On the Riemann zeta-function and analytic characteristic functions

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Contents

Abstract

Use the Riemann zeta-function, $\zeta(s)$, to define the odd meromorphic function $f(s) := 1/(b(s)\,\zeta(1/2+s))$. Here $b(s) := \sin(\pi s/4)\,a(1/2+s)$ with $a(s) := \pi^{-s/2}\,2\Gamma(1+s/2)(s-1)$. The classical unresolved conjectures used herein concern the zeta function's non real (critical) zeros z. They include: RH, the Riemann hypothesis, $\mathrm{Re}\,(z) = 1/2$, and SZC, the simple zeros conjecture, $\zeta'(z)$ is nonzero. Let V_u be the vertical strip of s with $u < \mathrm{Re}\,(s) < u+4$. Say u is a multiple of four. RH and SZC, together with conjectures advanced by the author, are used to show that $f(s)$ on V_u provides an analytic characteristic function: $(-1)^{u/4}\,f(s) = \int_R (dy)\,e^{sy}\cdot P_u(y)$, with the density function $P_u(y)$ positive. The essential case with $u = 0$ implies RH. A formula is obtained for $P_u(y)$, which for y negative involves the critical zeros. An initial step is the determination of the Mittag-Leffler partial fraction expansion of $f(s)$. Numerical studies made using Odlyzko's tables of the critical zeros of the zeta function corroborated the basis of this work.

Analogous conjectures for the more recondite case of the Dirichlet L-functions and that of the Ramanujan tau Dirichlet L-function are stated.

Consider the zeta-function case with $u \geq 0$. Take $\mathrm{Re}\,(s) > 1$, if u is zero. An alternative formula is obtained for $P_u(y)$, without relying on RH, SZC or other unproven conjectures. It does not involve the critical zeros of zeta. First a formula for the inverse Fourier transform of $1/b(s)$ is found. Hypergeometric functions enter therein. $P_0(y) = 4\cdot\pi^{-3/4}\cdot\Sigma_{k\geq 1}(-1)^{k-1}(\pi e^{-2y})^{2k}/(\Gamma(5/4+2k)(2k-1/4)\,\zeta(1/2+4k))$ extends $P_0(y)$ to an entire function.

"Internal" metric norms are introduced as a generalization of the Fourier integral metric geometries of J. von Neumann and I. J. Schoenberg. The positive definiteness of $(-1)^{u/4}f(s)$ on V_u is expressed via an inner metric norm.

1 Introduction

Let s be a complex number with real part $\mathrm{Re}\,(s) > 1$. Set $\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}$. Euler established the product representation $\zeta(s) = \Pi_p(1-p^{-s})^{-1}$, with p any prime number. Thus the zeta-function, $\zeta(s)$, does not have any zeros with real part > 1.

The zeta-function, $\zeta(s)$, has a unique analytic continuation to the complex plane except at $s = 1$, where it has a simple pole.

Set $a(s) := \pi^{-s/2} \cdot 2\Gamma\left(1 + \frac{s}{2}\right)(s-1)$ and $q(s) := a(s)\,\zeta(s)$. Then $q(s)$ is an entire function. The Riemann zeta-function, $\zeta(s)$, satisfies the functional equation $q(-s) = q(s)$. Also $q(s^*) = (q(s))^*$.

Let $b(s) := \sin\left(\frac{\pi}{4} \cdot s\right) a\left(\frac{1}{2} + s\right)$, $n(s) := \sin\left(\frac{\pi}{4}\,s\right) q\left(\frac{1}{2} + s\right)$ and $f(s) := 1/n(s)$. Then $n(-s) = -n(s)$. Also $n(s^*) = (n(s))^*$. $f(s)$ behaves similarly. The non-real zeros of $\zeta\left(\frac{1}{2} + s\right)$, $q(s)$ and $n(s)$ are the same and for each such zero its multiplicity is the same for each of these functions. These zeros lie in the open vertical strip of s with $-\frac{1}{2} < \mathrm{Re}\,(s) < \frac{1}{2}$ and are symmetric about each of the real axis and the imaginary axis.

In 1859 B. Riemann [15] formulated the following conjecture.

Riemann Hypothesis, RH: The real part of each non-real zero of $\zeta(s)$ is one-half.

The Riemann hypothesis has not been resolved since its formulation about 150 years ago despite the determined efforts of generations of leading mathematicians. The intrinsic relation of the zeta-function with the primes expressed in its Euler factorization has led to the reduction of the solution of deep questions in multiplicative number theory, in particular concerning the distribution of the primes, to the resolution of RH. The Riemann hypothesis is one of the most celebrated unsolved problems of mathematics. RH is allied with the unresolved conjecture stated next.

Simple Zeros Conjecture, SZC. Each non-real zero, z, of $\zeta(s)$ is of order one, $\zeta'(z) \neq 0$.

RH enables the enumeration of the non-trivial zeros $z_k = \frac{1}{2} + i\,r_k$ of ζ, with k an integer, $r_{-1} < 0 < r_1$ and $r_k < r_{k+1}$. Then $r_{-k} = -r_k$.

The real zeros of $n(s)$ and the real poles of $f(s)$ are at the multiples of four, $4k$, and are simple.

Use will be made of RH, SZC and the conjectures C_1, C_2, C_3 which are supported by computer studies and are stated later herein. Take J to be the "joint" conjecture that each of RH, SZC and C_1, C_2 holds.

Let V_u be the open vertical strip of all s with $u < \mathrm{Re}\,(s) < u + 4$. Say k is an integer, $u = 4k$ and s is in V_u. We will obtain the representation $f(s) = (-1)^k \int_{-\infty}^{\infty} e^{sy}\, d\mu_u(y)$ with μ_u a positive measure on the real line. Thus $f(s)$ is constituted from analytic characteristic functions. The latter functions are studied in Lukacs [6].

The measure density $p_u(y) := d\mu_u(y)/d(y)$ must be real since $f(s^*) = (f(s))^*$. Also $p_{-(u+4)}(y) = p_u(-y)$, since f is odd.

Next an explicit formula for the density $p_u(y)$ is given. The imaginary parts r_k of the non-real zeros of the zeta-function are components of the formula for $p_u(y)$ when $y < 0$.

Definitions. $c(z) := 1/n'(z)$ for $z = i\,r_j$ or $z = 4w$ with w an integer.

$$c_j := (-1)^j\, c(i\,r_j) = 1/\left(\sin h\left(\frac{\pi}{4} \cdot r_j\right) \cdot i \cdot a\left(\frac{1}{2} + i\,r_j\right) \zeta'\left(\frac{1}{2} + i\,r_j\right)\right).$$

$$K := \frac{4(\pi^{-3/4})}{\Gamma(1/4)}, \qquad (v)_m := \prod_{j=0}^{m-1} (v + j).$$

$$b_w := 1/\left(\left(\frac{1}{4}\right)_{2w+1} \cdot (8w-1) \cdot \zeta\left(\frac{1}{2} + 4w\right)\right).$$

Then $c(4w) = K\,b_w(-(\pi^2))^w$. Set $t_w(y) := (-1)^w \cdot b_w \cdot (\pi\,e^{2y})^{2w}$.

The convergence of $\sum_{j=1}^{\infty} c_j$ will later be proven using conjecture C_1 with $\varepsilon_1 < 3/4$.

Set $\lambda(y) := 2 \sum_{j=1}^{\infty} (-1)^j \, c_j \cos(r_j \, y)$. Let $h(y) := \sum_{w=0}^{\infty} t_w(y)$. Say $u = 4k$ with k a non-negative integer. Take y to be positive. Define

$$P_u(-y) := (-1)^k \left(\lambda(y) + K \left(h(-y) + \sum_{w=1}^{k} t_w(y) \right) \right)$$

and

$$P_u(y) := K \cdot (-1)^{k+1} \cdot \sum_{w=k+1}^{\infty} t_w(-y) \, .$$

Assume the joint conjecture J.

Main Theorem.

(i) Say s is in V_{4k}. Then

$$f(s) := (-1)^k \int_{-\infty}^{\infty} e^{sy} \, P_{4k}(y) \, dy \, .$$

(ii) $P_{4k}(y) > 0$ for $y > 0$.

Conjecture $\mathbf{C}_3$. $P_{4k}(y) > 0$ for $y < 0$.

Assuming J and C_3 the Main Theorem indicates that $(-1)^k \cdot f\,(iw)$ is positive definite in w for w on the horizontal strip $-i\, V_{4k}$. This is the mecanism underlying the "ridge" property of f described next.

Let x, t be real. Say g maps the complex plane C into C extended by a point at infinity. It is said that g is a "ridge function" (with ridge along the real axis) if for t nonzero $|g(x)| > |g(x+it)|$ for each nonzero x and $|g(0)| \geq |g(it)|$. (See Lukacs [6], Ch. 7, p. 195). Reversing the prior inequalities defines when g has the "groove property".

The Main Theorem together with C_3 implies that the meromorphic function $f(s)$ is a ridge function. Thus the entire function $n(s)$ is a groove function.

The groove property of $n(s)$ on V_0 implies RH and SZC.

Say $x > 1$ and t is a nonzero real. Then $|\zeta(x+it)| < \zeta(x)$. However, computer calculations indicate that there exist x with $1/2 < x < 1$ for which $\zeta(x+it) = \zeta(x)$ for some nonzero t and so the groove property of $\zeta(s)$ for $\mathrm{Re}\,(s) > 1$ does not extend to the critical strip. This underscores the significance of the factor $b(s)$ in obtaining the groove property for $n(s) := b(s)\,\zeta\left(\frac{1}{2}+s\right)$.

Analytic number theory makes yet unresolved conjectures, analogous to RH and SLC, concerning the Dirichlet L-functions, $L(s,\chi)$ and the Ramanujan tau Dirichlet function, $r(s)$. Later appropriate modifications of $L(s,\chi)$ and $r(s)$ are specified which we conjecture are analytic characteristic functions. This is very surprising for the case of $L(s,\chi)$ with $\chi(n)$ neither 0 nor 1 for some n, since $L(s,\chi)$ is not even a groove (nor a ridge) function for $\mathrm{Re}\,(s) > 1$. That is, for $x > 1$ and any positive j, there exist real numbers (integers) y, y' greater than j, such that

$$|L(x+iy,\chi)| > |L(x,\chi)| > |L(x+iy',\chi)| \, .$$

The proof uses the Euler factorization for $L(s,\chi)$ and the Kronecker approximation theorems. There is computational evidence for certain χ that x, t exist, with $1/2 < x < 1$ and t a nonzero real, for which $L(x+it,\chi) = L(x,\chi)$.

Dirichlet L functions, $L(s,\chi)$

Assume χ is a primitive non-principal character mod k with $k > 1$. Set $q(s,\chi) := (k/\pi)^{s/2}\Gamma((s+a)/2)L(s,\chi)$. Here $a = 0$ if $\chi(-1) = 1$ whereas $a = 1$ if $\chi(-1) = -1$. The functional equation is $q(1-s,\chi) = w(\chi)q(s,\chi^*)$. Here $w(\chi) := jG(1,\chi)(k^{-1/2})$. This j is 1 if $\chi(-1) = 1$, whereas j is $-i$ if $\chi(-1) = -1$. Moreover $G(1,\chi)$ is the Gaussian sum $\Sigma_r \chi(r)\exp(2\pi i r/k)$, with $r = 1, 2, \ldots, k$. Note that $|w(\chi)| = 1$ and $w(\chi^*) = (w(\chi))^*$.

Dirichlet L-Function Conjecture (DLFC). The non-trivial zeros z of the L-function $L(s,\chi)$ have $\mathrm{Re}(z) = 1/2$.

(Simple Zeros) Conjecture (SZCD). The non-trivial zeros of $L(s,\chi)$ are all simple.

Set $f(s,-1) := \sin(\pi s/4)$ and $f(s,1) := s\cos(\pi s/4)$. Define $v(s,\chi) := f(s,\chi(-1))q((1/2)+s,\chi)$. Then $v(s,\chi)$ is an entire function of s. Also $v(-s,\chi) := -w(\chi)v(s,\chi^*)$.

Assume χ is real. Set $n(s,\chi) := v(s,\chi)$. The next conjecture restricted to $0 < x < 1/2$, implies that the Dirichlet L-function conjecture (DLFC) and SZCD hold when χ is real-valued.

Conjecture D1. $n(s,\chi)$ is a groove function in s.

Now allow χ to be complex.

Define $g(s,\chi) := v(s,\chi)v(s,\chi^*)$ and $h(s,\chi) := s\sin(\pi s/2)q((1/2)+s,\chi)q((1/2)+s,\chi^*)$. Then g, h are even functions in s. Fix $n = g$ or $n = h$. The following conjecture restricted to $0 < x < 1/2$, implies DLSC and SZC for complex χ.

Conjecture D2. $n(s,\chi)$ is a groove function in s.

Assume $x(n)$ is not real for some n. Set $j(s,\chi) := f(2s,\chi(-1))q((1/2)+s,\chi)\ q((1/2)+s,\chi^*)$. The possibility that $j(s,\chi)$ is a groove function in s is sensitive to any exceptional nearness of pairs z, z' of zeros of $q((1/2)+s,\chi)$, $q((1/2)+s,\chi^*)$ respectively.

Computer calculations support conjectures D1, D2. These conjectures are corollaries of the next conjecture.

Main Conjecture for $L(s,\chi)$. Let u, u' be successive real zeros z of $f(z,\chi)$ and σ be the sign of $n'(u,\chi)$. Restrict $s = iw$ to be on the open vertical strip $u < \mathrm{Re}\,(s) < u'$. Then $\sigma/n(iw,\chi)$ is positive definite in w.

The Ramanujan Tau Dirichlet Function, $r(s)$

Let $\tau(n)$ be generated by $v(\Pi_n(1-v^n))^{24} = \Sigma_n \tau(n)v^n$. The Ramanujan tau Dirichlet function, $r(s)$, for $\mathrm{Re}(s) > 7$, is given by $r(s) := \Sigma_n \tau(n)n^{-s} = \Pi_p(1-\tau(p)p^{-s}+p^{11-2s})^{-1}$, p prime, (Euler product). $r(s)$ extends to an entire function in s on C. Define $q(s) := (2\pi)^{-s}\Gamma(s)r(s)$. The functional equation is $q(12-s) = q(s)$.

