Quaternion Landau-Ginsburg models and noncommutative Frobenius manifolds

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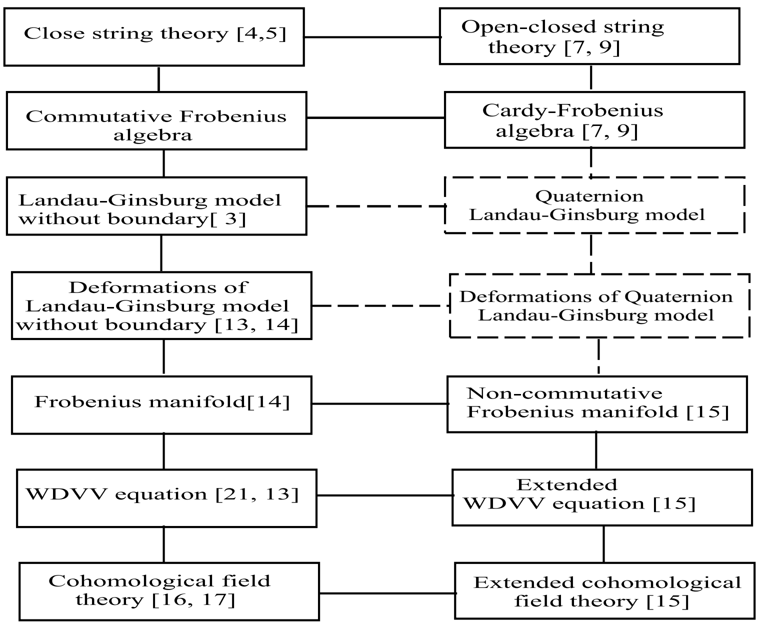
ABSTRACT. We extend topological Landau-Ginsburg models with boundaries to Quaternion Landau-Ginsburg models that satisfy the axioms for open-closed topological field theories. Later we prove that moduli spaces of Quaternion Landau-Ginsburg models are non-commutative Frobenius manifolds in means of [J. Geom. Phys, 51 (2003),387-403.].

1. Introduction

In this paper we construct the dotted cells and dotted lines in the next scheme:



Write this scheme more detailed.

Topological phases of classical Landau-Ginsburg models were found by Vafa [3]. The same mathematical construction appears also from some other mathematical-physical models (Dijkgraaf- Witten [1], Igusa-Yang [2]). In its simplest form (models without boundary), the topological Landau-Ginsburg models generate on the tangent space A_p to a polynomial $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ (in the space $Pol(n)$ of all such polynomials) some associative algebra A_p over $\mathbb{C}$ with unity and a linear functional $l_p : A_p \to \mathbb{C}$ such that bilinear form $(d_1, d_2) = l_p(d_1 d_2)$ is non degenerates. We call such pair by *Frobenius pairs*. Moreover, A_p is a commutative algebra.

The commutative Frobenius pairs one-to-one correspond to the topological field theories of the closed topological strings [4,5,14,6,9]. These topological field theories naturally extend up to open-closed topological field theories, describing strings with boundary [7, 8], and even up to Kleinian topological field theories, describing strings with arbitrary world surfaces [9]. In their turn, open-closed topological field theories one-to-one correspond to sums of commutative and non commutative Frobenius pairs with involutions, which connected by a Cardy condition [9]. In the present paper we call these structures by Cardi-Frobenius algebras. This class of algebras includes, for instance, the algebra of Hurwitz numbers of real algebraic curves [9].

Topological Landau-Ginsburg models without boundary also can be extended up to models with boundary [10, 11, 12, 23, 24, 25]. This extension is an analog of extension of closed topological field theories to open-closed owns. Thus it is natural to expect that topological Landau-Ginsburg models with boundary gives Cardi-Frobenius algebra. However, it is not the case for the extensions, suggested in [10, 11, 12, 23, 24, 25]. In the present paper we construct a "superisation" of the models [10, 11, 12, 23, 24, 25] and receive a new model that gives Cardy-Frobenius algebras. Our model can be interpreted also as "quaternionization" of the topological Landau-Ginsburg models without boundary. Thus we call it *Quaternion Landau-Ginsburg models.*

Further on, we prove that the family of all the Quaternion Landau-Ginsburg models over all polynomials $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ is a non-commutative Frobenius manifold. Let us explain this result in more detail.

The moduli space of the closed Landau-Ginsburg models coincides with the space $Pol(n)$ of the polynomials of type $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$. It should be noted, that $Pol(n)$ coincides also with the space of miniversale deformation for the singularity of type A_n [22].

A remarkable feature of the Landau-Ginsburg models is that the metrics of algebras A_p turn the space $Pol(n)$ into a Riemannian manifold with some additional properties [13,14]. The differential-geometric structure, arising here, is an important example of Frobenius manifolds

[14,17]. The theory of Frobenius manifolds has important applications, and joins various, at first glance rather different areas of mathematics (integrable systems, singularity theory, topology of symplectic manifolds, geometry of moduli spaces of algebraic curves etc.).

The Dubrovin's theory of Frobenius manifolds is built as a theory of flat deformations of commutative Frobenius algebras. As we discussed, the class of such algebras extends up to the class of Cardy-Frobenius algebras. This suggests an idea on an extension of Frobenius manifolds up to manifolds of special deformations of Cardy-Frobenius algebras. An approach to this problem has been elaborated in [15]. It is based on the following.

Frobenius manifold is determined by its potential F, that is a function on $A \cong \mathbb{R}^n$, whose third derivatives are the structure constants of the algebras, corresponding to points $r \in A$. Properties of the manifold are then encoded by differential equations (WDVV) on the potential. If the potential F develops in a Taylor series, then the WDVV equations can be represented as relations of the coefficients of the series. Kontsevich and Manin [16] found that the appropriate collections of coefficients are in a bijective correspondence with special systems $\{A^{\otimes r} \to H^*(\overline{M}_{0,r})\}$ of homomorphisms from the spaces $A^{\otimes r}$ into the cohomology groups of compactified spaces of the r - punctured Riemannian spheres.

In [15] such a system of homomorphisms is extended up to special homomorphisms from the spaces $A^{\otimes r} \otimes B^{\otimes \ell}$ into some vector spaces, constructed by means of subdivisions of punctured spheres and discs. These collections of homomorphisms one-to-one correspond to tensor series, satisfying "non-commutative" differential equations (further on referred to as *Extended WDVV equations*). Moreover, the third derivatives of such a tensor series F are the structure constants of Cardy-Frobenius algebras. Therefore, the solution to the Extended WDVV equations is naturally viewed as a potential of "non-commutative Frobenius manifolds".

In this paper we prove that *the constructed family of Quaternion Landau-Ginsburg models has a potential, satisfying the Extended WDVV equations.*

A structure of the paper is following.

In section 2, by combination of [7] and [9], we define Cardy-Frobenius algebras with graduation and involution, that in essence are equivalent to algebras of open-closed topological string theories. We construct also some importent for us examples of Cardy-Frobenius algebras.

In section 3, we describe various Landau-Ginsburg models corresponding to polynomials $p(z)$, and, in particular, our Quaternion models.

In section 4, we remain different definitions of (commutative) Frobenius manifolds and prove, following to [14], that the moduli space of the topological Landau-Ginsburg models without boundary form a (commutative) Frobenius manifold.

At last, in section 5, we construct the moduli space of the Quaternion Landau-Ginsburg models and prove that it is a non-commutative Frobenius manifold of some special type.

2. Cardy-Frobenius algebras.

We start with definitions of algebraic structures, that is important for this paper. Let $\mathbb{K}$ be a field.

2.1.Frobenius pairs.

Definition. By *Frobenius pair (with graduation and involution)* over $\mathbb{K}$ we call a set $(D, l^D, *_D)$ where

1. $D = D^0 + D^1$ is an $\mathbb{Z}_2$ graduated associative algebra over $\mathbb{K}$ with unity element 1^D. Put $|d| = i$ for $D \in D^i$.

2. $l^D : D \to \mathbb{K}$ is a $\mathbb{K}$ - linear functional such that bilinear form $(d_1, d_2)^D = l^D(d_1 d_2)$ is non degenerates on D, and $(d_2, d_1)^D = (-1)^{|d_1||d_2|}(d_1, d_2)^D$.

3. $*_D : D \to D$ is an involutive antiautomorphisme of algebras such that $(D^i)^* = D^i$, where $d^* = *_D(d)$, and $l^D(d^*) = l^D(d)$.

In this case $D = D^0$ is a Frobenius algebra [18]. A Frobenius pair is called *semisimple*, if D is a semisimple algebra.

Definition. By *orthogonal sum* of Frobenius pairs $(D_1, l^{D_1}, *_{D_1})$ and $(D_2, l^{D_2}, *_{D_2})$ we call the Frobenius pair $(D, l^D, *_D) = (D_1, l^{D_1}, *_{D_1}) \oplus (D_2, l^{D_2}, *_{D_2})$, where $D = D_1 \oplus D_2$ is the direct sum of algebras ($d_1 d_2 = 0$ for $d_i \in D_i$), $l^D = l^{D_1} \oplus l^{D_2}$, and $*_D|_{D_i} = *_{D_i}$.

For identical $*_D$ we put $(D, l^D) = (D, l^D, *_D)$.

Exampele 2.1. $\mathbb{K}$ - numbers.

$\mathbb{K}(\lambda) = (\mathbb{K}, l_\lambda)$, where $\lambda \neq 0 \in \mathbb{K}$ and $l_\lambda(z) = \lambda z$ for $z \in \mathbb{K}$.

Exampele 2.2. Matrixes over $\mathbb{K}$.

$\mathbb{M}(n, \mathbb{K})(\mu) = (\mathbb{M}(n, \mathbb{K}), l_\mu, *)$, where $\mu \neq 0 \in \mathbb{K}$, $l_\mu(z) = \mu \mathrm{tr}(z)$ for $z \in \mathbb{M}(n, \mathbb{K})$, and $*$ is the transposition of matrixes.

Exampele 2.3. Super matrixes 2×2.

$(\mathbb{SM}, Sl_\rho, *)$, where $\mathbb{SM} = \mathbb{M}(2, \mathbb{K})$, $\mathbb{SM}^0$ and $\mathbb{SM}^1$ are diagonal and antidiagonal matrixes, $\rho \neq 0 \in \mathbb{K}$, $l_\rho(z) = \frac{\rho}{2}\mathrm{Str}$, where

$\mathrm{Str}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{11} - b_{22}$ is the super trace, and $*$ is the transposition of matrixes.

Exampele 2.4. Quaternions over $\mathbb{K}$.

Let $\mathbb{R}_\mathbb{K} \in \mathbb{K}$ be a subfield, isomorphic to the field of real numbers. Let $\mathbb{Q}_\mathbb{R}$ be the algebra of quaternions, that is the algebra generated, as vector space over $\mathbb{R}$, by vectors $1^\mathbb{Q}$, I, J, K, where $IJ = K, JK = I, KI = J$. Use the isomorphism $\mathbb{R}_\mathbb{K} \cong \mathbb{R}$ for define $\mathbb{Q}_\mathbb{K} = \mathbb{Q}_\mathbb{R} \bigotimes_\mathbb{R} \mathbb{K}$. We will consider $1^\mathbb{Q}$, I, J, K also as a basis of $\mathbb{Q}_\mathbb{K}$ over $\mathbb{K}$. Put $\mathbb{Q}_\mathbb{K} = \mathbb{Q}^0_\mathbb{K} + \mathbb{Q}^1_\mathbb{K}$, where $\mathbb{Q}^0_\mathbb{K}$ generated by $1^\mathbb{Q}$, J and $\mathbb{Q}^1_\mathbb{K}$ generated by I, K.

