the form of Eq.(11). Let $\mathfrak{X} = y_1 y_2 \cdots y_m$ with $y_i \in X'$. We use induction on m. When $m = 1$, $\mathfrak{X} \in \tilde{X}$ so is in $\tilde{X}_{k+1}$. Assume this is true for $1 \leq m \leq j$, and let $\mathfrak{X} = y_1 \cdots y_{j+1}$. By the conditions of RBWs, y_1 can be either in X or is $\lfloor$. If $y_1 \in X$, then $y_2 = \lfloor$ and it is easy to see that all the conditions for a RBW are still satisfied for $y_2 \cdots y_m$. By the induction hypothesis, $y_2 \cdots y_m$ is in $\tilde{X}_{k+1}$. Since $y_2 = \lfloor$, the word $y_2 \cdots y_m$ must be of the form $\lfloor \mathfrak{X}_1\rfloor\cdots$ in $\tilde{X}_{k+1}$. Then $\mathfrak{X}=y_1\lfloor\mathfrak{X}_1\rfloor\cdots$ is still in $\tilde{X}_{k+1}$. Now if $y_1=\lfloor$, then by condition 1 in Definition 2.2, there is $1\le i\le j+1$ such that $d_i(\mathfrak{X})=0$. Let i be the minimum with this condition. Then $y_i=\rfloor$, $\mathfrak{X}=\lfloor y_2\cdots y_{i-1}\rfloor y_{i+1}\cdots$ and $y_2\cdots y_{i-1}$ is still a RBW. Since the depth of $\mathfrak{X}$ is $k+1$, the depth of $y_2\cdots y_{i-1}$ is k. So by the induction hypothesis on k, $y_2\cdots y_{i-1}$ is in $\tilde{X}_k$ and therefore $y_1\cdots y_i$ is in $\tilde{X}_{k+1}$. If $\mathfrak{X}=y_1\cdots y_i$, then we are done. Otherwise, $y_{i+1}\in X$, and $y_{i+1}\cdots y_m$ is still a RBW, of length not exceeding k. So $y_{i+1}\cdots y_m$ is in $\tilde{X}_{k+1}$. Then the concatenation $y_1\cdots y_iy_{i+1}\cdots y_m$ is in $\tilde{X}_{k+1}$. This completes the inductions, in n and m. □

We thus have $\text{Ш}^{\rm NC}(B)=\bigoplus_{\mathfrak{X}\in\tilde{X}_\infty}\mathbf{k}\mathfrak{X}$.

2.2. **The product.** We now define a product $\diamond$ on $\text{Ш}^{\rm NC}(B)$. Note that the product is different from the product in the free semigroup $M(X')$. Roughly speaking, the product of $\mathfrak{X}$ and $\mathfrak{X}'$ is defined to be the concatenation whenever $t(\mathfrak{X})\neq h(\mathfrak{X}')$. When $t(\mathfrak{X})=h(\mathfrak{X}')$, the product is defined by the product in B or by a shuffle relation, as in Eq. (13).

To be precise, we first define $\mathbf{1}$ to be the unit, that is,

$$\mathbf{1}\diamond\mathfrak{X}=\mathfrak{X}\diamond\mathbf{1}=\mathfrak{X},\ \forall\,\mathfrak{X}\in\text{Ш}^{\rm NC}(B). \tag{12}$$

Then we just need to define the product of $\mathfrak{X}$ and $\mathfrak{X}'$ when both are in X_∞. If $d(\mathfrak{X})=0$ or $d(\mathfrak{X}')=0$, then $\mathfrak{X}\in X$ or $\mathfrak{X}'\in X$. Let $\mathfrak{X}\in X$ and $\mathfrak{X}'=\mathfrak{X}'_1\mathfrak{X}'_2$ with $\mathfrak{X}'_1\in X$ or $\mathfrak{X}'_1\in\lfloor\mathfrak{M}(X)\rfloor$. Then define

$$\mathfrak{X}\diamond\mathfrak{X}'=\begin{cases}(\mathfrak{X}\cdot\mathfrak{X}'_1)\mathfrak{X}'_2, & \mathfrak{X}'_1\in X,\\ \mathfrak{X}\,\mathfrak{X}'_1\,\mathfrak{X}'_2, & \mathfrak{X}'_1\in\lfloor\mathfrak{M}(X)\rfloor.\end{cases}$$

Similarly define $\mathfrak{X}\diamond\mathfrak{X}'$ when $\mathfrak{X}=\mathfrak{X}_1\mathfrak{X}_2$ with $\mathfrak{X}_2\in X$ or $\mathfrak{X}_2\in\lfloor\mathfrak{M}(X)\rfloor$ and $\mathfrak{X}'\in X$. If both $\mathfrak{X}$ and $\mathfrak{X}'$ are in X, then define $\mathfrak{X}\diamond\mathfrak{X}'=\mathfrak{X}\cdot\mathfrak{X}'$.

We next consider the remaining case when $d(\mathfrak{X})\ge 1$ and $d(\mathfrak{X}')\ge 1$.

By Lemma 2.3, $\mathfrak{X}$ can be uniquely written as $\mathfrak{X}_1\mathfrak{X}_2$ with $\mathfrak{X}_1\in\mathfrak{M}(X)$ and $\mathfrak{X}_2\in X$ or $\mathfrak{X}_2=\lfloor\overline{\mathfrak{X}}_2\rfloor$ where $\overline{\mathfrak{X}}_2\in\mathfrak{M}(X)$. Likewise, $\mathfrak{X}'$ can be uniquely written as $\mathfrak{X}'_1\mathfrak{X}'_2$ with $\mathfrak{X}'_2\in\mathfrak{M}(X)$ and $\mathfrak{X}'_1\in X$ or $\mathfrak{X}'_1=\lfloor\overline{\mathfrak{X}}'_1\rfloor$ where $\overline{\mathfrak{X}}'_1\in\mathfrak{M}(X)$.

If $\mathfrak{X}_2\in X$ and $\mathfrak{X}'_1=\lfloor\overline{\mathfrak{X}}'_1\rfloor$ where $\overline{\mathfrak{X}}'_1\in\mathfrak{M}(X)$, then define $\mathfrak{X}\diamond\mathfrak{X}'$ to be the word in $M(X')$ by concatenation: $\mathfrak{X}\diamond\mathfrak{X}':=\mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}'_1\mathfrak{X}'_2$.

If $\mathfrak{X}_2=\lfloor\overline{\mathfrak{X}}_2\rfloor$ with $\overline{\mathfrak{X}}_2\in\mathfrak{M}(X)$ and $\mathfrak{X}'_1\in X$, then also define $\mathfrak{X}\diamond\mathfrak{X}'$ by concatenation.

Let $\mathfrak{X}_2\in X$ and $\mathfrak{X}'_1\in X$. Then in B, $\mathfrak{X}_2\cdot\mathfrak{X}'_1=\sum_{x\in X}c_xx$ with $c_x=c_x^{\mathfrak{X}_2,\mathfrak{X}'_1}\in\mathbf{k}$. We define

$$\mathfrak{X}\diamond\mathfrak{X}':=\mathfrak{X}_1\big(\sum_{x\in X}c_xx\big)\mathfrak{X}'_2=\sum_{x\in X}c_x\mathfrak{X}_1x\mathfrak{X}'_2.$$

It remains to define $\mathfrak{X}\diamond\mathfrak{X}'$ when $\mathfrak{X}_2=\lfloor\overline{\mathfrak{X}}_2\rfloor$ and $\mathfrak{X}'_1=\lfloor\overline{\mathfrak{X}}'_1\rfloor$ with $\overline{\mathfrak{X}}_2,\overline{\mathfrak{X}}'_1\in\mathfrak{M}(X)$. For this we use induction on $m:=d(\mathfrak{X}_2)+d(\mathfrak{X}'_1)$. Then $m\ge 2$. If $m=2$, then $d(\mathfrak{X}_2)=d(\mathfrak{X}'_1)=1$. So $\mathfrak{X}_2=\lfloor\overline{\mathfrak{X}}_2\rfloor$ and $\mathfrak{X}'_1=\lfloor\overline{\mathfrak{X}}'_1\rfloor$ with $\overline{\mathfrak{X}}_2,\overline{\mathfrak{X}}'_1\in X\cup\{\mathbf{1}\}$. We define

$$\begin{aligned}\mathfrak{X}\diamond\mathfrak{X}' &= \mathfrak{X}_1\lfloor\overline{\mathfrak{X}}_2\rfloor\diamond\lfloor\overline{\mathfrak{X}}'_1\rfloor\mathfrak{X}'_2\\ &= \mathfrak{X}_1\big\lfloor\lfloor\overline{\mathfrak{X}}_2\rfloor\diamond\overline{\mathfrak{X}}'_1+\overline{\mathfrak{X}}_2\diamond\lfloor\overline{\mathfrak{X}}'_1\rfloor+\lambda\overline{\mathfrak{X}}_2\diamond\overline{\mathfrak{X}}'_1\big\rfloor\mathfrak{X}'_2\\ &= \mathfrak{X}_1\big\lfloor\lfloor\overline{\mathfrak{X}}_2\rfloor\overline{\mathfrak{X}}'_1+\overline{\mathfrak{X}}_2\lfloor\overline{\mathfrak{X}}'_1\rfloor+\lambda\overline{\mathfrak{X}}_2\cdot\overline{\mathfrak{X}}'_1\big\rfloor\mathfrak{X}'_2.\end{aligned}\tag{13}$$

The right hand side of the equation is a well-defined element in $\text{Ш}^{\rm NC}(B)$.

Assume the product is defined for $m := d(\mathfrak{X}_2) + d(\mathfrak{X}'_1) \geq k \geq 2$ and let $\mathfrak{X}_2, \mathfrak{X}'_1 \in X$ and $m = k+1$. Then $\mathfrak{X}_2 = \lfloor \overline{\mathfrak{X}}_2 \rfloor$ and $\mathfrak{X}'_1 = \lfloor \overline{\mathfrak{X}}'_1 \rfloor$ with $\overline{\mathfrak{X}}_2, \overline{\mathfrak{X}}'_1 \in \mathfrak{M}(X)$. We recursively define below.

$$\begin{aligned}\mathfrak{X} \diamond \mathfrak{X}' &= \mathfrak{X}_1 \lfloor \overline{\mathfrak{X}}_2 \rfloor \diamond \lfloor \overline{\mathfrak{X}}'_1 \rfloor \mathfrak{X}'_2 \\ &= \mathfrak{X}_1 \big\lfloor \big(\lfloor \overline{\mathfrak{X}}_2 \rfloor \diamond \overline{\mathfrak{X}}'_1 + \overline{\mathfrak{X}}_2 \diamond \lfloor \overline{\mathfrak{X}}'_1 \rfloor + \lambda \overline{\mathfrak{X}}_2 \diamond \overline{\mathfrak{X}}'_1 \big) \big\rfloor \mathfrak{X}'_2\end{aligned} \tag{14}$$

in which the three products on the right hand side are defined by the induction hypothesis since we have

$$\begin{aligned} d(\mathfrak{X}_2) + d(\overline{\mathfrak{X}}'_1) &= d(\mathfrak{X}_2) + d(\mathfrak{X}'_1) - 1, \\ d(\overline{\mathfrak{X}}_2) + d(\mathfrak{X}'_1) &= d(\mathfrak{X}_2) - 1 + d(\mathfrak{X}'_1), \\ d(\overline{\mathfrak{X}}_2) + d(\overline{\mathfrak{X}}'_1) &= d(\mathfrak{X}_2) - 1 + d(\mathfrak{X}'_1) - 1. \end{aligned}$$

which are all less then or equal to k.

We record the following simple property of $\diamond$ for later applications.

Lemma 2.4. *Let $\mathfrak{X}, \mathfrak{X}' \in \mathfrak{M}(X)$.*

(1) *If $t(\mathfrak{X}) \neq h(\mathfrak{X}')$, then $\mathfrak{X} \diamond \mathfrak{X}' = \mathfrak{X}\mathfrak{X}'$ (concatenation).*

(2) *If $t(\mathfrak{X}) \neq h(\mathfrak{X}')$, then for any $\mathfrak{X}'' \in \mathfrak{M}(X)$,*

$$(\mathfrak{X}\mathfrak{X}') \diamond \mathfrak{X}'' = \mathfrak{X}(\mathfrak{X}' \diamond \mathfrak{X}''), \quad \mathfrak{X}'' \diamond (\mathfrak{X}\mathfrak{X}') = (\mathfrak{X}'' \diamond \mathfrak{X})\mathfrak{X}'.$$

(3) *Let $\mathfrak{X} = \mathfrak{X}_1\mathfrak{X}_2$ and $\mathfrak{X}' = \mathfrak{X}'_1\mathfrak{X}'_2$ with $\mathfrak{X}_1, \mathfrak{X}'_2 \in \mathfrak{M}(X)$ and $\mathfrak{X}_2, \mathfrak{X}'_1 \in X$ or $\lfloor \mathfrak{M}(X) \rfloor$. Then $\mathfrak{X} \diamond \mathfrak{X}' = \mathfrak{X}_1(\mathfrak{X}_2 \diamond \mathfrak{X}'_1)\mathfrak{X}'_2$.*

Extending $\diamond$ bilinearly, we obtain a binary operation

$$Ш^{NC}(B) \otimes Ш^{NC}(B) \to Ш^{NC}(B).$$

For $\mathfrak{X} \in \mathfrak{M}(X)$, define

$$R_B(\mathfrak{X}) = \lfloor \mathfrak{X} \rfloor. \tag{15}$$

Obviously $\lfloor \mathfrak{X} \rfloor$ is again in $\mathfrak{M}(X)$. Thus R_B extends to a linear operator R_B on $Ш^{NC}(B)$. Let

$$j_X : X \to \mathfrak{M}(X) \to Ш^{NC}(B)$$

be the natural injection which extends to an injection

$$j_B : B \to Ш^{NC}(B). \tag{16}$$

Theorem 2.5. *Assume that B satisfies Condition 2.1.*

(1) *The pair $(Ш^{NC}(B), \diamond)$ is an associative algebra.*

(2) *The triple $(Ш^{NC}(B), \diamond, R_B)$ is a Rota-Baxter algebra of weight λ.*

(3) *The quadruple $(Ш^{NC}(B), \diamond, R_B, j_B)$ is the free Rota-Baxter algebra on the algebra B of weight λ. More precisely, for any $A \in \mathbf{RB}_\lambda$ and $\mathbf{k}$-algebra homomorphism $f : B \to A$, there is a unique Rota-Baxter $\mathbf{k}$-algebra homomorphism $\bar{f} : Ш^{NC}(B) \to A$ such that $f = \bar{f} \circ j_B$.*

2.3. **The proof.**

Proof. (1). We just need to verify the associativity. For this we only need to verify

$$(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' = \mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''') \tag{17}$$

for $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in \mathfrak{M}(X)$. If at least one of them is $\mathbf{1}$, then the equation follows from Eq.(12). So we only need to verify for $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in X_\infty$.