Ramanujan Conjecture. (RC) The non-trivial zeros z of the Ramanujan function, $r(s)$, have $\mathrm{Re}(z) = 6$.

(Simple Zeros) Conjecture (SZCR). The non-trivial zeros of $r(s)$ are all simple.

The q associated with the Ramanujan function, $r(s)$, is used to define $n(s) := \sin(\pi s/2)q(6+s)$. This $n(s)$ is entire and odd.

Conjecture R. $n(s)$ is a groove function in s.

The latter conjecture limited to $0 < x < 1$ implies the Ramanujan conjecture and SZCR.

Next we derive the formula for the measure density $p_u(y)$.

Conjecture R is a corollary of the following.

Main Conjecture for $r(s)$. Let k be an integer. Restrict $s = iw$ to the open vertical strip $2k < \mathrm{Re}\,(s) < 2(k+1)$. Take σ to be the sign of $n'(2k)$. Then $\sigma/n(iw)$ is positive definite in w.

Return to the case of the zeta-function.

For fixed $x > 0$ the inverse Fourier transform of $f(x+it)$ exists provided $\int |f(x+it)|dt$ is finite.

We have set $a(s) := (\pi^{-s/2})2\Gamma(1+s/2)(s-1)$. Let $b(s) := \sin(\pi s/4)a(1/2+s)$. Then $f(s) := 1/n(s)$ with $n(s) := b(s)\zeta(1/2+s)$.

The Stirling formula for the gamma function is used in deriving the following approximation to a with $s = x+it$, x nonnegative, $\theta := \arg(1/2+s)$, $|\theta| < \pi/2$:

$$|a(1/2+s)| = (2^3\pi)^{1/4}|s|^{7/4}(|1/2+s|/(2\pi))^{x/2}e^{-uj(m)}(1+\varepsilon(s)/|s|)\,.$$

Here, $u = (1/2+x)/2$, $j(y) := y\arctan(y)+1$, $m := t/(1/2+x)$. Say $t \geq 0$. For $y \geq 1$, $j(y) = (\pi/2)\,y + (\varepsilon_1(y)/y)$. The $\varepsilon(s)$, $\varepsilon_1(y)$ vanish as $|s|$, y respectively become infinite. Thus for fixed x and t approaching infinity: $|a(1/2+s)| = 2^{1/2}(2\pi)^{1/4-x/2}t^{7/4+x/2}e^{-\pi t/4}(1+\varepsilon_2(t)/t)$, with $\varepsilon_2(t)$ approaching zero. So $1/a(1/2+s)$ grows like $ke^{\pi t/4}t^{-(7/4+x/2)}$, with k a positive constant.

Multiplying $a(1/2+s)$ by $\sin(\pi s/4)$ annihilates the exponential growth in t. This is evident from

$$|b(s)| = (\pi/2)^{1/4}|1-e^{-\pi t/2}e^{i\pi x/2}||s|^{7/4}(|1/2+s|/(2\pi))^{x/2}e^{uh(m)}(1+\varepsilon(s)/|s|)\,.$$

Here $h(y) := \pi y/2 - j(y)$. For $y \geq 1$, $h(y) = -\varepsilon_3(y)/(3y^2)$ with $\varepsilon_3(y)$ approaching 1 as y becomes infinite. For fixed nonnegative x and large t: $|b(s)| = (\pi/2)^{1/4}(2\pi)^{-x/2}t^{7/4+x/2}(1+\varepsilon_4(t)/t)$, with $\varepsilon_4(t)$ vanishing.

Therefore, for fixed positive x not a multiple of four, $\int_{-\infty}^{\infty}|f(x+it)|dt$ is finite provided: $\zeta(1/2+x+it)$ is nonzero for all t and $|\zeta(1/2+x+it)| \geq ct^{-(3/4+x/2)+\delta}$ for some positive constants c, δ and large t. This works when $x \geq 1/2$. Say $0 < x < 1/2$. Assume RH. Temporarily assume that the asymptotic lower bound stated for ζ holds at x for some $c(x)$, $\delta(x)$.

Next we give a heuristic derivation of the formula for the density function $P_{4k}(y)$ of the Main Theorem. We illustrate the method used with the simplified case of $f(s) := 1/n(s)$ with $n(s)$ a polynomial.

Let $a(s) := (-1)^M\Pi_w(s-\ell_w)$, with $1 \leq w \leq M$. Say the ℓ_w are real and ordered by magnitude. Let ℓ_0, ℓ_{M+1} be respectively $-$, $+$ infinity. $a(s)$ inherits the groove property from its linear factors. Set $f(s) := 1/a(s)$. Say x is strictly between ℓ_ω and its successor. Take $\sigma(w)$ to be 1, if $\omega \geq w$, and -1 otherwise. Let $V(\ell_\omega)$ be the open vertical strip $s = x+it$ of such x with t any real. $(-1)^\omega a(s) = \Pi_w\sigma(w)(s-\ell_w)$. Thus $(-1)^\omega a(x) > 0$. On $V(\ell_\omega)$ the function $(-1)^\omega f(s)$ is the analytic Fourier transform of a positive frequency function, $(-1)^\omega f(s) = \int e^{sy}p(y)dy$. This reduces to thus representing the reciprocal of each linear factor of $(-1)^\omega a(x)$. Apply: IFT (gh) is the convolution IFT (g), IFT (h). Represent each $1/(\sigma(w)(s-\ell_w))$ via the following. If $\mathrm{Re}(b) > 0$, then $1/b = \int e^{by}dy$, $(y \geq 0)$. If $\mathrm{Re}(b) < 0$, then $1/b = \int e^{by}dy$, $(y \geq 0)$.

The function $n(s)$ arising above from the zeta-function has zeros on the real and on the imaginary axis (and no others, assuming RH). It is enlightning to examine similar models for polynomials. Let $\alpha(s) := \Pi_w(s-\ell_w)$, $(1 \leq w \leq M)$, $\beta(s) := \Pi_w(s-v_k)$, $(1 \leq w \leq N)$ with the zeros positive and increasing with the index w, k respectively. Set $p(s) := s\alpha(s)\alpha(-s)$, $q(s) := \beta(is)\beta(-is)$, $n(s) := p(s)q(s)$ and $f(s) := 1/n(s)$.

q is even in s. Also p, n, f are odd. n' is even. Each of q, p, n, f, n' has real coefficients: $(q(s))^* = q(s^*)$, etc. $n'(s)$ is real on both the real and the imaginary axis. Set $\ell_0 := 0$. The derivatives $n'(\ell_w)$, $n'(iv_k)$ have signs $(-1)^w$, $(-1)^k$ respectively.

$n(s) = p(s)q(s)$ may not have the groove property, even though $p(s)$ does. Let x, v be fixed positive reals. $d(t) := |(x+it)-iv|$, decreases in t up to v and increases thereafter. The factor q of n causes problems as explained next. For all small x, $|n(x+iv)| < |n(x)|$, at each root iv of q for which $|n'(iv)|/n'(0) < 1$, since as x vanishes $n(x+iv)/n(x)$ approaches $n'(iv_1)/n'(0)$. A simple instance has $n(s) := s\gamma(s)\gamma(-s)$ with $\gamma(s) := (s-iv_1)(s-iv_2)$. Then $|n'(iv_1)|/n'(0) = 2(1-(v_1/v_2)^2)$. Thus n is not a groove function if $(v_2-v_1)/v_1 < 2^{1/2}-1$.

Return to the general case $n(s) := p(s)q(s)$ and $f(s) := 1/n(s)$. We now derive the Fourier representation of f from its partial fraction decomposition (pfd): $f(s) = \Sigma_z c(z)/(s-z)$, with z varying over the zeros of n and $c(z) := 1/(n'(z))$. Note that $n'(\ell_w) := p'(\ell_w)q(\ell_w)$ and $n'(iv_k) := p(iv_k)q'(iv_k)$.

Replace each term of the pfd with its Fourier representation. We state the result. Let $\ell_{-(M+1)}$, ℓ_{M+1} be respectively $-$, $+$ infinity. Say x is positive. Let x vary strictly between ℓ_ω and its successor. Say $s = x + it$. Then $(-1)^\omega f(s) = \int e^{sy} \rho_\omega(y) dy$, with ρ_ω as described next.

Set $c_w := 1/|p'(\ell_w) q(\ell_w)|$. Put $\gamma_k := 1/|p(iv_k) q'(iv_k)|$. $\lambda(y) := 2\Sigma_k (-1)^k \gamma_k \cos(v_k y)$, $(1 \le k \le N)$. Set $\tau_w(y) := (-1)^w c_w \exp(-\ell_w y)$. Let $h(y) := \Sigma_w \tau_w(y)$, $(0 \le w \le N)$.

For $y < 0$, define $\rho_\omega(y) := (-1)^\omega (\lambda(y) + \Sigma_w \tau_w(y))$, $(1 \le w \le \omega)$.

For $y \ge 0$, define $\rho_\omega(y) := (-1)^{\omega+1} \Sigma_w \tau_w(y)$, $(\omega + 1 \le w \le M)$.

Allow M, N to become infinite. Take $f(s) := 1/n(s)$, with $n(s) := b(s)\, \zeta\left(\frac{1}{2} + s\right)$.

The outlined derivation of the formula for $P_{4k}(y)$ depends upon the validity of the partial fraction decomposition $f(s) = \Sigma_z \frac{c(z)}{s-z}$, with z varying over the zeros of $n(s)$ and $c(z) := 1/n'(z)$. This validity is later established using the joint conjecture J.

The resulting formula for $P_{4k}(y)$ involves the ordinates r_j of the nonreal zeros of $\zeta(s)$. Another formula for $P_{4k}(y)$ in the representation of $f(s)$, with s now restricted to having $\mathrm{Re}\,(s) > \frac{1}{2}$, will be proven without using RH, SZC or any of the allied conjectures C_ℓ. This alternative formula for $P_{4k}(y)$ does not involve the nonreal zeros of the zeta-function. It does involve certain hypergeometric functions of Gauss and Kummer.

This alternative Fourier representation of $f(iw)$, with $f(s) := \frac{1}{b(s)} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+s}}$, is later obtained by first determining the inverse Fourier transform of $1/b(iw)$.

A semi-metric norm is a map, m, from an Abelian group $(A, +, 0)$ into the reals R with the properties: $m(0) = 0$, $m(x) > 0$ for x nonzero and $m(-x) = m(x)$. A metric norm is a semi-metric norm which is also subadditive, $m(x + y) \le m(x) + m(y)$.

Say x, t are real and $x > 1$. Fix x. $\zeta(x + it)/\zeta(x)$ is a characteristic function in t, since $\zeta(x + it) = \sum_{n=1}^{\infty} n^{-x} \cdot e^{-it \log n}$. This has the following geometric consequences. $z_{E,x}(t) := (1 - (|\zeta(x + it)|/\zeta(x))^2)^{1/2}$ and $z_{I,x}(t) := \left|1 - \frac{\zeta(x+it)}{\zeta(x)}\right|^{1/2}$ are metric norms in t. The norms $z_{E,x}(t)$ are instances of a type of metric geometry related to Fourier integrals studied by I.J. Schoenberg and J. von Neumann [16]. $z_{I,x}(t)$ is a special case of the "internal" metric norms presented later herein.

The metric norm property of $z_{\theta,x}(t)$ does not extend to the critical strip of $x + it$ with $0 < x < 1$, for $\theta = E$ or $\theta = I$. Computer calculations indicate the existence of x_0 with $0 < x_0 < \frac{1}{2}$ and of x_0 with $\frac{1}{2} < x_0 < 1$ such that there exist t for which $\zeta(x_0 + it) = \zeta(x_0)$. This leads to x_0, t with $z_{\theta,x_0}(2t) > 2z_{\theta,x_0}(t)$.

Nevertheless metric norms will be built from the zeta-function on the critical strip in a way which embodies each of RH and SZC in the norm property. This uses the positive definiteness in w of

$$f(s) := 1/n(s) = 1/\left(b(s)\, \zeta\left(\frac{1}{2} + s\right)\right)$$

with $s = iw$ in V_0, as in the Main Theorem with the assumptions J and C_3.

Say x, t are real and x is not an integral multiple of four. Set

$$m_{E,x}(t) := \left(1 - \left(\frac{n(x)}{|n(x+it)|}\right)^2\right)^{1/2}$$

and

$$m_{I,x}(t) := \left|1 - \frac{n(x)}{n(x+it)}\right|^{1/2}.$$

(If $|x| < \frac{1}{2}$ and $n(x + it) = 0$, take $m_{\theta,x}(t) = \infty$.) Define $d_{\theta,x}(t_1, t_2) := m_{\theta,x}(t_1 - t_2)$.

The Main Theorem, with J, C_3 assumed, has geometric consequences. Each $m_{\theta,x}(t)$ is a metric norm in t. Thus $d_{\theta,x}(t_1, t_2)$ is a translation invariant metric, on the real line.

$m_{E,x}(t)$ is an example of the "external" metric norms presented below as abstract versions of the Fourier metric norms of Schoenberg, et al. The hallmark of the external metrics is that each embeds in some Hilbert space H.

The $m_{I,x}(t)$ are instances of the more recondite "internal" norms introduced below.

Each external norm is an internal norm. However there are internal norms which are not embeddable in Hilbert space, probably even in the strong sense discussed in Appendix I.

Theorem 2.1, below, establishes a new class of metric spaces from which the (essential) inner metrics can be obtained as subspaces via the embedding of its Corollary 2.2.

In a similar manner metric norms can be obtained from the Dirichlet functions, $L(s, x)$, and from the Ramanujan function, $r(s)$, by employing the corresponding main conjecture.

Let T be the abelian group of the reals mod 2π under addition. For x in T set $m(x) = |a(x)|$ with $a(x) = 1 - e^{ix}$. Then m is a metric norm on T. The associated translation invariant metric $m(x - y)$ is embedded in the Euclidean plane by $m(x - y) = |a(x) - a(y)| = |e^{iy} - e^{ix}|$. Define $M(P, \mu, T)$ to be the set of μ-measurable maps from P into T. Identify v, w in M for which $v - w$ vanishes outside a set of measure zero.

