Kähler flat manifolds of low dimensions

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Abstract

We give a list of six dimensional flat Kähler manifolds. Moreover, we present an example of eight dimensional flat Kähler manifold M with finite $Out(\pi_1(M)$ group.

1 Introduction

We shall consider a flat manifolds of $\mathbb{R}$-dimension 4, 6 and 8 which have also a complex Kähler structure of dimension 2, 3 and 4 correspondigly. We would like to mention that any such manifold is also a Calabi-Yau manifold. In dimension four and six we shall give a complete list of such manifolds. That occupied the first parts of a paper. In the last one we consider an example of eight $\mathbb{R}$-dimensional flat manifold with Kähler structure and holonomy group $\mathbb{Z}_3 \times \mathbb{Z}_3$. Its fundamental group has a trivial centre and finite outer automorphism group. We finish with some open problems and questions.

2 Flat Kähler manifolds

In this section we shall introduce the basic definitions and conventions. We refer to [6]. A closed flat Riemannian manifold M is isometric to one of the form $M = \Gamma\backslash E(n)/O(n)$ where $E(n) = O(n) \ltimes R^n$ is the group of Euclidean motions of R^n and $\Gamma \subset E(n)$ is a cocompact, discrete and torsion free. From a Bieberbach theorem it is known that (cf. [10]) $\pi_1(M) = \Gamma$ and a subgroup of

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all translations T of Γ is of finite index and quotient group Γ/T is isomorphic to holonomy group of M. Hence we have a short exact sequence of groups

$$0 \to T \to \Gamma \to H \to 0,$$

where T is a torsion free abelian group $\mathbb{Z}^n$. By conjugation the above short exact sequence is defined a faithful (cf. [10]) holonomy representation $\phi : H \to GL(n, \mathbb{Z})$. We shall call such groups Γ a Bieberbach groups. Let us assume that n is an even number. We say that ϕ is essential complex if there exist some matrix $A \in GL(n, \mathbb{R})$ such that for any $h \in H, A\phi(h)A^{-1} \in GL(\frac{n}{2}, \mathbb{C})$. In [6, Theorem 3.1] is proved the following

Theorem 1 *([6, Theorem 3.1]) The following conditions on the group group $\Gamma \subset E(n)$ are equivalent*

(i) Γ is a fundamental group of Kähler flat manifold

(ii) Γ is a Bieberbach group and its holonomy representation is essential complex

(iii) Γ is a discrete cocompact torsion-free subgroup of $U(\frac{n}{2}) \ltimes \mathbb{C}^{\frac{n}{2}}$.

Moreover we have the following characterizations of essential complex representation (cf. [6, Proposition 3.2].)

Proposition 1 *Let G be a finite group and $h : G \to GL(m, \mathbb{R})$ some faithful representation. Then h is essentially complex if and only if m is even and each $\mathbb{R}$-irreducible summand of h which is also $\mathbb{C}$-irreducible occurs with even multiplicity.*

Proof Let W be a left $\mathbb{R}[G]$ module which is defined by the representation h. We have a direct summ $W = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ of $\mathbb{R}[G]$ irreducible modules V_i for $i = 1, 2, ..., k$. From definition (cf. [4, Theorem (73.9)]) there are three kinds of it: absolutely irreducible, "complex" and "quaternionic". A simple $\mathbb{R}[G]$ module V is "complex" if $End_{\mathbb{R}[G])}(V) = \mathbb{C}$ and is "quaternionic" if $End_{\mathbb{R}[G]}(V) = \mathbb{H}$. If $End_{\mathbb{R}[G])}(V) = \mathbb{R}$ then V is absolutely irreducible. If we consider the $\mathbb{C}[G]$-module $V \otimes \mathbb{C}$, then in the first case it is simple, in "complex" case it is the direct sum of two non-isomorphic simple, complex conjugate modules, or in "quaternionic" case is the direct sum of two isomorphic simple modules. Hence it is clear that "complex" and "quaternionic" summands of W are essential complex. Moreover if V is absolutely irreducible $\mathbb{R}[G]$- module then we can consider $(V \oplus V)$ as $V \otimes \mathbb{C}$ and from above an even number of abslutely irreducible modules has a complex structure.

□

Let us introduce a definition.