If $t(\mathfrak{X}') \neq h(\mathfrak{X}'')$, then by Lemma 2.4,

$$(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' = (\mathfrak{X}'\mathfrak{X}'') \diamond \mathfrak{X}''' = \mathfrak{X}'(\mathfrak{X}'' \diamond \mathfrak{X}''') = \mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''').$$

Similarly if $t(\mathfrak{X}'') \neq h(\mathfrak{X}''')$.

Thus we only need to verify the associativity when $a := t(\mathfrak{X}') = h(\mathfrak{X}'')$ and $b := t(\mathfrak{X}'') = h(\mathfrak{X}''')$. We first dispose of another simple case.

Lemma 2.6. *If $\ell(\mathfrak{X}'') \geq 2$, then for all $\mathfrak{X}', \mathfrak{X}'' \in X_\infty$, we have*

$$(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' = \mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''').$$

Proof. If $\ell(\mathfrak{X}'') \geq 2$, then $\mathfrak{X}'' = \mathfrak{X}''_1\mathfrak{X}''_2$ with $\mathfrak{X}''_1, \mathfrak{X}''_2 \in X_\infty$ and $t(\mathfrak{X}''_1) \neq h(\mathfrak{X}''_2)$. So using Lemma 2.4 repeatedly, we have

$$\begin{aligned}
(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' &= (\mathfrak{X}' \diamond (\mathfrak{X}''_1\mathfrak{X}''_2)) \diamond \mathfrak{X}''' \\
&= ((\mathfrak{X}' \diamond \mathfrak{X}''_1)\mathfrak{X}''_2) \diamond \mathfrak{X}''' \\
&= (\mathfrak{X}' \diamond \mathfrak{X}''_1)(\mathfrak{X}''_2 \diamond \mathfrak{X}''') \\
&= \mathfrak{X}' \diamond (\mathfrak{X}''_1(\mathfrak{X}''_2 \diamond \mathfrak{X}''')) \\
&= \mathfrak{X}' \diamond ((\mathfrak{X}''_1\mathfrak{X}''_2) \diamond \mathfrak{X}''') = \mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''').
\end{aligned}$$

□

Thus we can assume $\ell(\mathfrak{X}'') = 1$ without loss of generality. Further, if $\ell(\mathfrak{X}') \geq 2$, then $\mathfrak{X}' = \mathfrak{X}'_1\mathfrak{X}'_2$ with $\mathfrak{X}'_1 \in X_\infty$ and $\ell(\mathfrak{X}'_2) = 1$. Then by Lemma 2.4,

$$\begin{aligned}
(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' &= ((\mathfrak{X}'_1\mathfrak{X}'_2) \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' \\
&= (\mathfrak{X}'_1(\mathfrak{X}'_2 \diamond \mathfrak{X}'')) \diamond \mathfrak{X}''' = \mathfrak{X}'_1((\mathfrak{X}'_2 \diamond \mathfrak{X}'') \diamond \mathfrak{X}''').
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''') &= (\mathfrak{X}'_1\mathfrak{X}'_2) \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''') \\
&= \mathfrak{X}'_1(\mathfrak{X}'_2 \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''')).
\end{aligned}$$

Thus

$$(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' = \mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''')$$

whenever

$$(\mathfrak{X}'_2 \diamond \mathfrak{X}'') \diamond \mathfrak{X}''' = \mathfrak{X}'_2 \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''').$$

Therefore we can assume $\ell(\mathfrak{X}') = 1$. Similarly, we can assume $\ell(\mathfrak{X}''') = 1$.

To summarize, we have reduced to the special case when $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in X_\infty$ are of length one and $a := t(\mathfrak{X}') = h(\mathfrak{X}'')$ and $b := t(\mathfrak{X}'') = h(\mathfrak{X}''')$. Since $\ell(\mathfrak{X}'') = 1$, we have $h(\mathfrak{X}'') = t(\mathfrak{X}'')$. Therefore, either all the three elements are in X or they are all in $\lfloor \mathfrak{M}(X) \rfloor$.

Case 1. All of $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}'''$ are in X. Then the associativity follows from the associativity in B.

Case 2. All of $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}'''$ are in $\lfloor \mathfrak{M}(X) \rfloor$. For this, we use induction on $m := d(\mathfrak{X}') + d(\mathfrak{X}'') + d(\mathfrak{X}''')$. Then $m \geq 3$. We first check the case when $m = 3$. Then $\mathfrak{X}', \mathfrak{X}''$ and $\mathfrak{X}'''$ all have depth one. Then $\mathfrak{X}' = \lfloor \overline{\mathfrak{X}}' \rfloor, \mathfrak{X}'' = \lfloor \overline{\mathfrak{X}}'' \rfloor, \mathfrak{X}''' = \lfloor \overline{\mathfrak{X}}''' \rfloor$ where $\overline{\mathfrak{X}}', \overline{\mathfrak{X}}'', \overline{\mathfrak{X}}''' \in X$. We have

$$\begin{aligned}
(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}'' &= \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'' + \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor + \lambda \overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'' \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor \ \text{(by Eq. (13))} \\
&= \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'' \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'' \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor \\
&= \lfloor \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor (\lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'') \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor (\lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'') \diamond \overline{\mathfrak{X}}''' \rfloor \\
&\quad + \lfloor \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor \rfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor (\overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor) \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor (\overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor) \diamond \overline{\mathfrak{X}}''' \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor + \lambda \lfloor (\overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'') \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda^2 \lfloor (\overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'') \cdot \overline{\mathfrak{X}}''' \rfloor \text{(by Eq. (13))}.
\end{aligned}$$

Applying Lemma 2.4 to the fifth term, we get

$$(\overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor) \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor = \overline{\mathfrak{X}}' (\lfloor \overline{\mathfrak{X}}'' \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor).$$

Then using Eq. (13) again we get

$$\begin{aligned}
(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}'' &= \lfloor \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor (\lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'') \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor (\lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'') \diamond \overline{\mathfrak{X}}''' \rfloor \\
&\quad + \lfloor \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor \rfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor \overline{\mathfrak{X}}' \lfloor \lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor \rfloor \\
&\quad + \lambda \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor (\overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor) \overline{\mathfrak{X}}''' \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor + \lambda \lfloor (\overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'') \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda^2 \lfloor (\overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'') \cdot \overline{\mathfrak{X}}''' \rfloor.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\mathfrak{X}' \diamond (\mathfrak{X}'' \diamond \mathfrak{X}''') &= \lfloor \overline{\mathfrak{X}}' \rfloor \diamond (\lfloor \overline{\mathfrak{X}}'' \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor) \\
&= \lfloor \overline{\mathfrak{X}}' \rfloor \diamond \Big(\lfloor \lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor \overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor \overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''' \rfloor \Big) \\
&= \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \diamond (\lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''') \rfloor + \lfloor \overline{\mathfrak{X}}' \lfloor \lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' \diamond (\lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''') \rfloor \\
&\quad + \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \diamond (\overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor) \rfloor + \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' \diamond (\overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor) \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \diamond (\overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''') \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda^2 \lfloor \overline{\mathfrak{X}}' \diamond (\overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''') \rfloor \\
&= \lfloor \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor + \lfloor \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \rfloor \rfloor \overline{\mathfrak{X}}''' \rfloor + \lambda \lfloor \lfloor \overline{\mathfrak{X}}' \cdot \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor \\
&\quad + \lfloor \overline{\mathfrak{X}}' \lfloor \lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' (\lfloor \overline{\mathfrak{X}}'' \rfloor \overline{\mathfrak{X}}''') \rfloor \\
&\quad + \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor (\overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor) \rfloor + \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor \rfloor \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' \diamond (\overline{\mathfrak{X}}'' \lfloor \overline{\mathfrak{X}}''' \rfloor) \rfloor \\
&\quad + \lambda \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \diamond (\overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''') \rfloor + \lambda \lfloor \overline{\mathfrak{X}}' \lfloor \overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''' \rfloor \rfloor + \lambda^2 \lfloor \overline{\mathfrak{X}}' \cdot (\overline{\mathfrak{X}}'' \cdot \overline{\mathfrak{X}}''') \rfloor.
\end{aligned}$$

Then by the definition of $\diamond$ and Lemma 2.4, the i-th term on the left hand side matches with the $\sigma(i)$-th term on the right hand side. Here the permutation $\sigma \in \Sigma_{11}$ is

$$\left(\begin{array}{c} i \\ \sigma(i) \end{array} \right) = \left(\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 6 & 9 & 2 & 4 & 7 & 10 & 5 & 3 & 8 & 11 \end{array} \right). \tag{18}$$

Assume Eq. (17) holds when $3 \leq d(\mathfrak{X}') + d(\mathfrak{X}'') + d(\mathfrak{X}''') \leq k$ and take $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in \mathfrak{M}(X)$ with $d(\mathfrak{X}') + d(\mathfrak{X}'') + d(\mathfrak{X}''') = k + 1$. Then $\mathfrak{X}' = \lfloor \overline{\mathfrak{X}}' \rfloor, \mathfrak{X}'' = \lfloor \overline{\mathfrak{X}}'' \rfloor, \mathfrak{X}''' = \lfloor \overline{\mathfrak{X}}''' \rfloor$ with $\overline{\mathfrak{X}}', \overline{\mathfrak{X}}'', \overline{\mathfrak{X}}''' \in \mathfrak{M}(X)$. We have

$$(\mathfrak{X}' \diamond \mathfrak{X}'') \diamond \mathfrak{X}'' = \lfloor \lfloor \overline{\mathfrak{X}}' \rfloor \diamond \overline{\mathfrak{X}}'' + \overline{\mathfrak{X}}' \diamond \lfloor \overline{\mathfrak{X}}'' \rfloor + \lambda \overline{\mathfrak{X}}' \diamond \overline{\mathfrak{X}}'' \rfloor \diamond \lfloor \overline{\mathfrak{X}}''' \rfloor$$

$$\begin{aligned}
&= \lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}''\rfloor\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor\rfloor\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}''\rfloor\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\\
&= \lfloor\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lfloor(\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}'')\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor(\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}'')\diamond\overline{\mathfrak{X}}'''\rfloor\\
&\quad+\lfloor\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lfloor(\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor)\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor(\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor)\diamond\overline{\mathfrak{X}}'''\rfloor\\
&\quad+\lambda\lfloor\lfloor\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lambda\lfloor(\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}'')\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda^2\lfloor(\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}'')\diamond\overline{\mathfrak{X}}'''\rfloor.
\end{aligned}$$

Applying the induction hypothesis to the fifth term $(\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor)\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor$ and then use Eq. (13), we have

$$\begin{aligned}
(\mathfrak{X}'\diamond\mathfrak{X}'')\diamond\mathfrak{X}'' &= \lfloor\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lfloor(\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}'')\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor(\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}'')\diamond\overline{\mathfrak{X}}'''\rfloor\\
&\quad+\lfloor\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor\rfloor+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor\rfloor\\
&\quad+\lambda\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor(\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor)\diamond\overline{\mathfrak{X}}'''\rfloor\\
&\quad+\lambda\lfloor\lfloor\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lambda\lfloor(\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}'')\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda^2\lfloor(\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}'')\diamond\overline{\mathfrak{X}}'''\rfloor.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\mathfrak{X}'\diamond(\mathfrak{X}''\diamond\mathfrak{X}''') &= \lfloor\overline{\mathfrak{X}}'\rfloor\diamond\Big(\lfloor\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lfloor\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}'''\rfloor\Big)\\
&= \lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond(\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}''')\rfloor+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond(\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}''')\rfloor\\
&\quad+\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond(\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor)\rfloor+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond(\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor)\rfloor\\
&\quad+\lambda\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond(\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}''')\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda^2\lfloor\overline{\mathfrak{X}}'\diamond(\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}''')\rfloor\Big)\\
&= \lfloor\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lfloor\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\rfloor\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor+\lambda\lfloor\lfloor\overline{\mathfrak{X}}'\diamond\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor\\
&\quad+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond(\lfloor\overline{\mathfrak{X}}''\rfloor\diamond\overline{\mathfrak{X}}''')\rfloor\\
&\quad+\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond(\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor)\rfloor+\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor\rfloor\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond(\overline{\mathfrak{X}}''\diamond\lfloor\overline{\mathfrak{X}}'''\rfloor)\rfloor\\
&\quad+\lambda\lfloor\lfloor\overline{\mathfrak{X}}'\rfloor\diamond(\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}''')\rfloor+\lambda\lfloor\overline{\mathfrak{X}}'\diamond\lfloor\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}'''\rfloor\rfloor+\lambda^2\lfloor\overline{\mathfrak{X}}'\diamond(\overline{\mathfrak{X}}''\diamond\overline{\mathfrak{X}}''')\rfloor.
\end{aligned}$$

Now by induction, the i-th term on the left hand side of $(\mathfrak{X}'\diamond\mathfrak{X}'')\diamond\mathfrak{X}'''=\mathfrak{X}'\diamond(\mathfrak{X}''\diamond\mathfrak{X}''')$ matches with the $\sigma(i)$-th term on the right hand side. Here σ is given by Eq. (18). This completes the proof of the first part of Theorem 2.5.

(2). The proof is immediate from Eq. (13).