Put $\mathbb{Q}_{\mathbb{K}}(\rho) = (\mathbb{Q}_{\mathbb{K}}, l_\rho, *)$ where $\rho \neq 0 \in \mathbb{K}$ and $l_\rho : \mathbb{Q}_{\mathbb{K}} \to \mathbb{K}$ be the $\mathbb{K}$-linear functional, defined by $l_\rho(J) = \rho$, $l_\rho(1^{\mathbb{Q}}) = l^{\mathbb{Q}}(I) = l^{\mathbb{Q}}(K) = 0$ and $(1^{\mathbb{Q}})^* = 1^{\mathbb{Q}}$, $I^* = -I$, $J^* = J$, $K^* = K$.

The correspondance

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1^{\mathbb{Q}}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto J, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto K$$

define an isomorphism between $(\mathbb{SM}, Sl_\rho, *)$ and $\mathbb{Q}_{\mathbb{K}}(\rho)$.

2.2 Cardy-Frobenius algebras.

Definition. By *Cardy-Frobenius algebra (with graduation and involution)* over $\mathbb{K}$ we call a set $\{(A, l^A, *_A), (B, l^B, *_B), \phi\}$, where

1. $(A, l^A, *_A)$ and $(B, l^B, *_B)$ are Frobenius pairs over $\mathbb{K}$ where A is a commutative algebra;

2. $\phi : A \to B$ is a homomorphism of algebras, such that $\phi(A)$ belong to centre of B and $\phi(a^*) = (\phi(a))^*$;

3. Cardy axiom: $F_B^{ij} l^B(b_i b' b_j b") = F_A^{ij} l^B(\phi(a_i)b') l^B(\phi(a_j)b")$,where $b', b" \in B$, $(a_1, ...a_n) \in A$,$(b_1, ...b_m) \in B$ are bases of homogenous elements, and $\{F_A^{ij}\}$ (respectively $\{F_B^{ij}\}$) are matrixes, inverse to $\{F_{ij}^A\}$ (respectively to $\{F_{ij}^B\}$). Here $\{F_{ij}^X\} = \{l^X(x_i x_j^*)\}$ if $x_i, x_j \in X^0$ and $\{F_{ij}^X\} = \{l^X(x_i x_j)\}$ in other cases. (It is easy to prove that this definition is independent of choice of the basis.)

A Cardy-Frobenius algebra is called *semisimple*, if A and B are semisimple algebras.

Remark. It is possible to prove that for $A = A^0$, $B = B^0$ isomorphic classes of Cardy-Frobenius algebras one-to-one correspond to isomorphic classes of open-closed topological string theories [9]. Full classification of semisimple Cardy-Frobenius algebras for this case follow from [9].

Definition. By *orthogonal sum* of Cardy-Frobenius algebras $\{(A_1, l^{A_1}, *_{A_1}), (B_1, l^{B_1}, *_{B_1}), \phi_1\}$ and $\{(A_2, l^{A_2}, *_{A_2}), (B_2, l^{B_2}, *_{B_2}), \phi_2\}$ we call the Cardy-Frobenius algebra $\{(A, l^A, *_A), (B, l^B, *_B), \phi\} = \{(A_1, l^{A_1}, *_{A_1}), (B_1, l^{B_1}, *_{B_1}), \phi_1\} \oplus \{(A_2, l^{A_2}, *_{A_2}), (B_2, l^{B_2}, *_{B_2}), \phi_2\}$, where $(A, l^A, *_A) = (A_1, l^{A_1}, *_{A_1}) \oplus (A_2, l^{A_2}, *_{A_2})$, $(B, l^B, *_B) = (B_1, l^{B_1}, *_{B_1}) \oplus (B_2, l^{B_2} *_{B_2})$ and $\phi = \phi_1 \oplus \phi_2$.

Exampele 2.5. $\mathbb{K}$ - numbers and matrixes.

$\{\mathbb{K}(\mu^2), \mathbb{M}(n, \mathbb{K})(\mu), \phi_M\}$, where $\phi_M : \mathbb{K} \to \mathbb{M}(n, \mathbb{K})$ is the naturel homomorphism of numbers to diagonal matrixes.

Check the axiom 3. Choice a base of elementary matrixes E_{ij} as a base of $\mathbb{M}(n, \mathbb{K})$. Left and right parts of axiom 3 are equal to 0, if one of the matrixes b' or $b"$ are not diagonal. Check the axiom 3 for $b' = E_{ii}$, $b" = E_{jj}$. In this case left parts is equal to 1, and right parts is $\mu^{-2}\mu^2 = 1$.

Exampele 2.6. $\mathbb{K}$ - numbers and quaternions.

$\{\mathbb{K}(-\frac{\rho^2}{4}), \mathbb{Q}_{\mathbb{K}}(\rho), \phi_{\mathbb{Q}}\}$, where homomorphism $\phi_{\mathbb{Q}} : \mathbb{K} \to \mathbb{Q}$ is defined by $\phi_{\mathbb{Q}}(1) = 1^{\mathbb{Q}}$.

Check the axiom 3. Consider the basis $1^{\mathbb{Q}}, I, J, K$ of $\mathbb{Q}_{\mathbb{K}}$. Then $F_{1^{\mathbb{Q}}J} = F_{J1^{\mathbb{Q}}} = F_{KI} = -F_{IK} = \rho$ and $F_{kl} = 0$ in other cases. Thus $F^{1^{\mathbb{Q}}J} = F^{J1^{\mathbb{Q}}} = -F^{KI} = F^{IK} = \rho^{-1}$ and $F^{st} = 0$ in other cases.

The left part $f(b', b")$ of axiom 3 is equal to 0, if $b' \neq b"$. Moreover $f(I, I) = \rho^{-1}(l_\rho(1^{\mathbb{Q}}IJI) + l_\rho(JI1^{\mathbb{Q}}I) + l_\rho(IIKI) - l_\rho(KIII)) = \rho^{-1}(l_\rho(J) + l_\rho(-J) + l_\rho(-J) - l_\rho(-J)) = 0$. By analogy $f(K, K) = f(1^{\mathbb{Q}}, 1^{\mathbb{Q}}) = 0$. It is obviously that the right part of axiom 3 also equal to 0 for $b' \in \{1^{\mathbb{Q}}, I, K\}$ or $b" \in \{1^{\mathbb{Q}}, I, K\}$.

Check now axiom 3 for $b' = b" = J$. In this case $f(J, J) = \rho^{-1}(l_\rho(1^{\mathbb{Q}}JJJ) + l_\rho(JJ1^{\mathbb{Q}}J) + l_\rho(IJKJ) - l_\rho(KJIJ)) = \rho^{-1}(l_\rho(-J) + l_\rho(-J) + l_\rho(-J) - l_\rho(J)) = -4$ and $F_A^{ij}((\phi(a_i)), J)_B(\phi(a_j), J)_B = \lambda^{-1}(l_\rho(1^{\mathbb{Q}}J))^2 = \lambda^{-1}\rho^2$, where $\lambda^{-1}\rho^2 = -4$.

Definition. A set $\{(A, l^A, *_A), (B, l^B, *_B), \phi\}$ is called a *pseudoreal Cardy-Frobenius algebra* if $(A, l^A, *_A)$ and $(B, il^B, *_B)$ are real Frobenius pair, and the complexification of $\{(A, l^A, *_A), (B, l^B, *_B), \phi\}$ is complex Cardy-Frobenius algebra.

3. Landau-Ginsburg models.

3.1. Landau-Ginsburg models without boundary.

The simplest Landau-Ginsburg model [3] of degree n is generated by a complex polynomial p in the form $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ such that all roots $\alpha_1, ..., \alpha_n$ of its derivative $p'(z)$ are simple.

The set of all such polynomials form a complex manifold $\mathbb{C}Pol(n)$ of complex dimension n. Its tangent space A_p in a point p consist of all polynomials of degree $n-1$. Landau-Ginsburg model generates on $A_p = A_p^{\mathbb{C}}$ a structure of algebra, where the multiplication $q = q_1 *_p q_2$ for polynomials $q_1 = q_1(z)$ and $q_2 = q_2(z) \in A_p$ is defined by condition $q(z) = q_1(z)q_2(z)(\mathtt{mod} p'(z))$.

Moreover a Landau-Ginsburg model generate on A_p a non-degenerated bilinear form $(q_1, q_2)_p = l_p(q_1 *_p q_2)$, where $l_p(q) = \frac{1}{2\pi i}\oint \frac{q(z)dz}{p'(z)}$. (Here and late the formula $\frac{1}{2\pi i}\oint$ means "minus residue in ∞") Thus A_p has a structure of Frobenius algebra.

The polynomials $e_{p,\alpha_i}(z) = \prod_{j\neq i} \frac{z-\alpha_j}{\alpha_i - \alpha_j}$ form a basis of idempotents of A_p, that is $e_{p,\alpha_i} e_{p,\alpha_j} = \delta_{ij} e_{p,\alpha_i}$. Thus A_p is a semi-simple algebra.

Put $\mu_{\alpha_i} = l_p(e_{p,\alpha_i}) = \frac{1}{n+1}\prod_{j\neq i}\frac{1}{\alpha_i - \alpha_j}$.

Let A_{p,α_j} be the complex vector space generated by e_{p,α_j} and $l^A_{p,\alpha_j} = l_p|_{A_{p,\alpha_j}}$. Then $(A_{p,\alpha_j}, l^A_{p,\alpha_j})$ is a Frobenius pair, that is isomorphic to $\mathbb{C}(\mu_{\alpha_j})$. Thus $(A_p, l^A_p) = \bigoplus_{i=1}^n (A_{p,\alpha_j}, l^A_{p,\alpha_j}) \cong \bigoplus_{i=1}^n \mathbb{C}(\mu_{\alpha_i})$.

3.2. Landau-Ginsburg models with boundary.

The simplest Landau-Ginsburg models with boundary [10,11,12] is generated by pair (p, α) where $p \in Pol(n)$ and α is a root of order 2 of $p = p(z)$. Thus $p(z) = (z-\alpha)^2\nu(z)$ and $\nu(\alpha) \neq 0$. Put $A = A_{p,\alpha}$

The number α belong to the set $\{\alpha_j\}$ of roots of p'. We shall consider that $\alpha = \alpha_1$. Put $p_\alpha(z) = \frac{p'(z)}{z-\alpha}$. The set of roots of the polynomials p_α and $e = e_{p,\alpha}$ are coincide. Moreover $p_\alpha(\alpha) = (n+1)\prod_{j\neq 1}(\alpha-\alpha_j) = \mu_\alpha^{-1}$ and $e(\alpha) = \prod_{j\neq 1}\frac{\alpha-\alpha_j}{\alpha-\alpha_j} = 1$. Thus $p_\alpha = \mu_\alpha^{-1}e$. On the other hands $p'(z) = 2\nu(z)(z-\alpha)+\nu'(z)(z-\alpha)^2$ and thus $p_\alpha(\alpha) = 2\nu(\alpha)$. Therefore $\nu(\alpha) = \frac{1}{2}p_\alpha(\alpha) = \frac{1}{2}\mu_\alpha^{-1}$.

Let $\widetilde{B}$ be the algebra over $\mathbb{C}$ of all matrixes 2×2, with elements in the form of complex polynomials.

