

**On non-commutative analytic spaces over  
non-archimedean fields**

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## 1 Introduction

### 1.1

So far two main classes of spaces have been studied in non-commutative geometry. First one is the class of non-commutative smooth manifolds (see [Co]). Here we treat any real or complex unital associative algebra as an algebra of *smooth* functions on some “non-commutative manifold”. By adding extra structures one restricts the class of spaces. In particular,  $C^*$ -algebras correspond to non-commutative topological spaces, von Neumann algebras correspond to non-commutative measurable spaces, etc. Main source of examples in this approach are “bad” quotients and foliations.

Second class of spaces consists of non-commutative schemes and their generalizations. Here the starting point is to treat any associative unital ring as an algebra of *regular* functions on a “non-commutative affine scheme  $\text{Spec}(A)$ ”. The localization procedure of non-commutative associative rings is a complicated task. An attempt to glue general non-commutative schemes from the affine ones, leads to a replacement of the naively defined category of non-commutative affine schemes by a more complicated category (see e.g. [R1]). Good source of examples in this approach is the representation theory (e.g. theory of quantum groups).

There is an obvious contradiction between two points of view discussed above. Namely, associative algebras over  $\mathbf{R}$  or  $\mathbf{C}$  are treated as algebras of functions on different types of non-commutative spaces (smooth manifolds in non-commutative differential geometry and affine schemes in non-

commutative algebraic geometry). To my knowledge there is no coherent approach to non-commutative geometry which resolves this contradiction. In other words, one cannot start with, say, non-commutative complex smooth algebraic variety, obtain a non-commutative complex manifold from it and subsequently introduce a non-commutative version of smooth structure on the latter. It looks like there are many more types of non-commutative spaces than the commutative ones. Perhaps the traditional terminology (schemes, manifolds, algebraic spaces, etc.) has to be modified in the non-commutative world. Since non-commutative spaces resist an attempt to classify them, it is interesting to study non-commutative analogs of “conventional” classes of commutative spaces. Most interesting examples arise if one considers algebras which are close to commutative (e.g. deformation quantization, quantum groups) or algebras which are very far from commutative ones (like free algebras). In a sense these are two “extremal” cases, and for some reason the corresponding non-commutative geometry is richer than the “general” one.

In this paper we discuss non-commutative analytic spaces. The list of “natural” examples is non-empty (see e.g. [SoVo] and [KoSo1]). Analytic non-commutative tori from [SoVo] are different from  $C^\infty$  non-commutative tori of Connes and Rieffel. Although a non-commutative elliptic curve over  $\mathbf{C}$  (or a non-archimedean valuation field, see [SoVo]) appears as a “bad” quotient of an analytic space, it carries more structures than the corresponding “smooth” bad quotient in Connes theory. Non-commutative deformation of a non-archimedean K3 surface was mentioned in [KoSo1] as an example of “quantization”, which is not formal with respect to deformation parameter. It seems that there is natural class of “quantum” non-archimedean analytic spaces related to cluster ensembles (see [FG]).

Present paper is devoted to examples of non-commutative spaces which can be called *non-commutative rigid analytic spaces*. General theory is far from being developed. We plan to discuss some of its aspects elsewhere (see [RSo]).

Almost all examples of non-commutative analytic spaces considered in present paper are treated from the point of view of the approach to rigid analytic geometry offered in [Be1], [Be2]. The notion of spectrum of a commutative Banach ring plays a key role in the approach of Berkovich.

Recall that the spectrum  $M(A)$  introduced in [Be1] has two equivalent descriptions:

- a) the set of multiplicative continuous seminorms on  $A$ ;
- b) the set of equivalence classes of representations of  $A$  in 1-dimensional

Banach spaces over complete Banach fields (i.e. continuous characters).

The space  $M(A)$  is compact and carries natural Hausdorff topology. There is a canonical map  $\pi : M(A) \rightarrow \text{Spec}(A)$  which assigns to a multiplicative seminorm its kernel, a prime ideal. The spectrum  $M(A)$  is a natural generalization of the Gelfand spectrum of a unital commutative  $C^*$ -algebra.

The above definition is very general and does not require  $A$  to be an affinoid algebra (i.e. an admissible quotient of the algebra of analytic functions on a non-archimedean polydisc). In the affinoid case one can make  $M(A)$  into a ringed space (affinoid space). General analytic spaces are glued from affinoid ones similarly to gluing of general schemes from affine schemes. Classical Tate theory of rigid analytic spaces, which is based on the maximal spectrum of (strictly) affinoid algebras, is a part of Berkovich theory. The advantage of the latter lies in local properties of analytic spaces (e.g. they are locally arcwise connected, see [Be3], while in classical rigid analytic geometry the topology is totally disconnected).

Affinoid spaces play the same role in non-archimedean analytic geometry as affine schemes play in the algebraic geometry. For example, localization of a finite Banach  $A$ -module  $M$  on an affinoid subset  $V \subset M(A)$  is achieved by the topological tensoring of  $M$  with an affinoid algebra  $A_V$ , which is the localization of  $A$  on  $V$  (warning: this localization *is not* an essentially surjective functor, differently from the case of algebraic geometry). In order to follow this path one needs a large supply of “good” multiplicative subsets of  $A$ . If  $A$  is commutative this is indeed the case, and, moreover, all algebras  $A_V$  are Banach. It is natural to ask what needs to be changed in the above approach in non-commutative case.

## 1.2

If  $A$  is a non-commutative unital Banach ring (or non-commutative affinoid algebra, whatever this means) then there might be very few “good” multiplicative subsets of  $A$ . Consequently, the supply of open sets (affinoid sets) can be insufficient to produce a rich theory of non-commutative analytic spaces. This problem is already known in non-commutative algebraic geometry, and one can try to look for a possible solution there. It was suggested in [R1] that instead of localizing *rings*, one should localize *categories of modules over rings*, e.g. using the notion of *spectrum* of an abelian category introduced in [R1]. Spectrum is a topological space equipped with Zariski-type topology. For an associative unital ring  $A$  the category of left modules

$A - mod$  gives rise to a sheaf of local categories on the spectrum of  $A - mod$ . If  $A$  is commutative, then the spectrum of  $A - mod$  coincides with the usual spectrum  $Spec(A)$ . In the commutative case the fiber of the sheaf of categories over  $p \in Spec(A)$  is the category of modules over the local ring which is a localization of  $A$  at  $p$ . In the non-commutative case the fiber is not a category of modules over a ring. Nevertheless one can glue more general non-commutative spaces from such, calling them *non-commutative schemes*. In this approach non-commutative schemes are defined as topological spaces equipped with sheaves of local categories (see [R1] for details).

There are more general classes of non-commutative spaces than non-commutative schemes (see e.g. [KoR]). In particular, there might be no underlying topological space (i.e. no “spectrum”). One axiomatizes the notion of covering instead. Basic idea is that if  $X = \cup_{i \in I} U_i$  is a “good” covering of a scheme, then the algebra of functions  $C := \mathcal{O}(\sqcup_{i,j \in I} (U_i \cap U_j))$  is a coalgebra in the monoidal category of  $A - A$ -bimodules, where  $A := \mathcal{O}(\sqcup_{i \in I} U_i)$ . In order to have an equivalence of the category of descent data with the category of quasicohherent sheaves on  $X$  one deals with the flat topology, which means that  $C$  is a (right) flat  $A$ -module. In this approach the category of non-commutative spaces is defined as a localization of the category of coverings with respect to a class of morphisms called refinements of coverings. This approach can be generalized to non-commutative case ([KoR]). One problem of [KoR] is the absence of good definition of morphisms of non-commutative spaces defined by means of coverings. The authors developed another approach based on the same idea, which deals with derived categories of quasi-coherent sheaves rather than with the abelian categories of quasi-coherent sheaves. In analytic case this means that one works with inductive limits of coherent sheaves.

### 1.3

Let me mention some difficulties one meets trying to construct a general theory of non-commutative analytic spaces (some of them are technical but other are conceptual).

1) It is typical in non-commutative geometry to look for a point-independent (e.g. categorical) description of an object or a structure in the commutative case, and then take it as a definition in the non-commutative one. For example, an affine morphism of schemes can be characterized by the property that the direct image functor is faithful and exact. This is taken as a definition

(see [R1], VII.1.4) of an affine morphism of non-commutative schemes. Unfortunately this approach does not always work in analytic category. As was already mentioned, the localization procedure is not pure algebraic. I don't know how to characterize categorically the embedding  $V \rightarrow X = M(A)$ , where  $V$  is an affinoid domain.

2) The above remark means that one cannot simply repeat constructions of non-commutative algebraic geometry. All rings are now topological (e.g. Banach). Topology should be involved already in the definition of the non-commutative version of Berkovich spectrum  $M(A)$  as well as in the localization procedure.

3) It is not clear what is a non-commutative analog of the notion of affinoid algebra. In the commutative case a typical example of an affinoid algebra is the Tate algebra, i.e. the completion  $T_n$  of the polynomial algebra  $K[x_1, \dots, x_n]$  with respect to the Gauss norm  $\|\sum_{l \in \mathbf{Z}_+} a_l x^l\| = \sup_l |a_l|$ , where  $|a_l|$  is the norm of the Taylor coefficient taken in the valuation field  $K$ . It is important (at least at the level of proofs) that  $T_n$  is noetherian. If we relax the condition that variables  $x_i, 1 \leq i \leq n$  commute, then the noetherian property can fail. This is true, in particular, if one starts with the polynomial algebra  $K\langle x_1, \dots, x_n \rangle$  of free non-commuting variables, equipped with the same Gauss norm as above (it is easy to see that the norm is still multiplicative). In "classical" rigid analytic geometry many proofs are based on the properties of  $T_n$  (e.g. all maximal ideals have finite codimension, Weierstrass division theorem, Noether normalization theorem, etc.). There is no "universal" replacement of  $T_n$  in non-commutative world which enjoys the same properties. On the other hand, there are some candidates which are good for particular classes of examples. We discuss them in the main body of the paper.

## 1.4

About the content of the paper. In order to make exposition more transparent I have borrowed from [RSo] few elementary things, in particular the definition of a non-commutative analog of the Berkovich spectrum  $M_{NC}(A)$  (here  $A$  is a unital Banach ring). Our definition is similar to the algebro-geometric definition of the spectrum  $Spec(A) := Spec(A - mod)$  from [R1]. Instead of the category of  $A$ -modules in [R1] we consider here the category of continuous  $A$ -modules complete with respect to a seminorm. We prove that  $M_{NC}(A)$  is non-empty. There is a natural map  $\pi : M_{NC}(A) \rightarrow Spec(A)$ . Then we

equip  $M_{NC}(A)$  with a natural Hausdorff topology. The set of bounded multiplicative seminorms  $M(A)$  is the usual Berkovich spectrum. Even if  $A$  is commutative, the space  $M_{NC}(A)$  is larger than Berkovich spectrum  $M(A)$ . This phenomenon can be illustrated in the case of non-commutative algebraic geometry. Instead of considering  $k$ -points of a commutative ring  $A$ , where  $k$  is a field, one can consider matrix points of  $A$ , i.e. homomorphisms  $A \rightarrow Mat_n(k)$ . Informally speaking, such homomorphisms correspond to morphisms of a non-commutative scheme  $Spec(Mat_n(k))$  into  $Spec(A)$ . Only the case  $n = 1$  is visible in the “conventional” algebraic geometry. Returning to unital Banach algebras we observe that  $M(A) \subset M_{NC}(A)$  regardless of commutativity of  $A$ . I should say that only  $M(A)$  will play a role in this paper.

