An algebraic proof of a cancellation theorem for surfaces

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Abstract

Let $\mathbf{k}$ be an algebraically closed field of arbitrary characteristic. We give a self-contained algebraic proof of the following statement: If V is an affine surface over $\mathbf{k}$ such that $V \times \mathbf{k} \cong \mathbf{k}^3$, then $V \cong \mathbf{k}^2$. To achieve this, we first prove that if A is a finitely generated domain with $\mathrm{AK}(A) = A$, then $\mathrm{AK}(A[x]) = A$.

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1 Introduction

Let us set the following notations. Let $\mathbf{k}$ denote a field of arbitrary characteristic, and $\mathbf{k}^* = \mathbf{k} \setminus 0$. Let $\mathrm{trdeg}_{\mathbf{k}}$ denote transcendence degree over $\mathbf{k}$. For a ring A, let $A^{[n]}$ denote the polynomial ring in n indeterminates over A.

In this note we are concerned with the following

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Cancellation Theorem. *If V is an affine surface over an algebraically closed field $\mathbf{k}$ such that $V \times \mathbf{k} \cong \mathbf{k}^3$, then $V \cong \mathbf{k}^2$.*

This result is a special case of a theorem due to the combined work of Takao Fujita, Masayoshi Miyanishi, and Tohru Sugie in zero characteristic and Peter Russell in prime characteristic [F, MS, Ru]. In their theorem, $\mathbf{k}$ and $\mathbf{k}^3$ are replaced by $\mathbf{k}^n$ and $\mathbf{k}^{n+2}$, respectively, where $n > 0$.

Even for the special case we are considering, the only known proofs are the original one and a recent proof of Rajendra Gurjar [G] which relies on the topological methods of Mumford–Ramanujam [M, Ra]. These are beautiful proofs which use many ideas, making them not quite self-contained for some readers. So our intention is to present a more self-contained purely algebraic proof of the Cancellation Theorem and to narrow the gap between the formulation and the proof. We also hope that the algebraic approach will be easier to use in the case of higher dimensions.

Let us restate the Cancellation Theorem in algebraic terms. If A is a finitely generated domain over an algebraically closed field $\mathbf{k}$ such that $A^{[1]} \cong \mathbf{k}^{[3]}$, then $A \cong \mathbf{k}^{[2]}$. To obtain this result we first prove a statement on the AK invariant of such domains. Namely, we show that if A is a finitely generated domain with $\mathrm{AK}(A) = A$ then $\mathrm{AK}(A^{[1]}) = A$. This was known in zero characteristic [BML, ML] but here we give a characteristic-free proof.

The AK invariant, defined below, has already helped to recover and generalize other similar cancellation results by purely algebraic means. In [CML] the authors generalize the following cancellation theorem for curves of Shreeram Abhyankar, Paul Eakin, and William Heinzer [AEH]: If A and B are finitely generated domains with transcendence degree 1 over an algebraically closed field $\mathbf{k}$ such that $A^{[n]} \cong B^{[n]}$, then $A \cong B$.

The analogous statement for $\mathrm{trdeg}_{\mathbf{k}}(A) = \mathrm{trdeg}_{\mathbf{k}}(B) = 2$ is false. The first counterexample over the complex numbers is due to Wlodzimierz Danielewski in [D]. In [C] the AK invariant is used to demonstrate that Danielewski's surfaces provide a counterexample over any field of any characteristic, not necessarily algebraically closed. Connections to the cancellation problem not withstanding, the AK invariant seems to be a useful tool in its own right for studying rings.

2 Exponential maps and the AK invariant

Definitions

Let A be a commutative ring with identity. Suppose $\delta : A \to A^{[1]}$ is a homomorphism. We write $\delta = \delta_t : A \to A[t]$ if we wish to emphasize an indeterminate t. We call δ an *exponential map on A* if

(i) $\varepsilon_0\delta_t$ is the identity on A, where $\varepsilon_0 : A[t] \to A$ is evaluation at $t = 0$, and

(ii) $\delta_s\delta_t = \delta_{s+t}$, where δ_s is extended to a homomorphism $A[t] \to A[s,t]$ by $\delta_s(t) = t$.