Consider the matrix with polynomial coefficients

$$Q = Q_{p,\alpha} = \begin{pmatrix} 0 & (z-\alpha)\nu(z) \\ (z-\alpha) & 0 \end{pmatrix}$$

The Landau-Ginsburg model with boundary generate the bilinear form $\widetilde{(.,.)} : A \times \widetilde{B} \to \mathbb{C}$, where

$$\widetilde{(a,b)} = \frac{1}{2\pi i}\oint a\frac{STr((dQ)b)}{p'}$$

and STr is the supertrace.

The algebra $\widetilde{B}$ is the direct sum $\widetilde{B} = \widetilde{B}^0 \bigoplus \widetilde{B}^1$ of diagonal $\widetilde{B}^0$ and anti-diagonal $\widetilde{B}^1$ matrixes. For $b \in \widetilde{B}^i$ we put $|b| = i$.

The matrix Q generate the operator $D = D_{p,\alpha} : \widetilde{B}^0 \bigcup \widetilde{B}^1 \to \widetilde{B}^0 \bigcup \widetilde{B}^1$, where $D(b) = Qb - (-1)^{|b|}bQ$. Extend him to the linear operator on $\widetilde{B}$.

Theorem 3.1. $D^2 = 0$. The structure of algebra over $\mathbb{C}$ and the structure of module over the ring of polynomials for $\widetilde{B}$ generate a structure of algebra over $\mathbb{C}$ and a structure of module over the ring of polynomials for $\widehat{B} = \widehat{B}_{p,\alpha} = KerD/ImD$ such that the natural projection $\widetilde{B} \to \widehat{B}$ is an homomorphism. The bilinear form $\widetilde{(.,.)}$ generate the bilinear form $(.,.) : A \times \widehat{B} \to \mathbb{C}$.

Proof. Put $Ker_1 = KerD \bigcap \widetilde{B}^0$, $Im_1 = ImD \bigcap \widetilde{B}^0$, $Ker_I = KerD \bigcap \widetilde{B}^1$, $Im_I = ImD \bigcap \widetilde{B}^1$

If

$$b^0 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \in \widetilde{B}^0$$

then

$$Db^0 = (z-\alpha)\begin{pmatrix} 0 & (b_2-b_1)\nu(z) \\ (b_1-b_2) & 0 \end{pmatrix}.$$

Thus the set Ker_1 is the set of matrixes in the form of

$$f(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

, and the set Im_I is the set of matrixes in the form of

$$(z-\alpha)f(z)\begin{pmatrix} 0 & -\nu(z) \\ 1 & 0 \end{pmatrix}$$

where $f(z)$ is a polynomial.

If

$$b^1 = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} \in \widetilde{B}^1$$

then

$$Db^1 = (z-\alpha)\begin{pmatrix} b_1 + b_2\nu(z) & 0 \\ 0 & b_1 + b_2\nu(z) \end{pmatrix}.$$

Thus the set Ker_I is the set of matrixes in the form of

$$f(z)\begin{pmatrix} 0 & -\nu(z) \\ 1 & 0 \end{pmatrix}$$

and the set Im_1 is the set of matrixes in the form of

$$(z-\alpha)f(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $f(z)$ is a polynomial.

Thus $D^2 = 0$ and the structure of module over the ring of polynomials in $\widetilde{B}$ generate a structure of module over the ring of polynomials in $\widehat{B}$.

Moreover $Ker_1 Ker_1 = Ker_1, Ker_1 Im_1 = Im_1 Ker_1 = Im_1$, $Ker_1 Ker_I = Ker_I, Ker_1 Im_I = Im_1 Ker_I = Im_I$, $Ker_I Ker_I \subset Ker_1, Ker_I Im_I \subset Im_1$ and $Im_I Ker_I \subset Im_1$. Thus the multiplication of matrix generate a multiplication in $\widehat{B}$ turn $\widehat{B}$ into an algebra over $\mathbb{C}$.

If $b \in Im_1$, then

$$b = (z-\alpha)f(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $STr((dQ)b) = 0$. Thus $\widetilde{(a, Im_1)} = 0$

If $b \in Im_I$, then

$$b = (z-\alpha)f(z)\begin{pmatrix} 0 & -\nu(z) \\ 1 & 0 \end{pmatrix}$$

and $STr((dQ)b) = (z-\alpha)f(z)(2\nu(z) + (z-\alpha)\nu'(z)) = f(z)p'(z)$. Thus $\widetilde{(a, Im_I)} = 0$. Therefore the bilinear form $\widetilde{(.,.)}$ generate a bilinear form $(.,.) : A \times \widehat{B} \to \mathbb{C}$.

□

Denote by $\widehat{\psi} : \widetilde{B} \to \widehat{B}$ the natural projection. Then $\widehat{B} = \widehat{B}^0 \bigoplus \widehat{B}^1$, where $\widehat{B}^i = \widehat{B}^i_{p,\alpha} = \widehat{\psi}(\widetilde{B}^i)$.

The sets $\widehat{B}^0$ and $\widehat{B}^1$ are one-dimensional vector spaces. They generated by $e^i = e^i_{p,\alpha} = \psi(\widetilde{e}^i)$ where

$$\widetilde{e}^0 = e\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \subset \widetilde{B}^0$$

and

$$\widetilde{e}^1 = e\begin{pmatrix} 0 & -\nu(\alpha) \\ 1 & 0 \end{pmatrix} \subset \widetilde{B}^1.$$

In this case

$$e^1 = \psi(e\begin{pmatrix} 0 & -\nu(z) \\ 1 & 0 \end{pmatrix}),$$

e^0 is the unity element of the algebra $\widehat{B}$ and $e^1e^1 = -\nu(\alpha)e^0$. Moreover, $(e, e^0) = 0$ and

$$(e, e^1) = \frac{1}{2\pi i}\oint \frac{e \cdot STr(P)\,dz}{p'},$$

where

$$P = \begin{pmatrix} 0 & (z-\alpha)\nu(z) \\ (z-\alpha) & 0 \end{pmatrix}' e\begin{pmatrix} 0 & -\nu(z) \\ 1 & 0 \end{pmatrix} =$$

$$= e\begin{pmatrix} \nu(z) + (z-\alpha)\nu(z)' & 0 \\ 1 & -\nu(z) \end{pmatrix}.$$

Thus,

$$(e, e^1) = \frac{1}{2\pi i}\oint \frac{e^2(2\nu(z) + (z-\alpha)\nu'(z))\,dz}{p'}.$$

By $p(z) = (z-\alpha)^2\nu(z)$ we have

$$(e, e^1) = \frac{1}{2\pi i}\oint \frac{e\,dz}{(z-\alpha)} = e(\alpha) = 1.$$

Let $\phi : A \to \widehat{B}$ be the natural monomorphism, such that $\phi(re) = re^0$. Then the condition $(a, b) = l^{\widehat{B}}(\phi(a)b)$ define a linear functional $l^{\widehat{B}} = l^{\widehat{B}}_{p,\alpha} : \widehat{B} \to \mathbb{C}$. Thus $l^{\widehat{B}}(e^0) = 0$ and $l^{\widehat{B}}(e^1) = 1$.

Nevertheless, $\{(A, l^A, *_A), (\widehat{B}, l^{\widehat{B}}, *_{\widehat{B}}), \phi\}$ is not a Cardy-Frobenius algebra. Really, e generate A, $*_A$ is identical, $F^{11}_A = \mu^{-1}$, and the pair e^0 ,e^1 generate $\widehat{B}$.

If $*_{\widehat{B}}$ is also identical or, if e^1 is odd, then $F^{00}_{\widehat{B}} = F^{11}_{\widehat{B}} = 0$, $F^{01}_{\widehat{B}} = F^{10}_{\widehat{B}} = 1$, and, thus, for $b' = b" = e^0$, we have $F^{ij}_{\widehat{B}} l^{\widehat{B}}(b'b_ib"b_j) = l^{\widehat{B}}(e^0e^0e^0e^1) + l^{\widehat{B}}(e^0e^1e^0e^0) = 2 \neq 0 = \mu^{-1}l^{\widehat{B}}(e^0e^0)l^{\widehat{B}}(e^0e^0) = F^{ij}_A l^{\widehat{B}}(\phi(a_i)b')l^{\widehat{B}}(\phi(a_j)b")$.

If $*_{\widehat{B}}$ is not identical, and e^1 is even, then $(e^1)^* = -e^1$ and $F^{00}_{\widehat{B}} = F^{11}_{\widehat{B}} = 0$, $F^{01}_{\widehat{B}} = 1$, $F^{10}_{\widehat{B}} = -1$. Thus, for $b' = b" = e^1$, we have $F^{ij}_{\widehat{B}} l^{\widehat{B}}(b'b_ib"b_j) = l^{\widehat{B}}(e^1e^0e^1e^1) - l^{\widehat{B}}(e^1e^1e^1e^0) = 0 \neq \mu^{-1} = \mu^{-1}l^{\widehat{B}}(e^0e^1)l^{\widehat{B}}(e^0e^1) = F^{ij}_A l^{\widehat{B}}(\phi(a_i)b')l^{\widehat{B}}(\phi(a_j)b")$.

In the next section we "extend" Landau-Ginsburg models with boundary to some model that generate a Cardy-Frobenius algebra.

3.3. Quaternion Landau-Ginsburg model.
Let us consider, that any matrix $b \in \tilde{B}$ exists in two variants : $\check{B}^0$ with entiers, grading by 0 and $\check{B}^1$ with entiers, grading by 1. Full grading of $b \in \check{B} = \check{B}^0 \bigcup \check{B}^1$ is the sum the grading i of $\check{B}^i$ and the grading b as an element of $\tilde{B}$. The operator D generate an operator $\check{D} : \check{B} \to \check{B}$, changing the grading. Thus we can consider cohomology $B = Ker\check{D}/Im\check{D}$ of $\check{D}$ instead of $\hat{B}$.

For what to investigate B let us consider in parallel with operator $D(b) = Qb - (-1)^{|b|}bQ$ a "dual" operator $D^* = D^*_{p,\alpha} : \widetilde{B}^0 \bigcup \widetilde{B}^1 \to \widetilde{B}^0 \bigcup \widetilde{B}^1$, where $D^*(b) = Qb + (-1)^{|b|}bQ$. Extend him to the linear operator on $\widetilde{B}$.

Put $Ker_J = KerD^* \bigcap \widetilde{B}^0$, $Im_J = ImD^* \bigcap \widetilde{B}^0$, $Ker_K = KerD^* \bigcap \widetilde{B}^1$, $Im_K = ImD^* \bigcap \widetilde{B}^1$

Calculation similar to theorem 3.1 prove that $(D^*)^2 = 0$, Ker_J is the set of matrixes in the form of

$$f(z)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

the set Im_K is the set of matrixes in the form of

$$(z-\alpha)f(z)\begin{pmatrix} 0 & \nu(z) \\ 1 & 0 \end{pmatrix},$$

the set Ker_K is the set of matrixes in the form of

$$f(z)\begin{pmatrix} 0 & \nu(z) \\ 1 & 0 \end{pmatrix}$$

and the set Im_J is the set of matrixes in the form of

$$(z-\alpha)f(z)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $f(z)$ is a polynomial.