Most of the paper is devoted to examples. We start with the elementary ones (non-commutative affine spaces and quantum tori). Less trivial example of a non-commutative analytic K3 surface is the main subject of the paper. It can be also called *quantum* K3 surface, because it is a flat deformation (quantization) of the analytic K3 surface constructed in [KoSo1]. We follow the approach of [KoSo1], where the “commutative” analytic K3 surface was constructed by gluing it from “flat” pieces, each is an analytic analog of a Lagrangian torus fibration in symplectic geometry. General scheme of the construction is explained in Section 7.1. Main idea is based on the relationship between non-archimedean analytic Calabi-Yau manifolds and real manifolds with integral affine structure discussed in [KoSo1], [KoSo2]. Roughly speaking, to such a Calabi-Yau manifold  $X$  we associate a PL manifold  $Sk(X)$ , its skeleton. There is a continuous map  $\pi : X \rightarrow Sk(X)$  such that the generic fiber is an analytic torus. Moreover, there is an embedding of  $Sk(X)$  into  $X$ , so that  $\pi$  becomes a retraction. For the elliptically fibered K3 surface  $Sk(X)$  is a 2-dimensional sphere  $S^2$ . It has an integral affine structure outside of the set of 24 points. It is analogous to the integral affine structure on the base of a Lagrangian torus fibration in symplectic geometry (Liouville theorem). Fibers of  $\pi$  are Stein spaces. Hence in order to construct the sheaf  $\mathcal{O}_X$  of analytic functions on  $X$  it suffices to construct  $\pi_*(\mathcal{O}_X)$ .

I should say that almost all examples in this paper should be called *quantum non-commutative analytic spaces*. They are based on the version of Tate algebra in which the commutativity of variables  $z_i z_j = z_j z_i$  is replaced by the  $q$ -commutativity  $z_i z_j = q z_j z_i, q \in K^\times, |q| = 1$ . In particular our non-commutative analytic K3 surface is defined as a ringed space, with the underlying topological space being a K3 surface equipped with the natural

Grothendieck topology introduced in [Be3] and the sheaf of non-commutative algebras which is locally isomorphic to a quotient of the above-mentioned “quantum” Tate algebra. The construction of a quantum K3 surface uses a non-commutative analog of the map  $\pi$ . In other words, the skeleton  $Sk(X)$  survives the deformation procedure. For that reason even the “Berkovich part” of the whole non-commutative K3 surface gives a non-trivial example of non-commutative non-archimedean analytic space.

## 1.5

As a side remark we mention that the theory of non-commutative analytic spaces should have applications to mirror symmetry in the spirit of [KoSo1],[KoSo2]. More precisely, it looks plausible that certain deformation of the Fukaya category of a maximally degenerate (see [KoSo1], [KoSo2]) hyperkahler manifold can be realized as the derived category of coherent sheaves on the non-commutative deformation of the dual Calabi-Yau manifold (which is basically the same hyperkahler manifold).

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## 2 Non-commutative Berkovich spectrum

### 2.1 Preliminaries

We refer to [Be1], Chapter 1 for the terminology of seminormed groups, etc. Here we recall few terms for convenience of the reader.

Let  $A$  be an associative unital Banach ring. Then, by definition,  $A$  carries a norm  $|\bullet|_A$ , and moreover,  $A$  is complete with respect to this norm. The norm is submultiplicative, i.e.  $|ab|_A \leq |a|_A|b|_A$ ,  $a, b \in A$  and unital, i.e.  $|1|_A = 1$ . We call the norm *non-archimedean* if, instead of the usual inequality  $|a +$



$|b|_A \leq |a|_A + |b|_A$ ,  $a, b \in A$ , we have a stronger one  $|a + b|_A \leq \max\{|a|_A, |b|_A\}$ . A *seminormed module* over  $A$  is (cf. with [Be1]) a left unital (i.e. 1 acts as  $id_M$ )  $A$ -module  $M$  which carries a seminorm  $|\bullet|$  such that  $|am| \leq C|a|_A|m|$  for some  $C > 0$  and all  $a \in A, m \in M$ . Seminormed  $A$ -modules form a category  $A-mod^c$ , such that a morphism  $f : M \rightarrow N$  is a homomorphism of  $A$ -modules satisfying the condition  $|f(m)| \leq \text{const}|m|$  (i.e.  $f$  is bounded). Clearly the kernel  $Ker(f)$  is closed with respect to the topology defined by the seminorm on  $M$ . A morphism  $f : M \rightarrow N$  is called *admissible* if the quotient seminorm on  $Im(f) \simeq M/Ker(f)$  is equivalent to the one induced from  $N$ . We remark that the category  $A-mod^c$  is not abelian in general. Following [Be1] we call *valuation field* a commutative Banach field  $K$  whose norm is multiplicative, i.e.  $|ab| = |a||b|$ . If the norm is non-archimedean, we will call  $K$  a *non-archimedean valuation field*. In this case one can introduce a valuation map  $val : K^\times \rightarrow \mathbf{R} \cup +\infty$ , such that  $val(x) = -\log|x|$ .

## 2.2 Spectrum of a Banach ring

Let us introduce a partial order on the objects of  $A-mod^c$ . We say that  $N \geq_c M$  if there exists a closed admissible embedding  $i : L \rightarrow \bigoplus_{I\text{-finite}} N$  and an admissible epimorphism  $pr : L \rightarrow M$ . We will denote by  $N \geq M$  a similar partial order on the category  $A-mod$  of left  $A$ -modules (no admissibility condition is imposed). We will denote by  $=_c$  (resp.  $=$  for  $A-mod$ ) the equivalence relation generated by the above partial order.

Let us call an object of  $A-mod^c$  (resp.  $A-mod$ ) *minimal* if it satisfies the following conditions (cf. [R1]):

- 1) if  $i : N \rightarrow M$  is a closed admissible embedding (resp. any embedding in the case of  $A-mod$ ) then  $N \geq_c M$  (resp.  $N \geq M$ );
- 2) there is an element  $m \in M$  such that  $|m| \neq 0$  (resp.  $m \neq 0$  for  $A-mod$ ).

We recall (see [R1]) that the spectrum  $Spec(A) := Spec(A-mod)$  consists of equivalence classes (w.r.t.  $=$ ) of minimal objects of  $A-mod$ . It is known that  $Spec(A)$  is non-empty and contains classes of  $A$ -modules  $A/m$ , where  $m$  is a left maximal ideal of  $A$ . Moreover,  $Spec(A)$  can be identified with the so-called *left spectrum*  $Spec_l(A)$ , which can be also described in terms of a certain subset of the set of left ideals of  $A$  (for commutative  $A$  it is the whole set of prime ideals).

Let  $A-mod^b$  be a full subcategory of  $A-mod^c$  consisting of  $A$ -modules which are complete with respect to their seminorms (we call them *Banach*

modules for short).

**Definition 1** *The non-commutative analytic spectrum  $M_{NC}(A)$  consists of classes of equivalence (w.r.t. to  $=_c$ ) of minimal (i.e. satisfying 1) and 2)) objects  $M$  of  $A - \text{mod}^b$  which satisfy also the following property:*

3) if  $m_0 \in M$  is such that  $|m_0| \neq 0$  then the left module  $Am_0$  is minimal, i.e. defines a point of  $\text{Spec}(A)$  (equivalently, this means that  $p := \text{Ann}(m_0) \in \text{Spec}_l(A)$ ), and this point does not depend on a choice of  $m_0$ .

Next proposition implies that  $M_{NC}(A) \neq \emptyset$ .

**Proposition 1** *Every (proper) left maximal ideal  $m \subset A$  is closed.*

*Proof.* We want to prove that the closure  $\overline{m}$  coincides with  $m$ . The ideal  $\overline{m}$  contains  $m$ . Since  $m$  is maximal, then either  $\overline{m} = m$  or  $\overline{m} = A$ . Assume the latter. Then there exists a sequence  $x_n \rightarrow 1, n \rightarrow +\infty, x_n \in m$ . Choose  $n$  so large that  $|1 - x_n|_A < 1/2$ . Then  $x_n = 1 + (x_n - 1) := 1 + y_n$  is invertible in  $A$ , since  $(1 + y_n)^{-1} = \sum_{l \geq 0} (-1)^l y_n^l$  converges. Hence  $m = A$ . Contradiction. ■

**Corollary 1**  $M_{NC}(A) \neq \emptyset$

*Proof.* Let  $m$  be a left maximal ideal in  $A$  (it does exist because of the standard arguments which use Zorn lemma). It is closed by previous Proposition. Then  $M := A/m$  is a cyclic Banach  $A$ -module with respect to the quotient norm. We claim that it contains no proper closed submodules. Indeed, let  $N \subset M$  be a proper closed submodule. We may assume it contains an element  $n_0$  such that annihilator  $\text{Ann}(n_0)$  contains  $m$  as a proper subset. Since  $m$  is maximal we conclude that  $\text{Ann}(n_0) = A$ . But this cannot be true since  $1 \in A$  acts without kernel on  $A/m$ , hence  $1 \notin \text{Ann}(n_0)$ . This contradiction shows that  $M$  contains no proper closed submodules. In order to finish the proof we recall that simple  $A$ -module  $A/m$  defines a point of  $\text{Spec}(A)$ . Hence the conditions 1)-3) above are satisfied. ■

Abusing terminology we will often say that an object belongs to the spectrum (rather than saying that its equivalence class belongs to the spectrum).