Definition 1 *A Calabi-Yau manifold is a Kähler one which admits a Ricci-flat metric.*

From [1, Proposition 10.29] the orientation is a sufficient condition for a flat Kähler manifold to be a Calabi-Yau one. By Proposition 1 it is always satisfied. In fact, the determinant of any $\mathbb{R}$-irreducible and $\mathbb{C}$ reducible representation is one. Moreover, the determinant of any even number of absolutely irreducible representations is also one. Hence we have.
Example All Kähler flat manifolds are Calabi-Yau.

In the next part, using the above results, we shall give a list of the Kähler flat manifolds in $\mathbb{R}$-dimension 4 and 6.

3 Dimension four and six

From well known classification (cf. [2] and [3]) of flat manifolds we have in $\mathbb{R}$-dimension four only 7 Kähler flat manifolds. All of them have cyclic holonomy. There two (03/01/01/ and 03/01/02) with $\mathbb{Z}_2$ holonomy group, two (07/02/01 and 07/02/02) with $\mathbb{Z}_4$ holonomy group, two (08/01/01/002 and 08/01/02/00) with $\mathbb{Z}_3$ holonomy group and one with $\mathbb{Z}_6$ holonomy group (09/01/01). Here we use notations from [2]. In $\mathbb{R}$-dimension 6 we we have much more cases.

Lemma 1 *The following finite groups are a holonomy groups of 3 dimensional Kähler flat manifolds:* $\mathbb{Z}_n$, *for* $n = 2, 3, 4, 5, 6, 8, 10, 12$, $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_6 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_6 \times \mathbb{Z}_3, \mathbb{Z}_6 \times \mathbb{Z}_4, \mathbb{Z}_6 \times \mathbb{Z}_6$.

Proof: In [3] is given a list of holonomy groups of six dimensional flat manifolds. We shall use the notations for the finite groups from [3] which is the same as in [2]. We have to proof that the above property for the holonomy representation, from Proposition 1, is satisfied only for the group from above list. For example any flat manifold with the first Betti number one cannot be Kähler. Let us consider first a 2-groups. A group $[64, 250]$ is holonomy group of the uniqe 6-dimensional Bieberbach group with trivial center. Moreover a faithful 4 dimensional representation of it is absolutelly irreducible. It means it cannot be double (cf.Proposition 1). In similar way we can consider the groups $[32,47],[32,46],[32,36],[32,33],[32,31],[16,13]$,$[16,11],[16,8]$. The faithful integral representation of the group $[16,9]$ is a direct product of two $\mathbb{R}$ irreducible representations (cf. [2, page 182]). Since it is nonabelian group then the above $\mathbb{R}$ irreducible components are $\mathbb{C}$ irreducible. Hence the conditions of the Proposition 1 cannot be satisfied. For a group D_{16}, we have $d(D_{16}) = 6$ and we have 5 Bieberbach groups of dimension 6 with it holonomy. Only one has trivial center, but two different one dimensional $\mathbb{R}$-irreducible representation as components of holonomy representation. The similar is case of a group D_{24}. Here $d(G)$ denotes a minimal dimension of a flat manifold with holonomy group G. For the groups $\mathbb{Z}_2 \times [80,52], [80,52]$, $\mathbb{Z}_2 \times D_{10}, D_{10}, \mathbb{Z}_6 \times D_8, \mathbb{Z}_3 \times D_8$, $\mathbb{Z}_2 \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2)$, $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2, \mathbb{Z}_3 \rtimes \mathbb{Z}_8, \mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$, $\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4), \mathbb{Z}_3 \rtimes \mathbb{Z}_4, \mathbb{Z}_3 \times Q_8, Q_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4 \times A_4, \mathbb{Z}_6 \times A_4, \mathbb{Z}_3 \times A_4$, $\mathbb{Z}_2^3 \times A_4, \mathbb{Z}_3 \times S_4, \mathbb{Z}_2 \times S_4, D_6^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4 \times D_6, \mathbb{Z}_4 \times D_6, \mathbb{Z}_2^3 \times D_6$, $\mathbb{Z}_3 \times D_8, \mathbb{Z}_6 \times D_8, \mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6, \mathbb{Z}_4 \times A_4, \mathbb{Z}_2 \times D_{24}$ we have the first Betti number one. Let us consider the dihedral groups of order 6 and 8, D_6 and D_8. They have a faithful absolutely irreducible representations of rank 2 and one dimensional irreducible representations. Hence it easy to see that there is not possible to define six dimensional Bieberbach group with holonomy group D_6 and D_8 which holonomy representation satisfies Proposition 1. Then we can eliminate the groups: $D_6, \mathbb{Z}_2 \times D_6, \mathbb{Z}_3 \times D_6$, $\mathbb{Z}_6 \times D_6, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_6, D_8, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_8, \mathbb{Z}_4 \times D_8$. Similar arguments works for the symmetric group of order 24, S_4. It has two $\mathbb{C}$-irreducible faithful representations of rank 3, one of rank two and two one-dimensional. Also a group A_4 has absolutely irreducible faithful representation of rank 3. Hence cannot be on our list. Similar the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4$ and $\mathbb{Z}_2 \times A_4$ can be eliminate. All irreducible representation of the group $\mathbb{Z}_2^n, n \geq 1$ are one-dimensional. Hence for $n \geq 3$ there non exist 6 diemensional Bieberbach group with holonomy $\mathbb{Z}_2^n, n \geq 3$. Then we eliminate a groups: $\mathbb{Z}_2^3 \times \mathbb{Z}_4, \mathbb{Z}_2^3 \times \mathbb{Z}_3, \mathbb{Z}_2^4 \times \mathbb{Z}_3, \mathbb{Z}_2^n, n = 3,4,5$. To eliminate a group $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ we have to consider any integer faithful representation of it. The minimal dimension of it is four, and $\mathbb{R}$-irreducible components have dimensions one and two. Hence once make conclusion that conditions of Proposition 1 imput existance of cyclic subgroup $C \subset \mathbb{Z}_2^2 \times \mathbb{Z}_4$ such that any six dimensional faithful representation V of $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ has property $V^C = 0$. But it proves that any crystallographic group with holonomy group $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ which satisfies conditions of Proposition 1 has a torsion (cf. [8]). To eliminate a groups $[24,11]$ and $\mathbb{Z}_2 \times [16,9]$ we observe that the first one has a subgroup $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ and second one has a subgroup $\mathbb{Z}_2^4$ which were excluded above. There is only one group of order sixteen $[16, 10]$ in notations of [3] which we do not consider yet. It is a holonomy group of of the 31 Bieberbach groups of rank 6 with non trivial center. We can prove that all of them have the first Betti number one (cf. [7, Lemma 1, page 194].) Moreover it is also a holonomy group of 3 Bieberbach groups of rank 6 with trivial center. In this case it is easy to see, for example from elementary representation theory, that conditions of the Proposition one are not satisfied. That finish a proof. $\square$