Let H be the Hilbert space $L_2(P, \mu, C)$ with norm $\| \|_2$. Let W be the set of x in $M(P, \mu, T)$ for which $\int |1 - \exp(ix(s))|^2 \, d\mu(s)$ is finite. Set

$$m_E(x) := \|a \circ x\|_2 = \left(\int |1 - \exp(ix(s))|^2 \, d\mu(s) \right)^{1/2} .$$

m_E is a metric norm. These m_E are the "external" or "outer" metric norms ($\int$ external to $| \; |$).

The metric $m_E(x - y)$ on W is isometrically imbedded in the Hilbert space H by mapping x into $a \circ x$.

A generalization of the outer metric norms, and thus in particular of Fourier metric geometry, is described next.

Consider the Banach space $L_1(P, \mu, C)$ with norm $\| \|_1$. Let W be the set of x in $M(P, \mu, T)$ for which $\|a \circ x\|_1 = \int |1 - \exp(ix(s))| \, d\mu(s)$ is finite. For x in W set

$$m_I(x) := \left| \int (1 - \exp(ix)(s))) \, d\mu(s) \right|^{1/2} .$$

$m_I(x)$ is a metric norm on W. This is shown in Corollary 2.2 below via an isometric embedding of the internal metric in a metric space which issues from Theorem 2.1.

The m_I are the "internal" or "inner" metric norms.

Each external norm in an internal norm, since $|1 - e^{ix}|^2 = 1 - e^{ix} + 1 - e^{-ix}$.

2 A Generalization of Fourier Metric Geometry

Definitions.

Let x, y be complex numbers, x, $y \in \mathbb{C}$,

$$g(x, y) := |x - y|^2 - i \cdot 2\,\mathrm{Im}(x\,\bar{y}) = |x|^2 + |y|^2 - 2x\bar{y} .$$

Let μ be a measure on Ω. Take M to be the linear space of μ-measurable functions on P with complex values. L_2 is the Hilbert space of x in M with $\|x\|^2 = \int_P (d\mu(s)) \, |x(s)|^2$ finite.

Given $x,\, y$ in L_2 set

$$G(x,\, y) := \|x - y\|^2 - i \cdot 2\,\mathrm{Im}\langle x,\, y\rangle\,.$$

Thus

$$G(x,\, y) = \int_P (d\mu(s))\, g(x(s),\, y(s)) = \|x\|^2 + \|y\|^2 - 2\langle x,\, y\rangle\,.$$

Set

$$d(x,\, y) := |G(x,\, y)|^{1/2}\,.$$

Then

$$d(x,\, y) = (\|x - y\|^4 + (2\,\mathrm{Im}\langle x,\, y\rangle)^2)^{1/4}\,.$$

Given $x,\, \alpha$ in M, let $x\alpha$ be their pointwise product: $(x\alpha)(s) = x(s)\,\alpha(s)$.

g has the following properties. Let $x,\, y,\, \alpha$ be complex numbers with $|\alpha| = 1$.

(Rotational Invariance) $g(x\alpha,\, y\alpha) = g(x,\, y)$.

(Homogeneity) $g(x\lambda,\, y\lambda) = \lambda^2\, g(x,\, y)$ for $\lambda \geq 0$.

(Special Translation Invariance) Let $t = r(x - y)$ with r real.

$$g(x + t,\, y + t) = g(x,\, y)$$

for any real r, since $(x + t) \cdot (y + t)^* = x\,\bar{y} + r(|x|^2 - |y|^2)$ implies

$$\mathrm{Im}((x + t) \cdot (y + t)^*) = \mathrm{Im}(x\,\bar{y})\,.$$

Therefore G, d inherit these properties from g. Let α be in M with $|\alpha|$ a constant, λ say (μ almost everywhere, μ-a.e.). Say $x,\, y$ are in L_2.

$$G(x\alpha,\, y\alpha) = \lambda^2\, G(x,\, y)\,.$$

Hence $d(x\alpha,\, y\alpha) = \lambda d(x,\, y)$. α is a "similarity transformation". Say r is in L^2 with r real-valued. Set $t = r(x - y)$. Then

$$G(x + t,\, y + t) = G(x,\, y)\,.$$

Hence

$$d(x + t,\, y + t) = d(x,\, y)\,.$$

$d(x,\, y)$ is a semi-metric since $d(x,\, y) \geq 0$, $d(x,\, y) = 0$ iff $x = y$ (μ-a.e.), $d(x,\, y) = d(y,\, x)$.

Say $x,\, y,\, z$ are in L_2. Define

$$\epsilon(x,\, y,\, z) := 2^4 |T| \cdot |L|$$

with $T = \mathrm{Im}\langle x,\, y\rangle$ and $L = \mathrm{Im}\langle z,\, x\rangle = \mathrm{Im}\langle r,\, x\rangle$, $r = z - x$.

$$\epsilon(x\alpha,\, y\alpha,\, z\alpha) = \lambda^4\, \epsilon(x,\, y,\, z)$$

for any α in M with $|\alpha| = \lambda$ (μ-a.e.).

Abbreviations: "a" for $d(x,\, y)$, "b" for $d(z,\, x)$, "c" for $d(z,\, y)$ and "ϵ" for $\epsilon(x,\, y,\, z)$.

Theorem 2.1. d has the strong triangle inequality property

$$(c^4 + \epsilon)^{1/4} \leq a + b\,.$$

Corollary 2.1. d is a metric on L_2.

Proof of Theorem. Set $J := c^4 - (a^4 + b^4)$. The strong triangle inequality holds iff

$$J + \epsilon \leq 4a^3 b + 6a^2 \cdot b^2 + 4a \cdot b^3\,.$$

The proof employs the following definitions.

$\alpha = \mathrm{Re}\langle x-y,\, r\rangle$. $B = \mathrm{Im}\langle y,\, r\rangle$. $K_1 := 2^2(\|x-y\|^2\,\alpha - 2\mathrm{BT})$.
$K_2 := 2\|x-y\|^2\,\|r\|^2 + 2^2(\alpha^2 + B^2 - L^2)$. $K_3 := 2^2\|r\|^2 \cdot \alpha$.

For fixed x, y each K_ℓ as a function of r is homogeneous, $K_\ell(\lambda r) = \lambda^\ell K_\ell(r)$, for real λ.

Say $x = x_1 + ix_2$ and $y = y_1 + iy_2$ with x_k, y_k real-valued. Take $j := 2^3\|x_2 - y_2\|\,\|r\|\,|L|$. $J_1 := K_3 + j$. $J_2 := K_2 - j + \frac{3}{2}\,\epsilon$. $J_3 := K_1 - \frac{3}{2}\,\epsilon$. Then $K_1 + K_2 + K_3 = J_1 + J_2 + J_3$.

Lemma 2.1.

(i) $J = K_1 + K_2 + K_3$

(ii) $J_1 \le 4ab^3$, $J_2 \le 6a^2b^2$, $J_3 + \epsilon \le 4a^3b$.

Proof of (i). $J = c^4 - (a^4 + b^4)$. J is a function of r. For real λ, $J(\lambda r)$ is a polynomial in λ: $J(\lambda r) = \sum_{\ell=1}^{3} k_\ell\,\lambda^\ell$. We show that $k_\ell = K_\ell$.

$c^4 = N^2 + E^2$ with

$$N = \|r + (x-y)\|^2 = \|x-y\|^2 + 2\alpha + \|r\|^2\,.$$

$$E := 2\,\mathrm{Im}\langle r + x,\, y\rangle = 2(T - B)\,.$$

$$a^4 = \|x-y\|^4 + (2T)^2\,.$$

Also $b^4 = \|r\|^4 + (2L)^2$. Algebraic manipulation yields $J = \Sigma\,K_\ell$.

Proof of (ii). The invariance of d, ϵ under any similarity transformation reduces the strong triangle inequality to the special case with $r = z - x$ real-valued (μ-a.e.) with $\|r\| = 1$.

Proof of (ii) for the special case.

Proof of $J_1 \le 4ab^3$: $ab^3 = (ab^2)b$. Now $b \ge \|r\|$ and $\|r\| = 1$.
$J_1 = 2^2(\alpha + 2\|x_2 - y_2\|\,|L|)$. $a \ge \|x-y\| = |v|$ with $v = (\|x_1 - y_1\|,\, \|x_2 - y_2\|)$ in $\mathbb{R}^2$. Also $b^2 = |w|$ with $w = (1,\, 2|L|)$. Therefore $a \cdot b^2 \ge |v| \cdot |w|$.

$|v|\,|w| \ge \langle v,\, w\rangle$ by the Cauchy-Schwartz inequality for $\mathbb{R}^2$.
$\langle v,\, w\rangle = \|x_1 - y_1\| + \|x_2 - y_2\|\,(2\,|L|)$. $\alpha = \mathrm{Re}\,\langle x-y,\, r\rangle$. Now r is real-valued. So $\alpha = \langle x_1 - y_1,\, r\rangle$. Then

$$\alpha \le |\alpha| \le \|x_1 - y_1\|$$

by the Cauchy-Schwartz inequality for L_2.

Proof of $J_2 \le 6a^2 \cdot b^2$:

$a^2 = |v|$ with $v = (\|x-y\|^2,\, 2|T|)$. $b^2 = |w|$ as above. $a^2\,b^2 = |v| \cdot |w|$.
$|v|\,|w| \ge \langle v,\, w\rangle$. $\langle v,\, w\rangle = \|x-y\|^2 + 2^2|T| \cdot |L|$.

So $J_2 \le 6a^2 \cdot b^2$ if $K_2 - j \le 6\|x-y\|^2$. Now $K_2 = 2\,\|x-y\|^2 + 2^2(\alpha^2 + E)$ with $E := B^2 - L^2$. Set $Q := \alpha^2 + E - 2\,\|x_2 - y_2\| \cdot |L|$.
We need to prove that $Q \le \|x-y\|^2$. Set $\theta_1 := \mathrm{Im}\langle y + x,\, r\rangle$.

Now

$$E = (B - L)(B + L) = \theta_1 \cdot \mathrm{Im}\langle y - x,\, r\rangle$$

$$\mathrm{Im}\langle y - x,\, r\rangle = \langle y_2 - x_2,\, r\rangle$$

$$E \le |E| \quad \text{and} \quad |E| = |\theta_1| \cdot |\langle y_2 - x_2,\, r\rangle|\,.$$

The Cauchy-Schwartz inequality on L_2 gives $|\langle y_2 - x_2,\, r\rangle| \le \|y_2 - x_2\|$. Therefore $Q \le \alpha^2 + H \cdot \|y_2 - x_2\|$ with $H := |\theta_1| - |\theta_2|$, where $\theta_2 = 2L$.

$$H \le |\theta_1 - \theta_2| \quad \text{with} \quad \theta_1 - \theta_2 = \mathrm{Im}\langle y - x,\, r\rangle\,.$$

As previously, $|\mathrm{Im}\langle y - x,\, r\rangle| \le \|y_2 - x_2\|$. Thus $H \le \|y_2 - x_2\|$.
However, $|\alpha| \le \|x_1 - y_1\|$ as shown above.

Thus $Q \le \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2$. But $\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 = \|x - y\|^2$.

Proof of $J_3 + \epsilon \le 4a^3 b$: $b \ge 1$. $a \ge \|x - y\|$. Now $\|x - y\| = |w|$ with $w = (\|x_1 - y_1\|,\, \|x_2 - y_2\|)$. Also $a^2 = |v|$ as above. $a^3 = a^2$. $a \ge |v| \cdot |w|$. On $\mathbb{R}^2 : |v| \cdot |w| \ge \langle v,\, w\rangle$.

Now

$$\langle v,\, w\rangle = \|x - y\|^2\, \|x_1 - y_1\| + 2|T|\, \|x_2 - y_2\|\,.$$

Thus

$$a^3 b \ge \|x - y\|^2\, \|x_1 - y_1\| + 2|T|\, \|x_2 - y_2\|$$

$$J_3 + \epsilon = 2^2(\|x - y\|^2 \cdot \alpha - 2(BT + |L| \cdot |T|))\,,$$

$\alpha \le \|x_1 - y_1\|$ as above. Thus it remains to prove that

$$-BT - |L| \cdot |T| \le |T| \cdot \|x_2 - y_2\|\,.$$

$-BT \le |B| \cdot |T|$. Thus it suffices to prove: $|B| - |L| \le \|x_2 - y_2\|$. Now $|B| - |L| \le |B + L|$. Also $B + L = \mathrm{Im}\langle y - x,\, r\rangle$. The proof is completed by again using $|\mathrm{Im}\langle y - x,\, r\rangle| \le \|y_2 - x_2\|$. $\square$

Corollary 2.2. *$m_I(x)$ is a metric norm on W.*

Proof. We may assume $\mu(P)$ is finite. Set $\tilde{x}(s) := e^{ix(s)}$. Then $m_I(x - y) = \frac{1}{\sqrt{2}}\, d(\tilde{x}, \tilde{y})$.

Appendix I

Internal form metrics strongly non-embeddable in Hilbert space.

Say m is a metric norm and $0 < \gamma < 1$. Then m^γ is a metric norm. Also (see Schoenberg [17]). If the metric $m(x - y)$ embeds in Hilbert space, then so does $(m(x - y))^\gamma$.

Set $a(y) := 1 - e^{iy}$ for y real. Let T be the abelian group of real numbers mod 2π under addition.

Let k be positive integer. Define $v(y, k)$, $j(y, k)$ to be $(k+1)\, f(ky) + k f(-(k+1)\, y)$ with the function $f(y)$ respectively $|a(y)|$, $a(y)$. Assume $b_k > 0$ for each k. Say x is a sequence with each term x_k in T. The x with $\Sigma\, b_k\, v(x_k, k)$ finite constitute an Abelian group, G. Then $g(y, k) := |j(y, k)|^{1/2}$ and $g(x) := |\Sigma\, b_k\, j(x_k, k)|^{1/2}$ are inner metric norms on T, G respectively.

Claim. There does not exist a positive γ (at most one) such that the metric $(g(x - x'))^\gamma$ embeds in Hilbert space.

This claim is a corollary of conjecture (i) below.