**Proposition 2** *If  $M \in A - \text{mod}^b$  belongs to  $M_{NC}(A)$  then its seminorm is, in fact, a Banach norm.*

*Proof.* Let  $N = \{m \in M \mid |m| = 0\}$ . Clearly  $N$  is a closed submodule. Let  $L$  be a closed submodule of the finite sum of copies of  $N$ , such that there exists an admissible epimorphism  $pr : L \rightarrow M$ . Then the submodule  $L$  must carry trivial induced seminorm, and, moreover, admissibility of the epimorphism  $pr : L \rightarrow M$  implies that the seminorm on  $M$  is trivial. This contradicts to 2). Hence  $N = 0$ . ■

**Remark 1** a) *By condition 3) we have a map of sets  $\pi : M_{NC}(A) \rightarrow \text{Spec}(A)$ .*

b) *If  $A$  is commutative then any bounded multiplicative seminorm on  $A$  defines a prime ideal  $p$  (the kernel of seminorm). Moreover,  $A/p$  is a Banach  $A$ -module, which belongs to  $M_{NC}(A)$ . Hence Berkovich spectrum  $M(A)$  (see [Be1]) is a subset of  $M_{NC}(A)$ . Thus  $M_{NC}(A)$  can be also called non-commutative Berkovich spectrum.*

### 2.3 Topology on $M_{NC}(A)$

If a Banach  $A$ -module  $M$  belongs to  $M_{NC}(A)$  then it is equivalent (with respect to  $=_c$ ) to the closure of any cyclic submodule  $M_0 = Am_0$ . Then for a fixed  $a \in A$  we have a function  $\phi_a : (M, |\bullet|) \mapsto |am_0|$ , which can be thought of as a real-valued function on  $M_{NC}(A)$ .

**Definition 2** *The topology on  $M_{NC}(A)$  is taken to be the weakest one for which all functions  $\phi_a, a \in A$  are continuous.*

**Proposition 3** *The above topology makes  $M_{NC}(A)$  into a Hausdorff topological space.*

*Proof.* Let us take two different points of the non-commutative analytic spectrum  $M_{NC}(A)$  defined by cyclic seminormed modules  $(M_0, |\bullet|)$  and  $(M'_0, |\bullet|')$ . If  $M_0$  is not equivalent to  $M'_0$  with respect to  $=$  (i.e. in  $A$ -mod), then there exist two different closed left ideals  $p, p'$  such that  $M_0 \simeq A/p$  and  $M'_0 \simeq A/p'$  (again, this is an isomorphism of  $A$ -modules only. Banach norms are not induced from  $A$ ). Then there is an element  $a \in A$  which belongs, to say,  $p$  and does not belong to  $p'$  (if  $p \subset p'$  we interchange  $p$  and  $p'$ ). Hence the function  $\phi_a$  is equal to zero on the closure of  $(M_0, |\bullet|)$  and  $\phi_a(\overline{(M'_0, |\bullet|')} = c_a > 0$ . Then open sets  $U_0 = \{x \in M_{NC}(A) \mid \phi_a(x) < c_a/2\}$  and  $U'_0 = \{x \in M_{NC}(A) \mid 3c_a/4 < \phi_a(x) < c_a\}$  do not intersect and separate two given points of the analytic spectrum.

Suppose  $M = M_0 = M'_0 = Am_0$ . Since the corresponding points of  $\text{Spec}(A)$  coincide, we have the same cyclic  $A$ -module which carries two different norms  $|\bullet|$  and  $|\bullet|'$ . Let  $am_0 = m \in M$  be an element such that  $|m| \neq |m|'$ . Then the function  $\phi_a$  takes different values at the corresponding points of  $M_{NC}(A)$  (which are completions of  $M$  with respect to the above norms), and we can define separating open subsets as before. This concludes the proof. ■

## 2.4 Relation to multiplicative seminorms

If  $x : A \rightarrow \mathbf{R}_+$  is a multiplicative bounded seminorm on  $A$  (bounded means  $x(a) \leq |a|_A$  for all  $a \in A$ ) then  $\text{Ker } x$  is a closed 2-sided ideal in  $A$ . If  $A$  is non-commutative, it can contain very few 2-sided ideals. At the same time, a bounded multiplicative seminorm  $x$  gives rise to a point of  $M_{NC}(A)$  such that the corresponding Banach  $A$ -module is  $A/\text{Ker } x$  equipped with the left action of  $A$ .

There exists a class of *submultiplicative* bounded seminorms on  $A$  which is contained in  $M(A)$ , if  $A$  is commutative. More precisely, let us consider the set  $P(A)$  of all submultiplicative bounded seminorms on  $A$ . By definition, an element of  $P(A)$  is a seminorm such that  $|ab| \leq |a||b|$ ,  $|1| = 1$ ,  $|a| \leq C|a|_A$  for all  $a, b \in A$  and some  $C > 0$  ( $C$  depends on the seminorm). The set  $P(A)$  carries natural partial order:  $|\bullet|_1 \leq |\bullet|_2$  if  $|a|_1 \leq |a|_2$  for all  $a \in A$ . Let us call *minimal seminorm* a minimal element of  $P(A)$  with respect to this partial order, and denote by  $P_{\min}(A)$  the subset of minimal seminorms. The latter set is non-empty by Zorn lemma. Let us recall the following classical result (see e.g. [Be1]).

**Proposition 4** *If  $A$  is commutative then  $P_{\min}(A) \subset M(A)$ , i.e. any minimal seminorm is multiplicative.*

Since there exist multiplicative bounded seminorms which are not minimal (take e.g. the trivial seminorm on the ring of integers  $\mathbf{Z}$  equipped with the usual absolute value Banach norm) it is not reasonable to define  $M(A)$  in the non-commutative case as a set of minimal bounded seminorms. On the other hand one can prove the following result.

**Proposition 5** *If  $A$  is left noetherian as a ring then a minimal bounded seminorm defines a point of  $M_{NC}(A)$ .*

*Proof.* Let  $p$  be the kernel of a minimal seminorm  $v$ . Then  $p$  is a 2-sided closed ideal. The quotient  $B = A/p$  is a Banach algebra with respect to the induced norm. It is topologically simple, i.e. does not contain non-trivial closed 2-sided ideals. Indeed, let  $r$  be a such an ideal. Then  $B/r$  is a Banach algebra with the norm induced from  $B$ . The pullback of this norm to  $A$  gives rise to a bounded seminorm on  $A$ , which is smaller than  $v$ , since it is equal to zero on a closed 2-sided ideal which contains  $p = \text{Ker } v$ . The remaining proof that  $A/p \in M_{NC}(A)$  is similar to the one from [R1]. Recall that it was proven in [R1] that if  $n$  is a 2-sided ideal in a noetherian ring  $R$  such that  $R/n$  contains no 2-sided ideals then  $A/n \in \text{Spec}(A - \text{mod})$ . ■

We will denote by  $M(A)$  the subset of  $M_{NC}(A)$  consisting of bounded multiplicative seminorms. It carries the induced topology, which coincides for a commutative  $A$  with the topology introduced in [Be1]. We will call the corresponding topological space *Berkovich spectrum* of  $A$ .

Returning to the beginning of this section we observe that the set  $P(A)$  of submultiplicative bounded seminorms contains  $M_{NC}(A)$ . Indeed if  $v \in P(A)$  then we have a left Banach  $A$ -module  $M_v$ , which is the completion of  $A/\text{Ker } v$  with respect to the norm induced by  $v$ . Thus  $M_v$  is a cyclic Banach  $A$ -module. A submultiplicative bounded seminorm which defines a point of the analytic spectrum can be characterized by the following property:  $v \in P(A)$  belongs to  $M_{NC}(A)$  iff  $A/\text{Ker } v \in \text{Spec}(A - \text{mod})$  (equivalently, if  $\text{Ker } v \in \text{Spec}_l(A)$ ).

### 3 Non-commutative affinoid algebras

Let  $K$  be a non-archimedean valuation field,  $r = (r_1, \dots, r_n)$ , where  $r_i > 0, 1 \leq i \leq n$ . In the “commutative” analytic geometry an affinoid algebra  $A$  can be defined as an admissible quotient of the unital Banach algebra  $K\{T\}_r$  of formal series  $\sum_{l \in \mathbf{Z}_+^n} a_l T^l$ , such that  $\max |a_l| r^l < \infty, l = (l_1, \dots, l_n)$ . The latter algebra is the completion of the ring of polynomials  $K[T] := K[T_1, \dots, T_n]$  with respect to the norm  $|\sum_l a_l T^l|_r = \max |a_l| r^l$ . In the non-commutative case we start with the free algebra  $K\langle T \rangle := K\langle T_1, \dots, T_n \rangle$  of  $n$  non-commuting variables and consider its completion  $K\langle T \rangle_r$  with respect to the norm  $|\sum_{\lambda \in P(\mathbf{Z}_+^n)} a_\lambda T^\lambda| = \max_\lambda |a_\lambda| r^{|\lambda|}$ . Here  $P(\mathbf{Z}_+^n)$  is the set of finite paths in  $\mathbf{Z}_+^n$  starting at the origin, and  $T^\lambda = T_1^{\lambda_1} T_2^{\lambda_2} \dots$  for the path which moves  $\lambda_1$  steps in the direction  $(1, 0, 0, \dots)$  then  $\lambda_2$  steps in the direction  $(0, 1, 0, 0, \dots)$ , and so on (repetitions are allowed, so we can have a monomial like  $T_1^{\lambda_1} T_2^{\lambda_2} T_1^{\lambda_3}$ ). We say that a Banach unital algebra  $A$  is *non-*

*commutative affinoid algebra* if there is an admissible surjective homomorphism  $K\langle T \rangle_r \rightarrow A$ . In particular,  $K\{T\}_r$  is such an algebra, and hence all commutative affinoid algebras are such algebras. We will restrict ourselves to the class of *noetherian* non-commutative affinoid algebras, i.e. those which are noetherian as left rings. All classical affinoid algebras belongs to this class. For such algebras one can prove the following result (the proof is similar to the proof of Prop. 2.1.9 from [Be1], see also [SchT]).

**Proposition 6** *Let  $A$  be a noetherian non-commutative affinoid algebra. Then the category  $A - \text{mod}^f$  of (left) finite  $A$ -modules is equivalent to the category  $A - \text{mod}^{fb}$  of (left) finite Banach  $A$ -modules.*

An important class of noetherian affinoid algebras consists of *quantum* affinoid algebras. By definition they are admissible quotients of algebras  $K\{T_1, \dots, T_n\}_{q,r}$ . The latter consists of formal power series  $f = \sum_{l \in \mathbf{z}_+^n} a_l T^l$  of  $q$ -commuting variables (i.e.  $T_i T_j = q T_j T_i, i < j$  and  $q \in K^\times, |q| = 1$ ) such that  $|f|_r := \max |a_l| r^l < \infty$ . Here  $T^l = T_1^{l_1} \dots T_n^{l_n}$  (the order is important now). It is easy to see that for any  $r = (r_1, \dots, r_n)$  such that all  $r_i > 0$  the function  $f \mapsto |f|_r$  defines a multiplicative norm on the polynomial algebra  $K[T_1, \dots, T_n]_{q,r}$  in  $q$ -commuting variables  $T_i, 1 \leq i \leq n$ . Banach algebra  $K\{T_1, \dots, T_n\}_{q,r}$  is the completion of the latter with respect to the norm  $|\bullet|_r$ .

