A group of diffeomorphisms of the interval with intermediate growth

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Abstract. We construct a group of C^1-diffeomorphisms of the interval whose growth is intermediate, that is, faster than any polynomial but slower than any exponential. We prove that such an example cannot arise as a group of C^2-diffeomorphisms of the interval. This generalises a classical result by J. Plante and W. Thurston.

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Introduction

A theory for groups of diffeomorphisms of the interval has been extensively developed by many authors (see for example [16, 18, 19, 24, 26, 28, 31]). One of the most interesting topics of this theory is the interplay between the differentiability class of the diffeomorphisms and the algebraic (as well as dynamical) properties of the group (and the action). For instance, as a consequence of the classical Bounded Distorsion Principle, groups of C^2-diffeomorphisms appear to have a very rigid behavior. This is no longer true for subgroups of $\mathrm{Diff}_+^1([0,1])$, as it is well illustrated in the literature [4, 22]. The aim of this work is to study this phenomenon for a remarkable class of groups, first introduced by R. Grigorchuk.

Given a finitely generated group (provided with a finite system of generators), the growth function assigns, to each positive integer n, the number of elements of the group that can be written as a product of no more than n generators or their inverses. One says that the group has polynomial, exponential or intermediate growth, if its growth function has the corresponding asymptotics. Those notions do not depend on the choice of the (finite) system of generators. A deep theorem by M. Gromov establishes that a group has polynomial growth if and only if it is almost nilpotent, i.e. if it contains a finite index nilpotent subgroup. (See [11] and references therein.) Typical examples of groups with exponential growth are those that contain free semigroups on two generators. (However, there exist groups with exponential growth and no free semigroup on two generators; see [21]). The difficult question (raised by J. Milnor [17]) concerning the existence of groups with intermediate growth was answered affirmatively by R. Grigorchuk in [8] (see also [9]). Some years later [10], one of his examples was realised (by R. Grigorchuk himself and A. Maki) as a subgroup of $\mathrm{Homeo}_+([0,1])$. The problem of improving the regularity of this embedding is at the core of this work. In the main part of this paper we prove the following result.

Theorem A. *There exists a finitely generated subgroup of* $\mathrm{Diff}^1_+([0,1])$ *with intermediate growth.*

Theorem A above answer a question from [10]. In fact, the group we consider turns out to be isomorphic to the group introduced by R. Grigorchuk in [9], and studied in more detail in [10]. In the sequel, we will denote this group by H. We will prove more generally that for any C^1-neighborhood V of the identity map of $[0,1]$, there exists an embedding $H \hookrightarrow \mathrm{Diff}^1_+([0,1])$ sending the generators of H to elements of V. This is an interesting issue; for instance, it is known that subgroups of $\mathrm{Diff}^2_+(\mathrm{S}^1)$ generated by elements near the identity have very restrictive dynamical properties [20].

The proof of Theorem A has two main technical ingredients. The first one is, instead of embedding directly H into $\mathrm{Diff}^1_+([0,1])$ (which seems to be very difficult), to construct a coherent sequence of embeddings of some almost nilpotent groups which in some sense converge to H. (The group H turns out to be residually almost nilpotent.) An equicontinuity argument allows us to obtain, at the limit, the desired embedding. However, in order to apply this argument, it is necessary to obtain a uniform control for the derivatives of the generators of each group in the afore mentioned sequence with respect to some fixed modulus of continuity. To do this, the second ingredient of the construction is to use a technique inspired by Chapter X of M. Herman's thesis [14]. This is related to the classical construction of $C^{1+\tau}$-Denjoy counter-examples, which seems go back to F. Sergeraert.

It is maybe possible that, by refining the technique of the construction above, it is possible to improve the regularity for the embedding in order to reach the class $C^{1+\tau}$ for any $\tau < 1$. However, a quite remarkable obstruction appears in class C^2. A weak version of the following result was already given in [18] for the description of the dynamics of a particular family of amenable groups of diffeomorphisms of the interval.

Theorem B. *Every subgroup of* $\mathrm{Diff}^2_+([0,1[)$ *without free semigroups on two generators is Abelian.*

The proof of Theorem B relies on the rigidity theory for centralisers of C^2-diffeomorphisms of the interval. The foundations of this theory are related to the so called Kopell's Lemma [16] (each Abelian subgroup of $\mathrm{Diff}^2_+([0,1[)$ either acts freely on $]0,1[$ or has a global fixed point in $]0,1[$), and Szekeres's Theorem [31] (the centraliser in $\mathrm{Diff}^1_+([0,1[)$ of every element of $\mathrm{Diff}^2_+([0,1[)$ without fixed points in $]0,1[$ is conjugated to a group of translations). For other classes of groups there is Plante-Thurston's Theorem [24] (nilpotent subgroups of $\mathrm{Diff}^2_+([0,1[)$ are Abelian), and the classification of solvable subgroups of $\mathrm{Diff}^2_+([0,1[)$ obtained by the author in [19]. Let us remark that all those results, as well as Theorem B, are still true for groups of C^1-diffeomorphisms of $[0,1[$ whose derivatives have finite total variation on compact subsets of $[0,1[$.

Concerning other 1-dimensional manifolds, let us mention that the methods used for the proof of Theorem B allow to demonstrate that subgroups of $\mathrm{Diff}^2_+(\mathbb{R})$ with subexponential growth are metabelian. Moreover, following a short argument given in [18, 20], Theorem B implies that groups of C^2-diffeomorphisms of the circle with subexponential growth are Abelian.

Remark. The example given for Theorem A can be seen as a sharp C^1-counter-example for Theorem B. Nevertheless, it must be mentioned that several examples of "pathological" subgroups of $\mathrm{Diff}^1_+([0,1])$ were already known. Indeed, D. Pixton constructed in [22] a C^1-counter-example for Kopell's Lemma and Szekeres's Theorem simultaneously. And recently, B. Farb and J. Franks [4] gave C^1-counter-examples for Plante-Thurston's Theorem; more precisely, they proved that $\mathrm{Diff}^1_+([0,1])$ contains all finitely generated torsion free nilpotent groups. Our construction for Theorem A is strongly inspired by the one given in [4]. Another very interesting source of information on this kind of topics is [29].

1 Continuous actions on the Cantor set and the interval

First Grigorchuk's group G can be seen in many different ways: as the group generated by a finite automaton, as a group acting on the binary rooted tree $\mathcal{T}_2$, and as a group acting isometrically on the Cantor set $\{0,1\}^{\mathbb{N}}$. This last two points of view are essentially the same, since the boundary at infinity of $\mathcal{T}_2$ can be identified with $\{0,1\}^{\mathbb{N}}$. Using the convention $(x_1,(x_2,x_3,\ldots)) = (x_1,x_2,x_3,\ldots)$ for $x_i \in \{0,1\}$, the generators of G are elements $\bar{a},\bar{b},\bar{c},\bar{d}$ whose actions on sequences $(x_1,x_2,x_3,\ldots)$ in $\{0,1\}^{\mathbb{N}}$ are recursively defined by

$$\bar{a}(x_1,x_2,x_3,\ldots) = (1-x_1,x_2,x_3,\ldots),$$

$$\bar{b}(x_1,x_2,x_3,\ldots) = \begin{cases} (x_1,\bar{a}(x_2,x_3,\ldots)), & x_1 = 0, \\ (x_1,\bar{c}(x_2,x_3,\ldots)), & x_1 = 1, \end{cases}$$

$$\bar{c}(x_1,x_2,x_3,\ldots) = \begin{cases} (x_1,\bar{a}(x_2,x_3,\ldots)), & x_1 = 0, \\ (x_1,\bar{d}(x_2,x_3,\ldots)), & x_1 = 1, \end{cases}$$

$$\bar{d}(x_1,x_2,x_3,\ldots) = \begin{cases} (x_1,x_2,x_3,\ldots), & x_1 = 0, \\ (x_1,\bar{b}(x_2,x_3,\ldots)), & x_1 = 1. \end{cases}$$

The action on $\mathcal{T}_2$ of the element $\bar{a} \in G$ consists on permuting the first two edges (and consequently, the trees rooted on the final vertex of each one of those edges). Elements $\bar{b}$, $\bar{c}$ and $\bar{d}$ fix the two first edges of $\mathcal{T}_2$, and their action on higher levels is illustrated by the figure below.