Thus $(D^*)^2 = 0$ and we can consider $\widehat{B}^* = \widehat{B}^*_{p,\alpha} = KerD^*/ImD^*$. Denote by $\widehat{\psi}^* : \widetilde{B} \to \widehat{B}^*$ the natural projection. Then $\widehat{B}^* = (\widehat{B}^0)^* \bigoplus (\widehat{B}^1)^*$, where $(\widehat{B}^i)^* = (\widehat{B}^i_{p,\alpha})^* = \widehat{\psi}^*(\widetilde{B}^i)$.

The sets $(\widehat{B}^0)^*$ and $(\widehat{B}^1)^*$ are one-dimensional vector spaces. They generated by $f^i = f^i_{p,\alpha} = \widehat{\psi}^*(\widetilde{f}^i)$ where

$$\widetilde{f}^0 = e\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \subset \widetilde{B}^0$$

and

$$\widetilde{f}^1 = e\begin{pmatrix} 0 & \nu(\alpha) \\ 1 & 0 \end{pmatrix} \subset \widetilde{B}^1.$$

Put now $B = B_{p,\alpha} = \widehat{B}_{p,\alpha} \bigoplus \widehat{B}^*_{p,\alpha}$ and consider the map $\psi = \psi_{p,\alpha} = \widehat{\psi} \bigoplus \widehat{\psi}^* : \widetilde{B} \to B$.

Theorem 3.2. The structure of algebra over $\mathbb{C}$ and the structure of module over the ring of polynomials for $\widetilde{B}$ generate a structure of algebra over $\mathbb{C}$ and a structure of module over the ring of polynomials for B such that the natural projection $\psi : \widetilde{B} \to \widehat{B}$ is an epimomorphism. Moreover, the correspondence $\xi(1^{\mathbb{Q}}) = e^0$, $\xi(I) = \kappa e^1$, $\xi(J) = if^0$, $\xi(K) = i\kappa f^1$ where $\kappa^{-2} = \nu(\alpha)$, generate the isomorphism of the $\mathbb{C}$-algebras $\xi : \mathbb{Q}_{\mathbb{C}} \to B$.

Proof. It is follow from the description of $KerD, ImD, KerD^*, ImD^*$ that B is a module over the ring of polynomials and $Ker_J Ker_1 = Ker_1 Ker_J = Ker_J$; $Ker_J Im_1 = Im_1 Ker_J = Im_J$; $Ker_1 Im_J = Im_J Ker_1 = Im_J$; $Ker_K Ker_1 = Ker_1 Ker_K = Ker_K$; $Ker_K Im_1 = Im_1 Ker_K = Im_K$; $Ker_1 Im_K = Im_K Ker_1 = Im_K$; $Ker_K Ker_J = Ker_J Ker_K = Ker_I$; $Ker_K Im_J = Im_J Ker_K = Im_I$; $Ker_J Im_K = Im_K Ker_J = Im_I$; $Ker_K Ker_I = Ker_I Ker_K \subset Ker_J$; $Ker_K Im_I = Im_I Ker_K \subset Im_J$; $Ker_I Im_K = Im_K Ker_I \subset Im_J$; $Ker_J Ker_I = Ker_I Ker_J = Ker_K$; $Ker_J Im_I = Im_I Ker_J = Im_K$; $Ker_I Im_J = Im_J Ker_I = Im_K$; $Ker_I Ker_I, Ker_J Ker_J, Ker_K Ker_K \subset Ker_1$; $Ker_I Im_I, Ker_J Im_J, Ker_K Im_K \subset Im_1$; $Im_I Ker_I, Im_J Ker_J, Im_K Ker_K \subset Im_1$. Therefore B is an algebra and ψ, ξ are homomorphisms.

□

Define a linear functional $l^B = l^B_{p,\alpha} : B \to \mathbb{C}$ by $l^B|_{\widehat{B}} = i\sqrt{2} l^{\widehat{B}}$ and $l^B|_{\widehat{B}^*} = 0$. Let B^0 be the vector space generated by e^i and B^1 be the vector space generated by f^i. Then $B = B^0 + B^1$. Define the antihomomorphism $*$ by $*(e^0) = e^0$, $*(e^1) = e^1$, $*(f^1) = f^1$, $*(f^0) = -f^0$,

Theorem 3.3. The set $\{(A_{p,\alpha}, l^A_{p,\alpha}), (B_{p,\alpha}, l^B_{p,\alpha}, *), \phi_{p,\alpha}\}$ is a graduated Cardy-Frobenius algebra isomorphic to $\{\mathbb{C}(-\frac{1}{4}\rho^2), \mathbb{Q}_{\mathbb{C}}(\rho), \phi_{\mathbb{Q}}\}$, where $-\frac{1}{4}\rho^2 = \mu_{p,\alpha}$.

Proof. Let $\pi : \mathbb{Q}_{\mathbb{C}} \to \mathbb{Q}_{\mathbb{C}}$ be the automorphisms of quaternion algebras where $\pi(I) = -J$, $\pi(J) = I$, $\pi(K) = K$. Put $\kappa = -\frac{i}{\sqrt{2}}\rho$. Then $\kappa^{-2} = \nu(\alpha)$ and the isomorphism $\xi\pi$ generate the isomorphism between the Frobenius pairs $\mathbb{Q}_{\mathbb{C}}(\rho)$ and $(B, l^B, *)$. Thus, according to example 2.6., we see that $\{\mathbb{C}(\mu_{p,\alpha}), (B_{p,\alpha}, l^B_{p,\alpha}, *), \phi_{\mathbb{C}}\}$, where $\phi_{\mathbb{C}}(r) = re^0$, is a Cardy-Frobenius algebra, isomorphic to $\{\mathbb{C}(\mu_{p,\alpha}), \mathbb{Q}_{\mathbb{C}}(\rho), \phi_{\mathbb{Q}}\}$. But, according to subsection 3.2., the Frobenius algebras $\mathbb{C}(\mu_{p,\alpha})$ and $\{(A_{p,\alpha}, l^A_{p,\alpha})\}$ are isomorphic.

□

3.4. Joint quaternion Landau-Ginsburg model.

Extend the definition of the algebra $\{(A_{p,\alpha}, l^A_{p,\alpha}), (B_{p,\alpha}, l^B_{p,\alpha}, *), \phi_{p,\alpha}\}$ on arbitrary polynomial p, such that α is a simple root of p'. We shall consider that $\{(A_{p,\alpha}, l^A_{p,\alpha}), (B_{p,\alpha}, l^B_{p,\alpha}, *), \phi_{p,\alpha}\} = \{(A_{\tilde{p},\alpha}, l^A_{\tilde{p},\alpha}), (B_{\tilde{p},\alpha}, l^B_{\tilde{p},\alpha}, *), \phi_{\tilde{p},\alpha}\}$, where $\tilde{p} = p - p(\alpha)$.

Put $\{(A_p, l^A_p), (B_p, l^B_p, *), \phi_p\} = \bigoplus_{i=1}^n \{(A_{p,\alpha_i}, l^A_{p,\alpha_i}), (B_{p,\alpha_i}, l^B_{p,\alpha_i}, *), \phi_{p,\alpha_i}\}$ for arbitrary complex polynomial p in the form $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ such that all roots $\alpha_1, ..., \alpha_n$ of $p'(z)$ are simple.

Give now independent description of the Cardy-Frobenius algebra $\{(A_p, l^A_p), (B_p, l^B_p, *), \phi_p\}$. Put $\widetilde{B}_p = \widetilde{B}/p'\widetilde{B}$. The monomorphism $f \mapsto \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$ generates a monomorphism $\widetilde{\phi}_p : A_p \to \widetilde{B}_p$. Define the linear forme $l^{\widetilde{B}}_p : \widetilde{B} \to \mathbb{C}$ by

$$l^{\widetilde{B}}_p \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \sqrt{2} i (l^A_p(b_{21} e^\nu_p) - l^A_p(b_{12})),$$

where $e^\nu_p = \sum \nu(\alpha_i) e_{p,\alpha_i}$. It generates the linear forme $\widetilde{l}^B_p : \widetilde{B}_p \to \mathbb{C}$.

Put

$$* \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{22} & b_{12} \\ b_{21} & b_{11} \end{pmatrix}$$

Theorem 3.4. The set $\{(A_p, l^A_p), (\widetilde{B}_p, \widetilde{l}^B_p, *), \widetilde{\phi}_p\}$ is a Cardy-Frobenius algebra isomorphic to $\{(A_p, l^A_p), (B_p, l^B_p, *), \phi_p\}$.

Proof. The ideal $p'\widetilde{B}$ is contained in Im_1, Im_I, Im_J, Im_K for all α_i. Therefore $\psi_p = \bigoplus_{i=1}^n \psi_{p,\alpha}$ generate a homomorphism $\widetilde{\psi}_p : \widetilde{B}_p \to B_p$ transforming $\widetilde{\phi}_p$ to ϕ_p. Moreover the set $\{\widetilde{\psi}_p(\widetilde{e}^0_{p,\alpha_i}), \widetilde{\psi}_p(\widetilde{e}^1_{p,\alpha_i}), \widetilde{\psi}_p(\widetilde{f}^0_{p,\alpha_i}), \widetilde{\psi}_p(\widetilde{f}^1_{p,\alpha_i}) | i = 1, ..., n\}$ form a basis of B_p. Thus $\widetilde{\psi}_p$ is an isomorphism. Besides $l^{\widetilde{B}}_p(\widetilde{e}^0_{p,\alpha_i}) = 0 = l^B_p(\psi_p(\widetilde{e}^0_{p,\alpha_i}))$, $l^{\widetilde{B}}_p(\widetilde{f}^0_{p,\alpha_i}) = 0 = l^B_p(\psi_p(\widetilde{f}^0_{p,\alpha_i}))$, $l^{\widetilde{B}}_p(\widetilde{f}^1_{p,\alpha_i}) = 0 = l^B_p(\psi_p(\widetilde{f}^1_{p,\alpha_i}))$, $l^{\widetilde{B}}_p(\widetilde{e}^1_{p,\alpha_i}) = i\sqrt{2} = l^B_p(\psi_p(\widetilde{e}^1_{p,\alpha_i}))$. Thus $\widetilde{\psi}_p$ transform l^B_p to $\widetilde{l}^B_p$.
□

3.5. Pseudoreal quaternion Landau-Ginsburg model.

Consider a real version of (complex) Landau-Ginsburg model without boundary, described in subsection 3.1. This is a result of restriction the complex Landau-Ginsburg model on the real manifold $\mathbb{R}Pol(n)$ of all polynomial p with real coefficients in the form of $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$. The tangent space $\mathbb{R}A_p$ for $p \in \mathbb{R}Pol(n)$ consist of all real polynomials of degree $n-1$. Moreover $e^\nu_p \in \mathbb{R}A_p$. The complex Landau-Ginsburg model generates on $\mathbb{R}A_p$ a structure of real Frobenius algebra $(\mathbb{R}A_p, \mathbb{R}l_p)$ where $\mathbb{R}l_p$ is the restriction l^A_p on $\mathbb{R}A_p$.

Similarly we can consider the real part $\mathbb{R}B_p$ of B_p as the image by ψ_p of real polynomial matrixes. Then $\phi_p(\mathbb{R}A_p) \subset$

$\mathbb{R}B_p$. Let $\mathbb{R}l_p^B$ be the restriction l_p^B on $\mathbb{R}B_p$. Then $\mathbb{R}\{(A_p, l_p^A), (B_p, l_p^B,), \phi_p\} = \{(\mathbb{R}A_p, \mathbb{R}l_p^A), (\mathbb{R}B_p, \mathbb{R}l_p^B, *), \phi_p|_{\mathbb{R}A_p}\}$ be a pseudoreal Cardy-Frobenius algebra that is a *real part* of Cardy-Frobenius algebra $\{(A_p, l_p^A), (B_p, l_p^B, *), \phi_p\}$.