## 4 Non-commutative analytic affine spaces

Let  $k$  be a commutative unital Banach ring. We start with the algebra  $k\langle\langle x_1, \dots, x_n \rangle\rangle$  of formal series in free variables  $x_1, \dots, x_n$ . For each  $r = (r_1, \dots, r_n)$  we define a subspace  $k\langle\langle x_1, \dots, x_n \rangle\rangle_r$  consisting of series such  $f = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} x_{i_1} \dots x_{i_m}$  that  $\sum_{i_1, \dots, i_m} |a_{i_1, \dots, i_m}| r_{i_1} \dots r_{i_m} < +\infty$ . Here the summation is taken over all sequences  $(i_1, \dots, i_m), m \geq 0$  and  $|\bullet|$  denotes the norm in  $k$ . In the non-archimedean case the convergency condition can be replaced by a weaker one:  $\max |a_{i_1, \dots, i_m}| r_{i_1} \dots r_{i_m} < +\infty$ . Clearly each  $k\langle\langle x_1, \dots, x_n \rangle\rangle_r$  is a Banach algebra (called the algebra of analytic functions on a non-commutative polydisc  $E_{NC}(0, r)$ , cf. [Be1], 1.5). We define a non-commutative  $n$ -dimensional analytic affine space  $\mathbf{A}_{NC}^n$  as the coproduct  $\cup_r E_{NC}(0, r)$ . By definition the algebra of analytic functions on the quantum affine space is given by the above series such that  $\max |a_{i_1, \dots, i_m}| r_{i_1} \dots r_{i_m} < +\infty$  for all  $r = (r_1, \dots, r_n)$ . In other words, analytic functions are given by the

series which are convergent in all non-commutative polydiscs with centers in the origin.

The algebra of analytic functions on the non-archimedean *quantum closed polydisc*  $E_q(0, r)$  is, by definition,  $K\{T_1, \dots, T_n\}_{q,r}$ . The algebra of analytic functions on non-archimedean *quantum affine space*  $\mathbf{A}_q^N$  consists of the series  $f$  in  $q$ -commuting variables  $T_1, \dots, T_n$ , such that for all  $r$  the norm  $|f|_r$  is finite. Equivalently, it is the coproduct of *quantum closed polydiscs*  $E_q(0, r)$ . There is an obvious generalization of this example based on the Banach algebra with  $T_i T_j = q_{ij} T_j T_i, i < j, |q_{ij}| = 1$ . We will keep the terminology unchanged for these, more general, objects.

## 5 Quantum analytic tori

Let  $K$  be a non-archimedean valuation field,  $L$  a free abelian group of finite rank  $d$ ,  $\varphi : L \times L \rightarrow \mathbf{Z}$  is a skew-symmetric bilinear form,  $h \in K$  satisfies the condition  $|h| < 1$ . Then  $|\exp(h\varphi(\lambda, \mu))| = 1$  for any  $\lambda, \mu \in L$ . We denote by  $A_h(T(L, \varphi))$  the algebra of regular functions on the quantum torus  $T_h(L, \varphi)$ . By definition, it is a  $K$ -algebra with generators  $e(\lambda), \lambda \in L$ , subject to the relation

$$e(\lambda)e(\mu) = \exp(h\varphi(\lambda, \mu))e(\lambda + \mu).$$

**Definition 3** *The space  $\mathcal{O}_h(T(L, \varphi, (1, \dots, 1)))$  of analytic functions on the quantum torus of multiradius  $(1, 1, \dots, 1) \in \mathbf{Z}_+^d$  consists of series  $\sum_{\lambda \in L} a(\lambda)e(\lambda), a(\lambda) \in K$  such that  $|a(\lambda)| \rightarrow 0$  as  $|\lambda| \rightarrow \infty$  (here  $|(\lambda_1, \dots, \lambda_d)| = \sum_i |\lambda_i|$ ).*

It is easy to check (see [SoVo]) the following result.

**Lemma 1** *Analytic functions  $\mathcal{O}_h(T(L, \varphi, (1, \dots, 1)))$  form a Banach  $K$ -algebra. Moreover, it is a non-commutative noetherian affinoid algebra.*

The above example admits the following generalization. Let us fix a basis  $e_1, \dots, e_n$  of  $L$  as well as positive numbers  $r_1, \dots, r_n$ . We define  $r^\lambda = r_1^{\lambda_1} \dots r_n^{\lambda_n}$  for any  $\lambda = \sum_{1 \leq i \leq n} \lambda_i e_i \in L$ . Then the algebra  $\mathcal{O}_h(T(L, \varphi, r))$  of analytic functions on the *quantum torus of multiradius*  $r = (r_1, \dots, r_n)$  is defined by the same series as before, with the only change that  $|a(\lambda)|r^\lambda \rightarrow 0$  as  $|\lambda| \rightarrow \infty$ . In the case  $h = 0$  these objects are also called (poly)annuli. We are going to denote the corresponding non-commutative analytic space by  $T_h^{an}(L, \varphi, r)$ . The coproduct  $T_h^{an}(L, \varphi) = \cup_r T_h^{an}(L, \varphi, r)$  is called the

*quantum analytic torus.* The algebra of analytic functions  $\mathcal{O}_h(T(L, \varphi))$  on it consists, by definition, of the above series such that for all  $r = (r_1, \dots, r_d)$  one has:  $|a(\lambda)|r^\lambda \rightarrow 0$  as  $|\lambda| \rightarrow \infty$ . To be consistent with the notation of the previous subsection we will often denote  $e_i$  by  $T_i$ .

## 5.1 Remark on Berkovich spectrum of quantum affine space and analytic torus

Assume for simplicity that the pair  $(L, \varphi)$  defines a simply-laced lattice of rank  $d$ , i.e. for some basis  $(e_i)_{1 \leq i \leq d}$  of  $L$  one has  $\varphi(e_i, e_i) = 1$ . In more familiar notation we have  $T_i T_j = q \bar{T}_j \bar{T}_i, i < j, |q| = 1$ . Any  $r = (r_1, \dots, r_d) \in \mathbf{R}_{>0}^d$  gives rise to a point  $\nu_r \in M(\mathcal{O}_h(T(L, \varphi)))$  such that  $\nu_r(f) = \max_{\lambda \in L} |a(\lambda)|r^\lambda$ . In this way we obtain a (continuous) embedding of  $\mathbf{R}_{>0}^d$  into Berkovich spectrum of quantum affine space. This is an example of a more general phenomenon, since  $\mathbf{R}_{>0}^d$  is the so-called *skeleton*  $S((G_m^{an})^d)$  (see [Be], [Be3]) of the  $d$ -dimensional (commutative) analytic torus  $(G_m^{an})^d$ . Skeleta exist for more general analytic spaces. For example, the skeleton of a  $d$ -dimensional Drinfeld upper-half space  $\Omega_K^d$  is the Bruhat-Tits building of  $PGL(d, K)$ .

We have constructed an embedding of  $S((G_m^{an})^d)$  into  $M(\mathcal{O}_h(T(L, \varphi)))$ . Moreover, if  $h = 0$  then there is a retraction  $(G_m^{an})^d \rightarrow S((G_m^{an})^d)$ . The pair  $((G_m^{an})^d, S((G_m^{an})^d))$  is an example of *analytic torus fibration* which plays an important role in mirror symmetry (see [KoSo1]).

We see in this example that the skeleton of an analytic space (if it exists) can survive in  $q$ -deformations as long as  $|q| = 1$ . An example of a quantum K3 surface considered below supports this idea. Even so, one cannot expect that for generic  $q$  with  $|q| = 1$  Berkovich spectrum of a quantum non-archimedean analytic space contains more than the skeleton. Surprisingly, as  $q$  gets close to 1, the situation improves, as we will show in examples of the quantum affine space and quantum torus. Let us fix  $a_1, \dots, a_d \in K$  and assume that  $|1 - q| < 1$ .

**Proposition 7** *Any  $r = (r_1, \dots, r_d) \in \mathbf{R}_{>0}^d$  gives rise to a point  $\nu_{a,r}$  of the Berkovich spectrum of the quantum affine space, such that for a polynomial  $f = \sum_{n \in \mathbf{Z}_+^d} c_n (T - a)^n$  one has  $\nu_{a,r}(f) := |\sum_{n \in \mathbf{Z}_+^d} c_n (T - a)^n|_r$ .*

*Proof.* We need to show that  $\nu_{a,r}(fg) = \nu_{a,r}(f)\nu_{a,r}(g)$  for any two polynomials  $f, g \in K[T_1, \dots, T_d]_q$ .



Let us introduce new variables  $t_i$  by the formulas  $T_i - a_i = t_i$ ,  $1 \leq i \leq d$ . Clearly  $|t_i| = r_i$ ,  $1 \leq i \leq d$ . By definition  $t_i t_j = q t_j t_i + (q-1)(a_i t_j + a_j t_i) + (q-1)a_i a_j$ . Therefore, for any multi-indices  $\alpha, \beta$  one has  $t^\alpha t^\beta = q^{\varphi(\alpha, \beta)} t^{\alpha+\beta} + D$ , where  $|D|_r < r^{|\alpha|+|\beta|}$ , and  $t^{(\alpha_1, \dots, \alpha_d)} = t_1^{\alpha_1} \dots t_d^{\alpha_d}$ . Notice that  $|t_i t_j| = |t_j t_i| = |t_i| |t_j|$ . Then any polynomial  $f \in K[T_1, \dots, T_d]_q$  can be written as a finite sum  $f = \sum_{n \in \mathbf{Z}_+^d} a_n t^n + B$  where  $|B|_r < |\sum_{n \in \mathbf{Z}_+^d} a_n t^n|_r$ . We see that  $\nu_{a,r}(f) = \max |a_n| r^n$ . It follows that  $\nu_{a,r}$  is multiplicative. This concludes the proof.  $\blacksquare$