Set $e(y, n) := j(y, n)/(2n + 1)$. Assume $0 < \gamma < 1$. Say $|e(y, n)|^{2\gamma} = \sum_{v \ge 0} K(n, \gamma, v) \cos(v\, y)$.

Conjecture.

(i) $K(n, 1/(2n - 1), 2n + 1) > 1/(273n)$.

(ii) $K(n, 1/(2n - 1), v) < 0$ *if* $1 \le v \le 2n$.

Corollary. $|e(y, n)|^\gamma$ does not embed in Hilbert space for $\gamma = 1/(2n - 1)$.

This follows from conjecture (i) using the main theorem in J. von Neumann, I.J. Schoenberg [18].

Appendix II

A vector α in R^N is said to be "invasive" (respectively: discretely invasive or d-invasive) when its coordinates (resp. together with 2π) are independent over the integers. An infinite sequence α of reals α_k is invasive (resp. d-invasive) when each of its initial sections $P_N := (\alpha_1, \dots, \alpha_N)$ is invasive (resp. d-invasive). The sequence ω of logarithms of the successive primes is invasive by reason of the uniqueness of the decomposition of any positive integer into prime factors. It is also discretely invasive since each positive even power of e^π is irrational, e^π being transcendental.

Let $\ell = 1, 2$ and $J > 0$. If α is invasive but not d-invasive take $S_\ell(\alpha, J)$ to be the interval (J, ∞). When α is d-invasive take $S_1(\alpha, J) := (J, \infty)$ and $S_2(\alpha, J)$ to be the set of integers greater than J.

Define the distance d_N on the torus T^N by $d_N(u, v) := \operatorname*{Max}_{1 \le k \le N} |s_k|$ with $s_k \equiv (u_k - v_k) \bmod 2\pi$ and $-\pi < s_k \le \pi$.

Let z be an infinite sequence of complex numbers z_k. Henceforth assume $|z_k| < 1$ and $\sum_{k=1}^{\infty} |z_k|$ converges. Set $g_k(\theta) := |1 - z_k\, e^{i\theta}|$. Say $y = (y_1, \dots, y_N)$ is in T^N. Take $G_N(y) := \prod_{k=1}^{N} g_k(y_k)$. Given y in T^∞ take $G(y) := \prod_{k=1}^{\infty} g_k(y_k)$.

The sequence of functions G_N with $N \ge 1$ is equi-uniformly continuous: given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $N \ge 1$ and u, v on the compact $N-D$ torus T^N with $d_N(u, v) < \delta_\varepsilon$, one has $|G_N(u) - G_N(v)| < \varepsilon$.

As N becomes infinite $G_N(P_N(y))$ converges to $G(y)$ uniformly in y on T^∞.

Say y, y' are in T^∞ and $N \ge 1$. Set $V_N(y) := G(y) - G_N(P_N(y))$ and $D_N(y', y) := G_N(P_N(y')) - G_N(P_N(y))$.

Lemma. *Let $\varepsilon > 0$. There exists a positive integer N_ε and a positive $\delta(\varepsilon)$ such that for any $N \ge N_\varepsilon$ and x, y in T^∞, $d_N(P_N(x), P_N(y)) < \delta(\varepsilon)$ implies $|G(x) - G(y)| < \varepsilon$.*

Proof. Take N_ε such that for any $N \ge N_\varepsilon$ and x in T^∞ one has $|v_N(x)| < \varepsilon/3$. Take $\delta(\varepsilon) > 0$ such that for any $N \ge N_\varepsilon$ and u, v in T^N with $d_N(u, v) < \delta(\varepsilon)$ one has $|G_N(u) - G_N(v)| < \varepsilon/3$.

Say x, y in T^∞ have $d_N(P_N(x), P_N(y)) < \delta(\varepsilon)$. Then $|G(x) - G(y)| < \varepsilon$, as we now show. Set $r_1 := v_N(x)$, $r_2 := D_N(x, y)$ and $r_3 := -v_N(y)$. Then $G(x) - G(y) = r_1 + r_2 + r_3$. Now $|\Sigma\, r_k| \le \Sigma\, |r_k|$ with $|r_k| < \varepsilon/3$. □

Use will be made of Krönecker's approximation theorem (see Apostle [1]) in the following form. If α in T^N is invasive (respectively: d-invasive) and $J > 0$ then the $t\alpha$ arising from the real (resp. integral) t with $t > J$ are dense on the $N-D$ torus T^N.

Corollary. *Assume α is invasive. Say ε, J are positive. There exist N_0, δ such that for any y in T^∞ and $N \ge N_0$ the following holds.* (i) *There exists t in $S_\ell(\alpha, J)$ with $d_N(t\, P_N(\alpha), P_N(y)) < \delta$, and* (ii) *For any t as in* (i) *one has $|G(t\alpha) - G(y)| < \varepsilon$.*

Lemma. *Assume θ is real with $\Delta := g_{k_0}(\theta) - g_{k_0}(0)$ nonzero. Say α is invasive and $J > 0$. There exists t in $S_\ell(\alpha, J)$ with $\Delta' := G(t\alpha) - G(0)$ nonzero and of the same sign as Δ.*

Proof. Let $\delta(k_0, k) := 0$ except for $\delta(k_0, k_0) := 1$. $G(\theta\, \delta(k_0, k)) - G(0) = A\, \Delta$ with $A := \prod_{k \ne k_0} g_k(0)$ positive. The previous corollary gives *a fortiori* that for any positive ε, J there exists t in $S_\ell(\alpha, J)$ for which $|\Delta''| < \varepsilon$ with $\Delta'' := G(t\alpha) - G(\theta\, \delta(k_0, k))$. Now $\Delta' = \Delta'' + A\Delta$. Take ε with $\varepsilon < A\, |\Delta|$. □

Corollary. *Say α is invasive. Assume at least one of* (i), (ii) *as follows holds.*

(i) *There is a k_0 with z_{k_0} non real.*

(ii) *There exist k_0, k_1 with $z_{k_0} < 0$ and $z_{k_1} > 0$.*

Then for any $J > 0$ there exist t, t' in $S_\ell(\alpha, J)$ with $G(t\alpha) > G(0) > G(t'\alpha)$.

Next consider $L(s,\chi)$. Assume that $\chi(p)$ is normal for some prime p or there are primes p_0, p_1 with $\chi(p_0) = 1$ and $\chi(p) = -1$. Say $x > 1$. The previous Corollary implies that for any $J > 0$ there exist t, t' real (moreover integral) $t, t' > J$ with $|L(x+it,\chi)| > |L(x,\chi)| > |L(x+it',\chi)|$.

Kronecker's theorems are also used to establish the following.

Lemma. *Say $N \leq \infty$ and $m : T^N \to \mathbb{R}$. Assume α in T^N is invasive (respectively: d-invasive). Take $J > 0$. Assume $m(t\alpha)$ is subadditive in the real (resp. integral) t with $t > J$. Then $m(x)$ is subadditive in x on T^N provided either* (i) *N is finite and m is continuous or* (ii) *N is infinite and there are continuous $m_K : T^K \to \mathbb{R}$, for $K \geq 1$, with $m_K(P_K(x))$ converging to $m(x)$ uniformly in x on T^∞ as $K \to \infty$.*

Proof. m is subadditive on T^N if for any x_1, x_2 in T^N and $\varepsilon > 0$ one has $\Delta(x_1, x_2) := m(x_1) + m(x_2) - m(x_1 + x_2) > -\varepsilon$. Take $x_3 := x_1 + x_2$.

Case (i). There exists $\delta > 0$ such that for each of $k = 1, 2, 3$ one has $|m(v) - m(x_k)| < \varepsilon/3$ for any v on T^N with $d_N(v, x_k) < \delta$. For α invasive (resp. d-invasive) there are real (resp. integral) $t_1, t_2 > J$ with $d_N(t_k\,\alpha, x_k) < \delta/2$ for $k = 1, 2$. Set $t_3 := t_1 + t_2$ and $r_k := m(x_k) - m(t_k\,\alpha)$. Then for $k = 1, 2, 3$ one has $|r_k| < \varepsilon/3$. Now $\Delta(x_1, x_2) = \Delta(t_1\,\alpha, t_2\,\alpha) + r_1 + r_2 - r_3$. So $\Delta(t_1\,\alpha, t_2\,\alpha) \geq 0$ gives $\Delta(x_1, x_2) > -\varepsilon$.

Case (ii). Set $v_K(z) := m(z) - m_K(P_K(z))$. For any $\varepsilon > 0$ there exists a K for which $|v_K(z)| < \varepsilon/q$ for all z on T^∞. As in the proof for case (i) there are real (rep. integral) $t_1, t_2 > J$ with $|w_{k,2}| < \varepsilon/q$, for $k = 1, 2, 3$, where $t_3 := t_1 + t_2$ and $w_{k,2} := D_K(x_k, t_k\,\alpha)$. Define Δ, r_k as in case (i). Set $w_{k,1} := v_K(x_k)$ and $w_{k,3} := -v_K(t_k\,\alpha)$. Then $r_k = \sum_{h=1}^{3} w_{k,h}$.

$\Delta(x_1, x_2) = \Delta(t_1\,\alpha, t_2\,\alpha) + \sum_{1 \leq k,h \leq 3} \sigma_k\, w_{k,h}$ with $\sigma_1, \sigma_2 = 1$ and $\sigma_3 = -1$. Now $\Delta(t_1\,\alpha, t_2\,\alpha) \geq 0$. Also $|w_{k,h}| < \varepsilon/q$. Thus $\Delta(x_1, x_2) > -\varepsilon$. □

3 On the Partial Fraction Expansion of $\left(\sin\left(\frac{\pi}{4}s\right) q\left(\frac{1}{2} + s\right)\right)^{-1}$, with $q(s) := \pi^{-s/2} \cdot 2\Gamma\left(1 + \frac{s}{2}\right)(s-1)\,\zeta(s)$

Computer studies made by the author suggest that:

Conjecture. *There exist positive K, K' such that for $k \geq 1$,*

$$K'(\log r_k)(\log\log r_k)^2 > \left|\zeta'\left(\frac{1}{2} + i\,r_k\right)\right| > K \cdot \frac{1}{\log\log r_k}\,.$$

We assume the following weaker conjecture, C_1.

Conjecture 3.1. *There exists $\varepsilon_1 > 0$ such that:*

(i) *There exist positive λ, N with*

$$\left|\zeta'\left(\frac{1}{2} + i\,r_k\right)\right| > \lambda\, r_k^{-\varepsilon_1} \quad \textit{for} \quad k \geq N$$

(ii) $\varepsilon_1 < \frac{1}{8}$.

Assume Conjecture 3.1(i) and $\varepsilon_1 < \frac{3}{4}$.

Corollary 3.1. $\sum_{k=1}^{\infty} |c\,(i\,r_k)|$ *is finite.*

Proof.

$$c\,(i\,r_k) := \frac{1}{n'(i\,r_k)} \cdot \qquad n'(i\,r_k) = \zeta'\left(\frac{1}{2} + i\,r_k\right) b(i\,r_k)\,.$$

There exist $\lambda, N > 0$ for which $|b(it)| \geq \lambda\, t^{7/4}$ for $t > N$. So, Conjecture 3.1(i) implies that there exist λ, N such that for $k \geq N$: $|c\,(i\,r_k)| < \lambda\, r_k^{-(1+\varepsilon)}$ with $\varepsilon = \frac{3}{4} - \varepsilon_1$. We assumed $\varepsilon_1 < \frac{3}{4}$

$$\sum_{k=N}^{\infty} |c\,(i\,r_k)| < \lambda \sum_{k=N}^{\infty} r_k^{-(1+\varepsilon)} < \infty\,.$$

□

$$\sum_{w=1}^{\infty} |c\,(4w)| \leq \frac{2^2\,\pi^{-7/4}}{\Gamma\left(\frac{1}{4}\right)}\,(e^{\pi} - e^{-\pi})\,.$$

This follows from: $(\theta)_n \geq n!$, for $\theta \geq 1$; and $\zeta(\sigma) \geq 1$, for $\sigma \geq 1$.

Set $p_I(s) := \sum_{k=1}^{\infty} \left(\frac{1}{s - i\,r_k} + \frac{1}{s + i\,r_k}\right) c\,(i\,r_k)$

$p_R(s) := \frac{1}{s}\,c(0) + \sum_{w=1}^{\infty} \left(\frac{1}{s-4w} + \frac{1}{s+4w}\right) c(4w)$ and $p(s) := p_I(s) + p_R(s)$.

$p_I(s)$, $p_R(s)$ are analytic except for simple poles at $\pm\, i\,r_k$ $(k \geq 1)$, $\pm\, 4w$ $(w \geq 0)$, respectively. So $p(s)$ is analytic except for simple poles at $i\,r_k$, $4w$ with k, w integers.

We will prove that the partial fraction expansion of $f(s)$ is $p(s)$. This will involve making certain very plausible assumptions along the way.

$\Delta(s) := f(s) - p(s)$ is an entire function. So $\Delta(s) = 0$ for all s, if

(i) $\Delta(s)$ is bounded on the "critical" vertical strip $|\mathrm{Re}\,(s)| < \frac{1}{2}$ and

(ii) $\lim\limits_{\substack{|s|\to\infty \\ |\mathrm{Re}\, s| \geq \frac{1}{2}}} \Delta(s) = 0$.

Property (ii) without property (i) does not insure that the entire function $\Delta(s)$ is bounded. There exist entire functions $E(z)$ of infinite order with $\lim\limits_{\substack{\lambda\to\infty \\ \lambda\geq 0}} E(\lambda\, v) = \infty$ for a unique complex v of unit length, but with $\lim\limits_{\substack{\lambda\to\infty \\ \lambda\geq 0}} E(\lambda\, w) = 0$ for any unit vector w other than v. See Malmquist [7] and Lindelöf [5].

Each of $\theta(s) = f$, p_R, p_I, p, Δ has the symmetries $\theta(-s) = \theta(s)$ and $\theta(s^*) = (\theta(s))^*$. Thus in (i), (ii) it can be assumed that s is in the first quadrant Q of $s = x + it$ with $x, t \geq 0$.

Assume RH. Set $d_k := r_k - r_{k-1}$ and $\delta_k := \min\{d_k, d_{k+1}\}$.