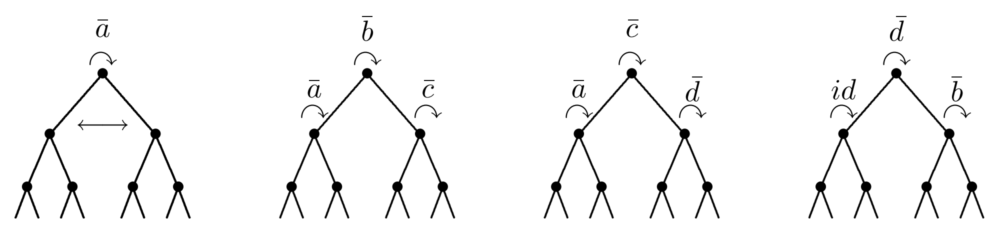

Figure 1

It can be proven that G is a torsion group: each element has order a power of 2 (see [7] or [12]). The first example of a torsion free group with intermediate growth was given in [9]. Geometrically, the idea consists on replacing $\mathcal{T}_2$ by a rooted tree having vertices of infinite (countable) degree. So, we consider the group H acting the space $\Omega = \mathbb{Z}^{\mathbb{N}}$ which is generated by elements a, b, c and d recursively defined by

$$a(x_1, x_2, x_3, \ldots) = (1 + x_1, x_2, x_3, \ldots),$$

$$b(x_1, x_2, x_3, \ldots) = \begin{cases} (x_1, a(x_2, x_3, \ldots)), & x_1 \text{ even}, \\ (x_1, c(x_2, x_3, \ldots)), & x_1 \text{ odd}, \end{cases}$$

$$c(x_1, x_2, x_3, \ldots) = \begin{cases} (x_1, a(x_2, x_3, \ldots)), & x_1 \text{ even}, \\ (x_1, d(x_2, x_3, \ldots)), & x_1 \text{ odd}, \end{cases}$$

$$d(x_1, x_2, x_3, \ldots) = \begin{cases} (x_1, x_2, x_3, \ldots), & x_1 \text{ even}, \\ (x_1, b(x_2, x_3, \ldots)), & x_1 \text{ odd}. \end{cases}$$

The group H preserves the lexicographic order in Ω. It is then an orderable group [6], and so it can be realised as a group of orientation preserving homeomorphisms of the interval. Those facts where first established (by an indirect method) in [10].

Now we give an elementary proof of the fact that H can be realised as a group of bilipschitz homeomorphisms of $[0, 1]$.[1] Fix a sequence $(l_i)_{i\in\mathbb{Z}}$ of positive numbers such that $\sum l_i = 1$ and

$$\max\left\{\frac{l_{i+1}}{l_i}, \frac{l_i}{l_{i+1}}\right\} \leq M < \infty \quad \text{for all} \quad i \in \mathbb{Z}.$$

Denote by I_i the interval $[\sum_{j<i} l_j, \sum_{j\leq i} l_j]$. Let $f : [0, 1] \to [0, 1]$ be the orientation preserving homeomorphism sending each interval I_i onto I_{i+1} affinely. Let g be the orientation preserving affine homeomorphism sending $[0, 1]$ onto I_0, and denote by $\lambda = 1/l_0$ the (constant) value of its derivative. Consider the maps A, B, C and D defined recursively on a dense subset of $[0, 1]$ by putting $A(x) = f(x)$ and, for $x \in I_i$,

$$B(x) = \begin{cases} f^i g A g^{-1} f^{-i}(x), & i \text{ even}, \\ f^i g C g^{-1} f^{-i}(x), & i \text{ odd}, \end{cases}$$

$$C(x) = \begin{cases} f^i g A g^{-1} f^{-i}(x), & i \text{ even}, \\ f^i g D g^{-1} f^{-i}(x), & i \text{ odd}, \end{cases}$$

$$D(x) = \begin{cases} x, & i \text{ even}, \\ f^i g B g^{-1} f^{-i}(x), & i \text{ odd}. \end{cases}$$

[1] This fact is not surprising at all. For instance, a simple argument using the harmonic measure allows to show that if Γ is a finitely generated subgroup of $\mathrm{Homeo}_+(\mathrm{S}^1)$ having all its orbits dense, then Γ is topologically conjugated to a group of bilipschitz homeomorphisms of the circle [5].

We claim that A, B, C and D are bilipschitz homeomorphisms with Lipschitz constant bounded above by M. Indeed, this is clear for A. For B, C and D, this fact can be easily verified by induction. For example, if $x \in I_i$ for an even integer i, then

$$B'(x) = \frac{(f^i)'(gAg^{-1}f^{-i}(x))}{(f^i)'(f^{-i}(x))} \cdot \frac{g'(Ag^{-1}f^{-i}(x))}{g'(g^{-1}f^{-i}(x))} \cdot A'(g^{-1}f^{-i}(x)),$$

and since $g'|_{[0,1]} = \lambda$ and $(f^i)'|_{I_0} = l_i/l_0$, we obtain $B'(x) = A'(g^{-1}f^{-i}(x)) \leq M$. The maps A, B, C and D extend to bilipschitz homeomorphisms of the whole interval $[0, 1]$, and it is geometrically clear that they generate a group isomorphic to H. Remark finally that the constant M can be chosen as near to 1 as we want.

The preceding idea is not appropriated to obtain an embedding of H into $\mathrm{Diff}^1_+([0, 1])$. Indeed, the discontinuities for the derivative repeat at each level of the action of H. In the following section we will give a constructive method to obtain such an embedding; for that construction we will have to renormalise suitably the geometry at each step. Note that this idea appears already in [18], and we will indeed perform the construction given there. Denoting by H_n the stabiliser of the level n of the tree $\mathcal{T}_\infty$ for the action of H, we will construct embeddings of H/H_n into $\mathrm{Diff}^1_+([0, 1])$ in a coherent way and keeping some uniform control for the derivatives of generators. Using Arzela-Ascoli's Theorem, we will pass to the limit and obtain the desired embedding. Unfortunatelly, this method of construction will involve some technical issues. As a mather of fact, we emphasize that we have not been able to carry out a similar construction to obtain an embedding $H \hookrightarrow \mathrm{Diff}^{1+\tau}_+(\mathrm{S}^1)$ for any $0 < \tau < 1$. Moreover, the action we will obtain is only semiconjugated, but not conjugated, to the bilipschitz action constructed above. (However, this seems to be a necessary condition for C^1-actions of H.)

Remark 1.1. In [8], R. Grigorchuk gives a general procedure to construct groups of intermediate growth as groups acting on the dyadic rooted infinite tree $\mathcal{T}_2$. It is very plausible that, by considering the induced groups acting on the tree $\mathcal{T}_\infty$ by order preserving maps, one still obtains groups with intermediate growth. For all of these induced groups, the methods of this work lead to realisations as subgroups of $\mathrm{Diff}^1_+([0, 1])$.

2 Embeddings using equivariant families of homeomorphisms

Henceforth, we will deal only with *orientation preserving* homeomorphisms of the interval. A family of such homeomorphisms $\{\varphi_{a,b} : [0, a] \to [0, b]; a > 0, b > 0\}$ will be called *equivariant* if for all $a > 0, b > 0$ and $c > 0$, one has $\varphi_{b,c} \circ \varphi_{a,b} = \varphi_{a,c}$. Given such a family and two non degenerated intervals $I = [x_1, x_2]$ and $J = [y_1, y_2]$, we will denote by $\varphi(I, J) : I \to J$ the homeomorphism defined by

$$\varphi(I, J)(x) = \varphi_{x_2-x_1,y_2-y_1}(x - x_1) + y_1.$$

Note that $\varphi(I, I)$ is forced to be the identity map.

The simplest family of equivariant homeomorphisms is the one formed by the affine maps $\varphi_{a,b}(x) = bx/a$. However, this family is not adequate if we want to fit together maps smoothly. So, let us introduce a general and simple procedure to construct families of equivariant homeomorphisms as follows. Let $\{\varphi_a : \mathbb{R} \to]0, a[; a > 0\}$ be any family of homeomorphisms. Define $\varphi_{a,b} :]0, a[\to]0, b[$ by $\varphi_{a,b} = \varphi_b \circ \varphi_a^{-1}$. One has

$$\varphi_{b,c} \circ \varphi_{a,b} = (\varphi_c \circ \varphi_b^{-1}) \circ (\varphi_b \circ \varphi_a^{-1}) = \varphi_c \circ \varphi_a^{-1} = \varphi_{a,c}.$$

So, extending continuously $\varphi_{a,b}$ to the whole interval $[0, a]$, we obtain the desired equivariant family.

Example 2.1. Let $\varphi_a : \mathbb{R} \to]0, a[$ given by

$$\varphi_a(u) = \frac{1}{\pi} \int_{-\infty}^{u} \frac{ds}{s^2 + (1/a)^2} = \frac{a}{2} + \frac{a}{\pi} \arctan(au).$$

The corresponding equivariant family $\{\varphi_{a,b} : [0, a] \to [0, b]; a > 0, b > 0\}$ will be essential in what follows. It was introduced by J. C. Yoccoz, and it was already used in [4]. The regularity properties of the maps $\varphi_{a,b}$ will be studied in §3.