Let us consider the case of quantum tori. Now we fix  $a = (a_1, \dots, a_d) \in (K^*)^d$  and  $r = (r_1, \dots, r_d) \in \mathbf{R}_{>0}^d$  such that  $r_i < |a_i|$ ,  $1 \leq i \leq d$ . Let  $f = \sum_{n \in \mathbf{Z}^d} c_n T^n$  be an analytic function on  $T_h^{an}(L, \varphi)$ . In the notation of the above proof we can rewrite  $f$  as  $f = \sum_{n \in \mathbf{Z}^d} c_n (t+a)^n$ , where  $(t_i + a_i)^{-1} := a_i^{-1} \sum_{m \geq 0} a_i^{-m} t_i^m$ . Therefore  $f = \sum_{m \in \mathbf{Z}_+^d} d_m t^m$ . Suppose we know that  $|d_m| r^m \rightarrow 0$  as  $|m| \rightarrow +\infty$ . Then similarly to the proof of the previous proposition one sees that  $\nu_{a,r}(f) := \max_m |d_m| r^m$  is the multiplicative seminorm (here we use the condition  $|q-1| < 1$ ). Let us estimate  $|d_m| r^m$ . It is easy to see that  $d_m = \sum_{l \in \mathbf{Z}^d} b_l^{(m)} c_l a^{l-m}$ , where  $|b_l^{(m)}|_r \leq 1$ . Therefore  $|d_m| r^m \leq (\max_{l \in \mathbf{Z}^d} |c_l| |a|_1^{l_1} \dots |a_d|^{l_d}) r^m |a_1|^{m_1} \dots |a_d|^{m_d} = |f|_{(|a_1|, \dots, |a_d|)} (ra^{-1})^m$ , where  $ra^{-1} = (r_1/a_1, \dots, r_d/a_d)$ . Since  $f$  is analytic on the quantum torus, the norm  $|f|_{(|a_1|, \dots, |a_d|)}$  is finite. On the other hand  $(ra^{-1})^m \rightarrow 0$  as  $|m| \rightarrow +\infty$ . This concludes our estimate of  $\nu_{a,r}(f)$  and shows that it is a multiplicative seminorm on the algebra of analytic functions on the quantum torus. The latter is a Frechet algebra. Its Berkovich spectrum is the union of Berkovich spectra of quantum analytic tori  $T_h(L, \varphi, r)$  of all multiradii  $r$ . Thus we see that the pair  $(a, r)$  as above defines a point of Berkovich spectrum of the quantum analytic torus. It seems plausible that any point of the analytic space  $(G_m^{an})^d$  appears as a point of the Berkovich spectrum of the quantum analytic torus, as long as we choose  $q$  sufficiently close to 1. Same is true for the quantum affine space. These claims is easy to prove for  $d = 1$ . The above Proposition shows in a toy-model the drastic difference with formal deformation quantization, when the deformed algebras typically have no “commutative” points at all. In non-archimedean picture it might be possible to “reconstruct” the “commutative” space (at least as a set) from its quantum deformation for finite  $q$ , step by step.

## 6 Non-commutative Stein spaces

This example is borrowed from [SchT].

Let  $K$  be a non-archimedean valuation field, and  $A$  be a unital Frechet  $K$ -algebra. We say that  $A$  is *Frechet-Stein* if there is a sequence  $v_1 \leq v_2 \leq \dots \leq v_n \leq \dots$  of continuous seminorms on  $A$  which define the Frechet topology and:

- a) the completions  $A_{v_n}$  of the algebras  $A/\text{Ker } v_n$  are left notherian for all  $n \geq 1$ ;
- b) each  $A_{v_n}$  is a flat  $A_{v_{n+1}}$ -module,  $n \geq 1$ .

Here we do not require that  $v_n$  are submultiplicative, only the inequalities  $v_n(xy) \leq \text{const } v_n(x)v_n(y)$ . Clearly the sequence  $(A_{v_n})_{n \geq 1}$  is a projective system of algebras and its projective limit is isomorphic to  $A$ .

A *coherent sheaf* for a Frechet-Stein algebra  $(A, v_n)_{n \geq 1}$  is a collection  $M = (M_n)_{n \geq 1}$  such that each  $M_n$  is a finite Banach  $A_{v_n}$ -module, and for each  $n \geq 1$  one has natural isomorphism  $A_{v_n} \otimes_{A_{v_{n+1}}} M_{n+1} \simeq M_n$ .

The inverse limit of the projective system  $M_n$  is an  $A$ -module called the module of *global sections* of the coherent sheaf  $M$ . Coherent sheaves form an abelian category. It is shown in [SchT] that the projective limit functor from coherent sheaves to finite Banach  $A$ -modules is exact (this is an analog of  $A$  and  $B$  theorems of Cartan).

One can define the non-commutative analytic spectrum of Frechet-Stein algebras in the same way as we did for Banach algebras, starting with the category of coherent sheaves.

It is shown in [SchT] that if  $G$  is a compact locally analytic group then the the strong dual  $D(G, K)$  to the space of  $K$ -valued locally analytic functions is a Frechet-Stein algebra. In the case  $G = \mathbf{Z}_p$  it is isomorphic to the (commutative) algebra of power series converging in the open unit disc in the completion of the algebraic closure of  $K$ . In general, the Frechet-Stein algebra structure on  $D(G, K)$  is defined by a family of norms which are submultiplicative only. Coherent sheaves for the algebra  $D(G, K)$  give rise to coherent sheaves on the non-commutative analytic spectrum  $M_{NC}(D(G, K))$ , rather than on Berkovich spectrum  $M(D(G, K))$ .

## 7 Non-commutative analytic K3 surface

### 7.1 General scheme

The following way of constructing a non-commutative analytic K3 surface  $X$  over the field  $K = \mathbf{C}((t))$  was suggested in [KoSo1].

1) We start with a 2-dimensional sphere  $B := S^2$  equipped with an integral affine structure outside of a finite subset  $B^{sing} = \{x_1, \dots, x_{24}\}$  of 24 distinct points (see [KoSo1] for the definitions and explanation why  $|B^{sing}| = 24$ ). We assume that the monodromy of the affine structure around each point  $x_i$  is conjugate to the  $2 \times 2$  unipotent Jordan block (it is proved in [KoSo1] that this restriction enforces the cardinality of  $B^{sing}$  to be equal to 24).

2) In addition to the above data we have an infinite set of “trees” embedded in  $S^2$ , called *lines* in [KoSo1]. Precise definition and the existence of such a set satisfying certain axioms can be found in [KoSo1], Sections 9, 11.5.

3) The non-commutative analytic space  $X_h^{an}$  will be defined by a pair  $(X_0, \mathcal{O}_{X_0, h})$ , where  $h \in K$  is an arbitrary element satisfying the condition  $|h| < 1$ ,  $X_0$  is a topological K3 surface and  $\mathcal{O}_{X_0, h}$  is a sheaf on (a certain topology on)  $X_0$  of non-commutative noetherian algebras over the field  $K$ .

4) There is a natural continuous map  $\pi : X_0 \rightarrow S^2$  with the generic fiber being a two-dimensional torus. The sheaf  $\mathcal{O}_{X_0, h}$  is uniquely determined by its direct image  $\mathcal{O}_{S^2, h} := \pi_*(\mathcal{O}_{X_0, h})$ . Hence the construction of  $X_h$  is reduced to the construction of the sheaf  $\mathcal{O}_{S^2, h}$  on the sphere  $S^2$ .

5) The sheaf  $\mathcal{O}_{S^2, h}$  is glued from two sheaves: a sheaf  $\mathcal{O}_h^{sing}$  defined in a neighborhood  $W$  of the “singular” subset  $B^{sing}$ , and a sheaf  $\mathcal{O}_h^{nonsing}$  on  $S^2 \setminus W$ .

6) The sheaf  $\mathcal{O}_h^{sing}$  is defined by an “ansatz” described below, while the sheaf  $\mathcal{O}_h^{nonsing}$  is constructed in two steps. First, with any integral affine structure and an element  $h \in K, |h| < 1$  one can associate canonically a sheaf  $\mathcal{O}_h^{can}$  of non-commutative algebras over  $K$ . Then, with each line  $l$  one associates an automorphism  $\varphi_l$  of the restriction of  $\mathcal{O}_h^{can}$  to  $l$ . The sheaf  $\mathcal{O}_h^{nonsing}$  is obtained from the restriction of  $\mathcal{O}_h^{can}$  to the complement to the union of all lines by the gluing procedure by means of  $\varphi_l$ .

The above scheme was realized in [KoSo1] in the case  $h = 0$ . In that case one obtains a sheaf  $\mathcal{O}_{X_0, 0}$  of *commutative* algebras which is the sheaf of analytic functions on the non-archimedean analytic K3 surface. In this

section we will explain what has to be changed in [KoSo1] in order to handle the case  $h \neq 0, |h| < 1$ . As we mentioned above, in this case  $\mathcal{O}_{X_0,h}$  will be a sheaf of non-commutative algebras, which is a flat deformation of  $\mathcal{O}_{X_0,0}$ . It was observed in [KoSo1] that  $\mathcal{O}_{X_0,0}$  is a sheaf of Poisson algebras. The sheaf  $\mathcal{O}_{X_0,h}$  is a *deformation quantization* of  $\mathcal{O}_{X_0,0}$ . It is an analytic (not a formal) deformation quantization with respect to the parameter  $h$ . The topology on  $X_0$  will be clear from the construction.

## 7.2 $\mathbf{Z}$ -affine structures and the canonical sheaf

Let  $K$  be a non-archimedean valuation field. Fix an element  $h \in K, |h| < 1$ . Let us introduce invertible variables  $\xi, \eta$  such that

$$\eta\xi = \xi\eta \exp(h).$$

Then we define a sheaf  $\mathcal{O}_h^{can}$  on  $\mathbf{R}^2$  such that for any open connected subset  $U$  one has

$$\mathcal{O}_h^{can}(U) = \left\{ \sum_{n,m \in \mathbf{Z}} c_{n,m} \xi^n \eta^m \mid \forall (x,y) \in U \sup_{n,m} (\log(|c_{n,m}|) + nx + my) < \infty \right\}.$$

The above definition is motivated by the following considerations.

Recall that an integral affine structure ( $\mathbf{Z}$ -affine structure for short) on an  $n$ -dimensional topological manifold  $Y$  is given by a maximal atlas of charts such that the change of coordinates between any two charts is described by the formula

$$x'_i = \sum_{1 \leq j \leq n} a_{ij} x_j + b_i,$$

where  $(a_{ij}) \in GL(n, \mathbf{Z}), (b_i) \in \mathbf{R}^n$ . In this case one can speak about the sheaf of  $\mathbf{Z}$ -affine functions, i.e. those which can be locally expressed in affine coordinates by the formula  $f = \sum_{1 \leq i \leq n} a_i x_i + b, a_i \in \mathbf{Z}, b \in \mathbf{R}$ . Another equivalent description:  $\mathbf{Z}$ -affine structure is given by a covariant lattice  $T^{\mathbf{Z}} \subset TY$  in the tangent bundle (recall that an affine structure on  $Y$  is the same as a torsion free flat connection on the tangent bundle  $TY$ ).

Let  $Y$  be a manifold with  $\mathbf{Z}$ -affine structure. The sheaf of  $\mathbf{Z}$ -affine functions  $Aff_{\mathbf{Z}} := Aff_{\mathbf{Z},Y}$  gives rise to an exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathbf{R} \rightarrow Aff_{\mathbf{Z}} \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 0,$$

where  $(T^*)^{\mathbf{Z}}$  is the sheaf associated with the dual to the covariant lattice  $T^{\mathbf{Z}} \subset TY$ .

Let us recall the following notion introduced in [KoSo1], Section 7.1.