(3). Let (A, R) be a Rota-Baxter algebra of weight λ and let $*$ be the product in A. Let $f : B \to A$ be a $\mathbf{k}$-algebra morphism. We will construct a $\mathbf{k}$-linear map $\bar{f} : \text{Ш}^{\mathrm{NC}}(B) \to A$ by defining $\bar{f}(\mathfrak{X})$ for $\mathfrak{X} \in \mathfrak{M}(X)$. We achieve this by using induction on n for $\mathfrak{X} \in X_n$. For $\mathfrak{X} \in X_1 := X$, define $\bar{f}(\mathfrak{X}) = f(\mathfrak{X})$. Suppose $\bar{f}(\mathfrak{X})$ has been defined for $\mathfrak{X} \in X_n$ and consider $\mathfrak{X} \in X_{n+1}$ which is, by definition and Eq. (11),

$$\begin{aligned}
A(X,\tilde{X}_n) &= \Big(\dot{\bigcup}_{r\geq 1}(X\lfloor\tilde{X}_n\rfloor)^r\Big)\,\dot{\bigcup}\,\Big(\dot{\bigcup}_{r\geq 0}(X\lfloor\tilde{X}_n\rfloor)^r X\Big)\\
&\quad\dot{\bigcup}\,\Big(\dot{\bigcup}_{r\geq 0}\lfloor\tilde{X}_n\rfloor(X\lfloor\tilde{X}_n\rfloor)^r\Big)\,\dot{\bigcup}\,\Big(\dot{\bigcup}_{r\geq 0}\lfloor\tilde{X}_{n-1}\rfloor(X\lfloor\tilde{X}_n\rfloor)^r X\Big).
\end{aligned}$$

Let $\mathfrak{X} \in \dot{\bigcup}_{r\geq 1}(X\lfloor\tilde{X}_n\rfloor)^r$. Then

$$\mathfrak{X} = \prod_{i=1}^{r}(\mathfrak{X}_{2i-1}\lfloor\mathfrak{X}_{2i}\rfloor)$$

for $\mathfrak{X}_{2i-1} \in X$ and $\mathfrak{X}_{2i} \in \tilde{X}_n$, $1 \leq i \leq r$. By the construction of the multiplication $\diamond$ and the Rota-Baxter operator R_B, we have

$$\mathfrak{X} = \diamond_{i=1}^{r} (\mathfrak{X}_{2i-1} \diamond R_B(\mathfrak{X}_{2i})).$$

Thus there is only one possible way to define $\bar{f}(\mathfrak{X})$ in order for $\bar{f}$ to be a Rota-Baxter homomorphism:

$$\bar{f}(\mathfrak{X}) = *_{i=1}^{r} (\bar{f}(\mathfrak{X}_{2i-1}) * R\big(\bar{f}(\mathfrak{X}_{2i})\big)\big). \tag{19}$$

$\bar{f}(\mathfrak{X})$ can be similarly defined if $\mathfrak{X}$ is in the other unions. This proves the existence of $\bar{f}$ as a map and its uniqueness.

We next prove that the map $\bar{f}$ defined in Eq. (19) is indeed a Rota-Baxter algebra homomorphism. First of all, for any $\mathfrak{X} \in \mathfrak{M}(X)$, we have $R_B(\mathfrak{X}) = \lfloor \mathfrak{X} \rfloor \in \mathfrak{M}(X)$, and by definition (Eq. (19)) we have

$$\bar{f}(\lfloor \mathfrak{X} \rfloor) = R(\bar{f}(\mathfrak{X})). \tag{20}$$

So $\bar{f}$ commutes with the Rota-Baxter operators.

To prove that $\bar{f}$ is an algebra homomorphism, we only need to check that

$$\bar{f}(\mathfrak{X} \diamond \mathfrak{X}') = \bar{f}(\mathfrak{X}) * \bar{f}(\mathfrak{X}'), \forall\, \mathfrak{X}, \mathfrak{X}' \in \mathfrak{M}(X). \tag{21}$$

First let $\mathfrak{X}$ and $\mathfrak{X}'$ be of length one, and apply the induction on $m := d(\mathfrak{X}) + d(\mathfrak{X}')$. If $m = 0$, then $\mathfrak{X}, \mathfrak{X}' \in X$. So $\mathfrak{X} \diamond \mathfrak{X}'$ is the product $\mathfrak{X} \cdot \mathfrak{X}'$ in B and

$$\bar{f}(\mathfrak{X} \diamond \mathfrak{X}') = \bar{f}(\mathfrak{X} \cdot \mathfrak{X}') = f(\mathfrak{X} \cdot \mathfrak{X}') = f(\mathfrak{X}) * f(\mathfrak{X}') = \bar{f}(\mathfrak{X}) * \bar{f}(\mathfrak{X}').$$

Suppose it holds for $m \geq k \geq 0$ and consider $\mathfrak{X}, \mathfrak{X}' \in \mathfrak{M}(X)$ with $d(\mathfrak{X}) + d(\mathfrak{X}') = m+1$. If $d(\mathfrak{X}) = 0$, then $d(\mathfrak{X}') > 0$. Note that we assume that $\mathfrak{X}'$ has length one. So $\mathfrak{X}' = \lfloor \overline{\mathfrak{X}}' \rfloor$ with $\overline{\mathfrak{X}}' \in \mathfrak{M}(X)$. Then $\mathfrak{X} \diamond \mathfrak{X}'$ is the concatenation, and by the definition of $\bar{f}$,

$$\bar{f}(\mathfrak{X} \diamond \mathfrak{X}') = \bar{f}(\mathfrak{X}\mathfrak{X}') = \bar{f}(\mathfrak{X}) * \bar{f}(\mathfrak{X}').$$

The same arguments hold if $d(\mathfrak{X}') = 0$.

If $d(\mathfrak{X}) > 0$ and $d(\mathfrak{X}') > 0$. Then $\mathfrak{X} = \lfloor \overline{\mathfrak{X}} \rfloor$ and $\mathfrak{X}' = \lfloor \overline{\mathfrak{X}}' \rfloor$. Then

$$\begin{aligned}
\bar{f}(\mathfrak{X} \diamond \mathfrak{X}') &= \bar{f}(\lfloor \overline{\mathfrak{X}} \rfloor \diamond \lfloor \overline{\mathfrak{X}}' \rfloor) \\
&= \bar{f}(\lfloor \mathfrak{X} \diamond \overline{\mathfrak{X}}' \rfloor + \lfloor \overline{\mathfrak{X}} \diamond \mathfrak{X}' \rfloor + \lambda \lfloor \overline{\mathfrak{X}} \diamond \overline{\mathfrak{X}}' \rfloor) \quad \text{(by Eq. (13))} \\
&= R\big(\bar{f}\big(\mathfrak{X} \diamond \overline{\mathfrak{X}}' + \overline{\mathfrak{X}} \diamond \mathfrak{X}' + \lambda \overline{\mathfrak{X}} \diamond \overline{\mathfrak{X}}'\big)\big) \quad \text{(by Eq. (20))} \\
&= R(\bar{f}(\mathfrak{X}) * \bar{f}(\overline{\mathfrak{X}}')) + R(\bar{f}(\overline{\mathfrak{X}}) * \bar{f}(\mathfrak{X}')) + \lambda R(\bar{f}(\overline{\mathfrak{X}}) * \bar{f}(\overline{\mathfrak{X}}')) \\
&\qquad \text{(by induction hypothesis)} \\
&= R(\bar{f}(\overline{\mathfrak{X}})) * R(\bar{f}(\overline{\mathfrak{X}}')) \quad \text{(by Eq. (1))} \\
&= \bar{f}(R_B(\overline{\mathfrak{X}})) * \bar{f}(R_B(\overline{\mathfrak{X}}')) \quad \text{(by Eq. (20))} \\
&= \bar{f}(\mathfrak{X}) * \bar{f}(\mathfrak{X}').
\end{aligned}$$

This completes the proof when $m := \ell(\mathfrak{X}) + \ell(\mathfrak{X}') = 2$. Now assume Eq. (21) holds for all $\mathfrak{X}, \mathfrak{X}'$ with $\ell(\mathfrak{X}) + \ell(\mathfrak{X}') \geq j \geq 2$. Let $\ell(\mathfrak{X}) + \ell(\mathfrak{X}') = j+1$. If $t(\mathfrak{X}) \neq h(\mathfrak{X}')$ then $\mathfrak{X} \diamond \mathfrak{X}'$ is the concatenation. So by Eq.(19), we have

$$\bar{f}(\mathfrak{X} \diamond \mathfrak{X}') = \bar{f}(\mathfrak{X}\mathfrak{X}') = \bar{f}(\mathfrak{X}) * \bar{f}(\mathfrak{X}').$$

Now let $t(\mathfrak{X}) = h(\mathfrak{X}')$. Since $j \geq 2$, we have $j+1 \geq 3$, so at least one of $\mathfrak{X}$ and $\mathfrak{X}'$ have length greater or equal 2. Let $\ell(\mathfrak{X}) \geq 2$. Write $\mathfrak{X} = \mathfrak{X}_1\mathfrak{X}_2$ with $\mathfrak{X}_1 \in \mathfrak{M}(X)$ and $\ell(\mathfrak{X}_2) = 1$. Since $t(\mathfrak{X}) = t(\mathfrak{X}_2) = h(\mathfrak{X}')$, the product $\mathfrak{X}_2 \diamond \mathfrak{X}'$ has length less than $\ell(\mathfrak{X}_2)+\ell(\mathfrak{X}')$. Then using the associativity of the product and the induction hypothesis, we have

$$\begin{aligned}
\bar{f}(\mathfrak{X} \diamond \mathfrak{X}') &= \bar{f}((\mathfrak{X}_1\mathfrak{X}_2) \diamond \mathfrak{X}') \\
&= \bar{f}(\mathfrak{X}_1(\mathfrak{X}_2 \diamond \mathfrak{X}')) \quad \text{(Lemma 2.4)} \\
&= \bar{f}(\mathfrak{X}_1 \diamond (\mathfrak{X}_2 \diamond \mathfrak{X}')) \quad \text{(Lemma 2.4)} \\
&= \bar{f}(\mathfrak{X}_1) * \bar{f}(\mathfrak{X}_2 \diamond \mathfrak{X}') \quad \text{(induction hypothesis)} \\
&= \bar{f}(\mathfrak{X}_1) * (\bar{f}(\mathfrak{X}_2) * \bar{f}(\mathfrak{X}')) \quad \text{(induction hypothesis)} \\
&= (\bar{f}(\mathfrak{X}_1) * \bar{f}(\mathfrak{X}_2)) * \bar{f}(\mathfrak{X}') \quad \text{(associativity of } * \text{)} \\
&= (\bar{f}(\mathfrak{X}_1 \diamond \mathfrak{X}_2)) * \bar{f}(\mathfrak{X}') \quad \text{(induction hypothesis)} \\
&= \bar{f}(\mathfrak{X}) * \bar{f}(\mathfrak{X}').
\end{aligned}$$

This completes the induction on $\ell(\mathfrak{X}) + \ell(\mathfrak{X}')$ and thus the proof of part (3). □

2.4. **Free Rota-Baxter algebra over a general algebra.** We now construct free Rota-Baxter algebra over a $\mathbf{k}$-algebra B without the restriction on B in Condition 2.1. By definition, it is a Rota-Baxter $\mathbf{k}$-algebra $F(B)$ with a $\mathbf{k}$-algebra map $j_B : B \to F(B)$ with the property that, for any Rota-Baxter $\mathbf{k}$-algebra (A, R) and $\mathbf{k}$-algebra map $f : B \to A$, there is a unique morphism $\bar{f} : F(B) \to A$ of Rota-Baxter $\mathbf{k}$-algebras such that $f = \bar{f} \circ j_B$.

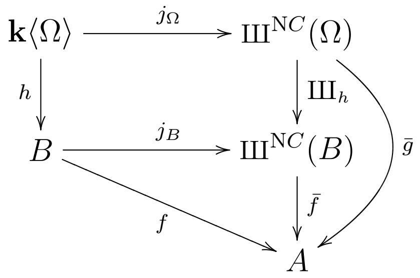
Let Ω be a generating set of B as $\mathbf{k}$-algebras. Then there is an algebra homomorphism $h : \mathbf{k}\langle\Omega\rangle \to B$ restricting to the identity map on Ω. Here $\mathbf{k}\langle\Omega\rangle$ is the free non-commutative algebra generated by Ω. Since $\mathbf{k}\langle\Omega\rangle$ satisfies Condition 2.1, Theorem 2.5 applies. we denote $Ш^{\mathrm{NC}}(\Omega) = Ш^{\mathrm{NC}}(\mathbf{k}\langle\Omega\rangle)$ and $j_\Omega = j_{\mathbf{k}\langle\Omega\rangle}$. Let I be the kernel of h, (I) be the Rota-Baxter algebra ideal of $Ш^{\mathrm{NC}}(\Omega)$ generated by I, $Ш^{\mathrm{NC}}(B)$ be the quotient Rota-Baxter $\mathbf{k}$-algebra $Ш^{\mathrm{NC}}(\Omega)/(I)$ and $Ш_h$ be the quotient map $Ш^{\mathrm{NC}}(\Omega) \to Ш^{\mathrm{NC}}(B)$. Since $\ker h \subseteq \ker(Ш_h \circ j_\Omega)$, there is a natural map $j_B : B \to Ш^{\mathrm{NC}}(B)$, making the following diagram commutative.

$$\begin{array}{ccc}
\mathbf{k}\langle\Omega\rangle & \xrightarrow{\ j_\Omega\ } & Ш^{\mathrm{NC}}(\Omega) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Ш_h} \\
B & \xrightarrow{\ j_B\ } & Ш^{\mathrm{NC}}(B)
\end{array}$$

Then $Ш_h$ is also the unique Rota-Baxter algebra morphism induced by the algebra morphism $j_B \circ h$.

Proposition 2.7. *The Rota-Baxter algebra $Ш^{\mathrm{NC}}(B)$, together with the natural map $j_B : B \to Ш^{\mathrm{NC}}(B)$, is the free Rota-Baxter algebra on B.*

Proof. Let A be a Rota-Baxter algebra and let $f : B \to A$ be a $\mathbf{k}$-algebra homomorphism. Consider the composite map $g = f \circ h : \mathbf{k}\langle\Omega\rangle \to B \to A$. By the freeness of $Ш^{\mathrm{NC}}(\Omega)$, there is a homomorphism $\bar{g} : Ш^{\mathrm{NC}}(\Omega) \to A$ of Rota-Baxter algebras such that $g = \bar{g} \circ j_\Omega$. Since $\ker(g) \supseteq \ker(h)$, the Rota-Baxter ideal I_g of $Ш^{\mathrm{NC}}(\Omega)$ generated by $\ker(g)$ contains the Rota-Baxter ideal I_h generated by $\ker(h)$. So $\ker(\bar{g})$, being a Rota-Baxter ideal containing $\ker(g)$, contains I_g and hence contains I_h which is $\ker(Ш_h)$ by definition. Thus $\ker(\bar{g}) \supseteq \ker(\text{Ш}_h)$. So there is an induced map $\bar{f} : \text{Ш}^{\mathrm{N}C}(B) \to A$ such that in the diagram



the outer trapezoid is commutative. Further, since Ш_h is a surjective Rota-Baxter algebra homomorphism, we have $\ker \bar{f} = \text{Ш}_h(\ker \bar{g})$ which is a Rota-Baxter ideal. Thus $\bar{f}$ is a Rota-Baxter homomorphism. The uniqueness of $\bar{f}$ follows from the uniqueness of $\bar{g}$ and the surjectivity of Ш_h. $\square$

Remark 2.1. *Assuming B to be a commutative $\mathbf{k}$-algebra, we obtain the free commutative Rota-Baxter algebra of weight λ on the algebra B, with Rota-Baxter map R_B, $(\text{Ш}(B), \diamond, R_B)$,* [**G-K1**] *by symmetrization $(\text{Ш}^{\mathrm{N}C}(B), \diamond) \xrightarrow{sym} (\text{Ш}(B), \diamond)$.*

Remark 2.2. *Following* [**EF2**] *the above construction of the non-commutative Rota-Baxter algebra can be used to construct the free non-commutative associative Nijenhuis algebra. The same is true for Leroux's TD operator relation* [**Le2**]*. Both give interesting new generalized shuffle relations.*

2.5. **Free nonunitary Rota-Baxter algebras.** We will find it more convenient to relate nonunitary Rota-Baxter algebras to dendriform dialgebras and trialgebras in the next two sections. This situation is similar to relating associative algebras with Lie algebras. While a non-unitary associative algebra $A \in \mathbf{Alg}^0$ and its unitarization $\tilde{A}$ are simply connected by $\tilde{A} = \mathbf{k1} \oplus A$ where the product is defined by $(m, a) \cdot (n, b) = (m+n, mb+na+a\cdot b)$; the unitarization of a non-unitary Rota-Baxter algebra $A \in \mathbf{RB}^0$ is more involved and will be treated in detail here. For this and later applications, we construct free non-unital (non-commutative) Rota-Baxter algebras. For commutative algebras, this has been done in [**G-K2**].