Now fix any equivariant family of homeomorphisms $\{\varphi_{a,b} : [0, a] \to [0, b]; a > 0, b > 0\}$. For each $n \in \mathbb{N}$ and each $(x_1, \ldots, x_n) \in \mathbb{Z}^n$, let us consider a non degenerated closed interval $I_{x_1,\ldots,x_n} = [a_{x_1,\ldots,x_n}, b_{x_1,\ldots,x_n}]$ and a (maybe degenerated) closed interval $J_{x_1,\ldots,x_n} = [c_{x_1,\ldots,x_n}, d_{x_1,\ldots,x_n}]$, both contained in some interval $[0, T]$. Let us suppose that the following conditions are satisfied (see Figure 2):

(i) $\sum_{x_1 \in \mathbb{Z}} |I_{x_1}| = T$ (where $|\cdot|$ denotes the length of the corresponding interval),

(ii) $a_{x_1,\ldots,x_n} < c_{x_1,\ldots,x_n} \leq d_{x_1,\ldots,x_n} = b_{x_1,\ldots,x_n}$, so in particular $J_{x_1,\ldots,x_n} \subset I_{x_1,\ldots,x_n}$,

(iii) $b_{x_1,\ldots,x_{n-1},x_n} = a_{x_1,\ldots,x_{n-1},1+x_n}$,

(iv) $\lim_{x_n \to -\infty} a_{x_1,\ldots,x_{n-1},x_n} = a_{x_1,\ldots,x_{n-1}}$,

(v) $\lim_{x_n \to \infty} a_{x_1,\ldots,x_{n-1},x_n} = c_{x_1,\ldots,x_{n-1}}$,

(vi) $\lim_{n \to \infty} \sup_{(x_1,\ldots,x_n) \in \mathbb{Z}^n} |I_{x_1,\ldots,x_n}| = 0$.

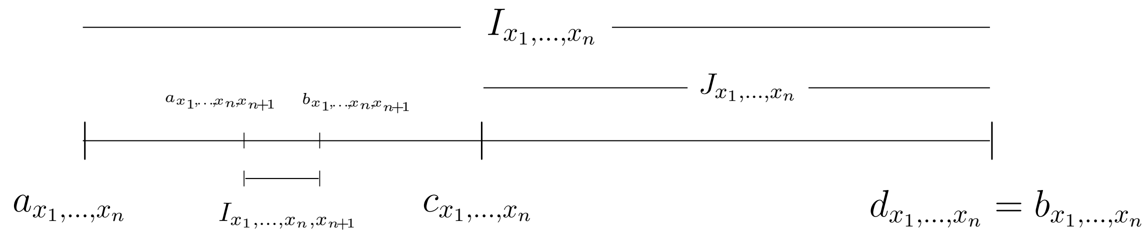

Figure 2

Note that

$$|J_{x_1,\ldots,x_n}| + \sum_{x_{n+1}\in\mathbb{Z}} |I_{x_1,\ldots,x_n,x_{n+1}}| = |I_{x_1,\ldots,x_n}|. \tag{1}$$

For each $n \in \mathbb{N}$ we will define homeomorphisms A_n, B_n, C_n and D_n, in such a way that the group generated by them will be isomorphic to H/H_n. To do this, let us consider the homomorphisms ϕ_0 and ϕ_1 from the subgroup of H generated by b, c and d into H defined by

$$\phi_0(b) = a, \quad \phi_0(c) = a, \quad \phi_0(d) = id, \qquad \text{and} \qquad \phi_1(b) = c, \quad \phi_1(c) = d, \quad \phi_1(d) = b.$$

<u>Definition of A_n</u>

– If $p \in J_{x_1,\ldots,x_i}$ for some $i < n$, let $A_n(p) = \varphi(J_{x_1,x_2,\ldots,x_i}, J_{1+x_1,x_2,\ldots,x_i})(p)$.

– If $p \in I_{I_{x_1,\ldots,x_n}} \setminus \cup_{i<n} \cup_{(x_1,\ldots,x_i)\in\mathbb{Z}^i} J_{x_1,\ldots,x_i}$, let $A_n(p) = \varphi(I_{x_1,x_2,\ldots,x_n}, I_{1+x_1,x_2,\ldots,x_n})(p)$.

<u>Definition of B_n</u>

Suppose that $p \in]0,1[$ belongs to $I_{x_1,\ldots,x_n}$, and denote by $(\bar{x}_1,\ldots,\bar{x}_n) \in \{0,1\}^n$ the corresponding sequence reduced modulo 2.

– If $\phi_{\bar{x}_1}(b)$, $\phi_{\bar{x}_2}\phi_{\bar{x}_1}(b)$, $\ldots$, $\phi_{\bar{x}_n}\cdots\phi_{\bar{x}_2}\phi_{\bar{x}_1}(b)$ are well defined, let $B_n(p) = p$.

– Otherwise, denote by $i = i(p) \leq n$ the smallest integer such that $\phi_{\bar{x}_i}\cdots\phi_{\bar{x}_2}\phi_{\bar{x}_1}(b)$ is not defined.

  · If $p \in J_{x_1,\ldots,x_j}$ for some $j < i$, let $B_n(p) = p$.

  · If $p \in J_{x_1,\ldots,x_i}$, let $B_n(p) = \varphi(J_{x_1,\ldots,x_{i-1},x_i}, J_{x_1,\ldots,x_{i-1},1+x_i})(p)$.

  · If $p \in J_{x_1,\ldots,x_i,\ldots,x_j}$ for some $i < j < n$, let $B_n(p) = \varphi(J_{x_1,\ldots,x_i,\ldots,x_j}, J_{x_1,\ldots,1+x_i,\ldots,x_j})(p)$.

  · If $p \in I_{x_1,\ldots,x_n} \setminus \cup_{j<n} J_{x_1,\ldots,x_j}$, let $B_n(p) = \varphi(I_{x_1,\ldots,x_i,\ldots,x_n}, I_{x_1,\ldots,1+x_i,\ldots,x_n})(p)$.

The definitions of C_n and D_n are similar to that of B_n. Clearly, the maps A_n, B_n, C_n and D_n extend to homeomorphisms of $[0,T]$. The fact that they generate a group isomorphic to H/H_n is geometrically clear and follows easily from the equivariant properties of the maps $\varphi_{a,b}$. Moreover, condition (vi) implies that the sequences of maps A_n, B_n, C_n and D_n converge to limit homeomorphisms A, B, C and D respectively, which generate a group isomorphic to H.

Example 2.2. Given a sequence $(l_i)_{i\in\mathbb{Z}}$ of positive numbers such that $\sum l_i = 1$, define $|I_{x_1,\ldots,x_n}|$ and $|J_{x_1,\ldots,x_n}|$ by

$$|J_{x_1,\ldots,x_n}| = 0, \qquad |I_{x_1,\ldots,x_n}| = l_{x_1}\cdots l_{x_n}.$$

If one carries out the preceding construction (for $T = 1$) using the equivariant family of affine maps $\varphi_{a,b}(x) = bx/a$, then one recovers the embedding of H into the group of bilipschitz homeomorphisms of the interval constructed at the end of §1 (under the assumptions that $l_{i+1}/l_i \leq M$ and $l_i/l_{i+1} \leq M$ for all $i \in \mathbb{Z}$).

3 Modulus of continuity for derivatives

Let $\omega : [0,1] \to [0, \omega(1)]$ be an increasing homeomorphism. A continuous map $\psi : [0,1] \to \mathbb{R}$ is ω-continuous if there exists $M < \infty$ such that, for all $x \neq y$ in $[0,1]$,

$$\left| \frac{\psi(x) - \psi(y)}{\omega(|x-y|)} \right| \leq M.$$

We denote by $\|\psi\|_\omega$ the supremum of the left hand expression, and we call it the ω-norm of ψ. The interest on the notion of ω-continuity relies on the obvious fact that, if (ψ_n) is a sequence of functions defined on $[0,1]$ such that

$$\sup_{n \in \mathbb{N}} \|\psi_n\|_\omega < \infty,$$

then (ψ_n) is an equicontinuous sequence.

Example 3.1. For $\omega(s) = s^\tau$, $0 < \tau < 1$, the notions of ω-continuity and τ-Hölder continuity coincide.

Example 3.2. For $\varepsilon > 0$ suppose that $\omega = \omega_\varepsilon$ is such that $\omega_\varepsilon(s) = s \log(1/s)^{1+\varepsilon}$ for s small. If a map φ is ω_ε-continuous, then it is τ-Hölder continuous for all $0 < \tau < 1$. Indeed, it is easy to verify that

$$s \left(\log \left(\frac{1}{s} \right) \right)^{1+\varepsilon} \leq C_{\varepsilon,\tau} s^\tau, \qquad C_{\varepsilon,\tau} = \frac{1}{e^{1+\varepsilon}} \left(\frac{1+\varepsilon}{1-\tau} \right)^{1+\varepsilon}.$$

Remark that the map $s \mapsto s \log(1/s)^{1+\varepsilon}$ is not Lipschitz. As a consequence, ω_ε-continuity for a function does not implies that the function is Lipschitz.

Example 3.3. A modulus of continuity ω satisfying $\omega(s) = 1/\log(1/s)$ for s small is weaker than any Hölder modulus $s \mapsto s^\tau$, $\tau > 0$. Nevertheless, such a modulus will be essential for our construction.