**Definition 4** *A  $K$ -affine structure on  $Y$  compatible with the given  $\mathbf{Z}$ -affine structure is a sheaf  $Aff_K$  of abelian groups on  $Y$ , an exact sequence of sheaves*

$$1 \rightarrow K^\times \rightarrow Aff_K \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 1,$$

together with a homomorphism  $\Phi$  of this exact sequence to the exact sequence of sheaves of abelian groups

$$0 \rightarrow \mathbf{R} \rightarrow Aff_{\mathbf{Z}} \rightarrow (T^*)^{\mathbf{Z}} \rightarrow 0,$$

such that  $\Phi = id$  on  $(T^*)^{\mathbf{Z}}$  and  $\Phi = val$  on  $K^\times$ .

Since  $Y$  carries a  $\mathbf{Z}$ -affine structure, we have the corresponding  $GL(n, \mathbf{Z}) \times \mathbf{R}^n$ -torsor on  $Y$ , whose fiber over a point  $x$  consists of all  $\mathbf{Z}$ -affine coordinate systems at  $x$ .

Then one has the following equivalent description of the notion of  $K$ -affine structure.

**Definition 5** *A  $K$ -affine structure on  $Y$  compatible with the given  $\mathbf{Z}$ -affine structure is a  $GL(n, \mathbf{Z}) \times (K^\times)^n$ -torsor on  $Y$  such that the application of  $val^{\times n}$  to  $(K^\times)^n$  gives the initial  $GL(n, \mathbf{Z}) \times \mathbf{R}^n$ -torsor.*

Assume that  $Y$  is oriented and carries a  $K$ -affine structure compatible with a given  $\mathbf{Z}$ -affine structure. Orientation allows us to reduce to  $SL(n, \mathbf{Z}) \times (K^\times)^n$  the structure group of the torsor defining the  $K$ -affine structure. One can define a higher-dimensional version of the sheaf  $\mathcal{O}_h^{can}$  in the following way. Let  $z_1, \dots, z_n$  be invertible variables such that  $z_i z_j = z_j z_i \exp(h)$ , for all  $1 \leq i < j \leq n$ . We define the sheaf  $\mathcal{O}_h^{can}$  on  $\mathbf{R}^n$ ,  $n \geq 2$  by the same formulas as in the case  $n = 2$ :

$$\mathcal{O}_h^{can}(U) = \left\{ \sum_{I=(I_1, \dots, I_n) \in \mathbf{Z}^n} c_I z^I, \mid \forall (x_1, \dots, x_n) \in U \sup_I \left( \log(|c_I|) + \sum_{1 \leq m \leq n} I_m x_m \right) < \infty \right\},$$

where  $z^I = z_1^{I_1} \dots z_n^{I_n}$ . Since  $|\exp(h)| = 1$  the convergency condition does not depend on the order of variables.

The sheaf  $\mathcal{O}_h^{can}$  can be lifted to  $Y$  (we keep the same notation for the lifting). In order to do that it suffices to define the action of the group  $SL(n, \mathbf{Z}) \times (K^\times)^n$  on the canonical sheaf on  $\mathbf{R}^n$ . Namely, the inverse to an element  $(A, \lambda_1, \dots, \lambda_n) \in SL(n, \mathbf{Z}) \times (K^\times)^n$  acts on monomials as

$$z^I = z_1^{I_1} \dots z_n^{I_n} \mapsto \left( \prod_{i=1}^n \lambda_i^{I_i} \right) z^{A(I)} .$$

The action of the same element on  $\mathbf{R}^n$  is given by a similar formula:

$$x = (x_1, \dots, x_n) \mapsto A(x) - (val(\lambda_1), \dots, val(\lambda_n)) .$$

Notice that the stalk of the sheaf  $\mathcal{O}_h^{can}$  over a point  $y \in Y$  is isomorphic to a direct limit of algebras of functions on quantum analytic tori of various multiradii.

Let  $Y = \cup_\alpha U_\alpha$  be an open covering by coordinate charts  $U_\alpha \simeq V_\alpha \subset \mathbf{R}^n$  such that for any  $\alpha, \beta$  we are given elements  $g_{\alpha, \beta} \in SL(n, \mathbf{Z}) \times (K^\times)^n$  satisfying the 1-cocycle condition for any triple  $\alpha, \beta, \gamma$ . Then the lifting of  $\mathcal{O}_h^{can}$  to  $Y$  is obtained via gluing by means of the transformations  $g_{\alpha, \beta}$ .

Let  $G_m^{an}$  be the analytic space corresponding to the multiplicative group  $G_m$  and  $(T^n)^{an} := (G_m^{an})^n$  the  $n$ -dimensional analytic torus. Then one has a canonically defined continuous map  $\pi_{can} : (T^n)^{an} \rightarrow \mathbf{R}^n$  such that  $\pi_{can}(p) = (-val_p(z_1), \dots, -val_p(z_n)) = (log|z_1|_p, \dots, log|z_n|_p)$ , where  $|a|_p$  (resp.  $val_p(a)$ ) denotes the seminorm (resp. valuation) of an element  $a$  corresponding to the point  $p$ .

For an open subset  $U \in \mathbf{R}^n$  we have a topological  $K$ -algebra  $\mathcal{O}_h^{can}(U)$  defined by the formulas above. Notice that a point  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$  defines a multiplicative seminorm  $|\sum_I c_I z^I|_{exp(x)}$  on the algebra of formal series of  $q$ -commuting variables  $z_1, \dots, z_n$  (here  $exp(x) = (exp(x_1), \dots, exp(x_n))$ ).

Let  $M(\mathcal{O}_h^{can}(U))$  be the set of multiplicative seminorms  $\nu$  on  $\mathcal{O}_h^{can}(U)$  extending the norm on  $K$ . We have defined an embedding  $U \rightarrow M(\mathcal{O}_h^{can}(U))$ , such that  $(x_1, \dots, x_n)$  corresponds to a seminorm with  $|z_i| = exp(x_i)$ ,  $1 \leq i \leq n$ . The map  $\pi_{can} : |\bullet| \mapsto (log|z_1|, \dots, log|z_n|)$  is a retraction of  $M(\mathcal{O}_h^{can}(U))$  to the image of  $U$ .

Let  $S_1^n \subset (K^\times)^n$  be the set of such  $(s_1, \dots, s_n)$  that  $|s_i| = 1$ ,  $1 \leq i \leq n$ . The group  $S_1^n$  acts on  $\mathcal{O}_h^{can}(U)$  in such a way that  $z_i \mapsto s_i z_i$ . Clearly the map  $\pi_{can}$  is  $S_1^n$ -invariant. For this reason we will call  $\pi_{can}$  a *quantum analytic torus fibration* over  $U$ . More precisely, we reserve this name for a pair  $(U, \mathcal{O}_h^{can}(U))$ , where the algebra is equipped with the  $S_1^n$ -action. We suggest to think about such a pair as of the algebra  $\mathcal{O}_h(\pi_{can}^{-1}(U))$  of analytic functions on the open

subset  $\pi_{can}^{-1}(U)$  of the non-commutative analytic torus  $(T_h^n(\mathbf{Z}^n)^{an}, \varphi_0)$ , where  $\varphi_0((a_1, \dots, a_n), (b_1, \dots, b_n)) = a_1 b_1 + \dots + a_n b_n$ .

We can make a category of the above pairs, defining a morphism  $(U, \mathcal{O}_h^{can}(U)) \rightarrow (V, \mathcal{O}_h^{can}(V))$  as a pair  $(f, \phi)$  where  $f : U \rightarrow V$  is a continuous map and  $\phi : \mathcal{O}_h^{can}(f^{-1}(V)) \rightarrow \mathcal{O}_h^{can}(V)$  is a  $S_1^n$ -equivariant homomorphism of algebras. In particular we have the notion of isomorphism of quantum analytic torus fibrations.

Let  $U \subset \mathbf{R}^n$  be an open set and  $A$  be a non-commutative affinoid  $K$ -algebra equipped with a  $S_1^n$ -action. We say that a pair  $(U, A)$  defines a quantum analytic torus fibration over  $U$  if it is isomorphic to the pair  $(U, \mathcal{O}_h^{can}(U))$ . Notice that morphisms of quantum analytic torus fibrations are compatible with the restrictions on the open subsets. Therefore we can introduce a topology on  $(T^n)^{an}$  taking  $\pi_{can}^{-1}(U), U \subset \mathbf{R}^2$  as open subsets, and make a ringed space assigning the algebra  $\mathcal{O}_h(U)$  to the open set  $\pi_{can}^{-1}(U)$ . We will denote the sheaf by  $(\pi_{can})_*(\mathcal{O}_{(T^n)^{an}, h})$ . Its global sections (for  $U = \mathbf{R}^n$ ) coincides with the projective limit of algebras of analytic functions on quantum tori of all possible multiradii. Slightly abusing the terminology we will call the above ringed space a quantum analytic torus.

**Remark 2** *The above definition is a toy-model of a  $q$ -deformation of the natural retraction  $X^{an} \rightarrow Sk(X)$  of the analytic space  $X^{an}$  associated with the maximally degenerate Calabi-Yau manifold  $X$  onto its skeleton  $Sk(X)$  (see [KoSo1]). Analytic torus fibrations introduced in [KoSo1] are “rigid analytic” analogs of Lagrangian torus fibrations in symplectic geometry. Moreover, the mirror symmetry functor (or rather its incarnation as a Fourier-Mukai transform) interchanges these two types of torus fibrations for mirror dual Calabi-Yau manifolds.*

It turns out that the sheaf  $\mathcal{O}_h^{can}$  is not good for construction of a non-commutative analytic K3 surface. We will explain later how it should be modified. Main reason for the complicated modification procedure comes from Homological Mirror Symmetry, as explained in [KoSo1]. In a few words, the derived category of coherent sheaves on a non-commutative analytic K3 surface should be equivalent to a certain deformation of the Fukaya category of the mirror dual K3 surface. If the K3 surface is realized as an elliptic fibration over  $\mathbf{CP}^1$  then there are fibers (they are 2-dimensional Lagrangian tori) which contain boundaries of holomorphic discs. Those discs give rise to an infinite set of lines on the base of the fibration. In order to have the above-

mentioned categorical equivalence one should modify the canonical sheaf for each line.

### 7.3 Model near a singular point

Let us fix  $h \in K, |h| < 1$ .

We start with the open covering of  $\mathbf{R}^2$  by the following sets  $U_i, 1 \leq i \leq 3$ . Let us fix a number  $0 < \varepsilon < 1$  and define

$$\begin{aligned} U_1 &= \{(x, y) \in \mathbf{R}^2 \mid x < \varepsilon|y|\} \\ U_2 &= \{(x, y) \in \mathbf{R}^2 \mid x > 0, y < \varepsilon x\} \\ U_3 &= \{(x, y) \in \mathbf{R}^2 \mid x > 0, y > 0\} \end{aligned}$$

Clearly  $\mathbf{R}^2 \setminus \{(0, 0)\} = U_1 \cup U_2 \cup U_3$ . We will also need a slightly modified domain  $U'_2 \subset U_2$  defined as  $\{(x, y) \in \mathbf{R}^2 \mid x > 0, y < \frac{\varepsilon}{1+\varepsilon}x\}$ .