Finally we have.

Theorem 2 *There are 172 3 dimensional Kähler (Calabi - Yau) flat manifolds.*

Proof: We shall use the results about the holonomy groups proved in Lemma 1 and list of six dimensional Bieberbach group from CARAT, [3]. To prepare the final list we shall use mainly the fact that, the first Betti number of any Kähler flat manifold is an even integer. It is an easy consequence of the Proposition 1, that each $\mathbb{R}$-irreducible summands of the holonomy representation which is also $\mathbb{C}$-irreducible has even multiplicity. For cyclic groups we have to consider the cases of order $2, 3, 4, 5, 6, 8, 10$ and 12. From the classification [3] we have 11 Bieberbach groups with $\mathbb{Z}_2$ holonomy. Five of them are fundamental groups of Kähler flat manifolds. In the case of the group $\mathbb{Z}_3$ all groups (4) satisfied conditions of the Proposition 1. In similar way we recocnize that there are 22 3 dimensional Kähler flat manifolds with holonomy $\mathbb{Z}_4$, 2 with holonomy group $\mathbb{Z}_5$, 14 with holonomy group $\mathbb{Z}_6$, one with holonomy group $\mathbb{Z}_8$ and $\mathbb{Z}_{10}$. Finally there are 6 Kähler flat manifolds with holonomy group $\mathbb{Z}_{12}$. For non cyclic holonomy groups we have 4 Bieberbach groups with holonomy group $\mathbb{Z}_2 \times \mathbb{Z}_2$, 45 with holonomy group $\mathbb{Z}_2 \times \mathbb{Z}_4$, 7 with holonomy $\mathbb{Z}_6 \times \mathbb{Z}_2$, 6 with holonomy $\mathbb{Z}_6 \times \mathbb{Z}_3$, 4 with holonomy $\mathbb{Z}_6 \times \mathbb{Z}_4$, 1 with holonomy $\mathbb{Z}_6 \times \mathbb{Z}_6$, 13 with holonomy $\mathbb{Z}_3 \times \mathbb{Z}_3$ and 8 with holonomy $\mathbb{Z}_4 \times \mathbb{Z}_4$ which are a fundamental groups of Kähler flat manifolds. $\square$