Define

$$A^\delta = \{a \in A \mid \delta(a) = a\},$$

a subring of A called the *ring of δ-invariants*. Let $\mathrm{EXP}(A)$ denote the set of all exponential maps on A. We define the *AK invariant*, or *ring of absolute constants of A* as

$$\mathrm{AK}(A) = \bigcap_{\delta \in \mathrm{EXP}(A)} A^\delta.$$

Any isomorphism $\varphi : A \to B$ of rings restricts to an isomorphism $\varphi : \mathrm{AK}(A) \to \mathrm{AK}(B)$. Indeed, if $\delta \in \mathrm{EXP}(A)$ then $\varphi\delta\varphi^{-1} \in \mathrm{EXP}(B)$. Remark that $\mathrm{AK}(A) = A$ if and only if the only exponential map on A is the standard inclusion $\delta(a) = a$ for all $a \in A$.

For $a \in A$, write

$$\delta(a) = \sum_{i=0}^{\infty} \delta^{(i)}(a)t^i.$$

Since $\delta(a)$ is a polynomial, the sequence $\{\delta^{(i)}(a)\}_{i=0}^{\infty}$ has finitely many nonzero elements for each $a \in A$. Since δ is a homomorphism, we see that $\delta^{(i)} : A \to A$ is linear for all i, and that the Leibniz rule

$$\delta^{(n)}(ab) = \sum_{i+j=n} \delta^{(i)}(a)\delta^{(j)}(b)$$

holds for all n and all $a, b \in A$. The above properties (i) and (ii) of the exponential map δ translate into the following properties: (i) $\delta^{(0)}$ is the identity map, and (ii) the "iterative property"

$$\delta^{(i)}\delta^{(j)} = \binom{i+j}{i}\delta^{(i+j)}$$

for all natural numbers i, j. The sequence $\{\delta^{(i)}\}_{i=0}^{\infty}$ is called a *locally finite iterative higher derivation on A*. The notion of higher derivations is due to H. Hasse and F.K. Schmidt [HS]. Note that $\delta^{(1)}$ is a locally nilpotent derivation on A. When the characteristic of A is zero, we can express $\delta^{(i)} = \frac{1}{i!}(\delta^{(1)})^i$ for each i, so that $\delta = \exp(t\delta^{(1)})$ and $A^\delta = \ker \delta^{(1)}$.

If A is a $\mathbf{k}$-algebra, we additionally assume that an exponential map on A is $\mathbf{k}$-linear. (Alternatively, one could distinguish between exponential maps and "$\mathbf{k}$-exponential maps" on A, but we shall not require this distinction.) In this case, the exponential maps on A correspond to algebraic actions of the additive group $\mathbf{k}^+$ on $\mathrm{Spec}(A)$. Also note then that A^δ and $\mathrm{AK}(A)$ are subalgebras and the maps $\delta^{(i)}$ are $\mathbf{k}$-linear. Note that $\mathrm{AK}(\mathbf{k}^{[n]}) = \mathbf{k}$ for $n > 0$, because there are n "variable shift" exponential maps δ_i on $\mathbf{k}^{[n]} = \mathbf{k}[x_1, \ldots, x_n]$ given by $\delta_i(x_i) = x_i + t$ and $\delta_i(x_j) = x_j$ for $j \neq i$.

Exponential maps on a domain

Given an exponential map $\delta : A \to A[t]$ on a domain A, we can define the δ*-degree* of an element $a \in A$ by $\deg_\delta(a) = \deg_t(\delta(a))$ (where $\deg_t(0) = -\infty$). Note that A^δ consists of all elements of A with non-positive δ-degree. The function $\deg_\delta$ is a degree function on A, i.e. for all $a, b \in A$ it satisfies (i) $\deg_\delta(ab) = \deg_\delta(a) + \deg_\delta(b)$, and (ii) $\deg_\delta(a + b) \le \max\{\deg_\delta(a), \deg_\delta(b)\}$. With the help of this degree function we easily extract several useful facts.

Lemma 2.1. *Let δ be an exponential map on a domain A.*

(a) If $a, b \in A$ such that $ab \in A^\delta \setminus 0$, then $a, b \in A^\delta$. In other words, A^δ is factorially closed in A.

(b) A^δ is algebraically closed in A.

(c) For each $a \in A$, $\deg_\delta(\delta^{(i)}(a)) \le \deg_\delta(a) - i$. In particular, if $a \in A \setminus 0$ and $n = \deg_\delta(a)$, then $\delta^{(n)}(a) \in A^\delta$.