We can consider the manifold $\mathbb{R}Pol(n)$ is a subset of the manifold $\mathbb{Q}Pol(n)$ of all polynomial p in the form $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ with quaternion coefficients. Let $\mathbb{Q}A_p$ be the tangent space for $p \in \mathbb{R}Pol(n)$. As a vector space over $\mathbb{R}$ it has the naturel representations $\mathbb{Q}A_p = \mathbb{R}A_p \bigoplus I\mathbb{R}A_p \bigoplus J\mathbb{R}A_p \bigoplus K\mathbb{R}A_p$. Let $\mathbb{R}\phi_p : \mathbb{R}A_p \to \mathbb{Q}A_p$ be the naturel monomorphism. Consider a linear form $\mathbb{Q}l_p^A : \mathbb{Q}A_p \to i\mathbb{R}$ where $\mathbb{Q}l_p^A|_{\mathbb{R}A_p} = \mathbb{Q}l_p^A|_{I\mathbb{R}A_p} = 0$, and $\mathbb{Q}l_p^A(Jg) = \sqrt{2}i(l_p^A(ge_p^\nu) - l_p^A(g))$, $\mathbb{Q}l_p^A(Kg) = \sqrt{2}i(l_p^A(ge_p^\nu) + l_p^A(g))$ for $g \in \mathbb{R}A_p$.

Define the involution $* : \mathbb{Q}A_p \to \mathbb{Q}A_p$ by $*(f) = f$, $*(If) = If$, $*(Jf) = -Jf$, $*(Kf) = Kf$.

Theorem 3.5. The set $\{(\mathbb{R}A_p, \mathbb{R}l_p^A), (\mathbb{Q}A_p, \mathbb{Q}l_p^A, *), \mathbb{R}\phi_p\}$ is a pseudoreal Cardy-Frobenius algebra isomorphic to $\mathbb{R}\{(A_p, l_p^A), (B_p, l_p^B, *), \phi_p\}$.

Proof. The isomorphism is generated by the correspondence $f \mapsto \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$, $If \mapsto \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix}$, $Jf \mapsto \begin{pmatrix} -f & 0 \\ 0 & f \end{pmatrix}$, $Kf \mapsto \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}$ for $f \in \mathbb{R}A_p$

□

4. Frobenius manifolds.

4.1. Frobenius manifolds and WDVV equations.

In middle of 90 years of the last century B.Dubrovin investigate a class of "flat" deformation of commutative Frobenius algebras that appear in different domain of mathematics. He call this structure Frobenius manifolds and construct beautiful theory of this manifolds and its applications [14]. Present some equivalent definitions of the Frobenius manifolds.

Definition.([14]) Let M be a smooth real (respectively complex) manifold. *Dubrovin connection* on M is called is a family of commutative Frobenius pair (M_p, θ_p) on the tangent spaces $M_p = T_pM$ for any point $p \in M$ such that

1. Tensors $\theta = \{\theta_p | p \in M\}$, $g(a, b) = \theta(ab)$, $c(a, b, d) = \theta(abd)$ are real (respectively complex) smooth and $\mathbf{d}\theta = 0$.

2. The Levi-Civita connection ∇ of the matric g is flat and such that $\nabla e = 0$, where e is the field of unity elements of the algebras M_p.

3. Tensors $c(a, b, d)$ and $\nabla_f c(a, b, d)$ are symmetrical by variables a, b, d, f.

4. There exists a vector field E (it is called *Euler field*), such that $\nabla(\nabla E) = 0$.

It is possible to prove that if $\theta(e) = 0$ (this is the most importent case), then M has a spesial coordinate system $t = (t^1, ..., t^n)$, (it is called *flat quasi-homogeneous coordinate system*) and constants d^i, r_i, such that : a)$g = \sum_{ij} \delta_{i+j,n+1} \mathbf{d}t^i \bigotimes \mathbf{d}t^j$, $e = \partial/\partial t^1$; b) $d_n = 1$ and $d_i r_i = 0$, $d_i + d_{n+1-i} = \upsilon + 2$ for all i; c) $E = \sum_{i=1}^n (d_i t^i + r_i)(\partial/\partial t^i)$.

Manifolds with a Dubrovin connection are called *Frobenius manifold.* They are called semi-simple, if M_p is a semi-simple algebra for any point $p \in M$.

Recall, that any semi-simple commutative Frobenius algebra is a direct sum of one-dimensional [18] and thus they have a *basis of idempotents* $e_1, ..., e_n$. This is a basis with the properties $e_i e_j = \delta_{ij} e_i$, $l(e_i) \neq 0$. It is defined uniquely up to enumeration of its elements.

Definition.([19]) A semi-simple Dubrovin connection on a smooth real (respectively complex) manifold M is a family of commutative Frobenius pair (M_p, θ_p) on the tangent spaces $M_p = T_p M$ for any point $p \in M$ such that

1. Tensors $\theta = \{\theta_p | p \in M\}$, $g(a, b) = \theta(ab)$, $c(a, b, d) = \theta(abd)$ are real (respectively complex) smooth and $\mathbf{d}\theta = 0$

2. There exists a covering $M = \bigcup_\alpha U_\alpha$ by coordinate maps $(x^1_\alpha, ..., x^n_\alpha) : U_\alpha \to \mathbb{K}^n$, such that: a) the vectors $(\partial/\partial x^1_\alpha, ..., \partial/\partial x^n_\alpha)$ form a canonical basis in M_p for any $p \in U_\alpha$; b) the field $E = \sum_{i=1}^n x^i_\alpha (\partial/\partial x^i_\alpha)$ don't depend from α; c)$L_E \theta = (\upsilon + 1)\theta$, where L_E is the Li derivative by E. Such coordinate systems are call *canonical coordinate*.

3. The Levi-Civita connection ∇ of the matric g is flat. There exists coordinate systems $t = (t^1, ..., t^n)$ on M, such that $g = \sum_{ij} \delta_{i+j,n+1} \mathbf{d}t^i \bigotimes \mathbf{d}t^j$, $e = \partial/\partial t^1$ and $E = \sum_{i=1}^n (d_i t^i + r_i)(\partial/\partial t^i)$.

For describe the Frobenius manifolds in term of differential equations let us introduce next definitions.

Definition. Let $F(t^1, ..., t^n)$ be a function on a set $U \subset \mathbb{C}^n = (t^1, ..., t^n)$ and be $E = \sum_{i=1}^n (d_i t^i + r_i)(\partial/\partial t^i)$ a vector field ,such that $d_n = 1$, $d_i r_i = 0$ $d_i + d_{n+1-i} = \upsilon + 2$ for all i. Then the pair (F, E) is a solution of WDVV equations, if:

1.$\sum_{q=1}^n \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^q} \frac{\partial^3 F}{\partial t^k \partial t^l \partial t^{n+1-q}} = \sum_{q=1}^n \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^q} \frac{\partial^3 F}{\partial t^i \partial t^l \partial t^{n+1-q}}$
(associativity equations);

2.$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^n} = \delta_{i+j,n+1}$
(normalization condition);

3.$L_E F = (\upsilon + 3)F + \sum_{ij} A_{ij} t^i t^j + \sum_i B_i t^i + C$, where L_E is the Li derivative and A_{ij}, B_i, C are constants.
(quasi-homogeneous conditions).

This equations was found in works of Witten[21] and Dijkgraaf, E.Verlinde,H.Verlinde[13] for a description of spaces of deformations of topological fields theories.

Definition. By *potential* of Dubrovin connection (M_p, θ_p) with flat quasi-homogeneous coordinates $t = (t^1, ..., t^n)$, is called a function $F(t^1, ..., t^n)$ such that $\theta_p(\frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}\frac{\partial}{\partial t^k}) = \frac{\partial F}{\partial t^i \partial t^j \partial t^k}$.

According to [14], any solution of WDVV equations is a potential of some Dubrovin connection and moreover the correspondance (potential)$\mapsto$(Dubrovin connection) form one-to-one between solutions of WDVV equations and Frobenius manifolds such that $\theta(e) = 0$.

If a solution F of WDVV equations has a representation in the form of Tailor series $F(t) = \sum c(i_1, i_2, ..., i_k) t^{i_1} t^{i_2} ... t^{i_k}$ then associativity equations is equivalent to some relations between the coefficients $c(i_1, i_2, ..., i_k)$. M.Kontsevich, Yu.Manin [16,17] prove that all admissible set of these coefficients are constructed by some special system of homeomorphisms the vector spaces $\mathbb{C}^{\otimes l}$ to homology of compactifications of moduli spaces of spheres with l pictures. This gives some other method of description of Frobenius manifolds that is called *Cohomological Field Theory.*

4.2. Moduli spaces of Landau-Ginsburg models without boundary.

The first and very important example of complex Frobenius manifold is the manifold $Pol^{\mathbb{C}}(n)$ of Landau-Ginsburg models without boundary. This is the space of polynomials, $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$ without multiple roots, considering is section 3. This space appear also in the theory of singularity, the theory of Coxeter groups, the theory of moduli spaces of Riemann surfaces, in matrix models of mathematical physics and in integrable systems. We will discuss (following on the whole [14,19]) this important for us example more detailed

Theorem 4.1.[13] The structure of Frobenius pairs (A_p, l_p) for $p \in Pol(n)$, described in section 3, generates a complex Dubrovin connection on the space $Pol(n)$ of polynomials $p(z) = z^{n+1} + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_n$.

Proof. Axiom 1 directly follow from the definitions. Let $(\alpha_1, ..., \alpha_n)$ be set of roots of p'. Consider the functions $x^i(p) = p(\alpha_i)$ (defined up to enumeration) as coordinates on $Pol(n)$.

Lemma 4.1. The coordinates $(x^1, ..., x^n)$ are canonical, $\upsilon = \frac{2}{n+1} - 1$ and $E = \sum_{i=k}^n \frac{k+1}{n+1} a_k \partial/\partial a_k$.

Proof. Prove, that $\frac{\partial}{\partial x^i} = e_{p,\alpha_i}$. Really, $\delta_{ij} = \frac{\partial x^i}{\partial x^j} = \frac{\partial((\alpha_i)^{n+1} + a_1(\alpha_i)^{n-2} + ... + a_n)}{\partial x^j} = ((n+1)(\alpha_i)^n + (n-1)a_1(\alpha_i)^{n-2} + ... + a_{n-1})\frac{\partial \alpha_i}{\partial x^j} + \frac{\partial a_1}{\partial x^j}(\alpha_i)^{n-1} + ... + \frac{\partial a_n}{\partial x^j} = p'(\alpha_i)\frac{\partial \alpha_i}{\partial x^j} + \frac{\partial p}{\partial x^j}(\alpha_i) = \frac{\partial p}{\partial x^j}(\alpha_i)$. Thus $\frac{\partial p}{\partial x^j}(\alpha_i) = \delta_{ij} = e_{p,\alpha_i}(\alpha_i)$ and therefore $\frac{\partial p}{\partial x^j}(z) = e_{p,\alpha_i}(z)$. By definition, this means that the polynomial e_{p,α_i} correspond to the tangent vector $\frac{\partial}{\partial x^i}$. Thus $(\partial/\partial x^1_\alpha, ..., \partial/\partial x^n_\alpha)$ form a canonical basis.