Let B be a non-unital $\mathbf{k}$-algebra with a $\mathbf{k}$-basis X. Let $X^0 = X \cup \{\lfloor, \rfloor\}$ and let $M(X^0)$ be the free semigroup generated by X^0. A **nonunitary Rota-Baxter word (RBW)** is defined to be a word in $M(X^0)$ that satisfies conditions (1)-(4) in Definition 2.2. Let $X' = X \cup \{\mathbf{1}, \lfloor, \rfloor\}$ and identify $M(X^0)$ as a sub-semigroup of $M(X')$. Then a non-unital RBW can also be defined as a word in $M(X')$ satisfying (1)-(4) in Definition 2.2 with condition (5) replaced by

(5') $\mathbf{1}$ does not occur in the word.

Let $\mathfrak{M}^0(X)$ be the set of nonunitary RBWs of X. As in Eq. (6), define $R_0 = X$ and, for $n \geq 0$, define

$$R_{n+1} = A(X, R_n),\ R_\infty = \bigcup_{n\geq 0} R_n = \varinjlim R_n.$$

Where the last equation follows since $R_1 = A(X, R_0) \supseteq R_0$ and inductively,

$$R_{n+1} = A(X, R_n) \supseteq A(X, R_{n-1}) = R_n,\ n \geq 0.$$

We then have

Lemma 2.8. *$\mathfrak{M}^0(X) = R_\infty$. In fact, R_n consists of words in $\mathfrak{M}^0(X)$ of depth less or equal n.*

Proof. Just apply the proof of Lemma 2.3 to elements in $M(X^0)$. □

We then define $Ш^{NC,0}(B) = \bigoplus_{\mathfrak{X}\in R_\infty} \mathbf{k}\mathfrak{X}$, regarded as a submodule of $Ш^{NC}(\tilde{B})$ through the natural embedding $\phi_B : B \to \tilde{B}$ into the unitarization. Define $\diamond^0 : Ш^{NC,0}(B) \times Ш^{NC,0}(B) \to Ш^{NC}(\tilde{B})$ to be the restriction of $\diamond : Ш^{NC}(\tilde{B}) \times Ш^{NC}(\tilde{B}) \to Ш^{NC}(\tilde{B})$. Since B is closed under multiplication, from the construction of $\diamond$ we see that $\diamond^0$ has its image in $Ш^{NC,0}(B)$. Define $R_B : Ш^{NC,0}(B) \to Ш^{NC,0}(B)$ by $R_B(\mathfrak{X}) = \lfloor \mathfrak{X} \rfloor$. Let

$$j_B : B \to Ш^{NC,0}(B) \tag{22}$$

be the natural embedding $j_B(x) = x,\ x \in X$.

Theorem 2.9. (1) *The pair $(Ш^{NC,0}(B), \diamond^0)$ is a nonunitary associative algebra in $\mathbf{Alg}^0$.*

(2) *The triple $(Ш^{NC,0}(B), \diamond^0, R_B)$ is a nonunitary Rota-Baxter algebra of weight λ.*

(3) *The quadruple $(Ш^{NC,0}(B), \diamond^0, R_B, j_B)$ is the free nonunitary Rota-Baxter algebra on B of weight λ.*

Proof. (1) We only need to verify the associativity of $\diamond^0$ which follows since $\diamond^0$ is the restriction of the associative product $\diamond$.

(2) is also automatic since R_B is the restriction of the Rota-Baxter operator $R_{\tilde{B}}$.

(3) Let E be a nonunitary Rota-Baxter algebra and let $f : B \to E$ be a given morphism in $\mathbf{Alg}^0$. We just need to show that there is a unique morphism $\bar{f} : Ш^{NC,0}(B) \to E$ in $\mathbf{RB}^0$ such that $\bar{f} \circ j_B = f$. For this we adapt the proof for the unitary case, Theorem 2.5 (3). □

3. Universal enveloping algebras of dendriform trialgebras

3.1. **Background on dendriform trialgebras.** The category of dendriform trialgebras $(D, \prec, \succ, \cdot)$ is denoted by $\mathbf{DT}$. Recall that $\cdot$, as well as $\star$, is an associative product. The category $\mathbf{DD}$ of dendriform dialgebras can be identified with the subcategory of $\mathbf{DT}$ of objects with $\cdot = 0$.

We recall the following facts from the introduction. They hold regardless of the unitariness of the Rota-Baxter algebras, as can be verified directly from the definitions.

Theorem 3.1. (1) **(Aguiar [Ag2])** *A Rota-Baxter algebra of weight zero defines a dendriform dialgebra $(A, \prec_R, \succ_R)$, where*

$$x \prec_R y = xR(y),\ x \succ_R y = R(x)y. \tag{23}$$

(2) **(Ebrahimi-Fard [EF1])** *A Rota-Baxter algebra (A, R) of weight λ defines a dendriform trialgebra $(A, \prec_R, \succ_R, \cdot_R)$, where*

$$x \prec_R y = xR(y),\ x \succ_R y = R(x)y, x \cdot_R y = \lambda xy. \tag{24}$$

(3) **(Ebrahimi-Fard [EF1])** *A Rota-Baxter algebra (A, R) of weight λ defines a dendriform dialgebra $(A, \prec'_R, \succ'_R)$, where*

$$x \prec'_R y = xR(y) + \lambda xy,\ x \succ'_R y = R(x)y. \tag{25}$$

We note that (3) specialize to (1) when $\lambda = 0$. The same can be said of (2) since when $\lambda = 0$, the product $\cdot_R$ is zero and the relations of the trialgebra reduces to the relations of a dialgebra.

Thus we obtain functors

$$\mathcal{E} : \mathbf{RB}^0_\lambda \to \mathbf{DT},\ \mathcal{F} : \mathbf{RB}^0_\lambda \to \mathbf{DD}.$$

We will study their adjoint functors. The two functors $\mathcal{E}$ and $\mathcal{F}$ are related by the following simple observation.

Lemma 3.2. *Let $(D, \prec, \succ, \cdot)$ be in $\mathbf{DT}$. Then $(D, \prec', \succ')$ is in $\mathbf{DD}$. Here $\prec' = \prec + \cdot$ and $\succ' = \succ$.*

Proof. Let $\star' = \prec' + \succ$. Then we have $\star' = \star$. We have

$$\begin{aligned}
(a \prec' b) \prec' c &= (a \cdot b + a \prec b) \prec' c \\
&= (a \cdot b + a \prec b) \cdot c + (a \cdot b + a \prec b) \prec c \\
&= (a \cdot b) \cdot c + (a \prec b) \cdot c + (a \cdot b) \prec c + (a \prec b) \prec c \\
&= a \cdot (b \cdot c) + a \cdot (b \succ c) + a \cdot (b \prec c) + a \prec (b \star c) \quad \text{(by Eq. (3))} \\
&= a \prec' (b \star' c).
\end{aligned}$$

This verifies the first relation for the dendriform dialgebra. The other two relations are easy to verify.

$$(a \succ' b) \succ' c = (a \succ b) \succ c = a \succ (b \star c) = a \succ' (b \star' c).$$

$$(a \succ' b) \prec' c = (a \succ b) \cdot c + (a \succ b) \prec c = a \succ (b \cdot c) + a \succ (b \prec c) = a \succ' (b \prec' c).$$

□

Let $\mathcal{G} : \mathbf{DT} \to \mathbf{DD}$ be the functor obtained from Lemma 3.2. Then we have $\mathcal{F} = \mathcal{G} \circ \mathcal{E}$. So we have the following commutative diagram of functors where $\lambda \neq 0$.

$$\begin{array}{ccc}
\mathbf{RB}_\lambda & \xrightarrow{\ \mathcal{E}\ } & \mathbf{DT} \\
& \searrow^{\mathcal{F}} & \downarrow{\scriptstyle \mathcal{G}} \\
& & \mathbf{DD}
\end{array} \qquad (26)$$

Thus the adjoint functor of $\mathcal{F}$ is the composition of the adjoint functors of $\mathcal{E}$ and $\mathcal{G}$. We will mostly concentrate on the adjoint functor of $\mathcal{E}$. Results on $\mathcal{F}$ can be either derived in a similar form or maybe derived as the composite functor of the adjoint functors of $\mathcal{E}$ and $\mathcal{G}$.

3.2. **The existence.**

3.2.1. *The definitions.*

Definition 3.3. *Let $D \in \mathbf{DT}$ (resp. $\mathbf{DD}$) and let $\lambda \in \mathbf{k}$. A* **universal enveloping Rota-Baxter algebra** *of weight λ of D is a Rota-Baxter algebra $R(D) := R_\lambda(D) \in \mathbf{RB}^0_\lambda$ with a morphism $\rho : D \to R(D)$ in $\mathbf{DT}$ (resp. $\mathbf{DD}$) such that for any $A \in \mathbf{RB}^0_\lambda$ and morphism $f : D \to A$ in $\mathbf{DT}$ (resp. $\mathbf{DD}$), there is a unique $\check{f} : R(D) \to A$ in $\mathbf{RB}^0_\lambda$ such that $\check{f} \circ \rho = f$.*

By the universal property of $R(D)$, it is unique up to isomorphisms in $\mathbf{RB}^0_\lambda$.

3.2.2. *The trialgebra case.* Let $D = (D, \prec, \succ, \cdot) \in \mathbf{DT}$. Then $(D, \cdot)$ is a nonunitary algebra. Let $\lambda \in \mathbf{k}$ be given. Let $Ш^{\mathrm{NC},0}(D) := Ш^{\mathrm{NC},0}{}_\lambda(D)$ be the free Rota-Baxter algebra over D of weight λ constructed in §2.5. Identify D as a subalgebra of $Ш^{\mathrm{NC},0}(D)$ by the natural injection j_D in Eq.(22). Let I_R be the Rota-Baxter ideal of $Ш^{\mathrm{NC},0}(D))$ generated by the set

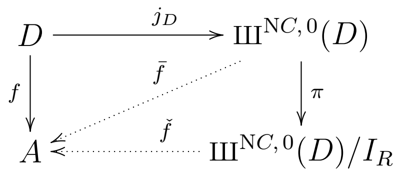
$$\big\{x \prec y - x\lfloor y\rfloor,\ x \succ y - \lfloor x\rfloor y \ \big|\ x, y \in D\big\} \tag{27}$$

Let $\pi : Ш^{\mathrm{NC},0}(D) \to Ш^{\mathrm{NC},0}(D)/I_R$ be the quotient map.

Theorem 3.4. *The quotient Rota-Baxter algebra $Ш^{\mathrm{NC},0}(D)/I_R$, together with $\rho := \pi \circ j_D$, is the universal enveloping Rota-Baxter algebra of D.*

Proof. Let $(A, R) \in \mathbf{RB}^0_\lambda$. It gives an object in $\mathbf{DT}$ by Theorem 3.1 which we still denote by A. Let $f : D \to A$ be a morphism in $\mathbf{DT}$. We will complete the following commutative diagram

(28)



By the freeness of $Ш^{\mathrm{NC},0}(D)$, there is a morphism $\bar{f} : Ш^{\mathrm{NC},0}(D) \to A$ in $\mathbf{RB}^0$ such that the upper left triangle commutes. So for any $x, y \in D$, we have

$$\begin{aligned} \bar{f}(x \prec y - x\lfloor y\rfloor) &= \bar{f}(x \prec y) - \bar{f}(x)R(\bar{f}(y)) \\ &= f(x \prec y) - f(x)R(f(y)) \\ &= f(x \prec y) - f(x) \prec_R f(y) \\ &= f(x \prec y) - f(x \prec y) = 0. \end{aligned}$$

Therefore, $x \prec y - x\lfloor y\rfloor$ is in $\ker(\bar{f})$. Similarly, $x \succ y - \lfloor x\rfloor y$ is in $\ker(\bar{f})$. Thus I_R is in $\ker(\bar{f})$ and there is a morphism $\check{f} : Ш^{\mathrm{NC},0}(D)/I_R \to A$ in $\mathbf{RB}^0$ such that $\bar{f} = \check{f} \circ \pi$. Then

$$\check{f} \circ \rho = \check{f} \circ \pi \circ j_D = \bar{f} \circ j_D = f.$$

This proves the existence of $\check{f}$.

Suppose $\check{f}' : Ш^{\mathrm{NC},0}(D)/I_R \to A$ is a morphism in $\mathbf{RB}^0$ such that $\check{f}' \circ \rho = f$. Then

$$(\check{f}' \circ \pi) \circ j_D = f = (\check{f} \circ \pi) \circ j_D.$$

By the universal property of the free Rota-Baxter algebra $Ш^{\mathrm{NC},0}(D)$ over D, we have $\check{f}' \circ \pi = \check{f} \circ \pi$ in $\mathbf{RB}$. Since π is surjective, we have $\check{f}' = \check{f}$. This proves the uniqueness of $\check{f}$. □

3.2.3. *The dialgebra case.* Now let $D = (D, \prec, \succ) \in \mathbf{DD}$ with a $\mathbf{k}$-basis Ω. Let $T(D) = \bigoplus_{n\geq 1} D^{\otimes n}$ be the tensor product algebra over D. Then $T(D)$ is the free nonunitary algebra generated by the $\mathbf{k}$-module D. A $\mathbf{k}$-basis of $T(D)$ is given by

$$X = \big\{x_1 \otimes \cdots \otimes x_n \big| x_i \in \Omega,\ 1 \leq i \leq n\big\}$$

with the product given by the tensor concatenation:

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots x_n \otimes y_1 \otimes \cdots y_m.$$

By Theorem 2.9,

$$Ш^{\mathrm{NC},0}(\Omega) := Ш^{\mathrm{NC},0}{}_\lambda(T(D))$$

is the free Rota-Baxter algebra over $T(D)$ of weight λ constructed in §2.5. Identify D as a $\mathbf{k}$-submodule of $Ш^{\mathrm{NC},0}(\Omega)$ by the natural injection $j_\Omega : D \to T(D) \to Ш^{\mathrm{NC},0}(\Omega)$.