Proof. Parts (a) and (b) follow immediately from the degree function properties. Part (c) follows immediately from the iterative property of $\{\delta^{(i)}\}$. □

Lemma 2.2. *Let δ be a nontrivial exponential map (i.e. not the standard inclusion) on a domain A with $\mathrm{char}(A) = p \ge 0$. Let $x \in A$ have minimal positive δ-degree n.*

(a) $\delta^{(i)}(x) \in A^\delta$ for each $i > 0$. Moreover, $\delta^{(i)}(x) = 0$ whenever $i > 1$ is not a power of p.

(b) If $a \in A \setminus 0$, then n divides $\deg_\delta(a)$.

(c) Let $c = \delta^{(n)}(x)$. Then A is a subring of $A^\delta[c^{-1}][x]$, where $A^\delta[c^{-1}] \subseteq \mathrm{Frac}(A^\delta)$ is the localization of A^δ at c.

(d) Suppose additionally that A is a $\mathbf{k}$-algebra with $\mathrm{trdeg}_{\mathbf{k}}(A) < \infty$. Then $\mathrm{trdeg}_{\mathbf{k}}(A^\delta) = \mathrm{trdeg}_{\mathbf{k}}(A) - 1$.

(e) Suppose additionally that A is finitely generated over $\mathbf{k}$. Then there exists an exponential map ε on A with $A^\delta = A^\varepsilon$ and $\varepsilon(x) = x + c^u t$ for some natural number u.

Proof. By part (c) of Lemma 2.1, $\deg_\delta(\delta^{(i)}(x)) < n$ if $i > 0$. So $\delta^{(i)}(x) \in A^\delta$ for all $i > 0$. From the relation $\delta_s \delta_t(x) = \delta_{s+t}(x)$ we have

$$\sum_{i=0}^{n} \delta_s(\delta^{(i)}(x)) t^i = \sum_{i=0}^{n} \delta^{(i)}(x)(s + t)^i$$

which therefore becomes

$$\sum_{i=0}^{n} \delta^{(i)}(x) s^i + \sum_{i=1}^{n} \delta^{(i)}(x) t^i = \sum_{i=0}^{n} \delta^{(i)}(x)(s + t)^i.$$

If $\delta^{(i)}(x) \neq 0$ for some $i > 0$, then $s^i + t^i = (s+t)^i$ which implies that i is a power of p. This proves (a). In particular note that n is a power of p.

Let $a \in A \setminus A^\delta$ with δ-degree m. By part (c) of Lemma 2.1 we conclude that $\delta^{(i)}(a)$ is δ-invariant for any $i > m-n$. Hence

$$\delta_s\delta_t(a) = \sum_{i=0}^{m-n} \delta_s(\delta^{(i)}(a))t^i + \sum_{i=m-n+1}^{m} \delta^{(i)}(a)t^i.$$

Examining the terms of total degree m in the relation $\delta_s\delta_t(a) = \delta_{s+t}(a)$, we therefore have

$$\begin{aligned}\delta^{(m)}(a)(s+t)^m =& \delta^{(m)}(a)t^m + \delta^{(n)}\delta^{(m-n)}(a)s^n t^{m-n} \\ &+ (\text{terms with } t\text{-degree smaller than } m-n)\end{aligned}$$

Now write $m = m_1 m_2$ where $m_1 = p^j$ for some $j \geq 0$ and m_2 is not divisible by p. Then

$$\begin{aligned}(s+t)^m &= (s^{m_1} + t^{m_1})^{m_2} \\ &= t^m + m_2 s^{m_1} t^{m-m_1} + (\text{terms with } t\text{-degree smaller than } m_1).\end{aligned}$$

It follows that $m_1 \geq n$ because $m_2 \neq 0$. Both m_1 and n are powers of p, and so n divides m. This proves (b).

Now we can write $\deg_\delta(a) = m = ln$ for some natural number l, and we note that $\deg_\delta((\delta^{(n)}(x))^l a - \delta^{(m)}(a)x^l) < \deg_\delta(a)$. (To verify it apply δ to this expression and observe that the coefficient with t^m is zero.) So by induction we can conclude that for any $a \in A$ there exists a nonnegative integer u dependent on a such that $(\delta^{(n)}(x))^u a \in A^\delta[x]$. This proves (c).

Part (d) follows from part (c), together with part (b) of Lemma 2.1 which states that A^δ is algebraically closed in A.