We will now investigate several upper bounds with respect to some moduli of continuity for the derivatives of maps in Yoccoz's family. Letting $u = \varphi_a^{-1}(x)$, we have

$$\varphi'_{a,b}(x) = \varphi'_b(u)(\varphi_a^{-1})'(x) = \frac{\varphi'_b(u)}{\varphi'_a(u)} = \frac{u^2 + 1/a^2}{u^2 + 1/b^2}.$$

Note that, as $x \to 0$ (resp. $x \to 1$), one has that $u \to -\infty$ (resp. $u \to +\infty$), and $\varphi'_{a,b}(x) \to 1$ (resp. $\varphi'_{a,b}(x) \to 1$). Therefore, the map $\varphi_{a,b}$ extends to a C^1-diffeomorphism from $[0,a]$ to $[0,b]$ which is tangent to the identity at the end points of $[0,a]$. Moreover, for $a \geq b$ (resp. $a \leq b$), the function $u \mapsto \frac{u^2+1/a^2}{u^2+1/b^2}$ attains its minimum (resp. maximum) value at $u = 0$. Since this value is equal to b^2/a^2, we have

$$\sup_{x \in [0,a]} |\varphi'_{a,b}(x) - 1| = \left| \frac{b^2}{a^2} - 1 \right|.$$

For the second derivative of $\varphi_{a,b}$ one has

$$\varphi''_{a,b}(x) = \frac{d\varphi'_{a,b}(x)}{du} \cdot \frac{du}{dx} = \frac{2u(u^2+1/b^2) - 2u(u^2+1/a^2)}{(u^2+1/b^2)^2} \pi(u^2+1/a^2) = \pi \frac{u^2+1/a^2}{(u^2+1/b^2)^2} \left[2u\Big(\frac{1}{b^2} - \frac{1}{a^2}\Big) \right].$$

Thus

$$|\varphi''_{a,b}(x)| = \pi \frac{u^2+1/a^2}{u^2+1/b^2} \cdot \frac{|2u(1/b^2 - 1/a^2)|}{u^2+1/b^2}.$$

Note that from this equality one deduces that $\varphi_{a,b}$ is a C^2-diffeomorphism, with $\varphi''_{a,b}(0) = \varphi''_{a,b}(a) = 0$. Moreover, the inequality $\frac{2|u|}{u^2+t^2} \leq \frac{1}{t}$ applied to $t = 1/b$ gives

$$\left|\varphi''_{a,b}(x)\right| \leq \pi \frac{u^2+1/a^2}{u^2+1/b^2} \left| \frac{1}{b^2} - \frac{1}{a^2} \right| b.$$

For $a \leq b$ this implies that

$$\left|\varphi''_{a,b}(x)\right| \leq \pi \frac{b^2}{a^2} \left(\frac{b^2 - a^2}{a^2 b^2} \right) b = \frac{\pi b}{a^2} \left(\frac{b^2}{a^2} - 1 \right).$$

So, if $a \leq b \leq 2a$, then

$$\left|\varphi''_{a,b}(x)\right| \leq 6\pi \left| \frac{b}{a} - 1 \right| \frac{1}{a}.$$

In an analogous way, if $2b \geq a \geq b$, then

$$\left|\varphi''_{a,b}(x)\right| \leq \frac{\pi}{b} \left(1 - \frac{b^2}{a^2} \right) \leq 2\pi \left| \frac{b}{a} - 1 \right| \frac{1}{b} \leq 4\pi \left| \frac{b}{a} - 1 \right| \frac{1}{a}.$$

So, in both cases,

$$\left|\varphi''_{a,b}(x)\right| \leq 6\pi \left| \frac{b}{a} - 1 \right| \frac{1}{a}. \tag{2}$$

This last inequality and the following elementary proposition show that the family of maps $\varphi_{a,b}$ is in some sense optimal.

Proposition 3.4. *If $\varphi : [0,a] \to [0,b]$ is a C^2-diffeomorphism such that $\varphi'(0) = \varphi'(a) = 1$, then there exists a point $s \in]0,a[$ such that*

$$|\varphi''(s)| \geq \frac{2}{a} \left| \frac{b}{a} - 1 \right|.$$

Proof. Let us suppose that $b \geq a$. (The case in which $b \leq a$ is similar.) Since $\varphi(0) = 0$ and $\varphi(a) = b$, there exists some point $x \in]0,a[$ such that $\varphi'(x) \geq b/a$. There are two cases:

(i) $x \geq a/2$: the Mean Value Theorem gives some point $s \in]x, a[$ such that

$$|\varphi''(s)| = \frac{\varphi'(x) - \varphi'(a)}{a - x} \geq \frac{2}{a}|\varphi'(x) - 1| \geq \frac{2}{a}\left|\frac{b}{a} - 1\right|;$$

(ii) $x \leq a/2$: again, there exists $s \in]0, x[$ such that

$$|\varphi''(s)| = \frac{\varphi'(x) - \varphi'(0)}{x} \geq \frac{2}{a}|\varphi'(x) - 1| \geq \frac{2}{a}\left|\frac{b}{a} - 1\right|.$$

The following elementary lemma will be useful to control the ω-norm of the derivative of a map obtained by fitting together many diffeomorphisms defined on subintervals.

Lemma 3.5. *Let $\{I_n : n \in \mathbb{N}\}$ be a family of closed intervals in $[0,1]$ having disjoint interiors and such that the complement of their union is countable. Suppose that φ is a homeomorphism of $[0,1]$ such that its restrictions to each interval I_n are $C^{1+\omega}$-diffeomorphisms which are C^1-tangent to the identity at both end points of I_n and whose derivatives have C^ω-norm bounded above by a constant M. Then φ is a $C^{1+\omega}$-diffeomorphism of the whole interval $[0,1]$, and the C^ω-norm of its derivative is less or equal than $2M$.*

Proof. Let $x < y$ be two points of $\cup_{n \in \mathbb{N}} I_n$. If they belong to the same interval I_n then, by hypothesis,

$$\left|\frac{\varphi'(y) - \varphi'(x)}{\omega(y - x)}\right| \leq M.$$

Suppose now that $x \in I_i = [x_i, y_i]$ and $y \in I_j = [x_j, y_j]$, with $y_i \leq x_j$. In that case,

$$\begin{aligned}
\left|\frac{\varphi'(y) - \varphi'(x)}{\omega(y - x)}\right| &= \left|\frac{(\varphi'(y) - 1) + (1 - \varphi'(x))}{\omega(y - x)}\right| \\
&\leq \left|\frac{\varphi'(y) - \varphi'(x_j)}{\omega(y - x)}\right| + \left|\frac{\varphi'(y_i) - \varphi'(x)}{\omega(y - x)}\right| \\
&\leq M\left[\frac{\omega(y - x_j)}{\omega(y - x)} + \frac{\omega(y_i - x)}{\omega(y - x)}\right] \\
&\leq 2M.
\end{aligned}$$

The map $x \mapsto \varphi'(x)$ is then uniformly continuous on the dense set $\cup_{n \in \mathbb{N}} I_n$, and so it extends to some continuous function on $[0,1]$ having C^ω-norm bounded above by $2M$. The complement of $\cup_{n \in \mathbb{N}} I_n$ being countable, the Fundamental Theorem of Calculus shows that this continuous function coincides (everywhere) with the derivative of φ. This finishes the proof of the lemma.

In what follows we will suppose, without loss of generality, that the map $s \mapsto \omega(s)/s$ is decreasing. Note that the moduli from examples 3.1, 3.2 and 3.3 can be taken satisfying this property. The notation φ will be used only for the maps of Yoccoz's family. The next lemma should be compared with §3.17 in [14].

Lemma 3.6. *If $a > 0$ and $b > 0$ are such that $a/b \leq 2$, $b/a \leq 2$ and*

$$\left| \frac{b}{a} - 1 \right| \frac{1}{\omega(a)} \leq M,$$

then the C^ω-norm of $\varphi'_{a,b}$ is less or equal than $6\pi M$.

Proof. By inequality (2) one has, for all $x \in [0, a]$,

$$|\varphi''_{a,b}(x)| \leq \frac{6\pi M \omega(a)}{a}.$$

If $y < z$ are points in $[0, a]$ then there exists $x \in [y, z]$ such that $\varphi'_{a,b}(z) - \varphi'(y) = \varphi''_{a,b}(x)(z - y)$. Since $s \mapsto \omega(s)/s$ is a decreasing function and $z - y \leq a$, this gives

$$\left| \frac{\varphi'_{a,b}(z) - \varphi'_{a,b}(y)}{\omega(z-y)} \right| = |\varphi''_{a,b}(x)| \left| \frac{z-y}{\omega(z-y)} \right| \leq |\varphi''_{a,b}(x)| \left| \frac{a}{\omega(a)} \right| \leq 6\pi M.$$

This proves our claim.