Recall that one has a canonical map  $\pi_{can} : (T_h^2)^{an} \rightarrow \mathbf{R}^2$ .

We define  $T_i := \pi_{can}^{-1}(U_i), i = 1, 3$  and  $T_2 := \pi_{can}^{-1}(U'_2)$  (see the explanation below). Then the projections  $\pi_i : T_i \rightarrow U_i$  are given by the formulas

$$\begin{aligned} \pi_i(\xi_i, \eta_i) &= \pi_{can}(\xi_i, \eta_i) = (\log|\xi_i|, \log|\eta_i|), \quad i = 1, 3 \\ \pi_2(\xi_2, \eta_2) &= \begin{cases} (\log|\xi_2|, \log|\eta_2|) & \text{if } |\eta_2| < 1 \\ (\log|\xi_2| - \log|\eta_2|, \log|\eta_2|) & \text{if } |\eta_2| \geq 1 \end{cases} \end{aligned}$$

In these formulas  $(\xi_i, \eta_i)$  are coordinates on  $T_i, 1 \leq i \leq 3$ . More pedantically, one should say that for each  $T_i$  we are given an algebra  $\mathcal{O}_h(T_i)$  of series  $\sum_{m,n} c_{mn} \xi_i^m \eta_i^n$  such that  $\xi_i \eta_i = q \eta_i \xi_i, q = \exp(h)$ , and for a seminorm  $|\bullet|$  corresponding to a point of  $T_i$  (which means that  $(\log|\xi_i|, \log|\eta_i|) \in U_i$ ) one has:  $\sup_{m,n} (m \log|\xi_i| + n \log|\eta_i|) < +\infty$ . In this way we obtain a sheaf of non-commutative algebras on the set  $U_i$ , which is the subset of the Berkovich spectrum of the algebra  $\mathcal{O}_h(T_i)$ . We will denote this sheaf by  $\mathcal{O}_{T_i, h}$ .

Let us introduce the sheaf  $\mathcal{O}_h^{can}$  on  $\mathbf{R}^2 \setminus \{(0, 0)\}$ . It is defined as  $(\pi_i)_*(\mathcal{O}_{T_i, h})$  on each domain  $U_i$ , with identifications

$$\begin{aligned} (\xi_1, \eta_1) &= (\xi_2, \eta_2) & \text{on } U_1 \cap U_2 \\ (\xi_1, \eta_1) &= (\xi_3, \eta_3) & \text{on } U_1 \cap U_3 \\ (\xi_2, \eta_2) &= (\xi_3 \eta_3, \eta_3) & \text{on } U_2 \cap U_3 \end{aligned}$$

Let us introduce the sheaf  $\mathcal{O}_h^{sing}$  on  $\mathbf{R}^2 \setminus \{(0, 0)\}$ . On the sets  $U_1$  and  $U_2 \cup U_3$  this sheaf is isomorphic to  $\mathcal{O}_h^{can}$  (by identifying of coordinates  $(\xi_1, \eta_1)$



and of glued coordinates  $(\xi_2, \eta_2)$  and  $(\xi_3, \eta_3)$  respectively). On the intersection  $U_1 \cap (U_2 \cup U_3)$  we identify two copies of the canonical sheaf by an automorphism  $\varphi$  of  $\mathcal{O}^{can}$ . More precisely, the automorphism is given (we skip the index of the coordinates) by

$$\varphi(\xi, \eta) = \begin{cases} (\xi(1 + \eta), \eta) & \text{on } U_1 \cap U_2 \\ (\xi(1 + 1/\eta), \eta) & \text{on } U_1 \cap U_3 \end{cases}$$

## 7.4 Lines and automorphisms

We refer the reader to [KoSo1] for the precise definition of the set of lines and axioms this set is required to obey. Roughly speaking, for a manifold  $Y$  which carries a  $\mathbf{Z}$ -affine structure a line  $l$  is defined by a continuous map  $f_l : (0, +\infty) \rightarrow Y$  and a covariantly constant nowhere vanishing integer-valued 1-form  $\alpha_l \in \Gamma((0, +\infty), f_l^*((T^*)^{\mathbf{Z}}))$ . A set  $\mathcal{L}$  of lines is required to be decomposed into a disjoint union  $\mathcal{L} = \mathcal{L}_{in} \cup \mathcal{L}_{com}$  of *initial* and *composite* lines. Each composite line is obtained as a result of a finite number of “collisions” of initial lines. A collision is described by a  $Y$ -shape figure, where the leg of  $Y$  is a composite line, while two other segments are “parents” of the leg. A construction of the set  $\mathcal{L}$  satisfying the axioms from [KoSo1] was proposed in [KoSo1], Section 9.3.

With each line  $l$  we can associate a continuous family of automorphisms of stalks of sheaves of algebras  $\varphi_l(t) : (\mathcal{O}_h^{can})_{Y, f_l(t)} \rightarrow (\mathcal{O}_h^{can})_{Y, f_l(t)}$ .

Automorphisms  $\varphi_l$  can be defined in the following way (see [KoSo1], Section 10.4).

First we choose affine coordinates in a neighborhood of a point  $b \in B^{nonsing} \setminus B^{sing}$ , identifying  $b$  with the point  $(0, 0) \in \mathbf{R}^2$ . Let  $l = l_+ \in \mathcal{L}_{in}$  be (in the standard affine coordinates) a line in the half-plane  $y > 0$  emerging from  $(0, 0)$  (there is another such line  $l_-$  in the half-plane  $y < 0$ , see [KoSo1] for the details). Assume that  $t$  is sufficiently small. Then we define  $\varphi_l(t)$  on topological generators  $\xi, \eta$  by the formula

$$\varphi_l(t)(\xi, \eta) = (\xi(1 + 1/\eta), \eta).$$

In order to extend  $\varphi_l(t)$  to the interval  $(0, t_0)$ , where  $t_0$  is not small, we cover the corresponding segment of  $l$  by open charts. Then a change of affine coordinates transforms  $\eta$  into a monomial multiplied by a constant from  $K^\times$ . Moreover, one can choose the change of coordinates in such a way that  $\eta \mapsto C\eta$  where  $C \in K^\times, |C| < 1$  (such change of coordinates preserve the

1-form  $dy$ . Constant  $C$  is equal to  $\exp(-L)$ , where  $L$  is the length of the segment of  $l$  between two points in different coordinate charts). Therefore  $\eta$  extends analytically in a unique way to an element of  $\Gamma((0, +\infty), f_l^*((\mathcal{O}_h^{can})^\times))$ . Moreover the norm  $|\eta|$  strictly decreases as  $t$  increases, and remains strictly smaller than 1. Similarly to [KoSo1], Section 10.4 one deduces that  $\varphi_l(t)$  can be extended for all  $t > 0$ . This defines  $\varphi_l(t)$  for  $l \in \mathcal{L}_{in}$ .

Next step is to extend  $\varphi_l(t)$  to the case when  $l \in \mathcal{L}_{com}$ , i.e. to the case when the line is obtained as a result of a collision of two lines belonging to  $\mathcal{L}_{in}$ . Following [KoSo1], Section 10, we introduce a group  $G$  which contains all the automorphisms  $\varphi_l(t)$ , and then prove the factorization theorem (see [KoSo1], Theorem 6) which allows us to define  $\varphi_l(0)$  in the case when  $l$  is obtained as a result of a collision of two lines  $l_1$  and  $l_2$ . Then we extend  $\varphi_l(t)$  analytically for all  $t > 0$  similarly to the case  $l \in \mathcal{L}_{in}$ .

More precisely, the construction of  $G$  goes such as follows. Let  $(x_0, y_0) \in \mathbf{R}^2$  be a point,  $\alpha_1, \alpha_2 \in (\mathbf{Z}^2)^*$  be 1-covectors such that  $\alpha_1 \wedge \alpha_2 > 0$ . Denote by  $V = V_{(x_0, y_0), \alpha_1, \alpha_2}$  the closed angle

$$\{(x, y) \in \mathbf{R}^2 \mid \langle \alpha_i, (x, y) - (x_0, y_0) \rangle \geq 0, i = 1, 2\}$$

Let  $\mathcal{O}_h(V)$  be a  $K$ -algebra consisting of series  $f = \sum_{n, m \in \mathbf{Z}} c_{n, m} \xi^n \eta^m$ , such that  $\xi\eta = q\eta\xi$  and  $c_{n, m} \in K$  satisfy the condition that for all  $(x, y) \in V$  we have:

1. if  $c_{n, m} \neq 0$  then  $\langle (n, m), (x, y) - (x_0, y_0) \rangle \leq 0$ , where we identified  $(n, m) \in \mathbf{Z}^2$  with a covector in  $(T_p^*Y)^{\mathbf{Z}}$ ;
2.  $\log |c_{n, m}| + nx + my \rightarrow -\infty$  as long as  $|n| + |m| \rightarrow +\infty$ .

For an integer covector  $\mu = adx + bdy \in (\mathbf{Z}^2)^*$  we denote by  $R_\mu$  the monomial  $\xi^a \eta^b$ . Then we consider a pro-unipotent group  $G := G(q, \alpha_1, \alpha_2, V)$  of automorphisms of  $\mathcal{O}_h(V)$  having the form

$$g = \sum_{n_1, n_2 \geq 0, n_1 + n_2 > 0} c_{n_1, n_2} R_{\alpha_1}^{-n_1} R_{\alpha_2}^{-n_2}$$

where

$$\log |c_{n, m}| - n_1 \langle \alpha_1, (x, y) \rangle - n_2 \langle \alpha_2, (x, y) \rangle \leq 0 \quad \forall (x, y) \in V$$

The latter condition is equivalent to  $\log |c_{n, m}| - \langle n_1 \alpha_1 + n_2 \alpha_2, (x_0, y_0) \rangle \leq 0$ .

Fixing the ratio  $\lambda = n_2/n_1 \in [0, +\infty]_{\mathbf{Q}} := \mathbf{Q}_{\geq 0} \cup \infty$  we obtain a subgroup  $G_\lambda := G_\lambda(q, \alpha_1, \alpha_2, V) \subset G$ . There is a natural map  $\prod_\lambda G_\lambda \rightarrow G$ , defined as in [KoSo1], Section 10.2. The above-mentioned factorization theorem states that this map is a bijection of sets.