Finally, suppose $\{a_i\}$ is a finite generating set of A over $\mathbf{k}$. Then we can choose a natural number u sufficiently large so that $(\delta^{(n)}(x))^u a_i \in A^\delta[x]$ for all i. Define $\varepsilon(x) = x + (\delta^{(n)}(x))^u t$ and $\varepsilon(a) = a$ for all $a \in A^\delta$. Extend ε to a homomorphism on A. By the choice of u we see that $\varepsilon(A) \subset A[t]$. It is then clear that ε is an exponential map on A with $A^\delta = A^\varepsilon$. This proves (e). $\square$

Homogenization of an exponential map

In this section A is a graded domain: $A = \bigoplus_{i=0}^\infty A_i$. An element of A is called homogeneous with respect to the grading if it belongs to some A_i. The grading induces a filtration $\{B_i\}_{i=0}^\infty$ on A given by $B_i = \bigoplus_{j=0}^i A_j$. Let us identify A_i with B_i/B_{i-1} for each i. Given $a \in A$ there exists a natural number k for which $a \in B_k \setminus B_{k-1}$. Define the *top part of* a to be the homogeneous element $\overline{a} = a + B_{k-1} \in A_k$, and define $\mathrm{grdeg}(a) = k$. Note that grdeg is a degree function.

Suppose that A has a finite set S of generators which are homogeneous with respect to the grading on A and that $\delta : A \to A[t]$ is a nontrivial exponential map on A. Define

$$\tau = \min \left\{ \frac{\operatorname{grdeg}(x) - \operatorname{grdeg}(\delta^{(i)}(x))}{i} \;\middle|\; x \in S, i > 0 \right\}. \qquad (\star)$$

We extend the function grdeg on $A[t]$ by assigning $\operatorname{grdeg}(t) = \tau$ (not necessarily a natural number) and defining

$$\operatorname{grdeg}\left(\sum_{i=0}^{n} a_i t^i\right) = \max_{0 \le i \le n} \{\operatorname{grdeg}(a_i) + i\tau\}.$$

Then it is possible to define

$$\overline{\sum_{i=0}^{n} a_i t^i} = \sum_{i \in I} \overline{a_i}\, t^i$$

where $i \in I$ if

$$\operatorname{grdeg}(a_i t^i) = \operatorname{grdeg}\left(\sum_{i=0}^{n} a_i t^i\right).$$

By the choice of τ we have $\operatorname{grdeg}(\delta^{(j)}(x)t^j) \le \operatorname{grdeg}(x)$ for all $x \in S$ and each natural number j. Now, the extended grdeg is a degree function so $\overline{\delta(ab)} = \overline{\delta(a)}\;\overline{\delta(b)}$ for any $a, b \in A$. Therefore $\operatorname{grdeg}(\delta^{(j)}(a)t^j) \le \operatorname{grdeg}(a)$ for all $a \in A$.

Define now a homomorphism $\overline{\delta}$ from A to $A[t]$:

$$\overline{\delta}(a) = \overline{\delta(a)}.$$

It is straightforward to verify that $\overline{\delta}$ is an exponential map on A. Moreover, by our choice of τ there exists some nonzero element $x \in S$ for which $\operatorname{grdeg}(\delta(x) - x) = \operatorname{grdeg}(x)$, and so $\overline{\delta}$ is nontrivial.

3 Exponential maps of $A[x]$

Observe that any exponential map on A extends trivially to an exponential map on $A^{[n]}$, and there also exist "variable shift" exponential maps on $A^{[n]}$ which are trivial when restricted to A. It follows that $\mathrm{AK}(A^{[n]}) \subseteq \mathrm{AK}(A)$.

It is interesting to study the relationship between $\mathrm{AK}(A)$ and $\mathrm{AK}(A^{[n]})$. When A is finitely generated over an algebraically closed field $\mathbf{k}$ and $\operatorname{trdeg}_{\mathbf{k}}(A) = 1$, then in fact $\mathrm{AK}(A^{[n]}) = \mathrm{AK}(A)$. This implies the Abhyankar-Eakin-Heinzer cancellation theorem for curves [AEH] as a corollary. See [CML] for proofs and a generalization. When $\operatorname{trdeg}_{\mathbf{k}}(A) > 1$, the situation is not yet fully understood, but the following theorem is one piece in the puzzle.