Later we will apply the next lemma to $E(s) := q\left(\frac{1}{2} + s\right)$ with RH assumed.

Let

$$E(s) := K\left(\prod_{\ell=1}^{N}\left(1 - \frac{s}{i\phi_\ell}\right)\right)\prod_{k=1}^{\infty}\left(1 + \left(\frac{s}{\theta_k}\right)^2\right)$$

with $0 \leq N \leq \infty$, $K \neq 0$, ϕ_ℓ a non-zero real, $\theta_k > 0$, $\displaystyle\sum_{\ell=1}^{N} \frac{1}{|\phi_\ell|} < \infty$ (when $N = \infty$) and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\theta_k^2} < \infty$. Say $s = x + it$ with x, t real. Fix t. Set $v = x^2$. Say $x \neq 0$.

Lemma 3.1. *$1/|E(s)|$ is a decreasing function of v.*

Proof. Say r is real. $\left|1 - \frac{s}{i\,r}\right|^2 = g(v,t,r)$ with $g(v,t,r) := r^{-2}(v + (t-r)^2)$. □

Corollary 3.2. *Assume* RH. $|f(x+it)|$ *decreases as* $|x|$ *increases from* 0 *to* 2.

Proof.

$$\left|\sin\left(\frac{\pi}{4}(x+it)\right)\right|^2 = \frac{1}{2}\left(\cos h\left(\frac{\pi}{2}\,t\right) - \cos\left(\frac{\pi}{2}\,x\right)\right).$$

□

Theorem 3.1. $\frac{1}{|E(x+it)|^2}$ *is a completely monotone function of* x^2.

Proof. Say $n \geq 1$. Set

$$A_n(v,t) := \left|K \prod_{\ell=1}^{M}\left(1 - \frac{s}{i\phi_\ell}\right) \cdot \prod_{k=1}^{n}\left(1 + \left(\frac{s}{\theta_k}\right)^2\right)\right|^{-2}$$

with $M = \min(n, N)$. Then

$$A_n(v,t) = |K|^2 \left(\prod_{\ell=1}^{M} \frac{1}{g(v,t,\phi_\ell)}\right) \prod_{k=1}^{\infty}\left(\frac{1}{g(v,t,\theta_k)} \cdot \frac{1}{g(v,t,-\theta_k)}\right).$$

Say $\operatorname{Re} v > 0$.

$$\frac{1}{v+a} = \int_0^\infty (dy)\, e^{-vy}\, e^{-ay}$$

for $a \geq 0$. Thus $A_n(v,t)$ is the Laplace transform in v of a function $p(y,t)$ which is the convolution of positive functions. Hence $p(y,t) > 0$ for $y \geq 0$. Thus $A_n(v,t)$ is completely monotone in v for $v > 0$.

Each $A_n(v,t)$ is analytic on the half-plane H of v with $\operatorname{Re} v > 0$. The sequence of $A_n(v,t)$, with $n \geq 1$, converges uniformly in v on compact subsets of H to

$$j(v,t) := |K|^2 \cdot \prod_{\ell=1}^{N}\left(\frac{1}{g(v,t,\varphi_\ell)}\right) \prod_{k=1}^{\infty}\left(\frac{1}{g(v,t,\theta_k)} \cdot \frac{1}{g(v,t,-\theta_k)}\right).$$

($|E(s)|^{-2} = j(x^2,t)$.) Thus $j(v,t)$ is analytic in v on H. Therefore

$$\lim_{n\to\infty}\left(\frac{d}{dv}\right)^m A_n(v,t) = \left(\frac{d}{dv}\right)^m j(v,t).$$

Each $A_n(v,t)$ is completely monotone for $v > 0 : (-1)^m \left(\frac{d}{dv}\right)^m A_n(v,t) \geq 0$. Thus so is $j(v,t)$. □

Corollary 3.3. *Assume the Riemann Hypothesis. Then* $\frac{1}{\left|q\left(\frac{1}{2}+x+it\right)\right|^2}$ *is a completely monotone function of* x^2 *for each fixed* t.

Assume Conjecture 3.1(i).

Lemma 3.2. *Say* $\varepsilon > 0$. *Then* $\delta_k \geq r_k^{-\left(\frac{1}{2}+\varepsilon_1+\varepsilon\right)}$ *for large* k.

Proof. Say $k \geq 1$. $\zeta\left(\frac{1}{2}+s\right)$ is analytic for $|h| < r_k$, with $h = s - i\,r_k$. Then

$$\zeta\left(\frac{1}{2}+s\right) = \left(\zeta'\left(\frac{1}{2}+i\,r_k\right) + F_2(s)\,h\right)h,$$

with

$$\sup_{|h|<\rho_k} |F_2(s)| \leq \frac{1}{\delta}\,\frac{1}{\delta-\rho_k}\,\max_{|h|=\delta}\left|\zeta\left(\frac{1}{2}+s\right)\right|$$

for any ρ_k, δ with $0 < \rho_k < \delta < r_k$. Now $\left|\zeta\left(\frac{1}{2}+s\right)\right| \geq r_k^{-\varepsilon_1}\,|h|\,|A_k - B_k(s)|$ with $A_k := r_k^{\varepsilon_1}\left|\zeta'\left(\frac{1}{2}+i\,r_k\right)\right|$ and $B_k(s) := r_k^{\varepsilon_1}\,|F_2(s)|\cdot|h|$, $A_k > \lambda > 0$ for large k, by Conjecture 3.1(i).

Say $-\frac{1}{2} < \delta' < \frac{1}{2}$. There exists $K(\delta') > 0$ such that

$$\left|\zeta\left(\frac{1}{2}+s\right)\right| \leq K(\delta')\,|t|^{\frac{1}{2}+\delta'}$$

for all $s = x + it$ with $\frac{1}{2} - \delta' \leq x \leq 2$ and $|t| \geq 1$. See Apostle [1].

Assume $\delta < \frac{1}{2}$. Then

$$\sup_{|h|<\rho_k} |F_2(s)| \leq \frac{1}{\delta(\delta-\rho_k)}\cdot 2\,K(\delta)\,r_k^{\frac{1}{2}+\delta}\,.$$

So

$$\sup_{|h|<\rho_k} |B_k(s)| < \frac{1}{\delta(\delta-\rho_k)}\cdot 2\,K(\delta)\,r_k^{\frac{1}{2}+\delta+\varepsilon_1}\,\rho_k\,.$$

Given $\varepsilon > 0$, take a positive δ with $\delta < \frac{1}{2}$ and $\delta < \frac{\varepsilon}{2}$. Take $\rho_k := r_k^{-\left(\frac{1}{2}+\varepsilon_1+\varepsilon\right)}$. Then for large k we have $\rho_k < \delta$. Also

$$r_k^{\frac{1}{2}+\delta+\varepsilon_1}\,\rho_k = r_k^{-(\varepsilon-\delta)} < r_k^{-\varepsilon/2}\,.$$

So $\lim\limits_{k\to\infty}\ \sup\limits_{|h|<\rho_k} B_k(s) = 0$.

Thus for large k: $\zeta\left(\frac{1}{2}+s\right) \neq 0$ if $|s - i\,r_k| < r_k^{-\left(\frac{1}{2}+\varepsilon_1+\varepsilon\right)}$. Therefore $\delta_k \geq r_k^{-\left(\frac{1}{2}+\varepsilon_1+\varepsilon\right)}$ for k large. □

Let $B(z, r)$ be the disc of s with $|s - z| < r$. Define $C(z, r)$ to be the circle of s with $|s - z| = r$. Say $\lambda \geq 0$. Set

$$B_k(\lambda) := B(i\,r_k, \lambda\,\delta_k)\,, \quad C_k(\lambda) := C(i\,r_k, \lambda\,\delta_k)$$

and

$$I_k(\lambda) := \{t : r_k + \lambda\,\delta_k \leq t \leq r_{k+1} - \lambda\,\delta_{k+1}\}\,.$$

Take $\theta(\lambda) := \bigcup\limits_{k=1}^{\infty} \theta_k(\lambda)$ with "θ" any of "B", "C", "I".

Say k is a positive integer. Let z_k, c_k be complex and $\ell_k > 0$. Set $G := \bigcup\limits_{k=1}^{\infty} B(z_k, \ell_k)$. Say $A := \sum\limits_{k=1}^{\infty} \dfrac{|c_k|}{\ell_k}$ is finite. Then $\sum\limits_{k=1}^{\infty} \dfrac{1}{s - z_k}\cdot c_k$ converges uniformly on the closed set $\mathbb{C} - G$ to a function $g(s)$ analytic on the interior of $\mathbb{C} - G$.

Lemma 3.3. *Say A is finite and $\mathbb{C} - G$ is unbounded. Then* $\lim\limits_{\substack{|s|\to\infty \\ s\in\mathbb{C}-G}} g(s) = 0$.

Proof. Given any $\varepsilon > 0$, there is an N with $\sum\limits_{k=N}^{\infty} \dfrac{|c_k|}{\ell_k} < \varepsilon$. Say s is in $\mathbb{C}-G$ and $|s| > m$, with $m := \max\limits_{1\leq k<N} |z_k|$. Then

$$|g(s)| \leq \frac{1}{|s| - m} \sum_{k=1}^{N-1} |c_k| + \varepsilon\,.$$

So $\lim\limits_{\substack{s\to\infty \\ s\in\mathbb{C}-G}} |g(s)| \leq \varepsilon$. □

Lemma 3.4. *Conjecture* 3.1 *implies that* $\sum_{k=1}^{\infty} \frac{|c(i\,r_k)|}{\delta_k}$ *converges.*

Proof. It was seen in the proof of Corollary 3.1 that Conjecture 3.1(i) implies that there exists a $\lambda > 0$ such that for large k : $|c(i\,r_k)| < \lambda\, r_k^{-(1+\varepsilon)}$ with $\varepsilon = \frac{3}{4} - \varepsilon_1$. Given any $\varepsilon' > 0$, for large k we have $\delta_k > r_k^{-\left(\frac{1}{2}+\varepsilon_1+\varepsilon'\right)}$, by Lemma 3.1. Thus, for large k,

$$\frac{|c(i\,r_k)|}{\delta_k} < \lambda\, r_k^{-(1+\theta)} \quad \text{with} \quad \theta = 2\left(\frac{1}{8} - \varepsilon_1\right) - \varepsilon' \,.$$

$\frac{1}{8} - \varepsilon_1 > 0$ by Conjecture 3.1(ii). $\theta > 0$ for $\varepsilon' < 2\left(\frac{1}{8} - \varepsilon_1\right)$. Also $\sum_{k=1}^{\infty} \frac{c(i\,r_k)}{\delta_k}$ converges since $\sum_{k=1}^{\infty} r_k^{-(1+\theta)}$ does. □

Littlewood (1924) proved the following theorem. There exist $A, T > 0$ such that for any $t > T$ there is at least one zero z of the zeta-function, $\zeta(z) = 0$, with

$$|t - \mathrm{Im}(z)| < \frac{A}{\log(\log(\log(t))} \,.$$

Corollary 3.4. *Assume R.H. Then* $\lim_{k\to\infty} \delta_k = 0$.

Set

$$j_k(t) := \left|\frac{\zeta\left(\frac{1}{2} + it\right)}{t - r_k}\right| \quad \text{and} \quad J_k(\alpha) := \min_{|t-r_k| \le \alpha\,\delta_k} j_k(t) \,.$$

Conjecture 3.2. (C_2) *There exist arbitrarily small positive* α *for which the following conditions (i), (ii) hold.*

(i) *Given* ε_1 *as in Conjecture* 3.1, *there is an* ε_2 *with* $0 < \varepsilon_2 < \frac{3}{4} - 2\varepsilon_1$ *such that for some* $\lambda' > 0$*:*

$$\min_{t\in I_k(\alpha)} \left|\zeta\left(\frac{1}{2} + it\right)\right| > \lambda'\, r_k^{-\varepsilon_2}$$

for all large k*;*

(ii) *There exists* $\lambda > 0$ *such that* $J_k(\alpha) > \lambda \cdot \frac{1}{\delta_k}\, r_k^{-\varepsilon_2}$ *for all large* k.

Say s is in $B(\alpha)$. There is a unique integer k with $|s - i\,r_k| < \alpha\,\delta_k$. Set

$$T(s) := \frac{1}{s - i\,r_k}\, c(i\,r_k) \quad \text{and} \quad \Delta_1(s) := f(s) - T(s) \,.$$

Let $s = x + it$ with x, t real.

Theorem 3.2.

$$\lim_{\substack{|t|\to\infty \\ s\in B(\alpha)}} \Delta_1(s) = 0 \,.$$

Proof. We may assume $t \ge 0$. Say $|s - i\,r_k| < \alpha\,\delta_k$.

$$\begin{aligned} \Delta_1(s) \quad &:= \quad f(s) - \frac{1}{s - i\,r_k} \cdot c(i\,r_k) \\ &= \quad \left(\frac{1}{\zeta\left(\frac{1}{2} + s\right)} - \frac{1}{s - i\,r_k} \cdot \frac{1}{\zeta'\left(\frac{1}{2} + i\,r_k\right)}\right) \frac{1}{b(s)} + \frac{1}{\zeta'\left(\frac{1}{2} + i\,r_k\right)} \cdot F_1(s) \end{aligned}$$

with

$$F_1(s) := \frac{1}{s - i\,r_k}\left(\frac{1}{b(s)} - \frac{1}{b(i\,r_k)}\right).$$

Therefore

$$\Delta_1(s) := \frac{1}{\zeta'\left(\frac{1}{2} + i\,r_k\right)}\left(-F_2(s)\cdot\frac{1}{\sin\left(\frac{\pi}{4}\,s\right)E(s)} + F_1(s)\right),$$

with

$$F_2(s) := \frac{\zeta\left(\frac{1}{2}+s\right) - (s - i\,r_k)\,\zeta'\left(\frac{1}{2} + i\,r_k\right)}{(s - i\,r_k)^2}$$

and

$$E(s) := \frac{q\left(\frac{1}{2}+s\right)}{(s - i\,r_k)}.$$

So

$$|\Delta_1(s)| \le \frac{1}{\left|\zeta'\left(\frac{1}{2} + i\,r_k\right)\right|}\left(|F_2(s)|\cdot\frac{1}{\left|\sin\left(\frac{\pi}{4}\,s\right)E(s)\right|} + F_1(s)\right).$$

Say $s = x + it$, with x, t real. Fix t. $\frac{1}{|E(s)|}$ decreases as $|x|$ increases, by Lemma 3.1. Thus $\frac{1}{\left|\sin\left(\frac{\pi}{4}\,s\right)E(s)\right|}$ increases as $|x|$ decreases from 2 to 0. $\alpha\,\delta_k \le 2$ for large k, by Corollary 3.4. $|t - r_k| < \alpha\,\delta_k$. So

$$\frac{1}{\left|\sin\left(\frac{\pi}{4}\,s\right)E(s)\right|} \le \frac{1}{J_k(\alpha)|b(it)|}.$$

Next use: Conjecture 3.2(ii); $|b(it)| > \lambda\,t^{7/4}$ for some $\lambda > 0$ for all large t; and Corollary 3.4. One obtains

$$\frac{1}{\left|\sin\left(\frac{\pi}{4}\,s\right)E(s)\right|} < \lambda\,\delta_k\,r_k^{\varepsilon_2 - \frac{7}{4}}$$

for some $\lambda > 0$ and all large k.