Prove now, that $l_p = \frac{\mathbf{d}a_1}{n+1}$. Really, considering the coefficients for z^{n-1} in $\frac{\partial a_1}{\partial x^i} z^{n-1} + ... + \frac{\partial a_n}{\partial x^i} = \frac{\partial p}{\partial x^i}(z) = e_{p,\alpha_i}(z)$ we find that $l_p(\partial/\partial x^i_\alpha) =$

$l_p(e_{p,\alpha_i}) = \frac{1}{n+1}\prod_{j\neq i}\frac{1}{(\alpha_i-\alpha_j} = \frac{1}{n+1}\frac{\partial a_1}{\partial x^i}$. Thus $l_p = \sum_{i=1}^n l_p(\partial/\partial x^i_\alpha)\mathbf{d}x^i_\alpha = \sum_{i=1}^n \frac{1}{n+1}\frac{\partial a_1}{\partial x^i}\mathbf{d}x^i_\alpha = \frac{\mathbf{d}a_1}{n+1}$.

Prove, that $E = \sum_{i=1}^n x^i_\alpha(\partial/\partial x^i_\alpha)$ don't depend from α and $L_E l_p = (\upsilon+1)l_p$. Prove at first, that $L_E p = p - \frac{z}{n+1}p'$. Really, these polynomials have the same degree $n-1$ and they are the same values in the points $\alpha_1,...,\alpha_n$, because $L_E(p)(\alpha_k) = (\sum_{i=1}^n x^i_\alpha(\partial/\partial x^i_\alpha)(p))(\alpha_k) = x^k = p(\alpha_k) = p(\alpha_k) - \frac{z}{n+1}p'(\alpha_k)$. Consider now the vector field $F = \sum_{i=k}^n \frac{k+1}{n+1}a_k\partial/\partial a_k$. Then $L_F(p)(z) = \sum_{i=k}^n \frac{k+1}{n+1}a_k z^{n-k} = \sum_{i=k}^n(1-\frac{n-k}{n+1})a_k z^{n-k} = p(z) - \frac{z}{n+1}p'(z) = L_E(p)(z)$. Thus $L_E(l_p) = L_F(\frac{\mathbf{d}a_1}{n+1}) = \frac{1}{n+1}\frac{2}{n+1}\mathbf{d}a_1 = \frac{2}{n+1}l_p$.
□

Construct now a coordinate systems $t = (t^1,...,t^n)$ on $Pol(n)$ by next specification. Consider the function $\omega = \omega(p,z)$ $Pol(n)\times\mathbb{C}$ such that $\omega^{n+1} = p(z)$ and $z = \omega + \frac{t^1}{\omega} + \frac{t^2}{\omega^2} + \frac{t^3}{\omega^3} + ...$ The equality $\omega^{n+1} = p(\omega + \frac{\tilde{t}^1}{\omega} + \frac{\tilde{t}^2}{\omega^2} + \frac{\tilde{t}^3}{\omega^3} + ...)$ gives a possibility to find $\tilde{t}^i$ as a polynomial of $\{a_j\}$. In particularity $\tilde{t}^1 = -\frac{a_1}{n+1}$ $\tilde{t}^2 = -\frac{a_2}{n+1}$. Put $t^1 = -(n+1)\tilde{t}^n$, $t^n = -\tilde{t}^1$ and $t^i = -\sqrt{n+1}\tilde{t}^{n+1-i}$ for $i = 2,...,n-1$.

Lemma 4.2. The coordinates $(t^1,...,t^n)$ are flat quasi-homogeneous and $d_i = \frac{n+2-i}{n+1}$.

Proof. Consider the set of polynomials $p(z,\tilde{t}) = z^{n+1} + a_1(\tilde{t})z^{n-1} + a_2(\tilde{t})z^{n-2} + ... + a_n(\tilde{t})$, where $\tilde{t} = (\tilde{t}^1,...,\tilde{t}^n)$. Consider the function $z(\omega,\tilde{t}) = \omega + \frac{\tilde{t}^1}{\omega} + \frac{\tilde{t}^2}{\omega^2} + \frac{\tilde{t}^3}{\omega^3} + ...$, such that $\omega^{n+1} = p(z(\omega,\tilde{t}),\tilde{t})$. We will consider z and $\{\tilde{t}^i\}$ as independent variables. Then $0 = \frac{\mathbf{d}\omega^{n+1}}{\mathbf{d}\tilde{t}^i} = \frac{\partial p}{\partial\tilde{t}^i} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial\tilde{t}^i} = \frac{\partial p}{\partial\tilde{t}^i} + p'\frac{1}{\omega^i}$. Thus $\frac{\partial p}{\partial\tilde{t}^i} = -p'\frac{1}{\omega^i}$ $g(\partial/\partial\tilde{t}^i,\partial/\partial\tilde{t}^j) = l_p(\partial/\partial\tilde{t}^i\partial/\partial\tilde{t}^j) = -\mathbf{Res}_{z=\infty}\frac{\frac{\partial p}{\partial\tilde{t}^i}\frac{\partial p}{\partial\tilde{t}^j}}{p'}\mathbf{d}z = -\mathbf{Res}_{z=\infty}\frac{p'\mathbf{d}z}{\omega^{i+j}} = -\mathbf{Res}_{\omega=\infty}\frac{\mathbf{d}p}{\omega^{i+j}} = -\mathbf{Res}_{\omega=\infty}\frac{\mathbf{d}\omega^{i+j}}{\omega^{i+j}} = (n+1)\delta_{i+j,n+1}$. Thus, $g(\partial/\partial t^i,\partial/\partial t^j) = \delta_{i+j,n+1}$.

Let $e = \sum_{i=1}^n \rho_i\frac{\partial}{\partial\tilde{t}^i}$ be the field of unity elements of the algebras A_p.Recall, that $\tilde{t}^1 = -\frac{a_1}{n+1}$. Using this fact and lemma 4.1, we find that $\delta_{\beta,1} = \mathbf{d}\tilde{t}^1(\partial/\partial\tilde{t}^\beta) = -\frac{a_1}{n+1}\mathbf{d}a_1(\partial/\partial\tilde{t}^\beta) = -l_p(\partial/\partial\tilde{t}^\beta) = -g(e,\partial/\partial\tilde{t}^\beta) = -g(\sum_{i=1}^n \rho_i\frac{\partial}{\partial\tilde{t}^i},\frac{\partial}{\partial\tilde{t}^\beta}) = -\sum_{i=1}^n \rho_i g(\frac{\partial}{\partial\tilde{t}^i},\frac{\partial}{\partial\tilde{t}^\beta}) = -(n+1)\sum_{i=1}^n \rho_i\delta_{i+\beta,n+1} = -(n+1)\rho_{n+1-\beta}$. Therefore, $e = -\frac{1}{n+1}\frac{\partial}{\partial\tilde{t}^n} = \frac{\partial}{\partial t^1}$

It is follow from lemma 4.1. that $L_E a_i = \frac{i+1}{n+1}a_i$. Thus, by the definition of $\tilde{t}^i$, we find that $L_E\tilde{t}^i = \frac{i+1}{n+1}\tilde{t}^i$. Therefore $L_E t^i = \frac{n+2-i}{n+1}t^i$, and $E = \sum_{i=1}^n \frac{n+2-i}{n+1}(\partial/\partial t^i)$.
□

Example 4.1. Find the numbers $\nu(\alpha_i)$ as some functions of flat quasi-homogeneous coordinates for $n = 2$. If $p(z) = z^3 + a_1 z + a_2$, then $p'(z) = 3z^2 + a_1$ and $\alpha_i = \pm\sqrt{-\frac{a_1}{3}}$. Thus, $\nu(\alpha_i) = \pm\frac{3}{2}2\sqrt{-\frac{a_1}{3}} = \pm\sqrt{-3a_1} = \pm3\sqrt{-t^1}$.

Example 4.2. The potential of the Frobenius manifold $Pol(n)$ is a polynomial F_n [14]. Its coefficients was found in [20]. In particulary $F_2 = \frac{1}{2}(t^1)^2t^2 + \frac{1}{24}(t^2)^4$.

5. Non-commutative Frobenius manifolds.

5.1. Extended WDVV equations.

A theory of deformations of closed strings is one of sours for the theory of Frobenius manifolds. Its mathematical equivalent is plane deformations of commutative Frobenius pairs. But the theory of closed strings is only part of more general open-closed strings theory. It is follow from [9](see also [7,8]), that a mathematica equivalent of a open-closed strings theory is a Cardy-Frobenius algebra. A theory of deformation for Cardy-Frobenius algebra was suggested in [15]. It continue the Kontsevich and Manin approach and gives some extension of WDVV equations to differential equations on series of non-commutative variables.

Describe more detail these equations. Let $t = (t^1, ..., t^n)$ (respectively $s = (s^1, ..., s^m)$ be the standard coordinates on $A \cong \mathbb{C}^n$ (respectively $B \cong \mathbb{C}^m$). Consider the algebras of formal tensor series $F = \sum c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l)t^{i_1} \otimes t^{i_2}...t^{i_k} \otimes s^{j_1} \otimes s^{j_2}...s^{j_k}$, $c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) \in \mathbb{C}$. Let F_A be the part of the series F that consists of all monomial without s^i .

Partial derivatives of F defined by partial derivatives of monomials.

We consider that $\frac{\partial(t^{i_1}\otimes t^{i_2}\otimes ...\otimes t^{i_k}\otimes s^{j_1}\otimes s^{j_2}\otimes...\otimes s^{j_k})}{\partial t^i}$ is the sum of monomials $t^{i_1} \otimes t^{i_2} \otimes ... \otimes t^{i_{p-1}} \otimes t^{i_{p+1}} ... \otimes t^{i_k} \otimes s^{j_1} \otimes s^{j_2} \otimes ... \otimes s^{j_k})$, such that $i_p = i$.

Reciprocally $\frac{\partial(t^{i_1}\otimes t^{i_2}\otimes ...\otimes t^{i_k}\otimes s^{j_1}\otimes s^{j_2}\otimes...\otimes s^{j_k})}{\partial s^j}$ is the sum of monomials $t^{i_1} \otimes t^{i_2} \otimes ... \otimes t^{i_k} \otimes s^{j_1} \otimes s^{j_2} \otimes ... \otimes s^{j_{p-1}} \otimes s^{j_{p+1}} ... \otimes s^{j_k}$, such that $j_p = j$.

Put $\frac{\partial^2}{\partial t^i \partial t^j} = \frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}$, $\frac{\partial^2}{\partial t^i \partial s^j} = \frac{\partial}{\partial t^i}\frac{\partial}{\partial s^j}$, $\frac{\partial^2}{\partial s^i \partial s^j} = \frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}$, $\frac{\partial^3}{\partial t^i \partial t^j \partial t^k} = \frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j}\frac{\partial}{\partial t^k}$.

The definition of the partial derivatives $\frac{\partial^3(t^{i_1}\otimes \cdots \otimes t^{i_k}\otimes s^{j_1}\otimes \cdots \otimes s^{j_\ell})}{\partial s^i \partial s^j \partial s^r}$ is more complicated. They are the sum of monomials $t^{i_1} \otimes \cdots \otimes t^{i_k} \otimes s^{k_2} \otimes \cdots \otimes s^{k_{p-1}} \otimes s^{k_{p+1}} \otimes \cdots \otimes s^{k_{q-1}} \otimes s^{k_{q+1}} \otimes s^{k_\ell}$ such that the sequences $s^i, s^{k_2}, \cdots, s^{k_{p-1}}, s^j, s^{k_{p+1}}, \cdots, s^{k_{q-1}}, s^r, s^{k_{q+1}}, \cdots, s^{k_\ell}$ and $(s^{j_1}, \cdots, s^{j_\ell})$ are the same after an cyclic transposition.