Let J_R be the Rota-Baxter ideal of $Ш^{\mathrm{NC},0}(\Omega)$ generated by the set

$$\Big\{x \prec y - x\lfloor y\rfloor - \lambda x \otimes y,\ x \succ y - \lfloor x\rfloor y \ \Big|\ x, y \in D\Big\} \tag{29}$$

Let $\pi : Ш^{\mathrm{NC},0}(\Omega) \to Ш^{\mathrm{NC},0}(\Omega)/J_R$ be the quotient map.

Theorem 3.5. *The quotient Rota-Baxter algebra $Ш^{\mathrm{NC},0}(\Omega)/J_R$, together with $\pi \circ j_D$, is the universal enveloping Rota-Baxter algebra of D of weight λ.*

Proof. Let (A, R) be a Rota-Baxter algebra of weight λ and let $f : D \to A$ be a morphism in $\mathbf{DD}$. More precisely, we have $f : D \to \mathcal{G}A$ where $\mathcal{G}A = (A, \prec'_R, \succ'_R)$ is the dendriform dialgebra in Theorem 3.1.

We will complete the following commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\ j_\Omega\ } & Ш^{\mathrm{NC},0}(\Omega) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar f} & \downarrow{\scriptstyle \pi} \\ A & \xleftarrow{\ \check f\ } & Ш^{\mathrm{NC},0}(\Omega)/J_R \end{array} \tag{30}$$

By the freeness of $Ш^{\mathrm{NC},0}(\Omega)$, there is a unique morphism $\bar f : Ш^{\mathrm{NC},0}(\Omega) \to A$ in $\mathbf{RB}_0$ such that the upper left triangle commutes. So for any $x, y \in D$, we have

$$\begin{aligned} \bar f(x \prec y - x\lfloor y\rfloor - \lambda x \otimes y) &= \bar f(x \prec y) - \bar f(x) R(\bar f(y)) - \lambda \bar f(x \otimes y) \\ &= f(x \prec y) - f(x) R(f(y)) - \lambda f(x) f(y) \\ &= f(x \prec y) - f(x) \prec'_R f(y) \\ &= f(x \prec y) - f(x \prec y) = 0. \end{aligned}$$

Therefore, $x \prec y - x\lfloor y\rfloor - \lambda x \otimes y$ is in $\ker(\bar f)$. Similarly, $x \succ y - \lfloor x\rfloor y$ is in $\ker(\bar f)$. Thus J_R is in $\ker(\bar f)$ and there is a morphism $\check f : Ш^{\mathrm{NC},0}(\Omega)/J_R \to A$ in $\mathbf{RB}$ such that $\bar f = \check f \circ \pi$. Then

$$\check f \circ \rho = \check f \circ \pi \circ j_D = \bar f \circ j_D = f.$$

This proves the existence of $\check f$.

The rest of the proof is the same as for Theorem 3.4. □

3.3. **The Poincaré-Birkhoff-Witt theorem.** Recall that the Poincaré-Birkhoff-Witt Theorem for the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ displays a basis of $U(\mathfrak{g})$ in terms of a basis of $\mathfrak{g}$. We prove an analog of the Poincaré-Birkhoff-Witt Theorem by displaying a basis of the universal enveloping Rota-Baxter algebra $R(D)$ of a dendriform trialgebra or dialgebra D in terms of D. As in the Lie algebra case, a consequence of our analog is that the dendriform trialgebra or dialgebra embeds into its universal enveloping Rota-Baxter algebra. We will only consider dendriform trialgebras. The result for dialgebras is similar.

Let D be a dendriform trialgebra and let X be a basis of D.

Let $\mathfrak{J}$ be the subset of X_∞ consisting of words not containing subwords of the form $x\lfloor y\rfloor$ or $\lfloor x\rfloor y$ where $x, y \in X$. For example, $\lfloor\lfloor y\rfloor\rfloor x$ is in $\mathfrak{J}$, but $\lfloor\lfloor y\lfloor x\rfloor\rfloor\rfloor$ is not.

Theorem 3.6. (Rota-Baxter analogue of the Poincaré-Birkhoff-Witt theorem) *Let $\pi : Ш^{\mathrm{NC},0}(D) \to R(D) := Ш^{\mathrm{NC},0}(D)/I_R$ be as defined in Theorem 3.4. Then $\pi\big|_{\mathfrak{J}}$ is injective and $\pi(\mathfrak{J})$ is a basis of $R(D)$.*

We immediately obtain the following "inverse" of Theorem 3.1.

Corollary 3.7. *The natural map $\rho = \pi \circ j_D : D \to R(D)$ is injective. Therefore, every dendriform trialgebra is a sub dendriform trialgebra of a $(A, \prec_R, \succ_R, \cdot_R)$ for a Rota-Baxter algebra (A, R).*

3.4. **The proof of the Poincaré-Birkhoff-Witt theorem.** The proof is similar to the proof of the classical Poincaré-Birkhoff-Witt theorem for Lie algebras, say in [**Hoc**]. Let $R'(D)$ be the **k**-vector space generated by $\mathfrak{J}$. The proof of Theorem 3.6 will be accomplished by

(1) defining a Rota-Baxter algebra structure on $R'(D)$;
(2) showing that the morphism $g : Ш^{NC,0}(D) \to R'(D)$ in **RB** induced by the natural map $j'_D : D \to R'(D)$ satisfies $I_R \subseteq \ker g$.

This forces $I_R = \ker g$, $R'(D) \cong R(D)$ and thus the basis $\mathfrak{J}$ of $R'(D)$ is a basis of $R(D)$.

The proof of the first step is similar to the proof for $Ш^{NC,0}(D)$ in §2.3 but more involved. For example, the proof of Lemma 2.4 is completely clear which is not the case with the corresponding Lemma 3.8. Since we have not found a uniform treatment of both, we will give the details with emphasis on the differences.

3.4.1. *Rota-Baxter algebra structure on $R'(D)$.* We let $\mathfrak{J}_n$ be the subset $\mathfrak{J} \cap R_n$. Then $\mathfrak{J} = \cup_{n\geq 0} \mathfrak{J}_n$. We will define a product $\check{\diamond}$ on $R'(D)$ by defining $\mathfrak{X} \check{\diamond} \mathfrak{X}'$ for $\mathfrak{X}, \mathfrak{X}' \in \mathfrak{J}$.

By Lemma 2.8, $\mathfrak{X}$ can be uniquely written as $\mathfrak{X}_1\mathfrak{X}_2$ with $\mathfrak{X}_1 \in \mathfrak{M}(X)$ and $\mathfrak{X}_2 \in X$ or $\mathfrak{X}_2 = \lfloor \overline{\mathfrak{X}}_2 \rfloor$ where $\overline{\mathfrak{X}}_2 \in X$. Likewise, $\mathfrak{X}'$ can be uniquely written as $\mathfrak{X}'_1\mathfrak{X}'_2$ with $\mathfrak{X}'_2 \in \mathfrak{M}(X)$ and $\mathfrak{X}'_1 \in X$ or $\mathfrak{X}'_1 = \lfloor \overline{\mathfrak{X}}'_1 \rfloor$ where $\overline{\mathfrak{X}}'_1 \in X$.

We use induction on $m := d(\mathfrak{X}_2) + d(\mathfrak{X}'_1)$. Then $m \geq 0$. If $m = 0$, then $d(\mathfrak{X}_2) = d(\mathfrak{X}'_1) = 0$. So $\mathfrak{X}_2$ and $\mathfrak{X}'_1$ are in X. Define

$$\mathfrak{X} \check{\diamond} \mathfrak{X}' = \mathfrak{X}_1(\mathfrak{X}_2 \cdot \mathfrak{X}'_1)\mathfrak{X}'_2. \tag{31}$$

If $m = 1$, one of $\mathfrak{X}_2$ and $\mathfrak{X}'_1$ is in X and the other one is in $\lfloor X \rfloor$. If $\mathfrak{X}_2 \in X$ and $\mathfrak{X}'_1 \in \lfloor X \rfloor$, so $\mathfrak{X}'_1 = \lfloor \overline{\mathfrak{X}}'_1 \rfloor$ with $\overline{\mathfrak{X}}'_1 \in X$. Then $\mathfrak{X}' = \lfloor \overline{\mathfrak{X}}'_1 \rfloor$ since otherwise the condition for $\mathfrak{X}' \in \mathfrak{J}$ will be violated. We then define

$$\mathfrak{X} \check{\diamond} \mathfrak{X}' = \mathfrak{X}_1(\mathfrak{X}_2 \prec \overline{\mathfrak{X}}'_1) \tag{32}$$

which is a well-defined element in $R'(D)$. Similarly for $\mathfrak{X}_2 = \lfloor \overline{\mathfrak{X}}_2 \rfloor \in \lfloor X \rfloor$ and $\mathfrak{X}' \in X$, define

$$\mathfrak{X} \check{\diamond} \mathfrak{X}' = \mathfrak{X}_1(\overline{\mathfrak{X}}_2 \succ \mathfrak{X}'_1). \tag{33}$$

Assume the product is defined for $m := d(\mathfrak{X}_2) + d(\mathfrak{X}'_1) \geq k \geq 1$. Let $\mathfrak{X}, \mathfrak{X}' \in \tilde{X}$ with $m = k+1 \geq 2$. If one of $d(\mathfrak{X}_2)$ and $d(\mathfrak{X}'_1)$ is 0, then $\mathfrak{X} \in X$ or $\mathfrak{X}' \in X$ and the other one is of the form $\lfloor \overline{\mathfrak{X}} \rfloor$ with $\overline{\mathfrak{X}} \in \mathfrak{J} \backslash X$. Therefore the concatenations $\mathfrak{X}_2\mathfrak{X}'_1$ and hence $\mathfrak{X}\mathfrak{X}'$ are in $\mathfrak{J}$. Then define

$$\mathfrak{X} \check{\diamond} \mathfrak{X}' = \mathfrak{X}\mathfrak{X}'.$$

If none of $d(\mathfrak{X}_2)$ and $d(\mathfrak{X}'_1)$ is 0, then $\mathfrak{X}_2 = \lfloor \overline{\mathfrak{X}}_2 \rfloor$ and $\mathfrak{X}'_1 = \lfloor \overline{\mathfrak{X}}'_1 \rfloor$ with $\overline{\mathfrak{X}}_2, \overline{\mathfrak{X}}'_1 \in \mathfrak{M}^0(X)$. We can recursively define

$$\begin{aligned} \mathfrak{X} \check{\diamond} \mathfrak{X}' &= \mathfrak{X}_1\big(\lfloor \overline{\mathfrak{X}}_2 \rfloor \check{\diamond} \lfloor \overline{\mathfrak{X}}'_1 \rfloor\big)\mathfrak{X}'_2 \\ &= \mathfrak{X}_1 \big\lfloor \big(\lfloor \overline{\mathfrak{X}}_2 \rfloor \check{\diamond} \overline{\mathfrak{X}}'_1 + \overline{\mathfrak{X}}_2 \check{\diamond} \lfloor \overline{\mathfrak{X}}'_1 \rfloor + \lambda \overline{\mathfrak{X}}_2 \check{\diamond} \overline{\mathfrak{X}}'_1\big) \big\rfloor \mathfrak{X}'_2 \end{aligned} \tag{34}$$

by using the induction hypothesis on the three products on the right hand side since we have

$$\begin{aligned} d(\lfloor\overline{\mathfrak{X}}_2\rfloor) + d(\overline{\mathfrak{X}}_1') &= d(\mathfrak{X}_2) + d(\mathfrak{X}_1') - 1, \\ d(\overline{\mathfrak{X}}_2) + d(\lfloor\overline{\mathfrak{X}}_1'\rfloor) &= d(\mathfrak{X}) - 1 + d(\mathfrak{X}'), \\ d(\overline{\mathfrak{X}}_2) + d(\overline{\mathfrak{X}}_1') &= d(\mathfrak{X}_2) - 1 + d(\mathfrak{X}_1') - 1. \end{aligned}$$

which are all less then or equal to k.

We record the following properties of $\check{\diamond}$ for later applications.

Lemma 3.8. *Let $\mathfrak{X} \in \mathfrak{J}$. If $\mathfrak{X} = \mathfrak{X}_1\mathfrak{X}_2$ with $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathfrak{J}$, then for any $\mathfrak{X}' \in \mathfrak{J}$,*

$$\mathfrak{X}\check{\diamond}\mathfrak{X}' = \mathfrak{X}_1\check{\diamond}(\mathfrak{X}_2\check{\diamond}\mathfrak{X}'), \mathfrak{X}'\check{\diamond}\mathfrak{X} = (\mathfrak{X}'\check{\diamond}\mathfrak{X}_1)\check{\diamond}\mathfrak{X}_2.$$

Proof. We only prove the first equation. The proof of the second equation is the same.

Case 1. First assume $t(\mathfrak{X}_2) = 0$. Then either $\mathfrak{X}_2 = \mathfrak{X}_{2,2} \in X$ or $\mathfrak{X}_2 = \mathfrak{X}_{2,1}\mathfrak{X}_{2,2}$ with $\mathfrak{X}_{2,1} \in \mathfrak{J}$ and $\mathfrak{X}_{2,2} \in X$. If $h(\mathfrak{X}') = 0$, then either $\mathfrak{X}' = \mathfrak{X}_1' \in X$ or $\mathfrak{X}' = \mathfrak{X}_1'\mathfrak{X}_2'$ with $\mathfrak{X}' = \mathfrak{X}_1' \in X$ and $\mathfrak{X}_2' \in \mathfrak{J}$. In either case, the product $\mathfrak{X}\check{\diamond}\mathfrak{X}'$ is given by taking the product $\mathfrak{X}_{2,2} \cdot \mathfrak{X}_1'$ and then taking the concatenation with the rest. This proves the equation.