Define $H_{k,\ell}(\alpha) := \sup_{s\in B_k(\alpha)} |F_\ell(s)|$, for $\ell = 1, 2$. Take β with $\alpha < \beta < 1$. Set

$$M_k(\beta) := \max_{s\in c_k(\beta)}\left|\zeta\left(\frac{1}{2} + s\right)\right|.$$

Assume Conjecture 3.2. Set

$$T_k := r_k^{-\frac{7}{4}+\varepsilon_1+\varepsilon_2}(\delta_k\,H_{k,2}(\alpha)).$$

Then

$$\sup_{s\in B_k(\alpha)} |\Delta_1(s)| < \lambda\,T_k + \lambda'\,r_k^{\varepsilon_1}\,H_{k,1}(\alpha),$$

for some $\lambda, \lambda' > 0$ and all large k.

Claim 1.

$$\lim_{k\to\infty} T_k = 0.$$

Proof of Claim 1. First we find an upper bound for $H_{k,2}(\alpha)$. $\zeta\left(\frac{1}{2}+s\right)$ is analytic on $B(i\,r_k, \delta_k)$. Hence

$$H_{k,2}(\alpha) \le \frac{1}{\beta(\beta-\alpha)}\,\delta_k^{-2}\,M_k(\beta).$$

Subclaim. *Say* $0 < \varepsilon < \frac{1}{2}$. *There exists* $B > 0$ *such that* $M_k(\beta) < B\,r_k^{\frac{1}{2}+\varepsilon}$ *for all large* k.

Proof of Subclaim. Say $-\frac{1}{2} < \delta < \frac{1}{2}$. There exists a positive $A(\delta)$ for which $\left|\zeta\left(\frac{1}{2}+s\right)\right| < A(\delta)\,|t|^{\frac{1}{2}+\delta}$ for all $s = x + it$ with $\frac{1}{2} - \delta \le x \le 2$ and $|t| \ge 1$ (as is shown in [1]). $\beta\,\delta_k \le \varepsilon$ for large k. So $B = 2A(\varepsilon)$ works. □

The Subclaim, Lemma 3.2 and Conjecture 3.2 together give: There exists $\theta > 0$ such that $T_k < r_k^{-\theta}$ for large k. Therefore $\lim_{k\to\infty} T_k = 0$. □

Claim 2.

$$\lim_{k\to\infty} r_k^{\varepsilon_1}\, H_{k,1}(\alpha) = 0\,.$$

Proof of Claim 2. $\frac{1}{b(s)}$ is analytic on $B(i\,r_k, r_k)$ for $k \ge 1$. So

$$H_{k,1}(\alpha) \le \frac{1}{\beta-\alpha}\,\frac{1}{\delta_k}\cdot\left(\min_{s\in C_k(\beta)} |b(s)|\right)^{-1}.$$

Say $0 < \delta < \frac{1}{2}$. There exist $K(\delta)$, $N(\delta)$ such that for any $s = x + it$ with $|x| < \delta$ and $|t| \ge N(\delta)$ we have

$$|b(s)| > K(\delta)\,|t|^{\frac{7}{4}+\frac{x}{2}}\,.$$

Now $\beta\,\delta_k < \delta$ for large k, by Corollary 3.4. Therefore there exists a $K'(\delta) > 0$ such that

$$\min_{s\in C_k(\beta)} |b(s)| > K'(\delta)\, r_k^{\frac{7}{4}-\delta}$$

for large k. Lemma 3.2 now gives that for any $\varepsilon > 0$ there is a $\lambda > 0$ such that for all large k: $r_k^{\varepsilon_1}\, H_{k,1}(\alpha) \le \lambda\, r_k^{-p}$, with $p := 2\theta - (\delta + \varepsilon)$, where $\theta := \frac{5}{8} - \varepsilon_1$. $\varepsilon_1 < 1/8$ by Conjecture 3.1(ii). So $\theta > 0$. Take $\varepsilon = \delta = \frac{\theta}{2}$. Then $r_k^{\varepsilon_1}\, H_{k,1}(\alpha) \le \lambda\, r_k^{-\theta}$ for large k. So $\lim_{k\to\infty} r_k^{\varepsilon_1}(\alpha)\, H_{k,1}(\alpha) = 0$. □

Claims 1, 2 together imply Theorem 3.1. □

Lemma 3.5.

$$\lim_{\substack{|t|\to\infty\\ s\in B(\alpha)}} \Delta(s) = 0\,.$$

Proof. Say s is in $B(\alpha)$.

$$\Delta(s) = \Delta_1(s) - (p_I(s) - T(s)) - p_R(s)\,.$$

Claim 1.

$$\lim_{\substack{|t|\to\infty\\ s\in B(\alpha)}} (p_I(s) - T(s)) = 0\,.$$

Proof of Claim 1. There is a unique k with $|s - i\,r_k| < \alpha\,\delta_k$.

$$p_I(s) - \frac{1}{s - i\,r_k}\,c(i\,r_k) = \frac{c(i\,r_k)}{s + i\,r_k} + \sum_{\substack{j=1\\ j\ne k}}^{\infty}\left(\frac{1}{s - i\,r_j} + \frac{1}{s + i\,r_j}\right) c(i\,r_j)\,.$$

$|s - i\,r_j| \ge \alpha\,\delta_j$ for all j other than k. So Conjecture 3.1 together with Lemmas 3.3, 3.4 yield Claim 1. □

$\lim_{|t|\to\infty} p_R(s) = 0$ is a special case of the next claim.

Say $\delta \ge 0$. Set

$$G(\delta) := \bigcup_{w=-\infty}^{\infty} B(4w, \delta)\,.$$

Claim 2. *Let δ be positive.*

$$\lim_{\substack{|s|\to\infty \\ s\in\mathbb{C}-G(\delta)}} p_R(s) = 0\,.$$

Proof. $\sum_{w=0}^{\infty} |c(4w)|$ is finite. So Lemma 3.3 yields Claim 2. □

Let v_2 be the vertical strip of $s = x + it$ with $-2 \le x \le 2$ and t real.

Lemma 3.5.

$$\text{(i)} \quad \lim_{\substack{|t|\to\infty \\ s\in V_2-B(\alpha)}} f(s) = 0 \qquad\qquad \text{(ii)} \quad \lim_{\substack{|t|\to\infty \\ s\in V_2-B(\alpha)}} p(s) = 0\,.$$

Thus

$$\lim_{\substack{|t|\to\infty \\ s\in V_2-B(\alpha)}} \Delta(s) = 0\,.$$

Proof. We may assume $x, t \ge 0$.

Proof of (i). Fix t. $|f(x+it)| \le |f(it)|$ for x with $-2 \le x \le 2$, by Corollary 3.2. Now $\alpha\,\delta_k \le 2$ for large k, by Corollary 3.4. So (i) holds if

$$\text{(a)} \quad \lim_{\substack{t\to\infty \\ t\in I(\alpha)}} f(it) = 0 \qquad \text{and} \qquad \text{(b)} \quad \lim_{\substack{t\to\infty \\ s\in C(\alpha) \\ x\ge 0}} f(s) = 0\,.$$

Proof of (a).

$$|f(it)| = \frac{1}{\left|\zeta\left(\frac{1}{2}+it\right)\right|} \cdot \frac{1}{|b(it)|}\,.$$

There exists a $\lambda > 0$ such that for large k $\min_{t\in I_k(\alpha)} |b(t)| > \lambda\, r_k^{-7/4}$. Conjecture 3.2(i) now gives $\max_{t\in I_k(\alpha)} |f(it)| < \frac{\lambda}{r_k}$ for some $\lambda > 0$ and all sufficiently large k. So (a) holds. □

Proof of (b). Say $s = x + it$ is in $C_k(\alpha)$. Set $E(s) := \frac{n(s)}{s - i\,r_k}$ for $s \ne i\,r_k$. Take $E(i\,r_k) = \zeta'\left(\frac{1}{2} + i\,r_k\right) b(it)$.

$$|f(s)| = \frac{1}{\alpha\,\delta_k}\ \frac{1}{|E(s)|}\,.$$

Now $\frac{1}{|E(s)|} \le \frac{1}{|E(it)|}$ if $\alpha\,\delta_k \le 2$, by Lemma 3.1. Now $|E(it)| = j(t)\,|b(it)|$. Thus

$$\max_{s\in C_k(\alpha)} |f(s)| \le \lambda \left((\delta_k\, J_k(\alpha))\, r_k^{7/4} \right)^{-1}$$

for some $\lambda > 0$ and all large k. Conjecture 3.2(ii) yields

$$\max_{s\in C_k(\alpha)} |f(s)| \le \frac{\lambda}{r_k}$$

for some $\lambda > 0$ and all large k. □

Proof of (ii).

$$p(s) := p_I(s) + p_R(s)\,.$$

$\lim_{\substack{|s|\to\infty \\ s\in\mathbb{C}-B(\alpha)}} p_I(s) = 0$ by Lemmas 3.3, 3.4.

$\lim\limits_{|t|\to\infty} p_R(s) = 0$ was derived in the proof of Lemma 3.5 from Claim 2 thereof. □

Corollary 3.5.

$$\lim_{\substack{|t|\to\infty \\ |x|\le 2}} |\Delta(x+it)| = 0\,.$$

The next lemma is proven without using RH, SZC or the above Conjectures 3.1, 3.2.

Let $s = x + it$, with x, t real.

Lemma 3.6.

$$\lim_{\substack{|t|\to\infty \\ |x|\ge \frac{1}{2}}} f(s) = 0\,.$$

Proof. We may assume $x \ge \frac{1}{2}$, $t \ge e$. There exists $\lambda > 0$ such that

$$(*) \qquad \frac{1}{|\zeta(\sigma+it)|} < \lambda(\log t)^7$$

for all $\sigma \ge 1$ and $t \ge e$ (Apostle [1]).

Thus $|f(s)| < \lambda(\log t)^7 \cdot \frac{1}{|b(s)|}$. Say $x \ge 0$, $|t| \ge 1$. Then

$$|b(s)| = \left(\frac{\pi}{2e}\right)^{1/4} |s|^{7/4} \left(\frac{\left|\frac{1}{2}+s\right|}{2\pi e}\right)^{x/2} \left(1 + \frac{\varepsilon(s)}{|s|}\right)$$

with $\varepsilon(s)$ bounded. Therefore: (i) For $x \ge \frac{1}{2}$, there is a $\lambda' > 0$ such that $|b(s)| > \lambda'\, t^2$ for large $|t|$.

So, for t large, $|f(s)| < \frac{\lambda}{\lambda'}\,\frac{(\log t)^7}{t^2}$. Say $\varepsilon > 0$. Then $\lim\limits_{t\to\infty}(t^{-\varepsilon}\log t) = 0$. □

The proof of the next lemma does not use the bound of Eq. $(*)$.

Lemma 3.7. *Assume* $\varepsilon > 0$. $\lim\limits_{\substack{|t|\to\infty \\ |x|\ge \frac{1}{2}+\varepsilon}} f(s) = 0$.

Proof. We may assume that $s = x + it$ has $x \ge \frac{1}{2} + \varepsilon$ and $t \ge 0$.

If $1 < \sigma \le \operatorname{Re} u$, then $\frac{1}{|\zeta(u)|} \le \frac{\zeta(\sigma)}{\zeta(2\sigma)}$. (This follows from the Euler factorization of ζ.)

So

$$|f(s)| \le \frac{\zeta(1+\varepsilon)}{\zeta(2(1+\varepsilon))} \cdot \frac{1}{|b(s)|}$$

$\lim\limits_{\substack{t\to\infty \\ x\ge\frac{1}{2}}} \dfrac{1}{|b(s)|} = 0$ by (i) of the previous proof. □

Say $0 < \delta \le 2$ and s is in $G(\delta)$. There exists a unique integer w with $|s-4w| < \delta$. Set $\Delta_2(s) := f(s) - \frac{c(4w)}{s-4w}$.

Lemma 3.8.

$$\lim_{\substack{|x|\to\infty \\ s\in G(\delta)}} \Delta_2(s) = 0\,.$$

Proof. We may assume $s = x + it$ has $x, t \ge 0$. Say $0 < \delta \le 2$ and $|s - 4w| < \delta$.

$$\Delta_2(s) = \theta(s)\,\frac{1}{q\left(\frac{1}{2}+s\right)} + \frac{1}{d}\cdot F(s)\,,$$

with $d := (-1)^w \frac{\pi}{4}$, $\theta(s) := \frac{1}{\sin\left(\frac{\pi}{4}\, s\right)} - \frac{1}{s-4w}\, \frac{1}{d}$ and

$$F(s) := \frac{1}{s-4w}\left(\frac{1}{q\left(\frac{1}{2}+s\right)} - \frac{1}{q\left(\frac{1}{2}+4w\right)}\right).$$

Thus

$$|\Delta_2(s)| \leq |\theta(s)| \cdot \frac{1}{\left|q\left(\frac{1}{2}+s\right)\right|} + \frac{4}{\pi}\,|F(s)|\,.$$

$\theta(s)$ is analytic for $|s| < 4$. $|\theta(s+4)| = |\theta(s)|$. Hence $\theta(s)$ is bounded on $G(\delta) : |\theta(s)| \leq B$.