We consider that a monomials $t^{i_1} \otimes \cdots \otimes t^{i_k} \otimes s^{j_1} \otimes \cdots \otimes s^{j_\ell}$ and $t^{\widetilde{i}_1} \otimes \cdots \otimes t^{\widetilde{i}_k} \otimes s^{\widetilde{j}_1} \otimes \cdots \otimes s^{\widetilde{j}_\ell}$ are *equivalent*, if $\cup_{r=1}^k i_r = \cup_{r=1}^k \widetilde{i}_r$ and $\cup_{r=1}^l j_r = \cup_{r=1}^l \widetilde{j}_r$. Let $[t^{i_1} \otimes \cdots \otimes s^{j_\ell}]$ be the equivalent class of $t^{i_1} \otimes \cdots \otimes s^{j_\ell}$. The tensor series $F = \sum c(i_1, ..., i_k|j_1, ..., j_\ell)a_{i_1} \otimes \cdots \otimes b_{j_\ell}$ generate the tensor series $[F] = \sum c[i_1, ..., i_k|j_1, ..., j_\ell][a_{i_1} \otimes \cdots \otimes b_{j_\ell}]$, where the sum is the sum by all equivalent classes of monomials and $c[i_1, ..., i_k|j_1, ..., j_\ell]$ $c(\widetilde{i}_1, ..., \widetilde{j}_\ell)$ is the sum of coefficients, corresponding monomials from equivalent class $[a_{i_1} \otimes \cdots \otimes b_{j_\ell}]$.

We say that a tensor series $F=\sum c(i_1\cdots i_k|j_1\cdots j_\ell)t^{i_1}\otimes\cdots\otimes t^k\otimes s^{j_1}\otimes\cdots\otimes s^{j_\ell}$ satisfy *extended WDVV equations* on a space $H=A\oplus B$, if next conditions satisfied

1. The coefficients $c(i_1\cdots i_k|j_1\cdots j_\ell)$ are invariant with respect to all transpositions of the indexes $\{i_r\}$.

2. The coefficients $c(i,j|)\ \ c(|i,j)$ form non-degenerated matrixes. Let $F_a^{t^it^j}\ \ F_b^{s^is^j}$ be the matrixes, invert to $c(i,j|)\ \ c(|i,j)$ reciprocally.

3.
$$[\sum_{p,q=1}^{n}\frac{\partial^3 F_A}{\partial t^i\partial t^j\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^3 F_A}{\partial t^q\partial t^k\partial t^\ell}]=[\sum_{p,q=1}^{n}\frac{\partial^3 F_A}{\partial t^k\partial t^j\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^3 F_A}{\partial t^q\partial t^i\partial t^\ell}].$$

4.
$$[\sum_{p,q=1}^{m}\frac{\partial^3 F}{\partial s^i\partial s^j\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3 F}{\partial s^q\partial s^k\partial s^\ell}]=\sum_{p,q=1}^{m}[\frac{\partial^3 F}{\partial s^\ell\partial s^i\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3 F}{\partial s^q\partial s^j\partial s^k}].$$

5.
$$[\sum\frac{\partial^2 F}{\partial t^k\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3 F}{\partial s^q\partial s^i\partial s^j}]=[\sum\frac{\partial^2 F}{\partial t^k\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3 F}{\partial s^q\partial s^j\partial s^i}].$$

6.
$$[\sum\frac{\partial^2 F}{\partial s^k\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^3 F}{\partial t^q\partial t^i\partial t^j}]=[\sum\frac{\partial^2 F}{\partial t^i\partial s^p}\otimes F_b^{s^ps^q}\frac{\partial^3 F}{\partial s^q\partial s^k\partial s^r}\otimes F_b^{s^rs^\ell}\frac{\partial^2 F}{\partial s^\ell\partial t^j}].$$

7.
$$[\sum\frac{\partial^2 F}{\partial s^u\partial t^p}\otimes F_a^{t^pt^q}\frac{\partial^2 F}{\partial t^q\partial s^v}]=[\sum\frac{\partial^3 F}{\partial s^u\partial s^p\partial s^r}F_b^{s^rs^l}\otimes F_b^{s^ps^q}\frac{\partial^3 F}{\partial s^l\partial s^v\partial s^q}].$$

In [15] are demonstrated that solutions of extended WDVV equations one-to-one correspond to potentials of some extension of Cohomological Field Theory and describe some class of deformations for Cardy-Frobenius algebras. Thus it is natural to consider solutions of extended WDVV equations as (non commutative) extension of Frobenius manifolds. Late we prove that deformation of quaternion Landau-Ginsburg models gives an example of such non commutative Frobenius manifolds.

5.2. Cardy-Frobenius bundles.

Definition. Let M be a smooth $\mathbb{K}$-manifold, were $\mathbb{K}$ is the real or the complex field. By *Cardy-Frobenius bundle* on M we call a pair of $\mathbb{K}$-bundles $\varphi_A:A\to M$ and $\varphi_B:B\to M$ with flat connection ∇_B, and a set of Cardy-Frobenius algebras $\{(A_p,l_p^A),(B_p,l_p^B,*),\phi_p\}$ over $\mathbb{K}$, where $A_p=\varphi_A^{-1}(p)$, $B_p=\varphi_B^{-1}(p)$, such that:

1. The algebras $\{(A_p,l_p^A)\}$ form a Dubrovin connection.

2. The connection ∇_B conserve the family of bilinear forms $\{(b_1,b_2)_p=l_p^B(b_1b_2)|p\in M\}$, graduation and involution $*$.

Call by *a flat system of coordinates* on B a family of linear coordinates systems $s=\{s_p=(s_p^1,...,s_p^m)|p\in M\}$ on bands B_p, that is invariant by ∇_B. It generates a basis $(\frac{\partial}{\partial s_p^1},...,\frac{\partial}{\partial s_p^m})$ on any vector space

B_p. Axiom 2. say that values $(\frac{\partial}{\partial s^i_p}, \frac{\partial}{\partial s^j_p})_p = l^B_p(\frac{\partial}{\partial s^i_p}\frac{\partial}{\partial s^j_p})$ are constants on M.

3. Let $s = (s^1, ..., s^m)$ be a flat system of coordinates on B. Then the tensor fields $c^B(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}, \frac{\partial}{\partial s^k}) = l^B_p(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}\frac{\partial}{\partial s^k})$, are smooth as functions on M and are called *B-structures tensors.*

4. The natural map $\phi = \{\bigcup \phi_p | p \in M\} : A \to B$ is smooth. It define smooth *transition tensors field* $c^{AB}(a, b) = l^B_p(\phi(a)b)$.

Theorem 5.1. Let M be a semi-simple Frobenius manifold with a Dubrovin connection $\{(A_p, l^A_p)|p \in M\}$. Then there exist Cardy-Frobenius bundles $\{(A_p, l^A_p), (B_p, l^B_p, *), \phi\}$.

Proof. Let $(x^1_\alpha, ..., x^n_\alpha)$ be canonical coordinates on M. Put $\lambda_{p,i} = l^A_p(\partial/\partial x^i_\alpha(p))$. Let $\mu_{p,i}$ be a smooth function on M, such that $\mu^2_{p,i} = \lambda_{p,i}$. Consider the family of Frobenius pair $(B_{p,i}, l^B_{p,i}, *) = \mathbb{M}(m, \mathbb{C})(\mu_{p,i})$ from example 2.2. Put $(B_p, l^B_p, *) = \bigoplus_i (B_{p,i}, l^B_{p,i}, *)$ and $B = \bigoplus_{p \in M} B_p$. Tautological isomorphisms between matrix algebras define a connection on B. Define a structure of smooth manifold on B considering that the projection $\varphi_B(B_p) = p$ is smooth. Define, finally, the homomorphism $\phi_p : A_p \to B_p$, considering that $\phi_p(\partial/\partial x^i_\alpha)$ is unit element of the algebra $B_{p,i}$. According to example 2.5. the structure that we constructed is Cardy-Frobenius bundles.

□

Define a potential for Cardy-Frobenius bundles.

Definition. Let $\varphi_A : A \to M$, $\varphi_B : B \to M$, ∇_B, $\{(A_p, l^A_p), (B_p, l^B_p, *), \phi_p | p \in M\}$ be a Cardy-Frobenius bundle on M. Let $t = (t^1, ..., t^n)$ be a system of flat quasi-homogenies coordinates of the Dubrovin connection (A_p, l^A_p) and let $s = (s^1, ..., s^m)$ be a system of flat coordinates on B. By *potential* of this Cardy-Frobenius bundle is called the formal tensor series $F(t|s) = \sum c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) t^{i_1} \otimes t^{i_2} ... t^{i_k} \otimes s^{j_1} \otimes s^{j_2} ... s^{j_k}$, where $c(i_1, i_2, ..., i_k|j_1, j_2, ..., j_l) \in \mathbb{C}$, such that

1. The matrixes $c(i, j|) = l^A_p(\frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j})$ $c(|i, j) = l^B_p(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j})$ are non-degenerated. Let $F^{t^i t^j}_a$ and $F^{s^i s^j}_b$ be the matrixes inverted to $c(i, j|)$ and $c(|i, j)$ respectively.

2. If F_A is the part of F that don't depend from s, that it pass to a potential of Dubrovin connection (M_p, l^A_p) after changing the tensor multiplication to the ordinary multiplication.

3. The formal tensor series $\frac{\partial^3 F}{\partial s^i \partial s^j \partial s^r}$ are not depend from s and, after changing the tensor multiplication to the ordinary multiplication, coincide with the B-structure tensors of the bundle.

4. The formal tensor series $\frac{\partial^2 F}{\partial t^i \partial s^j}$ are not depend from s and , after changing the tensor multiplication to the ordinary multiplication, coincide with the transition function of the bundle.

Theorem 5.2. Let a Dubrovin connection (A_p, l^A_p) on M has a potential in form of Taylor series. Then any Cardy-Frobenius bundle

$\varphi_A : A \to M$, $\varphi_B : B \to M$, ∇_B, $\{(A_p, l_p^A), (B_p, l_p^B, *), \phi_p | p \in M\}$ also has a potential.

Proof. Let $t = (t^1, ..., t^n)$ be a flat quasi-homogeneous coordinates of Dubrovin connection $\{(A_p, l_p^A) | p \in M\}$ and let $s = (s^1, ..., s^m)$ be be a flat coordinates system on B. Consider the potential of the Dubrovin connection $\{(A_p, l_p^A) | p \in M\}$. Changing the ordinary multiplication to the tensor multiplication we obtain a tensor series F_A. Let F_A^i be the tensor series, such that $\frac{\partial F_A^i}{\partial t^i} = F_A$. Then the tensor series $F = F_A + \frac{1}{2} \sum l^A(\frac{\partial}{\partial t^i} \frac{\partial}{\partial t^j}) t^i \otimes t^j + \sum l^B(\frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j}) s^i \otimes s^j + \sum F_A^i t^i \otimes s^i + \frac{1}{3} \sum l^B(\frac{\partial}{\partial s^i} \frac{\partial}{\partial s^j} \frac{\partial}{\partial s^k}) s^i \otimes s^j \otimes s^k$ is the potential of the Cardy-Frobenius bundle.