The preceding lemma allows to use the maps $\varphi_{a,b}$ to simplify the construction of Denjoy counter-examples in Chapter X of [14]. Indeed, consider any rotation R_α of irrational angle α and fix a point $x_0 \in \mathrm{S}^1$. Blow-up the orbit $(x_n = R_{n\alpha}(x_0))_{n\in\mathbb{Z}}$ of x_0 replacing each point x_n by an interval I_n of length l_n in such a way that $\sum l_n < \infty$. By this procedure one obtains another circle $\bar{\mathrm{S}}^1$ on which R_α induces a map $\bar{R}_\alpha$ by putting $\bar{R}_\alpha(x) = \varphi(I_n, I_{n+1})(x)$ for $x \in I_n$. It turns out that, if the lengths l_n are well chosen, then $\bar{R}_\alpha$ has good regularity properties. For example, for $k \geq 2$ and $\varepsilon > 0$, taking

$$l_n = \frac{1}{(|n| + k)[\log(|n| + k)]^{1+\varepsilon/2}},$$

it is easy to verify that $\bar{R}_\alpha$ is a C^1-diffeomorphism with ω_ε-continuous derivative, and that the ω_ε-norm of its derivative converges to zero as k goes to infinity. By example 3.2, $\bar{R}_\alpha$ is a $C^{1+\tau}$-Denjoy counter-example for any $0 \leq \tau < 1$. (See [15] for sharper Denjoy counter-examples.)

4 The embedding of H into $\mathrm{Diff}^1_+([0, 1])$

In this Section, ω will denote a fixed modulus of continuity satisfying $\omega(s) = 1/\log(1/s)$ for s small enough (namely, for $s \leq 1/e$), and such that the map $s \mapsto \omega(s)/s$ is decreasing. We will prove that there exist embeddings $H \hookrightarrow \mathrm{Diff}_+^{1+\omega}([0,1])$ sending the generators a, b, c, d of H (and their inverses) to diffeomorphisms as near as we want (in the $C^{1+\omega}$-topology) to the identity map. To do that, fix any number $M > 0$, and for each $k \in \mathbb{N}$ define $T_k = \sum_{i\in\mathbb{Z}} \frac{1}{(|i|+k)^2} < \infty$. Consider an increasing sequence (k_n) of positive integer numbers such that $k_1 \geq 4$. For $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ let

$$|I_{x_1,\ldots,x_n}| = \frac{1}{(|x_1| + \cdots |x_n| + k_n)^{2n}}.$$

Note that

$$\begin{aligned}
\sum_{x_{n+1}\in\mathbb{Z}} |I_{x_1,\ldots,x_n,x_{n+1}}| &= \sum_{x_{n+1}\in\mathbb{Z}} \frac{1}{(|x_1| + \cdots |x_n| + |x_{n+1}| + k_{n+1})^{2n+2}} \\
&\leq 2\int_{|x_1|+\cdots+|x_n|+k_{n+1}-1}^{\infty} \frac{ds}{s^{2n+2}} \\
&= \frac{2}{2n+1} \cdot \frac{1}{(|x_1| + \cdots + |x_n| + k_{n+1} - 1)^{2n+1}}.
\end{aligned}$$

Thus

$$\frac{\sum_{x_{n+1}\in\mathbb{Z}} |I_{x_1,\ldots,x_n,x_{n+1}}|}{|I_{x_1,\ldots,x_n}|} \leq \frac{2}{2n+1} \cdot \frac{(|x_1| + \cdots |x_n| + k_n)^{2n}}{(|x_1| + \cdots + |x_n| + k_{n+1} - 1)^{2n+1}} \leq \frac{2}{(2n+1)(|x_1| + \cdots + |x_n| + k_n)}.$$

In particular, we can define $|J_{x_1,\ldots,x_n}|$ by (1), that is

$$|J_{x_1,\ldots,x_n}| = |I_{x_1,\ldots,x_n}| - \sum_{x_{n+1}\in\mathbb{Z}} |I_{x_1,\ldots,x_n,x_{n+1}}|,$$

and for this choice we have

$$|I_{x_1,\ldots,x_n}| \geq |J_{x_1,\ldots,x_n}| \geq \left(1 - \frac{2}{(2n+1)(|x_1| + \cdots + |x_n| + k_n)}\right) |I_{x_1,\ldots,x_n}|. \tag{3}$$

The procedure of §2 (using Yoccoz' equivariant family of maps) gives subgroups of $\mathrm{Diff}_+^1([0, T_{k_1}])$ isomorphic to H/H_n and generated by elements A_n, B_n, C_n and D_n. Our next step is to estimate the C^ω-norm of the derivative of these maps.

Lemma 4.1. *If the sequence (k_n) satisfy the conditions*

$$\frac{(2n+1)k_n}{(2n+1)k_n - 2} \left(\frac{k_n+1}{k_n}\right)^{2n} \leq 2, \qquad \left(1 - \frac{2}{(2n+1)k_n}\right)\left(\frac{k_n - 1}{k_n}\right)^{2n} \geq \frac{1}{2}, \tag{4}$$

$$2n\log(k_n) \geq \log\left(\frac{(2n+1)k_n}{(2n+1)k_n - 2}\right), \tag{5}$$

$$\frac{\log(k_n)}{k_n}\left(n2^{2n+3} + \frac{32}{2n+1}\right) \leq \frac{M}{12\pi}, \tag{6}$$

then the C^ω-norm of the derivatives of A_n, B_n, C_n and D_n are less or equal than M for all $n \in \mathbb{N}$.

Proof. First of all, it is easy to verify that inequality (3) and hypothesis (4) imply that

$$\frac{1}{2} \leq \frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|} \leq 2, \qquad \frac{1}{2} \leq \frac{|J_{x_1,\dots,1+x_i,\dots,x_n}|}{|J_{x_1,\dots,x_i,\dots,x_n}|} \leq 2, \tag{7}$$

We also have

$$\frac{|x_1| + \cdots + |x_i| \cdots + |x_n| + k_n}{|x_1| + \cdots + |1 + x_i| + \cdots + |x_n| + k_n} \leq 2. \tag{8}$$

So, by construction of the maps, and thanks to inequalities (7), the problem reduces to estimate expressions of the form

$$\left| \frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|} - 1 \right| \frac{1}{\omega(|I_{x_1,\dots,x_i,\dots,x_n}|)} \qquad \text{and} \qquad \left| \frac{|J_{x_1,\dots,1+x_i,\dots,x_n}|}{|J_{x_1,\dots,x_i,\dots,x_n}|} - 1 \right| \frac{1}{\omega(|J_{x_1,\dots,x_i,\dots,x_n}|)}.$$

Indeed, if we verify that these expressions are bounded above by $M/12\pi$ for all possible choices of subindices, then Lemmas 3.5 and 3.6 will imply that the C^{ω}-norm of the derivatives of A_n, B_n, C_n and D_n are less or equal than M.

Now, using the identity $s^{2n} - 1 = (s-1)(s^{2n-1} + \cdots + 1)$ and (8), we obtain

$$\begin{aligned}
\left| \frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|} - 1 \right| &= \left| \left(\frac{|x_1| + \cdots + |x_i| + \cdots + |x_n| + k_n}{|x_1| + \cdots + |1 + x_i| + \cdots + |x_n| + k_n} \right)^{2n} - 1 \right| \\
&\leq \frac{\big| |x_i| - |1 + x_i| \big|}{|x_1| + \cdots + |1 + x_i| + \cdots + |x_n| + k_n} \cdot (2^{2n-1} + 2^{2n-2} + \cdots + 1) \\
&\leq \frac{2^{2n}}{|x_1| + \cdots + |1 + x_i| + \cdots + |x_n| + k_n}.
\end{aligned}$$

Since the map $s \mapsto \log(s)/s$ is decreasing for $s \geq e$, by (6) we have

$$\begin{aligned}
\left| \frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|} - 1 \right| \frac{1}{\omega(|I_{x_1,\dots,x_i,\dots,x_n}|)} &\leq \frac{2^{2n} \log([|x_1| + \cdots + |x_i| + \cdots + |x_n| + k_n]^{2n})}{|x_1| + \cdots + |1 + x_i| + \cdots + |x_n| + k_n} \\
&\leq \frac{n 2^{2n+1} \log(|x_1| + \cdots + |x_i| + \cdots + |x_n| + k_n)}{|x_1| + \cdots + |1 + x_i| + \cdots + |x_n| + k_n} \\
&\leq \frac{n 2^{2n+2} \log(k_n)}{k_n} \\
&\leq \frac{M}{12\pi}.
\end{aligned}$$

Let us now deal with the case of intervals $J_{x_1,\dots,x_n}$. First of all, a straightforward computation using (3) and (5) shows that

$$\omega(|J_{x_1,\dots,x_i,\dots,x_n}|) \geq \omega(|I_{x_1,\dots,x_i,\dots,x_n}|)/2. \tag{9}$$

Then, using (3), (6), (9) and the triangular inequality

$$\left|\frac{|J_{x_1,\dots,1+x_i,\dots,x_n}|}{|J_{x_1,\dots,x_i,\dots,x_n}|}-1\right| \leq \left|\frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|}-1\right| + \left|\frac{|J_{x_1,\dots,1+x_i,\dots,x_n}|}{|J_{x_1,\dots,x_i,\dots,x_n}|} - \frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|}\right|,$$

it is easy to verify that

$$\left|\frac{|J_{x_1,\dots,1+x_i,\dots,x_n}|}{|J_{x_1,\dots,x_i,\dots,x_n}|}-1\right| \frac{1}{\omega(|J_{x_1,\dots,x_i,\dots,x_n}|)}$$

is less or equal than

$$\frac{n2^{2n+3}\log(k_n)}{k_n} + \frac{8}{(2n+1)(|x_1|+\cdots+|x_i|+\cdots+|x_n|+k_n)} \cdot \frac{|I_{x_1,\dots,1+x_i,\dots,x_n}|}{|I_{x_1,\dots,x_i,\dots,x_n}|} \cdot \frac{1}{\omega(|I_{x_1,\dots,x_i,\dots,x_n}|)},$$

and the value of this expression is bounded above by

$$\frac{n2^{2n+3}\log(k_n)}{k_n} + \frac{32n}{2n+1} \cdot \frac{\log(|x_1|+\cdots+|x_i|+\cdots+|x_n|+k_n)}{|x_1|+\cdots+|x_i|+\cdots+|x_n|+k_n} \leq \frac{\log(k_n)}{k_n}\left(n2^{2n+3}+\frac{32n}{2n+1}\right) \leq \frac{M}{12\pi}.$$

This concludes the proof of the lemma.