Say $\sigma = \operatorname{Re} z$ and $\sigma > 1$. Then $|\zeta(z)| \geq 2 - \zeta(\sigma)$. Also $\zeta(\sigma)$ decreases from infinity to one as σ increases from one. Let σ_0 be the unique root of $\zeta(\sigma) = 2$ with $\sigma > 1$. Then $\frac{1}{q\left(\frac{1}{2}+s\right)}$ is analytic for $\operatorname{Re} s > \sigma_0 - \frac{1}{2}$.

Assume $0 < \delta \leq 2$ and $4w > \sigma_0 + \frac{5}{2}$. Then

$$\sup_{s\in B(4w,\delta)} |F(s)| \leq \max_{|s-4w|=3} \frac{1}{\left|q\left(\frac{1}{2}+s\right)\right|}\,.$$

Thus

$$\sup_{s\in B(4w,\delta)} |\Delta_2(s)| \leq \left(B + \frac{4}{\pi}\right) M(4w-3, 3)\,.$$

Here, for $y, c \geq 0$, we define

$$M(y,c) := \sup_{\substack{\operatorname{Re} s \geq y \\ |\operatorname{Im} s| \leq c}} \frac{1}{\left|q\left(\frac{1}{2}+s\right)\right|}\,.$$

Claim. $\lim\limits_{y\to\infty} M(y,c) = 0$.

Proof of Claim. Say $y > \sigma_0 - \frac{1}{2}$. Then

$$M(y,c) \leq \frac{1}{2 - \zeta\left(\frac{1}{2}+y\right)} \cdot \frac{1}{m(y,c)}$$

with

$$m(y,c) := \inf_{\substack{\operatorname{Re} s \geq y \\ |\operatorname{Im} s| \leq c}} \left| a\left(\frac{1}{2}+s\right)\right|\,.$$

$\lim\limits_{\sigma\to\infty} \zeta(\sigma) = 1$. Also for $s = x + it$, with x, t real, x large and t bounded:

$$\left| a\left(\frac{1}{2}+s\right)\right| = (2^3\,\pi)^{1/4}\, x^{7/4} \left(\frac{x}{2\pi e}\right)^{\frac{x}{2}} \left(1 + \frac{\varepsilon(s)}{x}\right),$$

with $\varepsilon(s)$ bounded. Thus $\lim\limits_{y\to\infty} m(y,c) = 0$. □

Therefore $\lim\limits_{w\to\infty} \sup\limits_{s\in B(4w,\delta)} |\Delta_2(s)| = 0$. □

Corollary 3.6. $\lim\limits_{\substack{|\operatorname{Re} s|\to\infty \\ s\in G(\delta)}} \Delta(s) = 0$, *if* $0 < \delta \leq 2$.

Proof. Say $|s - 4w| < \delta$, with w an integer

$$\Delta(s) = \Delta_2(s) - p_I(s) - \left(p_R(s) - \frac{c(4w)}{s-4w}\right).$$

$\lim\limits_{\substack{|\mathrm{Re}\, s| \to \infty \\ s \in G(\delta)}} \Delta_2(s) = 0$, by Lemma 3.8.

$\lim\limits_{|\mathrm{Re}\, s| \to \infty} p_I(s) = 0$ and $\lim\limits_{\substack{|w| \to \infty \\ s \in B(4w,\delta)}} \left(p_R(s) - \dfrac{c(4w)}{s-4w} \right) = 0$, by Lemma 3.3. □

Lemma 3.9. *Say* $\delta > 0$*,* $B \geq 0$ *and* $\delta^2 \leq B^2 + 2^2$*. Let* $s = x + iy$*, with* x, y *real. Let* T *be the set of* s *in* $\mathbb{C} - G(\delta)$ *with* $|t| \leq B$*. Then* $\lim\limits_{\substack{|x| \to \infty \\ s \in T}} f(s) = 0$.

Proof. We may assume $x \geq 0$,

$$|f(s)| = \frac{1}{\left|\sin\left(\frac{\pi}{4}\, s\right)\right|} \; \frac{1}{\left|q\left(\frac{1}{2} + s\right)\right|} \, .$$

Apply the claim of the proof of Lemma 3.8. Then $\lim\limits_{\substack{x \to \infty \\ s \in T}} f(s) = 0$, provided

$$\inf_{s \in T} \left|\sin\left(\frac{\pi}{4}\, s\right)\right| > 0 \, .$$

$\left|\sin\left(\frac{\pi}{4}\, s\right)\right|$ has period four. Let K be the set of $s = x + it$ with $|s| \geq \delta$, $|x| \leq 2$ and $|t| \leq B$. K is compact and $\sin\left(\frac{\pi}{4}\, s\right) \neq 0$ on K. Hence $\min\limits_{s \in K} \left|\sin\left(\dfrac{\pi}{4}\, s\right)\right| > 0$. □

Corollary 3.7. $\lim\limits_{\substack{|x| \to \infty \\ s \in T}} \Delta(s) = 0$.

We assemble the above results.

Corollary 3.8. $\lim\limits_{|s| \to \infty} \Delta(s) = 0$.

(Main) Theorem 3.3. $f(s) = p(s)$.

Proof. $\Delta(s)$ is analytic. $\Delta(s)$ is bounded. Hence $\Delta(s)$ is a constant, c say. $c = \lim\limits_{|s| \to \infty} \Delta(s) = 0$. □

4 On the Fourier Representation of $\left(\sin\left(\frac{\pi}{4}\, s\right) q\left(\frac{1}{2} + s\right)\right)^{-1}$ with $q(s) := \pi^{-s/2} \cdot 2 \cdot \Gamma\left(1 + \frac{s}{2}\right) \cdot (s-1)\, \zeta(s)$

Let s be complex. We define

$$q(s) := \pi^{-s/2}\, 2\Gamma\left(1 + \frac{s}{2}\right)(s-1)\, \zeta(s) \, ,$$

$$n(s) := \sin\left(\frac{\pi}{4}\, s\right) q\left(\frac{1}{2} + s\right) \quad \text{and} \quad f(s) := \frac{1}{n(s)} \, .$$

$f(s)$ is analytic on the half-plane $\mathrm{Re}(s) > \frac{1}{2}$ except for simple poles at the positive multiples of four.

Let A be one of the open vertical strips

$$\frac{1}{2} < \mathrm{Re}(s) < 4 \quad \text{or} \quad 4w < \mathrm{Re}(s) < 4(w+1)$$

with w a positive integer. We derive an explicit formula for the analytic inverse Fourier transform (AIFT), ρ_A, of $f(s)$ with s on A:

$$f(s) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \rho_A(y) \, .$$

This derivation does not involve the Riemann hypothesis or related conjectures about the non-real zeros of $\zeta(s)$.

Let n be a non-negative integer. Say $S = 2n+s$. Let $M = \frac{1}{2}\left(\frac{1}{2}+S\right)$, $m := \frac{1}{2}\left(\frac{1}{2}+s\right)$ and $z := \frac{1}{2}\left(\frac{3}{2}-s\right)$. Set

$$Q_n(m) := \prod_{j=0}^{n}(m+j) \quad \text{and} \quad \eta_n(s) := \frac{1}{2Q_n(m)} \cdot \Gamma(z)\,.$$

Claim 1.

$$\frac{1}{2\Gamma(1+M)} = \frac{1}{\pi}(\sin(\pi m))\,\eta_n(s)\,.$$

Proof. $M = n+m$. $\Gamma(1+n+m) = Q_n(m)\,\Gamma(m)$. $\frac{1}{\Gamma(m)} = \frac{1}{\pi}(\sin(\pi m))\,\Gamma(1-m)$. $z = 1-m$. $\square$

Set

$$e_n(u) := \sum_{j=0}^{n} \frac{u^j}{j!} \quad \text{and} \quad \tau_n(u) := (-u)^{-n}(e_n(-u) - e^{-u})\,.$$

Define

$$\tilde{p}_n(y) := \frac{1}{2}\, e^{y/2}\, \tau_n(e^{-2y})\,.$$

Let $x = \mathrm{Re}(s)$.

Claim 2. *Assume* $-\frac{1}{2} < x < \frac{3}{2}$. *Then*

$$\eta_n(s) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \tilde{p}_n(y)\,.$$

Proof.

$$\frac{1}{Q_n(m)} = \sum_{j=0}^{n} \frac{(-1)^j}{(n-j)!\,j!\,(m+j)}\,.$$

So

$$\eta_n(s) = \sum_{j=0}^{n} \frac{(-1)^j}{(n-j)!\,j!}\left(\frac{\Gamma(z)}{s+\frac{1}{2}+2j}\right)\,.$$

Assume $\mathrm{Re}(a) > -\frac{3}{2}$. Define

$$p_a(y) := \frac{1}{2}\, e^{ay}\, \gamma\left(\frac{1}{2}\left(\frac{3}{2}+a\right),\, e^{-2y}\right)\,.$$

($\gamma(u,v)$ is the incomplete gamma function)

The proof of Claim 2 will use:

Claim 3. *Let* $x = \mathrm{Re}(s)$ *and* $z = \frac{1}{2}\left(\frac{3}{2}+s\right)$. *Assume* $-a < x < \frac{3}{2}$. *Then*

$$\frac{\Gamma(z)}{s+a} = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, p_a(y)\,.$$

Proof of Claim 3. If $\mathrm{Re}(w) > 0$, then

$$\Gamma(w) = \int_{-\infty}^{\infty} (du)\, e^{wu} \exp(-e^{u})\,.$$

So $x < \frac{3}{2}$ implies

$$\Gamma(z) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, b(y) \quad \text{with} \quad b(y) := e^{-3y/2} \exp(-e^{-2y})\,.$$

If $x > -a$, then

$$\frac{1}{s+a} = \int_{-\infty}^{0} (dy)\, e^{sy}\, e^{ay}\,.$$

Say $-a < x < \frac{3}{2}$. Then

$$\frac{\Gamma(z)}{s+a} = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, (\alpha * b)(y) \quad \text{with} \quad \alpha(y) = e^{ay}\,.$$

Evaluating $\alpha * b$ yields $p_a(y)$. □

The proof of Claim 2 continues by applying Claim 3 with $-\frac{1}{2} < x < \frac{3}{2}$ to obtain

$$\eta_n(s) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \alpha(y) \quad \text{with} \quad \alpha(y) := \sum_{j=0}^{n} \frac{1}{(n-j)!}\, \frac{(-1)^j}{j!}\, p_{\frac{1}{2}+2j}(y)\,.$$

Now

$$p_{\frac{1}{2}+2j}(y) = \frac{1}{2}\, e^{y/2}\, J^{-j}\, \gamma(1+j, J) \quad \text{with} \quad J = e^{-2y}\,.$$

Also, for j a non-negative integer:

$$\gamma(1+j, u) = j!\,(1 - e_j(u) \cdot e^{-u})\,.$$

Therefore

$$\alpha(y) = \frac{1}{2}\, e^{y/2} (-J)^{-n} (e_n(-J) - \beta \cdot e^{-J})\,.$$

Here

$$\beta := \sum_{k=0}^{n} \frac{J^k}{k!}\, c_k \quad \text{with} \quad c_k := \sum_{v=0}^{k} \binom{k}{v} (-1)^v\,.$$

If $k \geq 1$, then $c_k = (1-1)^k = 0$. Therefore $\beta = 1$. □

$$\frac{1}{\sin\left(\frac{\pi}{4}\, S\right) \cdot 2\Gamma(1+M)} = \frac{1}{\pi}\, \frac{\sin(\pi m)}{\sin\left(\frac{\pi}{4}\, S\right)} \cdot \eta_n(s)$$

by Claim 1.

Claim 4. *Assume $n = 2k$. Then*

$$\frac{\sin(\pi m)}{\sin\left(\frac{\pi}{4}\, S\right)} = (-1)^k \left(\frac{1}{\sqrt{2}}\, \frac{1}{\sin\left(\frac{\pi}{4}\, s\right)} + 2\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\, s\right) \right)\,.$$

Assume $n = 2k+1$. Then

$$\frac{\sin(\pi m)}{\sin\left(\frac{\pi}{4}\, S\right)} = (-1)^k \left(-\frac{1}{\sqrt{2}}\, \frac{1}{\cos\left(\frac{\pi}{4}\, s\right)} + 2\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\, s\right) \right)\,.$$

Proof. If $n = 2k$, then

$$\sin\left(\frac{\pi}{4}\, S\right) = (-1)^k \sin\left(\frac{\pi}{4}\, s\right)\,.$$

If $n = 2k+1$, then

$$\sin\left(\frac{\pi}{4}\, S\right) = (-1)^k \cos\left(\frac{\pi}{4}\, s\right)\,.$$

The proof is completed using $\beta = \frac{\pi}{4}$ and $\alpha = \frac{\pi}{4}\, s$ in the identities:

$$\sin(\beta + 2\alpha) = \sin\beta + (\sin\alpha)\, 2\cos(\beta+\alpha)\,,$$

$$\sin(\beta + 2\alpha) = -\sin\beta + (\cos\alpha)\, 2\sin(\beta+\alpha)\,.$$

□

Define

$$\theta_{2k,1}(s) := \frac{1}{\sin\left(\frac{\pi}{4}\, s\right)}\, \eta_{2k}(s)$$

and

$$\theta_{2k+1,1}(s) := \frac{1}{\cos\left(\frac{\pi}{4}\, s\right)}\, \eta_{2k+1}(s)\,.$$

Set

$$\theta_{2k,2}(s) := 2\cos\left(\frac{\pi}{4} + \frac{\pi}{4}\, s\right) \eta_{2k}(s)$$

and

$$\theta_{2k+1,2}(s) := 2\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\, s\right) \cdot \eta_{2k+1}(s)\,.$$

Corollary 4.1. *If $n = 2k$ or $n = 2k+1$, then*

$$\frac{1}{\sin\left(\frac{\pi}{4}\, S\right) 2\Gamma(1+M)} = \frac{(-1)^k}{\pi} \left(\frac{(-1)^n}{\sqrt{2}}\, \theta_{n,1}(s) + \theta_{n,2}(s) \right)\,.$$

The AIFT, $h_n(y)$, of $\theta_{n,1}(s)$ (see Lemma 4.1) is built from the function $I_n(u,\alpha)$ decribed next.