Theorem 3.1. *Let A be a domain which is either finitely generated as a ring or finitely generated over a field $\mathbf{k}$. If $\mathrm{AK}(A) = A$ then $\mathrm{AK}(A[x]) = A$.*

Proof. We know that $\mathrm{AK}(A[x]) \subseteq A$. If A is contained in the ring of invariants of every exponential map on $A[x]$, then the reverse containment holds. Suppose $\delta : A[x] \to A[x,t]$ is an exponential map on $A[x]$ with $A \nsubseteq A[x]^\delta$. Let us obtain a contradiction.

$A[x]$ is graded by x-degree and we denote this degree function by grdeg. Let S be a finite generating set for A. The homogeneous elements in $A[x]$ are monomials, so $S \cup \{x\}$ is a finite set of homogeneous generators for $A[x]$. Hence we can define $\tau = \mathrm{grdeg}(t)$ by formula $(\star)$. There is an element $s \in S$ which is not δ-invariant since otherwise $\delta(a) = a$ for any $a \in A$. Because $\mathrm{grdeg}(s) = 0$ we must have $\tau \leq 0$.

Consider the nontrivial exponential map $\overline{\delta}$ on $A[x]$ described in the previous section. Suppose x is not $\overline{\delta}$-invariant, so that $\overline{\delta}^{(j)}(x) \neq 0$ for some $j > 0$. Since x is homogeneous we can write $\overline{\delta}^{(j)}(x) = bx^m$, where $b \in A \setminus 0$ and $m \geq 0$. By part (c) of Lemma 2.1, $\deg_{\overline{\delta}}(\overline{\delta}^{(j)}(x)) \leq \deg_{\overline{\delta}}(x) - j$. So $\deg_{\overline{\delta}}(\overline{\delta}^{(j)}(x)) = \deg_{\overline{\delta}}(b) + m\deg_{\overline{\delta}}(x) \leq \deg_{\overline{\delta}}(x) - j$, and $\deg_{\overline{\delta}}(b) + (m-1)\deg_{\overline{\delta}}(x) \leq -j$. Necessarily then $m = 0$ and $\overline{\delta}^{(j)}(x) = b$. By the construction of $\overline{\delta}$ we have

$$1 = \mathrm{grdeg}(x) = \mathrm{grdeg}(bt^j) = j\tau.$$

But $\tau \leq 0$. To avoid this contradiction, x must be $\overline{\delta}$-invariant.

Now take any element $a \in A$ which is not $\overline{\delta}$-invariant. Since a is homogeneous, for each i we can write $\overline{\delta}^{(i)}(a) = a_i x^{n_i}$, where $a_i \in A$ and $n_i \geq 0$. Then for each nonzero a_i we have

$$0 = \mathrm{grdeg}(a) = \mathrm{grdeg}(a_i x^{n_i} t^i) = n_i + i\tau.$$

Hence $n_i = -i\tau$, and so $\overline{\delta}(a) = \sum_i a_i(x^{-\tau}t)^i$. By setting $u = x^{-\tau}t$ and restricting $\overline{\delta}$ to A, we obtain a nontrivial exponential map $A \to A[u]$ on A. This contradicts the hypothesis that $\mathrm{AK}(A) = A$. □

Corollary 3.2. *Let A be a domain which is finitely generated over an algebraically closed field $\mathbf{k}$. If $A^{[1]} \cong \mathbf{k}^{[3]}$ then $A \cong \mathbf{k}^{[2]}$.*

Proof. Identify A with a subalgebra of $\mathbf{k}^{[3]}$ under the given isomorphism. Since $\mathrm{AK}(A^{[1]}) = \mathbf{k}$, by Theorem 3.1 there exists a nontrivial exponential map δ on A. Also A is the ring of invariants for the exponential map on $A^{[1]} = A[x]$ which sends x to $x + t$ and is the identity on A. So by part (a) of Lemma 2.1, A is a factorially closed subalgebra of the unique factorization domain $\mathbf{k}^{[3]}$. Thus A is a UFD, and by the same token so is A^δ. Next, since $\mathrm{trdeg}_{\mathbf{k}}(A) = 2$ we see that $\mathrm{trdeg}_{\mathbf{k}}(A^\delta) = 1$ by part (d) of Lemma 2.2. So A^δ is a transcendence degree 1 factorial subalgebra of $\mathbf{k}^{[3]}$. It is well known that then A^δ is a polynomial ring over $\mathbf{k}$. Here is a proof. Take $y \in A^\delta$ of minimal positive total degree. Clearly $y - c$ is irreducible in A^δ for each $c \in \mathbf{k}$. Suppose $r \in A^\delta$ is another irreducible element. The elements y and r are algebraically dependent over $\mathbf{k}$ since $\mathrm{trdeg}_{\mathbf{k}}(A^\delta) = 1$. Let $F(y,r) = 0$ be an irreducible polynomial dependence between them. Since $F(y,r) = F(y,0) + rG(y,r)$, we see that $F(y,0)$ is divisible by r. Since $\mathbf{k}$ is algebraically closed, we can factor $F(y,0)$ as a product of linear factors over $\mathbf{k}$, say $F(y,0) = \Pi_i(y - \lambda_i)$. Because r is irreducible, we have $r = \mu(y - \lambda)$ for some $\mu \in \mathbf{k}^*$ and $\lambda \in \mathbf{k}$. Each element of A^δ is a product of irreducible elements, and so $A^\delta = \mathbf{k}[y]$.