If $h(\mathfrak{X}') = 1$, then either $\mathfrak{X}' = \lfloor\overline{\mathfrak{X}}'\rfloor$ with $\overline{\mathfrak{X}}' \in \mathfrak{J}$ or $\mathfrak{X}' = \lfloor\overline{\mathfrak{X}}'\rfloor\mathfrak{X}_2'$ with $\overline{\mathfrak{X}}' \in \mathfrak{J}\backslash X$ and $\mathfrak{X}_2' \in \mathfrak{J}$. Then we have

$$\mathfrak{X}\check{\diamond}\mathfrak{X}' = \begin{cases} \mathfrak{X}_1(\mathfrak{X}_2 \prec \overline{\mathfrak{X}}'), & \mathfrak{X}' = \lfloor\overline{\mathfrak{X}}'\rfloor, \overline{\mathfrak{X}}' \in X, \\ \mathfrak{X}_1(\mathfrak{X}_2\lfloor\overline{\mathfrak{X}}'\rfloor), & \mathfrak{X}' = \lfloor\overline{\mathfrak{X}}'\rfloor, \overline{\mathfrak{X}}' \in \mathfrak{J}\backslash X, \\ \mathfrak{X}_1(\mathfrak{X}_2\lfloor\overline{\mathfrak{X}}'\rfloor)\mathfrak{X}_2', & \mathfrak{X}' = \lfloor\overline{\mathfrak{X}}_1'\rfloor\mathfrak{X}_2'. \end{cases}$$

So the first equation still holds.

Case 2. Now assume $t(\mathfrak{X}_2) = 1$. Then either $\mathfrak{X}_2 = \lfloor\overline{\mathfrak{X}}_2\rfloor$ with $\overline{\mathfrak{X}}_2 \in \mathfrak{J}\backslash X$ or $\mathfrak{X}_2 = \mathfrak{X}_{2,1}\lfloor\overline{\mathfrak{X}}_2\rfloor$ with $\mathfrak{X}_{2,1} \in \mathfrak{J}$ and $\mathfrak{X}_{2,2} \in \mathfrak{J}\backslash X$. If $h(\mathfrak{X}') = 0$, then $\mathfrak{X}\check{\diamond}\mathfrak{X}'$ is defined by the concatenation. So we have

$$\mathfrak{X}\check{\diamond}\mathfrak{X}' = \mathfrak{X}\mathfrak{X}' = \mathfrak{X}_1\mathfrak{X}_2\mathfrak{X}' = \mathfrak{X}_1(\mathfrak{X}_2\check{\diamond}\mathfrak{X}').$$

If $h(\mathfrak{X}') = 1$, then either $\mathfrak{X}' = \lfloor\overline{\mathfrak{X}}'\rfloor$ with $\overline{\mathfrak{X}}' \in \mathfrak{J}$ or $\mathfrak{X}' = \lfloor\overline{\mathfrak{X}}'\rfloor\mathfrak{X}_2'$ with $\overline{\mathfrak{X}}' \in \mathfrak{J}\backslash X$ and $\mathfrak{X}_2' \in \mathfrak{J}$. Then the product $\mathfrak{X}\check{\diamond}\mathfrak{X}'$ is defined by taking $\lfloor\overline{\mathfrak{X}}_2\rfloor\check{\diamond}\lfloor\overline{\mathfrak{X}}'\rfloor = \big\lfloor\lfloor\overline{\mathfrak{X}}_2\rfloor\check{\diamond}\overline{\mathfrak{X}}' + \overline{\mathfrak{X}}_2\check{\diamond}\lfloor\overline{\mathfrak{X}}'\rfloor + \lambda\overline{\mathfrak{X}}_2\check{\diamond}\overline{\mathfrak{X}}'\big\rfloor$ and take the concatenation with the rest, again giving the first equation in Lemma 3.8. ☐

Extending $\check{\diamond}$ bilinearly, we obtain a binary operation

$$\check{\diamond} : R'(B) \otimes R'(B) \to R'(B).$$

For $\mathfrak{X} \in \mathfrak{J}$, define

$$R_B'(\mathfrak{X}) = \lfloor\mathfrak{X}\rfloor. \tag{35}$$

Obviously $\lfloor\mathfrak{X}\rfloor$ is again in $\mathfrak{J}$. Thus R_B' extends to a linear operator R_B' on $R'(B)$. Let

$$j_X' : X \to R'(B)$$

be the natural injection and let

$$j_B' : B \to R'(B) \tag{36}$$

be the induced injection.

Theorem 3.9. (1) *The pair* $(R'(B), \check{\diamond})$ *is an associative algebra.*

(2) *The triple* $(R'(B), \check{\diamond}, R'_B)$ *is a Rota-Baxter algebra of weight* λ.

(3) *The Rota-Baxter morphism* $Ш^{\mathrm{NC},0}(B) \to R'(B)$ *contains* I_R *in its kernel.*

3.4.2. *The proof.*

Proof. (1). We just need to verify the associativity. For this we only need to verify

$$(\mathfrak{X}'\check{\diamond}\mathfrak{X}'')\check{\diamond}\mathfrak{X}''' = \mathfrak{X}'\check{\diamond}(\mathfrak{X}''\check{\diamond}\mathfrak{X}''') \tag{37}$$

for $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in \mathfrak{J}$.

Using Lemma 3.8 and similar arguments for Lemma 2.6, we have

Lemma 3.10. *If* $\ell(\mathfrak{X}'') \geq 2$*, the for all* $\mathfrak{X}', \mathfrak{X}'' \in X_\infty$*, we have*

$$(\mathfrak{X}'\check{\diamond}\mathfrak{X}'')\check{\diamond}\mathfrak{X}''' = \mathfrak{X}'\check{\diamond}(\mathfrak{X}''\check{\diamond}\mathfrak{X}''').$$

A similar proof as for Theorem 2.5 also allows us to assume $\ell(\mathfrak{X}') = 1$ and $\ell(\mathfrak{X}''') = 1$. Thus we can assume $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}'''$ to be in either X or $\lfloor \mathfrak{J} \rfloor$.

If $t(\mathfrak{X}') \neq h(\mathfrak{X}'')$, then by Lemma 3.8,

$$(\mathfrak{X}'\check{\diamond}\mathfrak{X}'')\check{\diamond}\mathfrak{X}''' = (\mathfrak{X}'\mathfrak{X}'')\check{\diamond}\mathfrak{X}''' = \mathfrak{X}'(\mathfrak{X}''\check{\diamond}\mathfrak{X}''') = \mathfrak{X}'\check{\diamond}(\mathfrak{X}''\check{\diamond}\mathfrak{X}''').$$

Similarly if $t(\mathfrak{X}'') \neq h(\mathfrak{X}''')$.

To summarize, we are reduced to the special case when $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in \mathfrak{J}$ are of length one and $a := t(\mathfrak{X}') = h(\mathfrak{X}'')$ and $b := t(\mathfrak{X}'') = h(\mathfrak{X}''')$. Since $\ell(\mathfrak{X}'') = 1$, we have $h(\mathfrak{X}'') = t(\mathfrak{X}'')$. Therefore, either all the three elements are in X or they are all in $\lfloor \mathfrak{J} \rfloor$.

Case 1. All of $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}'''$ are in X. Then the associativity follows from the associativity in B.

Case 2. All of $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}'''$ are in $\lfloor \mathfrak{J} \rfloor$. Then $\mathfrak{X}' = \lfloor \overline{\mathfrak{X}}' \rfloor, \mathfrak{X}'' = \lfloor \overline{\mathfrak{X}}'' \rfloor, \mathfrak{X}''' = \lfloor \overline{\mathfrak{X}}''' \rfloor$ where $\overline{\mathfrak{X}}', \overline{\mathfrak{X}}'', \overline{\mathfrak{X}}''' \in \mathfrak{J}$. For this, we use induction on $m := d(\mathfrak{X}') + d(\mathfrak{X}'') + d(\mathfrak{X}''')$. Then $m \geq 3$. We first check the case when $m = 3$. Then $\mathfrak{X}', \mathfrak{X}''$ and $\mathfrak{X}'''$ all have depth one. So $\overline{\mathfrak{X}}', \overline{\mathfrak{X}}'', \overline{\mathfrak{X}}'''$ are in X. By Eq. (34), (31), (32) and (33), we have

$$\mathfrak{X}'\check{\diamond}\mathfrak{X}'' = \lfloor \overline{\mathfrak{X}}' \star \overline{\mathfrak{X}}'' \rfloor, \ \mathfrak{X}''\check{\diamond}\mathfrak{X}''' = \lfloor \overline{\mathfrak{X}}'' \star \overline{\mathfrak{X}}''' \rfloor$$

where $\star = \prec + \succ + \cdot$. Then Eq. (37) follows from the associativity of $\star$ in Eq. (4). We recall that X is in the dendriform trialgebra B.

Assume Eq. (17) holds when $3 \leq d(\mathfrak{X}') + d(\mathfrak{X}'') + d(\mathfrak{X}''') \leq k$ and take $\mathfrak{X}', \mathfrak{X}'', \mathfrak{X}''' \in \mathfrak{J}$ with $d(\mathfrak{X}') + d(\mathfrak{X}'') + d(\mathfrak{X}''') = k+1$. Then the induction proof for Theorem 2.5 applies to show that Eq. 37 holds.

This completes the proof of the first part of Theorem 3.9.

(2). The proof is immediate from Eq. (34).

(3). By Eq. (31), (32) and (33), the natural injection $j'_B : B \to R'(B)$ of algebras is a morphism of dendriform trialgebras. Thus the canonical morphism $\tilde{j_B}' : Ш^{\mathrm{NC},0}(B) \to R'(B)$ induced by the natural injection j'_B annihilates the set in Eq. (27) and therefore I_R. $\square$

As explained at the beginning of § 3.4, this finishes the proof of Theorem 3.6.

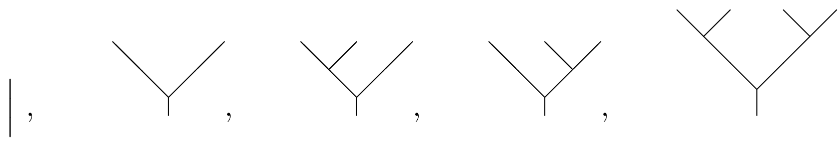
4. Free dendriform di- and trialgebras and free Rota-Baxter algebras

The results in this section can be regarded as more precise forms of results in §3 in special cases. Our emphasis here is to interpret free dendriform dialgebras and free dendriform trialgebras as natural subalgebras of free Rota-Baxter algebras. This interpretation also gives a planar tree structure on free Rota-Baxter algebras.

4.1. **The dialgebra case.**

4.1.1. *Free dendriform dialgebras.* Let $\mathbf{k}$ be a field. We briefly recall the construction of free dendriform dialgebra $\mathbf{DD}(V)$ over a $\mathbf{k}$-vector space V as colored planar binary trees. For details, see [**Lo1**, **Ron**].

Let X be a basis of V. For $n \geq 0$, let Y_n be the set of planar binary trees with $n+1$ leaves and one root such that the valence of each internal vertex is exactly two. Let $Y_{n,X}$ be the set of planar binary trees with $n+1$ leaves and with vertices decorated by elements of X. The unique tree with 1 leave is denoted by $|$. So we have $Y_0 = Y_{0,X} = \{|\}$. Let $\mathbf{k}[Y_{n,X}]$ be the $\mathbf{k}$-vector space generated by $Y_{n,X}$. Here are the first few of them without decoration.



For $T \in Y_{m,X}, U \in Y_{n,X}$ and $x \in X$, the grafting of T and U over x is $T \vee_x U \in Y_{m+n+1,X}$. Let $\mathbf{DD}(V)$ be the graded vector space $\bigoplus_{n\geq 1} \mathbf{k}[Y_{n,X}]$. Define binary operations $\prec$ and $\succ$ on $\mathbf{DD}(V)$ recursively by

(1) $| \succ T = T \prec | = T$ and $| \prec T = T \succ | = 0$ for $T \in Y_{n,X}, n \geq 1$;
(2) For $T = T^\ell \vee_x T^r$ and $U = U^\ell \vee_y U^r$, define

$$T \prec U = T^\ell \vee_x (T^r \prec U + T^r \succ U), \quad T \succ U = (T \prec U^\ell + T \succ U^\ell) \vee_y U^r.$$

Since $| \prec |$ and $| \succ |$ is not defined, the binary operations $\prec$ and $\succ$ are only defined on $\mathbf{DD}(V)$ though the operation $\star := \prec + \succ$ can be extended to $H_{\rm LR} := \mathbf{k}[Y_0] \oplus \mathbf{DD}(V)$ by defining $| \star T = T \star | = T$.

Theorem 4.1. (Loday)[**Lo1**] *$(\mathbf{DD}(V), \prec, \succ)$ is the free dendriform dialgebra over V.*

Theorem 4.2. *Let V be a $\mathbf{k}$-vector space. The free dendriform dialgebra over V is a sub dendriform dialgebra of the free Rota-Baxter algebra $Ш^{NC,0}(V)$ of weight zero.*

The proof will be given in the next subsection.

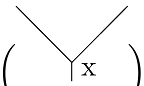
4.1.2. *Proof of Theorem 4.2.* For the given vector space V, make V into a $\mathbf{k}$-algebra without identity by given V the zero product. Let $Ш^{NC,0}(V)$ be the free nonunitary Rota-Baxter algebra of weight zero over V constructed in Theorem 2.9. Since $Ш^{NC,0}(V)$ is a dendriform dialgebra, the natural map $j_V : V \to Ш^{NC,0}(V)$ extends uniquely to a dendriform dialgebra morphism $D(j) : \mathbf{DD}(V) \to Ш^{NC,0}(V)$. We will prove that this map is injective and identifies $\mathbf{DD}(V)$ as a subalgebra of $Ш^{NC,0}(V)$ in the category of dendriform dialgebras. We first define a map

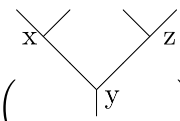
$$\phi : \mathbf{DD}(V) \to Ш^{NC,0}(V)$$

and then show in Theorem 4.4 below that it agrees with $D(j)$. We construct ϕ by defining $\phi(T)$ for $T \in Y_{n,X}, n \geq 1$, inductively on n. Any $T \in Y_{n,X}, n \geq 1$ can be uniquely written as $T = T^\ell \vee_x T^r$ with $x \in X$ and $T^\ell, T^r \in \cup_{0 \leq i < n} Y_{i,X}$. We then define

$$\phi(T) = \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r) \rfloor, & T^\ell \neq 1, T^r \neq 1, \\ x \lfloor \phi(T^r) \rfloor, & T^\ell = 1, T^r \neq 1, \\ \lfloor \phi(T^\ell) \rfloor x, & T^\ell \neq 1, T^r = 1, \\ x, & T^\ell = 1, T^r = 1. \end{cases} \tag{38}$$

For example,

$\phi\Big($ 

$\Big) = x,$ $\phi\Big($ 

$\Big) = x\lfloor y \rfloor z.$

We recall [**Lo1**] that $\mathbf{DD}(V)$ with the operation $\star := \prec + \succ$ is an associative algebra.

We now describe a submodule of $Ш^{NC,0}(V)$ to be identified with the image of ϕ in Theorem 4.4.