Assume $\mathrm{Re}(u) > 0$ and $-2 < \mathrm{Re}(a) < 1$. Define

$$I_n(u,a) := \int_0^\infty (dv)\, \frac{v^a}{1+v^2}\, \tau_n(uv)\,.$$

(Note that $\int_0^1 (dv)\, \frac{v^a}{1+v^2}\, \tau_n(uv)$ exists since $\lim_{\alpha\to 0} \frac{\tau_n(\alpha)}{\alpha} = \frac{1}{(n+1)!}$. Also $\int_1^\infty (dv)\, \frac{v^a}{1+v^2}\, \tau_n(uv)$ exists since $\lim_{\substack{w\in\mathbb{R} \\ w\to\infty}} \tau_n(we^{i\varphi}) = \frac{1}{n!}$ for $|\varphi| < \frac{\pi}{2}$.)

$I_n(u,a)$ is related to hypergeometric functions as described next. Say $\mathrm{Re}(b) > 0$. Set

$$g_1(b) := \frac{1}{b} \cdot \frac{1}{2}(F(1,b,b+1,i) + F(1,b,b+1,-i))\,,$$

with F the hypergeometric function of Gauss

$$\int_0^1 (dt)\, \frac{t^{b-1}}{1+t^2} = g_1(b)\,.$$

If $\mathrm{Re}\, b < 2$, then

$$\int_1^\infty (dt)\, \frac{t^{b-1}}{1+t^2} = g_1(2-b)\,.$$

The confluent hypergeometric function of Kummer, $U(a,a,z)$, with $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, z > 0$, satisfies

$$j^{-a}\, \Gamma(a)\, U(a,a,z/j) = \int_0^\infty (dt)\, t^{a-1}(1+jt)^{-1}\, e^{-zt}$$

for $j = 1$, hence for $j > 0$ and by analytic continuation for non zero j with $|\mathrm{Arg}\,(j)| < \pi$. Define

$$g_2(a,z) := \Gamma(a) \cdot \frac{1}{2}\, (e^{-i\frac{\pi}{2}a}\, U(a,a,-iz) + e^{i\frac{\pi}{2}a}\, U(a,a,iz))\,.$$

Then

$$\int_0^\infty (dt)\, t^{a-1} \cdot \frac{1}{1+t^2} \cdot e^{-zt} = g_2(a,z)\,.$$

Say $\operatorname{Re} a > -2$ and $\operatorname{Re}(u) > 0$. Define

$$\Delta_n(u,a) := \left(\int_0^1 (dv)\, \frac{v^a}{1+v^2} \cdot \tau_n(uv)\right) - (-u)^{-n} \int_1^\infty (dv)\, \frac{v^{a-n}}{1+v^2}\, e^{-uv}\,.$$

$\Delta_n(u,a)$ is the analytic extension from $\operatorname{Re}(a) > n-1$ to $\operatorname{Re}(a) > -2$ of

$$\left(\sum_{j=0}^{n} \frac{(-u)^{-j}}{(n-j)!}\, g_1(a-j+1)\right) - (-u)^{-n}\, g_2(u, a-n+1)\,.$$

Say $\operatorname{Re}(a) < 1$. Define

$$\lambda_n(u,a) := \sum_{j=0}^{n} \frac{(-u)^{-j}}{(n-j)!}\, g_1(j+1-a)\,.$$

Assume $-2 < \operatorname{Re}(a) < 1$. Then $I_n(u,a) = \Delta_n(u,a) + \lambda_n(u,a)$.

Let $x = \operatorname{Re}(s)$.

Lemma 4.1. $\theta_{n,1}(s) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, h_n(y)$ *holds in the following cases.*

Say n is even.

(i) *Assume* $0 < x < \frac{3}{2}$. *Then* $h_n(y) := \frac{1}{\pi}\, e^{y/2}\, I_n\left(e^{-2y}, \frac{-5}{4}\right)$.

(ii) *Assume* $-\frac{1}{2} < x < 0$. *Then* $h_n(y) := -\frac{1}{\pi}\, e^{y/2}\, I_n\left(e^{-2y}, \frac{3}{4}\right)$.

Say n is odd. Assume $-\frac{1}{2} < x < \frac{3}{2}$. *Then*

$$h_n(y) := \frac{1}{\pi}\, e^{y/2}\, I_n\left(e^{-2y}, -\frac{1}{4}\right)\,.$$

Proof. Claim 2 holds for $-\frac{1}{2} < x < \frac{3}{2}$. Say n is even. If $0 < x < 4$, then

$$\frac{1}{\sin\left(\frac{\pi}{4}\, s\right)} = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \alpha(y) \quad \text{with} \quad \alpha(y) := \frac{4}{\pi}\, \frac{1}{1+e^{4y}}$$

$$\left(\text{since} \quad \frac{\pi}{\sin(\pi\beta)} = \frac{1}{\beta} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{\beta-k} + \frac{1}{\beta+k}\right)\right)\,.$$

If $-4 < x < 0$, then

$$\frac{1}{\sin\left(\frac{\pi}{4}\, s\right)} = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \alpha(y) \quad \text{with} \quad \alpha(y) := -\frac{4}{\pi}\, \frac{1}{1+e^{-4y}}\,.$$

Thus cases (i), (ii) each have $\alpha * \operatorname{pr}_n = h_n$.

Say n is odd. If $-2 < x < 2$, then

$$\frac{1}{\cos\left(\frac{\pi}{4}\, s\right)} = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \alpha(y) \quad \text{with} \quad \alpha(y) := \frac{2}{\pi}\, \sec h(2y)\,.$$

Say $-\frac{1}{2} < x < \frac{3}{2}$. Then $\alpha * \operatorname{pr}_n = h_n$. $\square$

Say y is real. Define

$$j_n(y) := \frac{1}{2}\, e^{y/2}\, \mathrm{Re}(A) \quad \text{for } n \text{ even}$$

and

$$j_n(y) := \frac{1}{2}\, e^{y/2}\, \mathrm{Im}(A) \quad \text{for } n \text{ odd,}$$

with

$$A = e^{i\pi/8}\, \tau_n(i\, e^{-2y})\,.$$

Lemma 4.2. *Assume* $-\frac{1}{2} < \mathrm{Re}(s) < \frac{3}{2}$*. Then*

$$\theta_{n,2}(s) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, j_n(y)\,.$$

Proof. Say $s = x + it$ with x, t real. Set

$$a_n(u, s) = e^{su} \cdot 2\, \tilde{p}_n(u)\,.$$

Take

$$K_n = \sum_{k=0}^{n} \frac{1}{k!}\,.$$

Say α is real.

The proof of Lemma 4.2 employs Claims 4, 5 below.

Claim 4.

(i) *If* $y \geq 0$*, then*

$$|a_n(y + i\alpha, x + iy)| \leq (e - K_n)\, e^{|\alpha||t|}\, e^{\left(x - \frac{3}{2}\right) y}\,.$$

$\Big($*Therefore: If* $x < \frac{3}{2}$*, then*

$$\int_0^{\infty} (dy) |a_n(y + i\alpha, s)| \leq \frac{(e - K_n)}{\frac{3}{2} - x}\, e^{\frac{\pi}{4}|t|}\,.\Big)$$

(ii) *Assume* $|\alpha| \leq \frac{\pi}{4}$*. If* $y \leq 0$*, then*

$$|a_n(y + i\alpha, s)| \leq (1 + K_n)\, e^{\frac{\pi}{4}\,|t|}\, e^{\left(x + \frac{1}{2}\right) y}\,.$$

$\Big($*Therefore: If* $x > -\frac{1}{2}$*, then*

$$\int_{-\infty}^{0} (dy)\, |a_n(y + i\alpha, s)| \leq \frac{1 + K_n}{x + \frac{1}{2}} \cdot e^{\frac{\pi}{4}\,|t|}\,.\Big)$$

Proof of Claim 4.

$$|a_n(y + i\alpha, x + it)| = e^{-\alpha t}\, e^{\left(x + \frac{1}{2}\right) y}\, |\tau_n(w)| \quad \text{with} \quad w = e^{-2y}\, e^{i(-2\alpha)}\,.$$

(i) follows from:

$$|\tau_n(w)| \leq \left(\sum_{k=1}^{\infty} \frac{|w|^{k-1}}{(k+n)!} \right) |w|$$

(ii) follows from:

$$|\tau_n(w)| \leq \left(\sum_{j=0}^{n} \frac{|w|^{-(n-j)}}{j!}\right) + |w|^{-n} \cdot e^{-b} \quad \text{with} \quad b = e^{-2y} \cos(2\alpha)\,.$$

□

Fix $s = x + it$ with $-\frac{1}{2} < x < \frac{3}{2}$. Assume $|\alpha| \leq \frac{\pi}{4}$. Define

$$J_n(\alpha, s) := \int_{-\infty+i\alpha}^{\infty+i\alpha} (dy)\, e^{sy}\, 2\, \tilde{p}_n(y)\,.$$

Claim 5. *$J_n(\alpha, s)$ is constant in α for $|\alpha| \leq \frac{\pi}{4}$.*

Proof of Claim 5.

$$J_n(\alpha, s) = \int_{-\infty}^{\infty} (dy)\, a_n(y + i\alpha, s)\,.$$

$a_n(u, s)$ is analytic in u on $\mathbb{C}$. Let γ be a counter-clockwise rectangular path with successive components V_1, H_1, V_2, H_2. Here V_k is a vertical side on the line $\mathrm{Re}(u) = \ell_k$, with $\ell_1 \leq \ell_2$. Also, H_k is a horizontal side on the line $\mathrm{Im}(u) = \alpha_k$, with

$$-\frac{\pi}{4} \leq \alpha_1 \leq \alpha_2 \leq \frac{\pi}{4}\,.$$

$\int_\gamma a_n(\gamma, s) = 0$. Thus

$$\left|\int_{H_1} a_n + \int_{H_2} a_n\right| \leq L(\ell_1) + L(\ell_2)\,,$$

with

$$L(y) := \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (d\alpha)\, \mid a_n(y + i\alpha, s)$$

for real y. $L(y)$ vanishes as $y \to \pm\infty$, by Claim 4. Thus $J_n(\alpha_1, s) - J_n(\alpha_2, s)$, which is the limit as $\ell_1 \to -\infty$ and $\ell_2 \to \infty$ of $\int_{H_1} a_n + \int_{H2} a_n$, is zero.

We now complete the proof of Lemma 4.2. Let $\sigma = \pm 1$. Set

$$E_{n,\sigma}(s) = e^{i\sigma\left(\frac{\pi}{4} + \frac{\pi}{4}\, s\right)} \cdot J_n(0, s)\,.$$

If n is even,

$$\theta_{n,2}(s) = \frac{1}{2} \sum_{\sigma} E_{n,\sigma}(s)\,.$$

If n is odd,

$$\theta_{n,2}(s) = \frac{1}{2i} \sum_{\sigma} \sigma \cdot E_{n,\sigma}(s)\,.$$

Say $|\beta| \leq \frac{\pi}{4}$. Then $J_n(0, s) = J_n(\beta, s)$. So

$$E_{n,\sigma}(s) = e^{i\alpha s} \int_{-\infty}^{\infty} (dy)\, e^{sy} \cdot e^{y/2}\, e^{i\alpha/2}\, \tau_n(e^{i(-2\beta)}\, e^{-2y})\,,$$

with $\alpha = \sigma\, \frac{\pi}{2} + \beta$. Take $\beta = -\sigma\, \frac{\pi}{4}$. Then

$$E_{n,\sigma}(s) = \int_{-\infty}^{\infty} (dy)\, e^{sy} \cdot e^{y/2}\, e^{i\sigma\frac{\pi}{8}}\, \tau_n(\sigma i e^{-2y})\,.$$

□

Consider the cases of Lemma 4.1. Define

$$r_n(y) := \frac{(-1)^k}{\pi} \left(\frac{(-1)^n}{\sqrt{2}}\, h_n(y) + j_n(y) \right)$$

with $n = 2k$ or $2k+1$.

Corollary 4.1 and Lemmas 4.1, 4.2 together imply:

Corollary 4.2.

$$\frac{1}{\sin\left(\frac{\pi}{4}\, S\right) \cdot 2\Gamma(1+M)} = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, r_n(y)\,.$$

Let $S = 2n + s$ with $\mathrm{Re}(s) > \frac{1}{2}$ and n is non-negative integer. Set $M = \frac{1}{2}\left(\frac{1}{2} + S\right)$ and $a = \frac{1}{2} + 2n$.

$$f(S) = \pi^{a/2} \cdot \pi^{s/2} \cdot \frac{1}{\sin\left(\frac{\pi}{4}\, S\right)\, 2\Gamma(1+M)} \cdot \frac{1}{\left(S - \frac{1}{2}\right) \zeta\left(\frac{1}{2} + S\right)}\,.$$

Let $\mathrm{Re}(u) > 1$. $1/(\zeta(u)) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^u}$. Hence

$$\frac{1}{(u-1)\,\zeta(u)} = \int_{-\infty}^{0} (dy)\, e^{sy}\, \omega(e^{-y})\,,$$

with

$$\omega(v) := \sum_{1 \le n \le v} \frac{\mu(n)}{n}\,.$$

Define

$$\Omega_b(v) := v^{-b} \omega(v)\,.$$

Then

$$\frac{1}{\left(S - \frac{1}{2}\right)\, \zeta\left(\frac{1}{2} + S\right)} = \int_{-\infty}^{0} (dy)\, e^{sy}\, \Omega_a(e^{-y})\,.$$

Next applying Corollary 4.2 yields:

Theorem 4.1.

$$f(S) = \int_{-\infty}^{\infty} (dy)\, e^{sy}\, \rho_n(y) \quad \text{with} \quad \rho_n(y) := \pi^{\frac{1}{4}+n} (r_n * \Omega_a) \left(y - \frac{1}{2} \log \pi \right)\,.$$

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