Let us present a final table.

holonomy	number	CARAT symbols and β_1
$\mathbb{Z}_2$	5	$\beta_1 = 2$, 174.1.1, 174.1.2, $\beta_1 = 4$, 173.1.1, 173.1.2, 173.1.3,
$\mathbb{Z}_3$	4	$\beta_1 = 2$, 291.1.1, 291.1.2, $\beta_1 = 4$, 311.1.1, 311.1.2,
$\mathbb{Z}_4$	22	$\beta_1 = 2$, 201.1.1, 202.1.2, 225.1.1., 225.1.10, 225.1.11, 225.1.12 (2 groups), 225.1.13, 225.1.2, 225.1.3, 225.1.4 (2 groups), 225.1.5, 225.1.6 (2 groups), 225.1.7 (2 groups), 225.1.8 (2 groups), 225.1.9, $\beta_1 = 4$, 219.1.1, 219.1.2
$\mathbb{Z}_5$	2	$\beta_1 = 2$, 626.1.1, 626,1,2
$\mathbb{Z}_6$	14	$\beta_1 = 2$, 1611.1.1, 318.1.1, 318.1.2, 318.1.3, 318.1.5, 319.1.1, 319.1.2, 319.1.3, 319.1.5, 404.1.1, 404.1.2, 404.1.3, 404.1.4 $\beta_1 = 4$, 1694.1.1
$\mathbb{Z}_8$	1	$\beta_1 = 2$, 468.1.1
$\mathbb{Z}_{10}$	1	$\beta_1 = 2$, 7093.1.1
$\mathbb{Z}_{12}$	6	$\beta_1 = 2$, 359.1.1, 359.1.3, 359.1.4, 361.1.1, 361.1.2, 554.1.1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	33	$\beta_1 = 0$, 185.1 (4 groups), $\beta_1 = 2$, 186.1 (29 groups)
$\mathbb{Z}_2 \times \mathbb{Z}_4$	45	$\beta_1 = 2$, 257.1 (19 groups), 1135.1 (26 groups)
$\mathbb{Z}_3 \times \mathbb{Z}_3$	13	$\beta - 1 = 2$, 405.1.1 (5 groups), 405.1.2 (3 groups), 405.1.3 (3 groups), 405.1.4, 405.1.5
$\mathbb{Z}_6 \times \mathbb{Z}_2$	7	$\beta_1 = 2$, 1732.1 (5 groups), 2701.1 (2 groups)
$\mathbb{Z}_4 \times \mathbb{Z}_4$	8	$\beta_1 = 2$, 1264.1 (8 groups)
$\mathbb{Z}_6 \times \mathbb{Z}_3$	6	$\beta_1 = 2$, 2719.1 (2 groups), 2720.1 (4 groups)
$\mathbb{Z}_6 \times \mathbb{Z}_4$	4	$\beta_1 = 2$, 1920.1 (4 groups)
$\mathbb{Z}_6 \times \mathbb{Z}_6$	1	$\beta_1 = 2$, 2752.1

4 An example in dimension 8

We shall consider an eight dimensional flat manifold with holonomy group $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Q}$ multiplicity free holonomy representation. That is a case where all $\mathbb{Q}$-irreducible summands are $\mathbb{R}$-irreducible and $\mathbb{C}$-reducible. Hence satisfies condition of Proposition 1. That is the fundamental group $\pi_1(M)$

of minimal dimensional Kähler flat manifold M with the first Betti number zero and finite outer automorphism group (cf. [9]). We follow [5, Prop. 3.3]. Let a, b denote generators of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Get

$$a \to A = \begin{bmatrix} I_2 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi \end{bmatrix}, \qquad b \to B = \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & \xi^2 \end{bmatrix}.$$

Where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\xi = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. However it is possible to prove that, as a subgroup of $E(8) = O(8) \ltimes R^8, \pi_1(M)$ is generated by

$$(A, (-1/3, 1/3, 0, 0, 0, 0, -1/3, 1/3))$$

and

$$(B, (0, 0, -1/3, 1/3, -1/3, 1/3, 0, 0)).$$

Problems 1. What are the properties of the Calabi-Yau flat manifolds ?

2. Classify all holonomy groups of flat Kähler manifolds with finite outer automorphism of the fundamental group. Are there different than $\mathbb{Z}_3^n$?

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