Definition 4.3. An RBW $y \in \mathfrak{M}^0(X)$ is called a **dendriform diword (DW)** if it satisfies the following *additional* properties.

(1) y is not in $\lfloor \mathfrak{M}^0(X) \rfloor$;
(2) There is no subword $\lfloor \lfloor \mathfrak{X} \rfloor \rfloor$ with $\mathfrak{X} \in \mathfrak{M}^0(X)$ in the word;
(3) There is no subword of the form $\mathfrak{X}_1 \lfloor \mathfrak{X}_2 \rfloor \mathfrak{X}_3$ with $\mathfrak{X}_1, \mathfrak{X}_3 \in X$ and $\mathfrak{X}_2 \in \mathfrak{M}^0(X)$.

We let $DW(V)$ be the subspace of $Ш^{NC,0}(V)$ generated by the dendriform diwords.

For example

$$x_0 \lfloor x_1 \lfloor x_2 \rfloor \rfloor, \lfloor x_1 \rfloor x_0 \lfloor x_2 \rfloor$$

are dendriform diwords; while

$$\lfloor \lfloor x_1 \rfloor \rfloor, \lfloor \lfloor x_1 \rfloor x_2 \lfloor x_3 \rfloor \rfloor, x_1 \lfloor x_2 \rfloor x_3$$

are RBWs but not dendriform diwords.

Equivalently, $DW(V)$ can be characterized in terms of the decomposition (11). For subsets Y, Z of $\mathfrak{M}(X)$, define

$$D(Y, Z) = (Y \lfloor Z \rfloor) \bigcup (\lfloor Z \rfloor Y) \bigcup \lfloor Z \rfloor Y \lfloor Z \rfloor.$$

Then define $D_0(V) = X$ and, for $n \geq 0$, inductively define

$$D_{n+1}(V) = D(X, D_n(V)) = (X \lfloor D_n(V) \rfloor) \bigcup (\lfloor D_n(V) \rfloor X) \bigcup \lfloor D_n(V) \rfloor X \lfloor D_n(V) \rfloor. \tag{39}$$

Then $D_\infty := \cup_{n \geq 0} D_n(V)$ is the set of dendriform diwords and $DW(V) = \oplus_{\mathfrak{X} \in D_\infty} \mathbf{k}\mathfrak{X}$.

Theorem 4.2 follows from the following theorem.

Theorem 4.4. (1) *$\phi : \mathbf{DD}(V) \to Ш^{NC,0}(V)$ is a homomorphism of dendriform dialgebras.*

(2) *$\phi = D(j)$, the morphism of dendriform dialgebras induced by $j : V \to Ш^{NC,0}(V)$.*

(3) *$\phi(\mathbf{DD}) = DW(V)$.*

(4) *ϕ is injective.*

Proof. (1) we first note that the operations $\prec$ and $\succ$ can be equivalently defined as follows without using $| \prec T$, etc. For $T \in Y_{m,X}, U \in Y_{n,X}$ with $m \geq 1, n \geq 1$. Then $T = T^\ell \vee_x T^r, U = U^\ell \vee_y U^r$ with $x, y \in X$ and $T^\ell, T^r, U^\ell, U^r \in \cup_{i\geq 0} Y_{i,X}$. Define

$$T \prec U : \;= \begin{cases} T^\ell \vee_x (T^r \prec U + T^r \succ U), & \text{if } T^r \neq |, \\ T^\ell \vee_x U, & \text{if } T^r = |. \end{cases} \tag{40}$$

$$T \succ U : \;= \begin{cases} (T \prec U^\ell + T \succ U^\ell) \vee_y U^r, & \text{if } U^\ell \neq |, \\ T \vee_y U^r, & \text{if } U^\ell = |. \end{cases} \tag{41}$$

Thus we have

$$\begin{aligned}
\phi(T \prec U) &= \begin{cases} \phi(T^\ell \vee_x (T^r \prec U + T^r \succ U), & \text{if } T^r \neq |, \\ \phi(T^\ell \vee_x U), & \text{if } T^r = |. \end{cases} \\
&= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r \prec U + T^r \succ U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor \phi(T^r \prec U + T^r \succ U) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases} \\
&\quad \text{(by definition of } \phi) \\
&= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r) \prec_R \phi(U) + \phi(T^r) \succ_R \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor (\phi(T^r) \prec_R \phi(U) + \phi(T^r) \succ_R \phi(U)) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases} \\
&\quad \text{(by induction hypothesis)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\phi(T) \prec_R \phi(U) &= \phi(T^\ell \vee_x T^r) \lfloor \phi(U) \rfloor \\
&= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(T^r) \rfloor \lfloor \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor \phi(T^r) \rfloor \lfloor \phi(U) \rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases} \\
&\quad \text{(by definition of } \phi)
\end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \lfloor \phi(T^\ell) \rfloor x \big\lfloor \phi(T^r) \lfloor \phi(U) \rfloor + \lfloor \phi(T^r) \rfloor \phi(U) \big\rfloor, & \text{if } T^r \neq |, T^\ell \neq |, \\ x \lfloor \phi(T^r) \big\lfloor \phi(U) \rfloor + \lfloor \phi(T^r) \rfloor \phi(U) \big\rfloor, & \text{if } T^r \neq |, T^\ell = |, \\ \lfloor \phi(T^\ell) \rfloor x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell \neq |, \\ x \lfloor \phi(U) \rfloor, & \text{if } T^r = |, T^\ell = |. \end{cases} \\
&\quad \text{(by Rota} - \text{Baxter relation of } R(T) = \lfloor T \rfloor).
\end{aligned}$$

This proves $\phi(T \prec U) = \phi(T) \prec_R \phi(U)$. We similarly prove $\phi(T \succ U) = \phi(T) \succ_R \phi(U)$. Thus ϕ is a homomorphism in $\mathbf{DD}$.

(2) follows from the uniqueness of the dendriform dialgebra morphism $\mathbf{DD}(V) \to Ш^{\mathrm{NC},0}(V)$ extending the map $j_V : V \to Ш^{\mathrm{NC},0}(V)$.

(3) We only need to prove $DW(V) \subseteq \phi(\mathbf{DD}(V))$ and $\phi(\mathbf{DD}(V)) \subseteq DW(V)$. To prove the former, we prove $D_n \subseteq \phi(\mathbf{DD}(V))$ by induction on n.

When $n = 0$, $D_n = X$ so the inclusion is clear. Suppose the inclusion holds for n. Then by the definition of $D_{n+1}(V)$ in Eq. (39), an element of $D_{n+1}(V)$ is of the following three forms:

i) It is $\mathcal{X}\lfloor\mathcal{X}'\rfloor$ with $\mathcal{X} \in X$, $\mathcal{X}' \in D_n(V)$. Then it is $\mathcal{X} \prec_R \mathcal{X}'$ which is in $\phi(\mathbf{DD}(V))$ by the induction hypothesis and the fact that $\phi(\mathbf{DD}(V))$ is a sub dendriform algebra.

ii) It is $\lfloor\mathcal{X}\rfloor\mathcal{X}'$ with $\mathcal{X} \in D_n(V)$ and $\mathcal{X}' \in X$. Then the same proof works.

iii) It is $\lfloor\mathcal{X}\rfloor\mathcal{X}'\lfloor\mathcal{X}''\rfloor$ with $\mathcal{X}, \mathcal{X}'' \in D_n(V)$ and $\mathcal{X}' \in X$. Then it is

$$(\mathcal{X} \succ_R \mathcal{X}') \prec_R \mathcal{X}'' = \mathcal{X}' \succ_R (\mathcal{X}' \prec_R \mathcal{X}'').$$

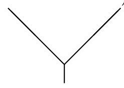
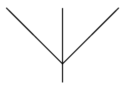
By induction, $\mathcal{X}$ and $\mathcal{X}''$ are in the sub dendriform dialgebra $\phi(\mathbf{DD}(V))$. So the element itself is in $\phi(\mathbf{DD}(V))$.

The second inclusion follows easily by induction on degrees of trees in $\mathbf{DD}(V)$.

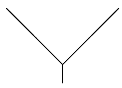
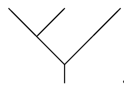
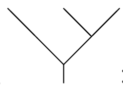
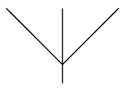
(4) By the definition of ϕ and part (3), ϕ gives a one-one correspondence between $\cup_{n\geq 0} Y_{n,X}$ as a basis of $\mathbf{DD}(V)$ and $DW(V)$ as a basis of $\phi(\mathbf{DD}(V))$. Therefore ϕ is injective. $\square$

4.2. **The trialgebra case.**

4.2.1. *Free dendriform trialgebras.* We describe the construction of free dendriform trialgebra $\mathbf{DT}(V)$ over a vector space V as colored planar trees. For details when V is of rank one over $\mathbf{k}$, see [**L-R1**].

Let X be a basis of V. For $n \geq 0$, let T_n be the set of planar trees with $n+1$ leaves and one root such that the valence of each internal vertex is at least two. Let $T_{n,X}$ be the set of planar binary trees with $n+1$ leaves and with vertices **valently decorated** by elements of X, in the sense that if a vertex has valence k, then the vertex is decorated by a vector in X^{k-1}. For example the vertex of  is decorated by $x \in X$ while the vertex of of  is decorated by $(x, y) \in X^2$. The unique tree with one leaf is denoted by $|$. So we have $T_0 = T_{0,X} = \{|\}$. Let $\mathbf{k}[T_{n,X}]$ be the $\mathbf{k}$-vector space generated by $T_{n,X}$.

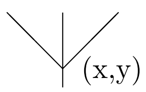
Here are the first few of them without decoration.

$T_0 = \{ \mid \}$, $T_1 = \Big\{$  $\Big\}$, $T_2 = \Big\{$  ,  ;  $\Big\}$

For $T^{(i)} \in T_{n_i,X}$, $0 \leq i \leq k$, and $x_i \in X$, $1 \leq i \leq k$, the grafting of $T^{(i)}$ over $(x_1, \cdots, x_k)$ is

$$T^{(0)} \vee_{x_1} T^{(1)} \vee_{x_2} \cdots \vee_{x_k} T^{(k)}.$$

Any tree can be uniquely expressed as such a grafting of lower degree trees. For example



$= | \vee_x | \vee_y |.$

Let $\mathbf{DT}(V)$ be the graded vector space $\bigoplus_{n\geq 1} \mathbf{k}[T_{n,X}]$. Define binary operations $\prec$, $\succ$ and $\cdot$ on $\mathbf{DT}(V)$ recursively by

(1) $| \succ T = T \prec | = T$, $| \prec T = T \succ | = 0$ and $| \cdot T = T \cdot | = 0$ for $T \in T_{n,X}, n \geq 1$;

(2) For $T = T^{(0)} \vee_{x_1} \cdots \vee_{x_m} T^{(m)}$ and $U = U^{(0)} \vee_{y_1} \cdots \vee_{y_n} U^{(n)}$, define

$$\begin{aligned} T \prec U &= T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U), \\ T \succ U &= (T \star U^{(0)}) \vee_{y_1} \cdots \vee_{y_n} U^{(n)}, \\ T \cdot U &= T^{(0)} \vee_{x_1} \cdots \vee_{x_m} (T^{(m)} \star U^{(0)}) \vee_{y_1} \cdots \vee_{y_n} U^{(n)}. \end{aligned}$$

Here $\star := \prec + \succ + \cdot$ Since $| \prec |$, $| \succ |$ and $| \cdot |$ are not defined, the binary operations $\prec$, $\succ$ and $\cdot$ are only defined on $\mathbf{DT}(V)$ though the operation $\star$ can be extended to $H_{\mathbf{DT}} := \mathbf{k}[T_0] \oplus \mathbf{DT}(V)$ by defining $| \star T = T \star | = T$.

Theorem 4.5. $(\mathbf{DT}(V), \prec, \succ, \cdot)$ *is the free dendriform trialgebra over* V.

Proof. The proof is given by Loday and Ronco in [**L-R1**] when V is of dimension one. The proof for the general case is similar. □

Our goal is to prove

Theorem 4.6. *Let* V *be a* $\mathbf{k}$*-vector space. The free dendriform trialgebra over* V *is a canonical sub dendriform trialgebra of the free Rota-Baxter algebra* $Ш^{NC,0}(V)$ *of weight one.*

We restrict the weight of the Rota-Baxter algebra to one to easy the notation. The proof will be given in the next subsection.

4.2.2. *Proof of Theorem 4.6.* Let V be the given $\mathbf{k}$-vector space with basis Ω. Let $T(V) = \bigoplus_{n\geq 1} V^{\otimes n}$ be the tensor product algebra over V. Then $T(V)$ is the free nonunitary algebra generated by the $\mathbf{k}$-space V. A $\mathbf{k}$-basis of $T(V)$ is given by

$$X = \big\{ x_1 \otimes \cdots \otimes x_n \big| x_i \in \Omega,\ 1 \leq i \leq n \big\}$$

with the product given by the tensor concatenation:

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_m) = x_1 \otimes \cdots x_n \otimes y_1 \otimes \cdots y_m.$$

By Theorem 2.9, $Ш^{NC,0}(\Omega) := Ш^{NC,0}{}_{\mathbf{1}}(T(V))$ is the free nonunitary Rota-Baxter algebra over $T(V)$ of weight $\mathbf{1}$ constructed in §2.5.

Since $Ш^{NC,0}(\Omega)$ is a dendriform trialgebra, the natural map $j_V : V \to Ш^{NC,0}(\Omega)$ extends uniquely to a dendriform trialgebra morphism $T(j) : \mathbf{DT}(V) \to Ш^{NC,0}(\Omega)$. We will prove that this map is injective and identifies $\mathbf{DT}(V)$ as a subalgebra of $Ш^{NC,0}(\Omega)$ in the category of dendriform trialgebras. We first define a map

$$\psi : \mathbf{DT}(V) \to Ш^{NC,0}(\Omega)$$

and then show in Theorem 4.8 below that it agrees with $T(j)$. We construct ψ by defining $\psi(T)$ for $T \in T_{n,X}, n \geq 1$, inductively on n. Any $T \in T_{n,X}, n \geq 1$, can be uniquely written as $T = T^{(0)} \vee_{x_1} \cdots \vee_{x_k} T^{(k)}$ with $x_i \in X$ and $T^{(i)} \in \cup_{0\leq i<n} T_{i,X}$. We then define

$$\psi(T) = \overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \overline{\lfloor \psi(T^{(1)}) \rfloor} \cdots \overline{\lfloor \psi(T^{(k-1)}) \rfloor} x_k \overline{\lfloor \psi(T^{(k)}) \rfloor}, \tag{42}$$

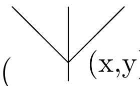
where $\overline{\lfloor \psi(T^{(i)}) \rfloor} = \lfloor \psi(T^{(i)}) \rfloor$ if $\psi(T^{(i)}) \neq |$. If $\psi(T^{(i)}) = |$, then the factor $\lfloor \psi(T^{(i)}) \rfloor$ is dropped when $i = 0$ or k, and is replaced by $\otimes$ when $0 < i < k$. For example,

$$\overline{\lfloor \psi(|) \rfloor} x_1 \overline{\lfloor \psi(T^{(1)}) \rfloor} x_2 \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor} = x_1 \overline{\lfloor \psi(T^{(1)}) \rfloor} x_2 \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor}$$

and

$$\overline{\lfloor \psi(T^{(0)}) \rfloor} x_1 \overline{\lfloor \psi(|) \rfloor} x_2 \overline{\lfloor \psi(T^{(2)}) \rfloor} \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor} = \overline{\lfloor \psi(T^{(0)}) \rfloor} (x_1 \otimes x_2) \overline{\lfloor \psi(T^{(2)}) \rfloor} \cdots x_k \overline{\lfloor \psi(T^{(k)}) \rfloor}.$$

In particular,

$\psi($  (x,y) $) = \psi(|\vee_x|\vee_y|) = \overline{\lfloor\psi(|)\rfloor} \vee_x \overline{\lfloor\psi(|)\rfloor} \vee_y \overline{\lfloor\psi(|)\rfloor} = x\otimes y.$

We now describe a submodule of $Ш^{NC,0}(\Omega)$ to be identified with the image of ψ in Theorem 4.8.