Remark that similar computations can be done to estimate the C^ω-norm of $(A_n^{-1})'$, $(B_n^{-1})'$, $(C_n^{-1})'$ and $(D_n^{-1})'$. Thus A_n', B_n', C_n' and D_n' converge to some C^ω-continuous maps (having C^ω-norm bounded above by M), which are the derivatives of $C^{1+\omega}$-diffeomorphisms A, B, C and D such that the group generated by them is isomorphic to H. Note however that this group acts on the interval $[0, T_{k_1}]$. In order to obtain a group acting on $[0,1]$, we can conjugate by the affine map $g : [0,1] \to [0, T_{k_1}]$, i.e. $g(x) = T_{k_1}x$. Since $k_1 \geq 4$ we have $T_{k_1} \leq 1$, and therefore this procedure does not increase C^ω-norms of derivatives. For example,

$$\begin{aligned}
\left|\frac{(g^{-1}Ag)'(x)-(g^{-1}Ag)'(y)}{\omega(|x-y|)}\right| &= \left|\frac{A'(g(x))-A'(g(y))}{\omega(|x-y|)}\right| \\
&= \left|\frac{A'(g(x))-A'(g(y))}{\omega(|g(x)-g(y)|)}\right| \cdot \left|\frac{\omega(T_{k_1}|x-y|)}{\omega(|x-y|)}\right| \\
&\leq M.
\end{aligned}$$

Since every (orientation preserving) diffeomorphism f of $[0,1]$ has a point on which the derivative is equal to 1, if the C^ω-norm of f' is bounded above by M then

$$\sup_{x\in[0,1]} |f'(x)-1| \leq M\omega(1).$$

So, if M is small, then f is near the identity in the $C^{1+\omega}$-topology. Finally, it is easy to construct sequences (k_n) of integer positive numbers satisfying (4), (5) and (6). This finishes the proof of Theorem A.

Remark 4.2. Note that the action we obtained is given by diffeomorphisms which are tangent to the identity at the end points of $[0,1]$. So, gluing together these two end points, we obtain a C^1-action on the circle. Using the classical procedure of suspension, this allows to construct a codimension-1 foliation (on a 3-dimensional compact manifold) which is transversaly of class C^1 and whose leaves have subexponential growth. It turns out that a countable number of leaves have polynomial growth and a continium of leaves have intermediate growth, a fact that should be compared with [13]. In [3] one can find other interesting examples of codimension-1 foliations which are transversally of class C^1 but not C^2.

5 On the non existence of embedding into $\mathrm{Diff}^2_+([0,1[)$

It is possible to give a proof for Theorem B following the lines of the proof sketched for Proposition 5.1 in [18]. To do this, one has to replace, along that proof, the use of Plante's Theorem [23] (all subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ having subexponential growth preserve a Radon measure of the real line) by Solodov's [27] or Beklaryan's [1] result (all finitely generated subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ with no free semigroup on two generators preserve a Radon measure). For the completeness of the present work, we will give in the sequel a self contained and direct proof of Theorem B, without any reference to Plante's Theorem or Solodov's and Beklaryan's result.

Definition 5.1. We say that two homeomorphisms f and g are *crossed* on an interval $[a,b]$ if they do not fix any point on $]a,b[$, one of them fixes a and b, and the other sends a or b into $]a,b[$.

The points a and b in the definition above can be equal to $-\infty$ and to $+\infty$ respectively. The following elementary criterion to obtain free semigroups on two generators is well known.

Lemma 5.2. *If a subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ contains two crossed elements, then Γ contains a free semigroup on two generators.*

Proof. Suppose that there exist $f,g \in \Gamma$ and an interval $[a,b]$ such that $Fix(f) \cap [a,b] = \{a,b\}$ and $g(a) \in]a,b[$. (The case in which $g(b) \in]a,b[$ is analogous.) Changing f by its inverse if necessary, we may assume that $f(x) < x$ for all $x \in]a,b[$. Let $c = g(a) \in]a,b[$, and let us fix a point $d' \in]c,b[$. Since $gf^n(a) = c$ for all $n \in \mathbb{N}$ and since $gf^n(d')$ converges to c as n goes to infinite, the map gf^n has a fixed point in $]a,d'[$ for $n \in \mathbb{N}$ large enough. Fix such an $n \in \mathbb{N}$ and let $d > c$ be the infimum of the fixed points of gf^n in $]a,b[$. For $m \in \mathbb{N}$ large enough we have $f^m(d) < c$, and so the Ping-Pong Lemma applied to the restrictions of f^m and fg^n to $[a,b]$ shows that the semigroup generated by these elements is free. (See [12], Chapter VII.)

Our second criterion to obtain free semigroups in $\mathrm{Diff}^2_+([0,1[)$ uses the C^2-differentiability of the maps as an essential hypothesis.

Lemma 5.3. *Let f and g be two elements of* $\mathrm{Diff}^2_+([0,1[)$ *such that $f(x) < x$ for all $x \in]0,1[$, and let $]u,v[$ be an interval contained in $]0,1[$ such that $g(x) < x$ for all $x \in]u,v[$, $g(u) = u$, $g(v) = v$ and $f(v) \leq u$. Let $[a,b] \subset]u,v[$ be an interval such that $g(b) \leq a$. Let us suppose that g fixes all the intervals $f^n([u,v])$, and that for each $n \in \mathbb{N}$ one has $f^{-n}g^if^n(]a,b[) = g^{i\epsilon_n}(]a,b[)$ for some $\epsilon_n \in \{-1,0,1\}$. Then the semigroup generated by f and g is free.*

Proof. For each $n \in \mathbb{Z}$ let $a_n = f^n(a)$. We claim that for all $n \in \mathbb{N}$ sufficiently large one has $g(a_n) = a_n$. Indeed, if this is not the case, then there exists an increasing sequence of positive numbers n_i such that $g(b_{n_i}) \leq a_{n_i}$ or $g(a_{n_i}) \geq b_{n_i}$ for all $i \in \mathbb{N}$. Let us consider the first case, the second being analogous. Following the proof of Kopell's Lemma in [2], let us fix a constant κ such that $1 < \kappa \leq 1 + (b-a)/(e^\delta(v-b))$, where

$$\delta = var\left(\log(f')|_{[0,v]}\right) = \int_0^v \left|\frac{f''(s)}{f'(s)}\right| ds.$$

For some $x_i, y_i \in [f^{n_i}(a), f^{n_i}(b)]$ one has

$$\frac{b_{n_i} - g(b_{n_i})}{f^{n_i}(v) - b_{n_i}} \geq \frac{b_{n_i} - a_{n_i}}{f^{n_i}(v) - b_{n_i}} \geq \frac{(f^{n_i})'(x_i)}{(f^{n_i})'(y_i)} \cdot \frac{b-a}{v-b}.$$

The well known Bounded Distorsion Principle then gives

$$\frac{b_{n_i} - g(b_{n_i})}{f^{n_i}(v) - b_{n_i}} \geq \frac{b-a}{e^\delta(v-b)} \geq \kappa - 1,$$

which implies, for all $i \in \mathbb{N}$,

$$\frac{g(f^{n_i}(v)) - g(b_{n_i})}{f^{n_i}(v) - b_{n_i}} = \frac{f^{n_i}(v) - g(b_{n_i})}{f^{n_i}(v) - b_{n_i}} \geq \kappa > 1.$$

Note that the left hand term of this inequality is equal to $g'(z_i)$ for some point $z_i \in [f^{n_i}(u), f^{n_i}(v)]$. But since $g'(z_i)$ tends to $g'(0)$, which is necessarily equal to 1, this leads to a contradiction.