By part (e) of Lemma 2.2 we can assume that there exist elements in A with δ-degree 1. Choose such an element $z \in A$ with $\delta(z) = z + ft$, where the element $f \in A^\delta = \mathbf{k}[y]$ has minimal possible y-degree. We claim that $A = \mathbf{k}[y,z]$. Suppose not, and let $a \in A \setminus \mathbf{k}[y,z]$. We know by part (c) of Lemma 2.2 that $A \subset A(y)[z]$, so a has a denominator which is a nonconstant polynomial in y. Because $\mathbf{k}$ is algebraically closed, we can multiply a by an appropriate polynomial from $\mathbf{k}[y]$ to obtain an element in A whose only fractional term is $g(z)/(y-k)$ for some $g(z) \in \mathbf{k}[z]$ and some $k \in \mathbf{k}$. Now A is a UFD, and $y-k$ is irreducible in A, and $g(z)$ splits into a product of linear factors, so this implies that $(z-l)/(y-k) \in A$ for some $l \in \mathbf{k}$. But then

$$\delta\left(\frac{z-l}{y-k}\right) = \frac{z-l}{y-k} + \frac{f}{y-k}t,$$

and $\deg_y(f/(y-k)) < \deg_y(f)$, contradicting our choice of f. $\square$

As a final remark, a very similar argument shows that if A is a UFD which is finitely generated over an algebraically closed field $\mathbf{k}$ and $\mathrm{AK}(A) = \mathbf{k}$, then $A \cong \mathbf{k}^{[2]}$.

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References

[AEH] S. Abhyankar, P. Eakin, and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring, *J. Algebra* **23** (1972), 310-342.

[BML] T. Bandman and L. Makar-Limanov, Non-stability of AK-invariant, Preprint MPI 2003-89 (2003), to appear in *Michigan Math. J.*.

[C] A. Crachiola, On automorphisms of Danielewski surfaces, to appear in *J. Algebraic Geometry*.

[CML] A. Crachiola and L. Makar-Limanov, On the rigidity of small domains, *J. Algebra* **284** (2005), 1-12.

[D] W. Danielewski, On the cancellation problem and automorphism groups of affine algebraic varieties, preprint, 1989, 8 pages.

[F] T. Fujita, On Zariski problem, *Proc. Japan Acad. Ser. A Math. Sci.* **55** (1979), 106-110.

[G] R. Gurjar, A topological proof of cancellation theorem for $\mathbf{C}^2$, *Math. Z.* **240** (2002), 83-94.

[HS] H. Hasse and F.K. Schmidt, Noch eine Bergründung der Theorie der höheren Differentialquotienten in einem algebraischen Functionenkörper einer Unbestimmten, *J. Reine Angew. Math.* **177** (1937), 215-237.

[ML] L. Makar-Limanov, Locally nilpotent derivations, a new ring invariant and applications, available at http://www.math.wayne.edu/∼lml.

[MS] M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets, *J. Math. Kyoto Univ.* **20** (1980), 11-42.

[M] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, *Publ. Math. Inst. Hautes Études Sci.* **9** (1961), 5-22.

[Ra] C. P. Ramanujam, A topological characterization of the affine plane as an algebraic variety, *Ann. of Math. (2)* **94** (1971), 69–88.

[Ru] P. Russell, On affine-ruled rational surfaces, *Math. Ann.* **255** (1981), 287-302.