Definition 4.7. An RBW $y\in\mathfrak{M}^0(X)$ is called a **dendriform triword (TW)** if it satisfies the following *additional* properties.

(1) y is not in $\lfloor\mathfrak{M}^0(X)\rfloor$;
(2) There is no subword $\lfloor\lfloor\mathfrak{X}\rfloor\rfloor$ with $\mathfrak{X}\in\mathfrak{M}^0(X)$ in the word;

We let $TW(V)$ be the subspace of $Ш^{NC,0}(\Omega)$ generated by the dendriform triwords.

For example

$$x_0\lfloor x_1\lfloor x_2\rfloor\rfloor, \lfloor x_0\rfloor x_1\lfloor x_2\rfloor, \lfloor x_0\rfloor x_1\lfloor x_2\rfloor x_3\lfloor x_4\rfloor, x_0\otimes x_1$$

are dendriform triwords; while

$$\lfloor\lfloor x_1\rfloor\rfloor, \lfloor x_1\lfloor x_2\rfloor x_3\rfloor$$

are RBWs but not dendriform triwords.

Equivalently, TWs can be characterized in terms of the decomposition (11). For subsets Y,Z of $\mathfrak{M}(X)$, define

$$\begin{aligned} S(Y,Z) &= \Big(\bigcup_{r\geq 1}(Y\lfloor Z\rfloor)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}(Y\lfloor Z\rfloor)^r Y\Big)\\ &\bigcup\Big(\bigcup_{r\geq 1}\lfloor Z\rfloor(Y\lfloor Z\rfloor)^r\Big)\bigcup\Big(\bigcup_{r\geq 0}\lfloor Z\rfloor(Y\lfloor Z\rfloor)^r Y\Big). \end{aligned} \tag{43}$$

Then define $S_0(V)=M(X)$, the free semigroup generated by X and identified as a basis of the free non-commutative nonunitary algebra $T(V)$ over V. For $n\geq 0$, inductively define

$$S_{n+1}(V) = S(M(X), S_n(V)). \tag{44}$$

Then $S_\infty := \cup_{n\geq 0}S_n(V)$ is the set of dendriform triwords and $TW(V)=\oplus_{\mathfrak{X}\in S_\infty}\mathbf{k}\mathfrak{X}$.

Theorem 4.6 follows from the following theorem.

Theorem 4.8. (1) *$\psi:\mathbf{DT}(V)\to Ш^{NC,0}(\Omega)$ is a homomorphism of dendriform trialgebras.*
(2) *$\psi=T(j)$, the morphism of dendriform trialgebras induced by $j:V\to Ш^{NC,0}(\Omega)$.*
(3) *$\psi(\mathbf{DT})=DT(V)$.*
(4) *ψ is injective.*

Proof. (1) we first note that the operations $\prec$ and $\succ$ can be equivalently defined as follows without using $|\prec T$, etc. For $T\in T_{i,X}, U\in T_{j,X}$ with $i\geq 1, j\geq 1$. Then $T=T^{(0)}\vee_{x_1}\cdots\vee_{x_m}T^{(m)}$ and $U=U^{(0)}\vee_{y_1}\cdots\vee_{y_n}U^{(n)}$, define

$$T\prec U = \begin{cases} T^{(0)}\vee_{x_1}\cdots\vee_{x_m}(T^{(m)}\star U), & \text{if } T^{(m)}\neq |,\\ T^{(0)}\vee_{x_1}\cdots\vee_{x_m}U, & \text{if } T^{(m)}=| \end{cases}$$

$$T\succ U = \begin{cases} (T\star U^{(0)})\vee_{y_1}\cdots\vee_{y_n}U^{(n)}, & \text{if } U^{(0)}\neq |,\\ T\vee_{y_1}\cdots\vee_{y_n}U^{(n)}, & \text{if } U^{(0)}=| \end{cases}$$

$$T\cdot U \;=\; \left\{ \begin{array}{ll} T^{(0)}\vee_{x_1}\cdots\vee_{x_m}(T^{(m)}\star U^{(0)})\vee_{y_1}\cdots\vee_{y_n}U^{(n)}, & \text{if } T^{(m)}\neq |, U^{(0)}\neq |,\\ T^{(0)}\vee_{x_1}\cdots\vee_{x_m}U^{(0)}\vee_{y_1}\cdots\vee_{y_n}U^{(n)}, & \text{if } T^{(m)}= |, U^{(0)}\neq |,\\ T^{(0)}\vee_{x_1}\cdots\vee_{x_m}T^{(m)}\vee_{y_1}\cdots\vee_{y_n}U^{(n)}, & \text{if } T^{(m)}\neq |, U^{(0)}= | \end{array}\right.$$

Now we use induction on $i+j$ to prove

$$\psi(T\prec U)=\psi(T)\prec_R\psi(U),\ \psi(T\succ U)=\psi(T)\succ_R\psi(U), \tag{45}$$

$$\psi(T\cdot U)=\psi(T)\cdot_R\psi(U). \tag{46}$$

Here $R:=R_\Omega$ is the Rota-Baxter operator on $Ш^{NC,0}(\Omega)$. Since $i+j\geq 2$, we first assume $i+j=2$. Then $T=|\vee_x|$, $U=|\vee_y|$. So by Eq. (42),

$$\psi(T\prec U)=\psi((|\vee_x|)\prec U)=\psi(|\vee_x U)=x\lfloor\psi(U)\rfloor=x\lfloor y\rfloor=x\prec_R y.$$

We similarly have $\psi(T\succ U)=x\succ_R y$ and

$$\psi(T\cdot U)=\psi((|\vee_x|)\cdot(|\vee_y|))=\psi(|\vee_x|\vee_y|)=x\otimes y=x\cdot_R y.$$

Assume Equations (46) hold for $T\in T_{i,X},\ U\in T_{j,X}$ with $i+j\geq k\geq 2$. Then we also have

$$\begin{array}{rcl}\psi(T\star U) &=& \psi(T\prec U+T\succ U+T\cdot U)\\ &=& \psi(T)\prec_R\psi(U)+\psi(T)\succ_R\psi(U)+\psi(T)\cdot_R\psi(U)\\ &=& \psi(T)\star_R\psi(U).\end{array} \tag{47}$$

Here $\star_R=\prec_R+\succ_R+\cdot_R$. Consider T,U with $m+n=k+1$. We consider two cases of $T=T^{(0)}\vee_{x_1}\cdots\vee_{x_m}T^{(m)}$. Since $U\neq|$, we have $\overline{\lfloor T^{(m)}\star U\rfloor}=\lfloor T^{(m)}\star U\rfloor$ if $T^{(m)}\neq|$, and $\overline{\lfloor U\rfloor}=\lfloor U\rfloor$ if $T^{(m)}\neq|$.

Case 1. If $T^{(m)}\neq|$, then

$$\begin{array}{rcl}\psi(T\prec U) &=& \phi(T^{(0)}\vee_{x_1}\cdots\vee_{x_m}(T^{(m)}\star U))\ \ \text{(definition of } \prec\text{)}\\ &=& \overline{\lfloor\psi(T^{(0)})\rfloor}x_1\cdots x_m\lfloor\psi(T^{(m)}\star U)\rfloor\ \ \text{(definition of }\psi\text{)}\\ &=& \overline{\lfloor\psi(T^{(0)})\rfloor}x_1\cdots x_m\lfloor\psi(T^{(m)})\star_R\psi(U)\rfloor\ \ \text{(induction hypothesis (47))}\\ &=& \overline{\lfloor\psi(T^{(0)})\rfloor}x_1\cdots x_m\lfloor\psi(T^{(m)})\rfloor\lfloor\psi(U)\rfloor\ \ \text{(relation (1))}\\ &=& \phi(T^{(0)}\vee_{x_1}\cdots\vee_{x_m}T^{(m)})\prec_R\psi(U)\ \ \text{(defintion of }\psi\text{)}\\ &=& \psi(T)\prec_R\psi(U).\end{array}$$

Case 2. If $T^{(m)}=|$, then

$$\begin{array}{rcl}\psi(T\prec U) &=& \phi(T^{(0)}\vee_{x_1}\cdots\vee_{x_m}U)\ \ \text{(definition of } \prec\text{)}\\ &=& \overline{\lfloor\psi(T^{(0)})\rfloor}x_1\cdots x_m\lfloor\psi(U)\rfloor\ \ \text{(definition of }\psi\text{)}\\ &=& \psi(T^{(0)}\vee_{x_1}\cdots\vee_{x_m}T^{(m)})\lfloor\psi(U)\rfloor\ \ \text{(defintion of }\psi\text{)}\\ &=& \psi(T)\prec_R\psi(U).\end{array}$$

This proves $\psi(T\prec U)=\psi(T)\prec_R\psi(U)$. We similarly prove $\psi(T\succ U)=\psi(T)\succ_R\psi(U)$ and $\psi(T\cdot U)=\psi(T)\cdot_R\psi(U)$. Thus ψ is a homomorphism in **DT**.

(2) follows from the uniqueness of the dendriform trialgebra morphism $\mathbf{DT}(V)\to Ш^{NC,0}(\Omega)$ extending the map $i:V\to Ш^{NC,0}(\Omega)$.

(3) We only need to prove $TW(V)\subseteq\psi(\mathbf{DT}(V))$ and $\psi(\mathbf{DT}(V))\subseteq TW(V)$. To prove the former, we prove $S_n\subseteq\psi(\mathbf{DT}(V))$ by induction on n.

When $n=0$, $S_n=X$ so the inclusion is clear. Suppose the inclusion holds for $1\leq n\leq k$. Then by the definition of $S_{k+1}(V)$ in Eq. (44), an element of $S_{k+1}(V)$ has length greater or equal to 2. We apply the induction on its length. If the length is 2, then it is one of the following two cases.

i) It is $\mathfrak{X}\lfloor\mathfrak{X}'\rfloor$ with $\mathfrak{X} \in X$, $\mathfrak{X}' \in S_n(V)$. Then it is $\mathfrak{X} \prec_R \mathfrak{X}'$ which is in $\psi(\mathbf{DT}(V))$ by the induction hypothesis and the consequence from part (1) that $\psi(\mathbf{DT}(V))$ is a sub dendriform algebra.

ii) It is $\lfloor\mathfrak{X}\rfloor\mathfrak{X}'$ with $\mathfrak{X} \in D_n(V)$ and $\mathfrak{X}' \in X$. Then the same proof works.

Suppose all elements of S_{k+1} with length $\leq q$ and ≥ 2 are in $\psi(\mathbf{DT}(V))$. Consider an element $\mathfrak{X}$ of S_{k+1} with length $q+1$. Then $q+1 \geq 3$. If $q+1=3$, we again have two cases.

i) $\mathfrak{X} = \lfloor\overline{\mathfrak{X}}_1\rfloor\mathfrak{X}_2\lfloor\overline{\mathfrak{X}}_3\rfloor$ with $\overline{\mathfrak{X}}_1, \overline{\mathfrak{X}}_2 \in S_n(V)$ and $\mathfrak{X}_1 \in X$. Then it is $(\overline{\mathfrak{X}}_1 \succ_R \mathfrak{X}_2) \prec_R \overline{\mathfrak{X}}_3$. By induction hypothesis on n, $\overline{\mathfrak{X}}_1$ and $\overline{\mathfrak{X}}_3$ are in the sub dendriform dialgebra $\psi(\mathbf{DT}(V))$. So the element itself is in $\psi(\mathbf{DT}(V))$.

ii) $\mathfrak{X} = \mathfrak{X}_1\lfloor\overline{\mathfrak{X}}_2\rfloor\mathfrak{X}_3$ with $\mathfrak{X}_1, \mathfrak{X}_3 \in X$ and $\overline{\mathfrak{X}}_2 \in S_n(V)$. Then $\mathfrak{X} = \mathfrak{X}_1 \cdot_R (\overline{\mathfrak{X}}_2 \succ \mathfrak{X}_3)$ which is in $\psi(\mathbf{DT}(V))$.

If $q+1 \geq 4$, then $\mathfrak{X}$ can be expressed as the concatenation of $\mathfrak{X}_1$ and $\mathfrak{X}_2$ of lengths at least two and hence are in $TW(V)$. By induction hypotheses, $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are in $\psi(\mathbf{DT}(V))$. Therefore $\mathfrak{X} = \mathfrak{X}_1 \cdot_R \mathfrak{X}_2$ is in $\psi(\mathbf{DT}(V))$.

This completes the proof of the first inclusion. The proof of the second inclusion follows from a similar induction on the degree of trees in $\mathbf{DT}(V)$.

(4) By the definition of ψ and part (3), ψ gives a one-one correspondence between $\cup_{n\geq 0} T_{n,X}$ as a basis of $\mathbf{DT}(V)$ and $TW(V)$ as a basis of $\psi(\mathbf{DT}(V))$. Therefore ψ is injective. □

Remark 4.1. *It is interesting to notice that Holtkamp and Foissy* [**Fo**, **Hol1**, **A-S1**] *showed that the non-commutative version of the Connes-Kreimer Hopf algebra* **NCK** *of rooted trees is isomorphic to Loday-Ronco's. Composing the injection ψ in Theorem 4.8 with such an isomorphism, as the one explicitly defined in* [**A-S1**]*, we obtain an injection from* **NCK** *into free noncommutative Rota-Baxter algebras.*

References

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