Now let us fix the smallest positive integer N such that $g(a_n) = a_n$ for all $n \geq N$. Let $A = f^n g^{m_r} f^{n_r} \cdots g^{m_1} f^{n_1}$ and $B = g^q f^{p_s} g^{q_s} \cdots f^{p_1} g^{q_1}$ be two words on f and g, where m_i, n_i, p_i, q_i are positive integers, $n \geq 0$ and $q \geq 0$. We have to prove that A and B do not represent the same diffeomorphism. Remark that, by the definition of N, one has $A(a_{N-1}) = a_M$, where $M = N - 1 + n + \sum_i n_i$. On the other hand, one has $B(a_{N-1}) = f^{M'}(g^{q_1}(a_{N-1}))$, where $M' = \sum_i p_i$. However, since $g^{q_1}(a_{N-1}) \neq a_{N-1}$, it is easy to verify that $f^{M'}(g^{q_1}(a_{N-1}))$ is not equal to $a_{M'+N-1}$, and so to any a_n. Thus $A(a_{N-1}) \neq B(a_{N-1})$, and this finishes the proof of the lemma.

Proof of Theorem B. Without loss of generality, we may assume that Γ has no global fixed point in $]0,1[$. By Lemma 5.2, Γ has no crossed elements, and it is then easy to see that there exists a generator f of Γ without fixed points in $]0,1[$ (recall that Γ is finitely generated). Changing f by f^{-1} if necessary, we can suppose that $f(x) < x$ for all $x \in]0,1[$. There are two different cases.

Case 1: the action of Γ on $]0,1[$ is semiconjugated to the action of a dense group of translations.

We claim that in this case the semiconjugacy is in fact a conjugacy (which implies in particular that Γ is Abelian). To prove this, let us suppose the contrary and let $[a,b] \subset]0,1[$ be a non degenerated closed interval which is sent to a single point by the semiconjugacy and which is maximal for this property. Without loss of generality, we may assume that f is mapped to the translation $T_{-1} : x \mapsto x-1$ by the homomorphism induced by the semiconjugacy. By the definition of $[a,b]$, there exists an increasing sequence (n_i) of positive integers numbers such that for each $i \in \mathbb{N}$ there exists $\bar{f}_i \in \Gamma$ satisfying, for all $n \in \mathbb{N}$,

$$\bar{f}_i^{n_i}(f^n(a)) \geq f^{n+1}(a), \qquad \bar{f}_i^{n_i+1}(f^n(a)) < f^{n+1}(a),$$

$$\bar{f}_i^{n_i}(f^n(b)) \geq f^{n+1}(b), \qquad \bar{f}_i^{n_i+1}(f^n(b)) < f^{n+1}(b).$$

Let $a_n = f^n(a)$ and $b_n = f^n(b)$. By passing to the limit as n goes to infinite in the inequality

$$\frac{\bar{f}_i^{n_i+1}(a_n)}{a_n} < \frac{f(a_n)}{a_n} \leq \frac{\bar{f}_i^{n_i}(a_n)}{a_n}$$

we obtain

$$(\bar{f}_i'(0))^{n_i+1} \leq f'(0) \leq (\bar{f}_i'(0))^{n_i}. \tag{10}$$

For fixed $n \geq 0$, the intervals $f^n(]\bar{f}_i(a), b[)$ are disjoint. For u, v in $]\bar{f}_i(a), b[$ and

$$\delta = var\left(\log(f')|_{[0,b]}\right) = \int_0^b \left|\frac{f''(s)}{f'(s)}\right| ds,$$

the Bounded Distorsion Principle gives $(f^n)'(v)/(f^n)'(u) \leq e^{\delta}$. Passing to the limit (as n goes to infinity) in the inequality

$$|\bar{f}_i([a,b])| = |f^{-n}\bar{f}_i f^n([a,b])| \geq \frac{\inf_{u\in[\bar{f}_i(a),b]}(f^n)'(u)}{\sup_{v\in[\bar{f}_i(a),b]}(f^n)'(v)} \cdot \inf_{x\in[f^n(a),f^n(b)]} \bar{f}_i'(x) \cdot |[a,b]|,$$

and using (10), we obtain for all $i \in \mathbb{N}$ and some uniform constant $C > 0$,

$$|\bar{f}_i([a,b])| \geq \exp(-\delta) \cdot (f'(0))^{1/n_i} \cdot |[a,b]| \geq C.$$

But this is impossible, because $|\bar{f}_i([a,b])|$ converges to zero as i goes to infinity.

Case 2: the action of Γ on $]0,1[$ is not semiconjugated to the action of a dense group of translations.

Note that, by Hölder's Theorem [6], the action of Γ on $]0,1[$ is not free. We will prove that this leads to a contradiction.

Claim (i): if $h \in \Gamma$ has a fixed point $a \in]0,1[$, then it also has fixed points in $]0,a[$ and $]a,1[$.

Indeed, since there is no global fixed point in $]0,1[$, this is a direct consequence of Lemma 5.2.

Let us denote by $\mathcal{F}$ the family of intervals $]p,q[$ strictly contained in $]0,1[$ such that for each $\varepsilon > 0$ there exist $h \in \Gamma$ and an interval $[a_\varepsilon, b_\varepsilon]$ such that $Fix(h) \cap [a_\varepsilon, b_\varepsilon] = \{a_\varepsilon, b_\varepsilon\}$, $p \leq a_\varepsilon \leq p + \varepsilon$ and $q - \varepsilon \leq b_\varepsilon \leq q$. Claim (i) implies that for any $]p,q[\in \mathcal{F}$, both p and q belong to $]0,1[$. Moreover, if $]p_1,q_1[$ and $]p_2,q_2[$ are in $\mathcal{F}$, then they are disjoint or one of them is contained in the other. Note also that $\mathcal{F}$ is invariant by the action of Γ.

Claim (ii): $\mathcal{F}$ has an element of maximal length.

Indeed, let $]p_n,q_n[$ be a sequence of elements of $\mathcal{F}$ such that $\lim_{n\to\infty} |[p_n,q_n]| = \sup_{]p,q[\in\mathcal{F}} |[p,q]|$. There must exist a subsequence (n_i) such that $]p_{n_i},q_{n_i}[\subset]p_{n_{i+1}},q_{n_{i+1}}[$ for all $i \in \mathbb{N}$. Let $p = \lim_{i\to\infty} p_{n_i}$ and $q = \lim_{i\to\infty} q_{n_i}$. Since $p_n \leq f(q_n)$ for all $n \in \mathbb{N}$, the points p and q belong to $]0,1[$ and $p \leq f(q)$. It follows that $]p,q[\in \mathcal{F}$, and obviously the length of $[p,q]$ is maximal.

Claim (iii): the orbit of any interval $[p,q] \in \mathcal{F}$ of maximal length is discrete in $]0,1[$.

Suppose that the opposite is true. Collapsing to a point each connected component of the complementary set of the closure of the orbit of $[p,q]$, we obtain a group $\bar{\Gamma}$ of homeomorphisms of the interval which is semiconjugated to Γ. Collapsing now to a point each interval of the orbit of $[p,q]$, we obtain a group $\hat{\Gamma}$ of homeomorphisms of the interval which is semiconjugated to $\bar{\Gamma}$ (and so to Γ), and such that the $\hat{\Gamma}$-orbit of the point x_0 corresponding to $[p,q]$ is dense. The action of $\hat{\Gamma}$ at the interior of the interval is free. Indeed, if not, one can find an element $\hat{h} \in \hat{\Gamma}$ and a closed interval $[\hat{a},\hat{b}]$ such that $x_0 \in]\hat{a},\hat{b}[$ and $Fix(\hat{h}) \cap [\hat{a},\hat{b}] = \{\hat{a},\hat{b}\}$. But this gives an element $h \in \Gamma$ and an interval $[a,b] \subset]0,1[$ such that $[p,q] \subset]a,b[$ and $Fix(h) \cap [a,b] = \{a,b\}$, contradicting the fact that the lenght of $]p,q[\in \mathcal{F}$ is maximal. Finally, the action of $\hat{\Gamma}$ being free, Hölder's Theorem implies that $\hat{\Gamma}$ (hence Γ) is semiconjugated to a (dense) group of translations, contradicting our initial hypothesis.

Claim (iv): the commutator group $\Gamma' = [\Gamma,\Gamma]$ fixes $[p,q]$.

Indeed, the action of Γ preserves the order between the intervals of the orbit of $]p,q[$. Since this orbit is discrete in $]0,1[$, the claim easily follows.

Denoting by $[p_n,q_n]$ the points of the orbit of $[p,q]$ ordered in such a way that $[p_0,q_0] = [p,q]$ and $p_{n+1} < p_n$ for all $n \in \mathbb{N}$, we have in greater generality that the commutator group $\Gamma' = [\Gamma,\Gamma]$ is contained in the stabiliser Γ^* of all (equivalently, of some) of the intervals $[p_{n+1},p_n]$.

Claim (v): the stabiliser Γ^* is trivial.

Let us suppose the contrary. The group Γ^* is normal in Γ and Γ/Γ^* is isomorphic to $(\mathbb{Z},+)$. If

$\bar{f} \in \Gamma$ is an element such that $\bar{f}\Gamma^*$ generates Γ/Γ^* and $\bar{f}(x) < x$ for some $x \in]0,1[$, then $p_n = \bar{f}^n(p_0)$ for all $n \in \mathbb{Z}$.