□

Theorem 5.3. The potential of any Cardy-Frobenius bundle satisfy the extended WDVV equations.

Proof. All relation follow from properties of Cardy-Frobenius algebras $\{(A_p l_p^A), (B_p, l_p^B, *), \phi_p\}$. In particulary relation 1 follow from the commutativity of A. Relation 2 follow from the non-degeneracy the bilinear forms on A and B. Relations 3 and 4 follow from the associativity of algebras A and B. Relation 5. is true because $\phi_p(A_p)$ belong to center of algebra B_p. Relation 6 is true because the map ϕ_p is a homomorphism. Relation 7 follow from Cardy axiom.

□

5.3. Moduli space of quaternion Landau-Ginsburg models.

Consider the family of Cardy-Frobenius algebras $\{(A_p, l_p^A), (B_p, l_p^B,), \phi_p | p \in Pol(n)\}$, of quaternion Landau-Ginsburg models from section 3. Let $\varphi_A^{LG} : A \to Pol(n)$ and $\varphi_B^{LG} : B \to Pol(n)$ be the natural projections$\varphi_A^{LG}(A_p) = p, \varphi_B^{LG}(B_p) = p$.

Construct a connection on $\varphi_B^{LG} : B \to Pol(n)$. Consider the vector space $\mathbb{C}^{4n}$ with the basis $\{e_i^0, e_i^1, f_i^0, f_i^1 | i = 1, , , n\}$. For any $p \in Pol(n)$ consider the map $\Phi_p : B_p \to Pol(n) \times \mathbb{C}^{4n}$ such that $\Phi_p(e^0_{p,\alpha_j}) = (p, e_j^0)$,$\Phi_p(e^1_{p,\alpha_j}) = (p, e_j^1)$, $\Phi_p(f^0_{p,\alpha_j}) = (p, f_j^0)$, $\Phi_p(f^1_{p,\alpha_j}) = (p, f_j^1)$. Put $B = \bigoplus B_p$, $\Phi = \bigoplus \Phi_p$. Define a complex structure on B, considering that Φ is a holomorphic. The map Φ pass the natural connection on $Pol(n) \times \mathbb{C}^{4n}$ to a connection ∇_B^{LG} on $\varphi_B^{LG} : B \to Pol(n)$.

Theorem 5.4. The set $\varphi_A^{LG} : A \to Pol(n)$, $\varphi_B^{LG} : B \to Pol(n)$, ∇_B^{LG}, $\{(A_p, l_p^A), (B_p, l_p^B), \phi_p | p \in Pol(n)\}$ is a complex Cardy-Frobenius bundle $LG(n)$.

Proof. Any vector $b_p \in B_p$ has a representation in the form $b_p = b_1 + ... + b_n$, where $b_i = s^{i1} e^0_{p,\alpha_i} + s^{i2} e^1_{p,\alpha_i} + s^{i3} f^0_{p,\alpha_i} + s^{i4} f^1_{p,\alpha_i}$. Therefore the functions s^{ij} form a flat coordinate system on B and $\frac{\partial}{\partial s^{i1}} = e^0_{p,\alpha_i}$, $\frac{\partial}{\partial s^{i2}} = e^1_{p,\alpha_i}$, $\frac{\partial}{\partial s^{i3}} = f^0_{p,\alpha_i}$, $\frac{\partial}{\partial s^{i4}} = f^1_{p,\alpha_i}$. Moreover $l^B(\frac{\partial}{\partial s^{i2}}) = i\sqrt{2}$ and $l^B(\frac{\partial}{\partial s^{ij}}) = 0$ in other cases. Thus all axioms of Cardy-Frobenius bundles are true.

□

Example 5.1. By our definitions, $\frac{\partial}{\partial s^{ik}}\frac{\partial}{\partial s^{jl}} = 0$ for $i \neq j$ and
$\frac{\partial}{\partial s^{i1}}\frac{\partial}{\partial s^{il}} = \frac{\partial}{\partial s^{il}}$,
$\frac{\partial}{\partial s^{i2}}\frac{\partial}{\partial s^{i2}} = -\nu(\alpha_i)\frac{\partial}{\partial s^{i1}}$,
$\frac{\partial}{\partial s^{i2}}\frac{\partial}{\partial s^{i3}} = -\frac{\partial}{\partial s^{i3}}\frac{\partial}{\partial s^{i2}} = -\frac{\partial}{\partial s^{i4}}$,
$\frac{\partial}{\partial s^{i2}}\frac{\partial}{\partial s^{i4}} = -\frac{\partial}{\partial s^{i4}}\frac{\partial}{\partial s^{i2}} = -\nu(\alpha_i)\frac{\partial}{\partial s^{i3}}$,
$\frac{\partial}{\partial s^{i3}}\frac{\partial}{\partial s^{i3}} = \frac{\partial}{\partial s^{i1}}$,
$\frac{\partial}{\partial s^{i3}}\frac{\partial}{\partial s^{i4}} = -\frac{\partial}{\partial s^{i4}}\frac{\partial}{\partial s^{i3}} = -\frac{\partial}{\partial s^{i2}}$,
$\frac{\partial}{\partial s^{i4}}\frac{\partial}{\partial s^{i4}} = \nu(\alpha_i)\frac{\partial}{\partial s^{i1}}$.

Thus we can to find the bilinear form $(\frac{\partial}{\partial s^{ij}}, \frac{\partial}{\partial s^{kl}})^B = l^B(\frac{\partial}{\partial s^{ij}}\frac{\partial}{\partial s^{kl}})$ and the structure tensor $c^B(\frac{\partial}{\partial s^{ij}}, \frac{\partial}{\partial s^{kl}}, \frac{\partial}{\partial s^{uv}}) = l^B(\frac{\partial}{\partial s^{ij}}\frac{\partial}{\partial s^{kl}}\frac{\partial}{\partial s^{uv}})$, by values $\nu(\alpha_i)$. Example 4.1. contain an algorithm for these calculations for $n = 2$. In this case $\nu(\alpha_i) = \pm 3\sqrt{-t^1}$.

Example 5.2. Coupling between canonical $x = (x^1, ..., x^n)$ and flat quasi-homogeneous coordinates $t = (t^1, ..., t^n)$ generate transition tensors. Demonstrate this for $n = 2$.

According to our definitions, $\frac{\partial}{\partial t^1} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ $\frac{\partial}{\partial t^2} = R_1\frac{\partial}{\partial x^1} + R_2\frac{\partial}{\partial x^2}$. Thus $\frac{\partial}{\partial t^2}\frac{\partial}{\partial t^2} = R_1^2\frac{\partial}{\partial x^1} + R_2^2\frac{\partial}{\partial x^2}$. On the other hand, according to example 4.2., $\frac{\partial}{\partial t^2}\frac{\partial}{\partial t^2} = t^2\frac{\partial}{\partial t^1} = t^2(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2})$. Thus $R_i = \pm\sqrt{t^2}$.

It is follow from example 4.2., that $F_A = \frac{1}{2}(t^1)^2t^2 + \frac{1}{24}(t^2)^4$. Therefore $F_A^1 = \frac{1}{6}(t^1)^3t^2 + \frac{1}{24}t^1(t^2)^4$ and $F_A^2 = \frac{1}{4}(t^1)^2(t^2)^2 + \frac{1}{120}(t^2)^5$.

Thus, we can to find the potential of the Cardy-Frobenius bundle $LG(2)$. According to theorem 5.2 it is the tensor series
$F = F_A + \frac{1}{2}\sum l^A(\frac{\partial}{\partial t^i}\frac{\partial}{\partial t^j})t^i \otimes t^j + \sum l^B(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j})s^i \otimes s^j + \sum F_A^i t^i \otimes s^i + \frac{1}{3}\sum l^B(\frac{\partial}{\partial s^i}\frac{\partial}{\partial s^j}\frac{\partial}{\partial s^k})s^i \otimes s^j \otimes s^k$.

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References

1.R.Dijkgraaf, E.Witten, Mean field theory theory, topological field theory theory and multi-matrix models, Nucl.Phys. B342(1990)486.

2. T.Eguchi, S.-K. Yang, N=2 super-conformal models and topological field theories, Tokyo-preprint UT-564.

3. C.Vafa, Topological Landau-Ginsburg models. Modern Physics Letters A Vol 6 N 4 (1991) 337-349.

4. M.Atiyah, Topological quantum field theories, Inst.Hautes Etudes Sci. Publ. Math., 68(1988) 175-186.

5. R.Dijkgraaf, Geometrical approach to two-dimensional conformal field theory, Ph.D. Thesis(Utrecht,1989).

6. V. Turaev, Homotopy field theory in dimension 2 and group-algebras. http://xxx.lanl.org/math.GT/9910010.

7. C.I.Lazorau, On the structure of open-closed topological field theory in two-dimension Nucl.Phys. B603(2001),497-530.

8. G.Moor, D-branes, RR-Fields and K-theory. http://online.itp.ucsb.edu/online/mp01/moore2/

9. A.Alekseevskii, S.Natanzon, Noncommutative two-dimensional topological field theories and Hurwitz numbers for real algebraic curves. http://xxx.lanl.org/math.GT/0202164.

10. A.Kapustin, Y.Li, Topological correlators in Landau-Ginsburg models with boundaries. Adv.Ther.Math.Phys. 7(2003) 727-749.

11. I.Brunner, M.Herbst, W.Lerche, B.Scheuner, Landau-Ginsburg realization of open string TFT. http://xxx.lanl.org/hep-th/0305133.

12. A.Kapustin, Y.Li, D-branes in topological minimal models: Landau-Ginsburg approach. http://xxx.lanl.org/hep-th/0306001.

13. R.Dijkgraaf, E.Verlinde, H.Verlinde, Note on topological string theory and 2D quantum gravity, Nucl. Phys.B352(1991) 59.

14. B.Dubrovin, Geometry of 2D topological field theories In: LNM, 1620(1996) 120-348.

15. S.Natanzon, Extended cohomological field theories and noncommutative Frobenius manifolds, J. Geom. Phys, 51 (2003), 387-403.

16. M.Kontsevich, Yu.Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, Commun. Math. Phys, 164 (1994), 525-562.

17. Yu.Manin, Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces, vol.47, American Mathematical Society, Colloquium Publication, 1999.

18. .Faith Algebra: rings, modules and categories I. Springer-Verlag Berlin Heidelberg New York 1973.

19. S.Natanzon Structures de Dubrovin. Preprint 1997/26 IRMA Strasbourg 1997.

20. S.Natanzon, Formulas for A_n and B_n - solutions of WDVV equations,J. Geom. Phys, 39 (2001), 323-336.

21. E.Witten, On the structure of the topological phase of two-dimensional gravity,Nucl.Phys. B340(1990) 281

22. V.Arnold, S. Gusein-Zade, A.Varchenko. Singularities of differentiable maps, vols I,II. Birkhauser,Boston,1985 and 1988.

23. M.Herbst, C.I.Lazorau, W.Lerche. Superpotentials, A_∞ relations and WDVV equations for open topological strings. //xxx.lanl.org/hep-th/0402110.

24. C.I.Lazorau. On the boundary coupling of topological Landay-Ginzburg models. //xxx.lanl.org/hep-th/0312286.

25. A.Kapustin, Y.Li, D-branes in Landay-Ginzburg models and algebraic geometry. JHEP 0312 (2003)005 http://xxx.lanl.org/hep-th/0210296.