Let us now assume that lines  $l_1$  and  $l_2$  collide at  $p = f_{l_1}(t_1) = f_{l_2}(t_2)$ , generating the line  $l \in \mathcal{L}_{com}$ . Then  $\varphi_l(0)$  is defined with the help of factorization theorem. More precisely, we set  $\alpha_i := \alpha_{l_i}(t_i)$ ,  $i = 1, 2$  and the angle  $V$  is the intersection of certain half-planes  $P_{l_1, t_1} \cap P_{l_2, t_2}$  defined in [KoSo1], Section 10.3. The half-plane  $P_{l, t}$  is contained in the region of convergence of  $\varphi_l(t)$ . By construction, the elements  $g_0 := \varphi_{l_1}(t_1)$  and  $g_{+\infty} := \varphi_{l_2}(t_2)$  belong respectively to  $G_0$  and  $G_{+\infty}$ . Then we have:

$$g_{+\infty}g_0 = \prod_{\rightarrow} ((g_\lambda)_{\lambda \in [0, +\infty]_{\mathbf{Q}}}) = g_0 \cdots g_{1/2} \cdots g_1 \cdots g_{+\infty}.$$

Each term  $g_\lambda$  with  $0 < \lambda = n_1/n_2 < +\infty$  corresponds to the newborn line  $l$  with the direction covector  $n_1\alpha_{l_1}(t_1) + n_2\alpha_{l_2}(t_2)$ . Then we set  $\varphi_l(0) := g_\lambda$ . This transformation is defined by a series which is convergent in a neighborhood of  $p$ , and using the analytic continuation we obtain  $\varphi_l(t)$  for  $t > 0$ , as we said above. Recall that every line carries an integer 1-form  $\alpha_l = adx + bdy$ . By construction,  $\varphi_l(t) \in G_\lambda$ , where  $\lambda$  is the slope of  $\alpha_l$ .

Having automorphisms  $\varphi_l$  assigned to lines  $l \in \mathcal{L}$  we proceed as in [KoSo1], Section 11, modifying the sheaf  $\mathcal{O}_h^{can}$  along each line. We denote the resulting sheaf  $\mathcal{O}_h^{nonsing}$ . By construction it is isomorphic to the sheaf  $\mathcal{O}_h^{sing}$  in a neighborhood of the point  $(0, 0)$ .

Let us now consider the manifold  $Y = S^2 \setminus B^{sing}$ , i.e. the complement of 24 points on the sphere  $S^2$  equipped with the  $\mathbf{Z}$ -affine structure, which has standard singularity at each point  $x_i \in B^{sing}$ ,  $1 \leq i \leq 24$  (see Section 7.1). Using the above construction (with any choice of set of lines on  $S^2$ ) we define the sheaf  $\mathcal{O}_{S^2, h}^{nonsing}$  on  $Y$ . Notice that in a small neighborhood of each singular point  $x_i$  the sheaf  $\mathcal{O}_{S^2, h}^{nonsing}$  is isomorphic to the sheaf  $\mathcal{O}_h^{sing}$  (in fact they become isomorphic after identification of the punctured neighborhood of  $x_i$  with the punctured neighborhood of  $(0, 0) \in \mathbf{R}^2$  equipped with the standard singular  $\mathbf{Z}$ -affine structure (see Section 7.1 and [KoSo1], Section 6.4 for the description of the latter). In the next subsection we will give an alternative description of the sheaf  $\mathcal{O}_h^{sing}$ . It follows from that description that  $\mathcal{O}_h^{sing}$  can be extended to the point  $(0, 0)$ . It gives a sheaf  $\mathcal{O}_{S^2, h}^{sing}$  in the neighborhood of  $B^{sing}$ . As a result we will obtain the sheaf  $\mathcal{O}_{S^2, h}$  of non-commutative  $K$ -algebras on the whole sphere  $S^2$  such that it is isomorphic to  $\mathcal{O}_{S^2, h}^{nonsing}$  on the

complement of  $B^{sing}$  and isomorphic to  $\mathcal{O}_{S^2, h}^{sing}$  in a neighborhood of  $B^{sing}$ .

## 7.5 About the sheaf $\mathcal{O}_h^{sing}$

We need to check that the sheaf  $\mathcal{O}_{S^2, h}$  is a flat deformation of the sheaf  $\mathcal{O}_{S^2}$  constructed in [KoSo1]. For the sheaf  $\mathcal{O}_{S^2, h}^{nonsing}$  this follows from the construction. Indeed, the algebra of analytic functions on the quantum analytic torus (of any multiradius) is a flat deformation of the algebra of analytic functions on the corresponding ‘‘commutative’’ torus, equipped with the Poisson bracket  $\{x, y\} = xy$ . The group  $G$  described in the previous subsection is a flat deformation of its ‘‘commutative’’ limit  $G_0$  defined in [KoSo1], Section 10. The group  $G_0$  preserves the above Poisson bracket.

In order to complete the construction of the non-commutative space analytic K3 surface  $X_h$  we need to investigate the sheaf  $\mathcal{O}_h^{sing}$  and prove that it is a flat deformation with respect to  $h$  of the sheaf  $\mathcal{O}^{model}$  introduced in [KoSo1], Section 8. First we recall the definition of the latter.

Let  $S \subset \mathbf{A}^3$  be an algebraic surface given by equation  $(\alpha\beta - 1)\gamma = 1$  in coordinates  $(\alpha, \beta, \gamma)$ , and  $S^{an}$  be the corresponding analytic space. We define a continuous map  $f : S^{an} \rightarrow \mathbf{R}^3$  by the formula  $f(\alpha, \beta, \gamma) = (a, b, c)$  where  $a = \max(0, \log |\alpha|_p)$ ,  $b = \max(0, \log |\beta|_p)$ ,  $c = \log |\gamma|_p = -\log |\alpha\beta - 1|_p$ . Here  $|\cdot|_p = \exp(-val_p(\cdot))$  denotes the multiplicative seminorm corresponding to the point  $p \in S^{an}$ .

Let us consider the embedding  $j : \mathbf{R}^2 \rightarrow \mathbf{R}^3$  given by formula

$$j(x, y) = \begin{cases} (-x, \max(x + y, 0), -y) & \text{if } x \leq 0 \\ (0, x + \max(y, 0), -y) & \text{if } x \geq 0 \end{cases}$$

One can easily check that the image of  $j$  coincides with the image of  $f$ . Let us denote by  $\pi : S^{an} \rightarrow \mathbf{R}^2$  the map  $j^{(-1)} \circ f$ . Finally, we denote  $\pi_*(\mathcal{O}_{S^{an}})$  by  $\mathcal{O}^{model} := \mathcal{O}_{\mathbf{R}^2}^{model}$ . It was shown in [KoSo1], Section 8, that  $\mathcal{O}^{model}$  is canonically isomorphic to the sheaf  $\mathcal{O}_{h=0}^{sing}$  (the latter is defined as a modification of the sheaf  $\mathcal{O}_{h=0}^{can}$  by means of the automorphism  $\varphi$ , given by the formula at the end of Section 7.3 for commuting variables  $\xi$  and  $\eta$ ).

Let us consider a non-commutative  $K$ -algebra  $A_h(S)$  generated by generators  $\alpha, \beta, \gamma$  subject to the following relations for  $q = \exp(h)$ :

$$\alpha\gamma = q\gamma\alpha, \beta\gamma = q\gamma\beta,$$

$$\beta\alpha - q\alpha\beta = 1 - q,$$

$$(\alpha\beta - 1)\gamma = 1.$$

For  $q = 1$  this algebra coincides with the algebra of regular functions on the surface  $X \subset \mathbf{A}_K^3$  given by the equation  $(\alpha\beta - 1)\gamma = 1$  and moreover, it is a flat deformation of the latter with respect to the parameter  $q$  (or  $h$ ).

Recall that in Section 7.3 we defined three open subsets  $T_i, 1 \leq i \leq 3$  of the two-dimensional quantum analytic torus  $(T^2)^{an}$ . The subset  $T_i$  is defined as a ringed space  $(\pi_i^{-1}(U_i), \mathcal{O}_{T_i, h})$ , where  $U_i$  are open subsets of  $\mathbf{R}^2$  and  $\mathcal{O}_{T_i, h}$  is a sheaf of non-commutative algebras, uniquely determined by the  $K$ -algebra  $\mathcal{O}_h(T_i)$  of its global sections.

We define morphisms  $g_i : T_i \hookrightarrow S, 1 \leq i \leq 3$  by the following formulas

$$\begin{aligned} g_1(\xi_1, \eta_1) &= (\xi_1^{-1}, \xi_1(1 + \eta_1), \eta_1^{-1}) \\ g_2(\xi_2, \eta_2) &= ((1 + \eta_2)\xi_2^{-1}, \xi_2, \eta_2^{-1}) \\ g_3(\xi_3, \eta_3) &= ((1 + \eta_3)(\xi_3\eta_3)^{-1}, \xi_3\eta_3, (\eta_3)^{-1}) \end{aligned}$$

Pedantically speaking this means that for each  $1 \leq i \leq 3$  we have a homomorphism of  $K$ -algebras  $A_h(S) \rightarrow \mathcal{O}_h(T_i)$  such that  $\alpha$  is mapped to the first coordinate of  $g_i$ ,  $\beta$  is mapped to the second coordinate and  $\gamma$  is mapped to the third coordinate. One checks directly that three coordinates obey the relations between  $\alpha, \beta, \gamma$ . Modulo  $(q - 1)$  these morphisms are inclusions. In the non-commutative case they induce embeddings of  $M(\mathcal{O}_h(T_i)), 1 \leq i \leq 3$  into the set  $M(A_h(S))$  of multiplicative seminorms on  $A_h(S)$ .

Notice that in the commutative case we have:  $j \circ \pi_i = f \circ g_i$  and  $f^{-1}(j(U_i)) = g_i(T_i)$  for all  $1 \leq i \leq 3$ . Using this observation we can decompose a neighborhood  $V$  of  $\pi^{-1}(0, 0)$  in  $S^{an}$  into three open analytic subspaces and describe explicitly algebras of analytic functions as series in coordinates  $(\alpha, \beta)$  or  $(\beta, \gamma)$  or  $(\alpha, \gamma)$  (choice of the coordinates depend on the domain) with certain growth conditions on the coefficients of the series. This gives explicit description of the algebra  $\pi_*(\mathcal{O}_{S^{an}}(\pi(V)))$ . Then we *declare* the same description in the non-commutative case to be the answer for the direct image. Non-commutativity does not affect the convergency condition because  $|q| = 1$ . This description, perhaps, can be obtained from the “general theory” which will be developed elsewhere. The direct check, as in the commutative case, shows the compatibility of this description of the direct image sheaf with the description of  $\mathcal{O}_h^{nonsing}$  in the neighborhood of  $(0, 0)$ . Therefore we can glue both sheaves together, obtaining  $\mathcal{O}_{S^2, h}$ . This concludes the construction.

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