Let us first suppose that the restriction of Γ^* to $]p_1, p_0[$ (equivalently, to each $]p_{n+1}, p_n[$) is Abelian. In that case, it is easy to see that Γ is solvable (and in fact metabelian). Since Γ has no free semigroup on two generators, the main result of [25] implies that Γ is almost nilpotent. By Plante-Thurston's Theorem [24], Γ is almost Abelian. Let Γ_0 be an Abelian finite index subgroup of Γ and let $[a, b[$ any interval which is fixed by Γ_0 and such that the corresponding restriction has no global fixed point. It is easy to see that $[a, b[$ has the same properties with respect to the whole group Γ, and so $[a, b[= [0, 1[$. By Szekeres's Theorem [31], Γ_0 is contained in the flow associated to any of its non trivial elements. If g belongs to Γ, then there exists $n \in \mathbb{N}$ such that $g^n \in \Gamma_0$. Thus the diffeomorphism g^n belongs to the afore mentioned flow, which implies that g also belongs to that flow. This shows that the whole group Γ is contained in a flow, and in particular it is Abelian.

Let us suppose now that the restriction of Γ^* to $]p_1, p_0[$ (equivalently, to each $]p_{n+1}, p_n[$) is not Abelian. Let $[u, v]$ be a non degenerated closed interval strictly contained in $[p_1, p_0]$ which is fixed by Γ^* and such that the restriction of Γ^* to $]u, v[$ has no global fixed point and is non Abelian. By Hölder's Theorem, the action of this restriction is not free. Thus there exist an interval $[a, b]$ strictly contained in $[u, v]$ and an element $h_0 \in \Gamma^*$ such that $Fix(h_0) \cap [a, b] = \{a, b\}$. Lemma 5.2 implies that for all $h \in \Gamma^*$, the intervals $]a, b[$ and $h(]a, b[)$ coincide or are disjoint. The orbit of $]a, b[$ by Γ^* must be discrete in $]u, v[$. Indeed, if not then it is easy to see that the action of Γ^* on $]u, v[$ would be semiconjugated to a group of translations, and the argument used in Case 1 would imply that the restriction of Γ^* to $]u, v[$ would be Abelian, contrary to our hypothesis. So, let us fix an element $g \in \Gamma^*$ such that $g(b) \leq a$ and such that the orbit of $]a, b[$ by Γ^* equals $\{g^i(]a, b[)) : i \in \mathbb{Z}\}$. Observe that $\bar{f}^n \Gamma^* \bar{f}^{-n} = \Gamma^*$ for all $n \in \mathbb{Z}$, since Γ^* is stable by conjugacy. It is then easy to verify that the elements $\bar{f}$ and g satisfy the hypothesis of Lemma 5.3, showing that Γ has a free semigroup on two generators. This contradiction finishes the proof of Theorem B.

Remark 5.4. Recently, J. Wilson gave a nice example of a group having non uniform exponential growth, i.e. its exponential rate of growth is positive but becomes arbitrarily small under suitable changes of the system of generators [30]. This group acts faithfully by automorphisms of a rooted tree. Nevertheless, we think that such an example cannot arise in our context, that is, we think that non Abelian subgroups of $\mathrm{Diff}^2_+([0,1[)$ have uniform exponential growth.

6 Some final remarks on centralisers

To prove that it is not possible to embed the group H into $\mathrm{Diff}^2_+([0,1])$ it is not completely necessary to use Theorem B. Indeed, one can give a direct and simple argument to prove that, for any homomorphism $\phi : H \to \mathrm{Diff}^2_+([0,1[)$, the image $\phi(H)$ is Abelian. This proof is based on the fact that the element $a^2 \in H$ belongs to the center of H.

Denote by $Fix_\phi(a)$ the set of points in $[0,1]$ which are fixed by $\phi(a^2)$ (equivalently, by $\phi(a)$). Fix any interval $[x_0,y_0[\subset[0,1[$ such that $Fix_\phi(a)\cap[x_0,y_0]=\{x_0,y_0\}$. Since a^2 and b commute, $\phi(b)^n(x_0)$ and $\phi(b)^n(y_0)$ belong to $Fix_\phi(a)$ for all $n\in\mathbb{Z}$, and so they are not in $]x_0,y_0[$. If $\phi(b)(x_0)\le x_0$ let $x=\lim_{n\to\infty}\phi(b)^n(x_0)\le x_0$ and $y=\lim_{n\to\infty}\phi(b)^{-n}(y_0)\ge y_0$. If $\varphi(b)(x_0)\ge x_0$ let $x=\lim_{n\to\infty}\phi(b)^{-n}(x_0)\le x_0$ and $y=\lim_{n\to\infty}\phi(b)^n(y_0)\ge y_0$. Note that x and y both belong to $Fix_\phi(a)$. Kopell's Lemma applied to the restrictions of $\phi(a^2)$ and $\phi(b)$ to $[x,y[$ shows that $x=x_0$ and $y=y_0$. Thus the restriction of $\phi(b)$ to $[x_0,y_0[$ is contained in the centraliser (in $\mathrm{Diff}^2_+([x_0,y_0[)$) of the restriction of $\phi(a)^2$, which by Szekeres's Theorem is an Abelian group. Similar arguments can be given for $\phi(c)$ and $\phi(d)$, concluding that $\phi(H)$ fixes $[x_0,y_0[$ and the corresponding restriction is an Abelian group. Since $[x_0,y_0[$ was an arbitrary interval satisfying $Fix_\phi(a)\cap[x_0,y_0]=\{x_0,y_0\}$, this finishes the proof of the commutativity of $\phi(H)$.

Because of the proof above and the construction of the first part of this work, it is natural to study the structure of C^1-centralisers of C^1-diffeomorphisms (or homeomorphisms). The following simple and nice remark (due to C. Bonatti, S. Crovisier and A. Wilkinson) goes in that direction. We include a proof for the completeness of this work.

Proposition 6.1. *Let h be a homeomorphism of $[0,1]$ without fixed points in $]0,1[$. If we denote by $C^1_+(h)$ the group of (orientation preserving) C^1-diffeomorphisms of $[0,1]$ commuting with h, then $C^1_+(h)$ does not contain crossed elements.*

Proof. By contradiction, let us suppose that f and g are elements of $C^1_+(h)$ which are crossed on some interval $[a,b]\subset[0,1]$. As in the proof of Lemma 5.2, we can reduce the problem to the case where $f(a)=a$, $f(b)\in]a,b[$, $g(a)\in]a,b[$ and $g(b)=b$. Moreover, changing f and g by some iterates, we may assume that $g(a)>f(b)$. All these conditions are preserved by conjugacy, that is f and g satisfy $f(h^n(a))=h^n(a)$, $f(h^n(b))\in]h^n(a),h^n(b)[$, $g(h^n(a))\in]h^n(a),h^n(b)[$, $g(h^n(b)=h^n(b)$ and $g(h^n(a))>f(h^n(b))$ for all $n\in\mathbb{Z}$. If $h(x)<x$ (resp. if $h(x)>x$) for all $x\in]0,1[$, then the sequences $(h^n(a))$ and $(h^n(b))$ (resp. $(h^{-n}(a))$ and $(h^{-n}(b))$) converge to the origin. Since f and g are of class C^1, this implies that $f'(0)=g'(0)=1$. However, since $g(h^n(a))>f(h^n(b))$, there must exist a sequence of points $x_n\in]a_n,b_n[$ such that for each $n\in\mathbb{N}$ one has $f'(x_n)<1/2$ or $g'(x_n)<1/2$. This contradiction finishes the proof.

The preceding proposition implies that the group $C^1_+(h)$ has some dynamically rigid properties. For instance, althought $C^1_+(h)$ may contain free semigroups on two generators, it is not possible to detect them using a Ping-Pong Argument as in the proof of Lemma 5.2. Nevertheless, it is not clear that the algebraic structure of $C^1_+(h)$ is also rigid in some sense. Indeed, the methods of the first part of this work allow easily to give examples of elements $h\in\mathrm{Diff}^1_+([0,1])$ without fixed points in $]0,1[$ and such that $C^1_+(h)$ contain (non Abelian) free groups on two generators.

Let us finally point out that all preceding examples are exceptional in some sense. In effect, it has been recently remarked by C. Bonatti, S. Crovisier and A. Wilkinson, that centralisers are generically trivial in $\mathrm{Diff}^1_+([0,1])$. That is, for a generic element $g\in\mathrm{Diff}^1_+([0,1])$ without fixed points in $]0,1[$, one has that $C^1_+(g) = \{g^n : n \in \mathbb{Z}\}$. A similar statement holds for C^2-centralisers of C^2-diffeomorphisms [16]. However, and quite paradoxally, the reason is completely different: in class C^2 the phenomenon is related to the possibility of controling distorsion, and in class C^1 it is due to the absence of control of distorsion...

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