On obstructions to asphericity of certain crossed modules

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ON OBSTRUCTIONS TO ASPHERICITY OF CERTAIN CROSSED MODULES

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The paper [Co] of D. Conduche exibited an interesting link between the structure of the second homotopy module for certain crossed modules and the intersection of the lower central series. In particular, due to [Co], the well-known Whitehead asphericity conjecture can be formulated in terms of residual nilpotence of certain crossed modules. Here we continue the research in this direction. The main results we prove here are the following (for notation see the next section):

Theorem. Let (M, ∂, F) be a crossed module with F free, $H_1(Coker(\partial))$ free Abelian and $H_2(B(M, \partial, F)) = 0$ then $ker(\partial) = \gamma_\omega(F, M)$.

Theorem. Let K be a 2-dimensional complex with its plus-construction K^+ aspherical. Then the following conditions are equivalent:
(i) $\pi_2(K, K^1) \rtimes \pi_1(K^1)$ is residually solvable;
(ii) K is aspherical.

For any 2-complex K, which is a subcomplex of a finite contractible 2-complex, the plus-construction K^+ is aspherical, hence the obstruction to the Whitehead asphericity conjecture lies in the intersecion of the derived series of fundamental cat^1-groups of such complexes.

Also we show that the obstructions to so-called LOT-asphericity problem (which is equivalent to the finite Whitehead asphericity problem modulo Andrews-Curtis conjecture) lies in the difference between the first limit term δ_ω of the derived series of some finitely generated groups and the next term $\delta_{\omega+1}$ (the commutator subgroup of δ_ω).

1. Homotopy 2-types

Category $\mathcal{H}o_n$ of homotopy n-types consists of connected CW-complexes X whose homotopy groups $\pi_i(X)$ are trivial in dimension $\geq n+1$. The map $p_n : \mathcal{T}op \to \mathcal{H}o_n$ is the n-th stage of Postnikov tower. It is well known that the category $\mathcal{H}o_1$ is equivalent to the category $\mathcal{G}r$ of groups. The corresponding equivalence map $p_1 : \mathcal{T}op \to \mathcal{H}o_1$ is the usual classifying space functor

$$p_1 : X \to K(\pi_1(X), 1),\ X \in \mathcal{T}op.$$

In the case of the category of homotopy 2-types, there are a lot of algebraic models. The following categories are equivalent:

Category $\mathcal{H}o_2$ of homotopy 2-types

Category $\mathcal{C}M$ of crossed modules

Category $\mathcal{C}at^1$ of cat^1- groups

Category $\mathcal{S}Gr_1$ of simplicial groups with Moore complex of length 1

Category $\mathcal{C}at(Gr)$ of internal categories in the category of groups

Recall that a *crossed module* is a triple (M, ∂, G), where G and M are groups, G acts on M (the action we will denote $g \circ m$ for $g \in G$, $m \in M$), and $\partial : M \to G$ is a group homomorphism, such that the following conditions are satisfied:

$$CM1: \quad \partial(g \circ m) = g\partial(m)g^{-1} \ (g \in G, m \in M);$$
$$CM2: \quad mnm^{-1} = \partial(m) \circ n \ (m, n \in M).$$

Sometimes a crossed module (M, ∂, G) is called a *G-crossed module*, fixing the acting group.

A *cat*^{1}-group consists of a group G with two endomorphisms $s, t : G \to G$, satisfying

$$ss = s, st = t, ts = s, [Ker(s), Ker(t)] = 1.$$

As we mentioned, category Cat^1 of *cat*^{1}-groups is equivalent to the category $\mathcal{CM}$ of crossed modules. Equivalence map $L : \mathcal{CM} \to Cat^1$ can be constructed by setting

$$L : (M, \partial, G) \mapsto (M \rtimes G, s : (m, g) \mapsto g, t : (m, g) \mapsto \partial(m)g).$$

The classifying functor $B : \mathcal{C}M \to \mathcal{T}op$ is defined by Loday. For a given crossed module (M, ∂, G), the space $B(M, \partial, G)$ has the following homotopy groups

$$\pi_1 B(M, \partial, G) = Coker(\partial),$$
$$\pi_2 B(M, \partial, G) = ker(\partial),$$
$$\pi_i B(M, \partial, G) = 0, \ i \geq 3$$

and has the property that $p_2 B(M, \partial, G)$ is weakly equivalent to the (M, ∂, G) in $\mathcal{C}M$. It is shown in [Lo] that such a functor can be constructed by taking the geometric realization of the diagonal of the bisimplicial set $\mathcal{N}\mathcal{N}L(M, \partial, G)$.

For a given CW-complex X, we shall write $\mathcal{L}^1(X)$ for its fundamental *cat*^{1}-group, i.e.

$$\mathcal{L}^1(X) = \pi_2(X, X^1) \rtimes \pi_1(X^1).$$

There are different ways to define (co)homologies in the above categories. G. Ellis defined homologies of the given crossed module (M, ∂, G) as $H_* B(M, \partial, G)$ [E]. In the work [CCG] the authors show that $\mathcal{C}M$ is a tripleable category over sets and define homology of crossed modules as cotriple homology in the sense of Barr-Beck [BB]. Homologies in the sence of [CCG] are functors $H_* : \mathcal{C}M \to \mathcal{A}b(\mathcal{C}M)$ from the category $\mathcal{C}M$ to the category $\mathcal{A}b(\mathcal{C}M)$ of Abelian crossed modules. That is, for any crossed module (M, ∂, G), we have

$$H_*(M, \partial, G) = (\xi H_*(M, \partial, G), h_*, kH_*(M, \partial, G)),$$

where $\xi H_*(M, \partial, G), kH_*(M, \partial, G)$ are Abelian groups with trivial action of $kH_*(M, \partial, G)$ on $\xi H_*(M, \partial, G)$. We shall use the following properties of these homologies:

1. [GLP] For any crossed module (M, ∂, G) there exists a natural long exact sequence

$$\cdots \to H_{n+1}B(M,\partial,G) \to \xi H_n(M,\partial,G) \to H_n(G) \to H_nB(M,\partial,G) \to \dots$$

2. (5-term sequence for crossed modules) [CCG] Let $(N, \mu, R) \to (Q, \mu, F) \to (T, \partial, G)$ be a short exact sequence of crossed modules. Then there exists a long exact sequence of Abelian crossed modules:

$$H_2(Q,\mu,F) \to H_2(T,\partial,G) \to (\frac{N}{[F,N][R,Q]}, \mu, R/[F,R]) \to H_1(Q,\mu,F) \to H_1(T,\partial,G) \to 0$$

1.1. **Lower central series of crossed modules.** Define the lower central series $\{\gamma_i(G,M)\}_{i\geq 1}$ for a crossed module $\partial : M \to G$ by induction: $\gamma_1(G,M) := M$ and $\gamma_{i+1}(G,M)$ is the subgroup of M, generated by the elements

$$[g,m] := (g \circ m)m^{-1},\ m \in \gamma_i(G,M), g \in G. \tag{1}$$

Denote the intersection of the series $\gamma_i(G,M), i \geq 1$ by $\gamma_\omega(G,M)$. A crossed module (M,∂,G) is called *residually nilpotent* if $\gamma_\omega(G,M) = \{1\}$. Also we will use the standard notation $\gamma_i(G), 1 \leq i \leq \omega$ for the lower central series of a group G and $[g,h] := ghg^{-1}h^{-1}, g,h \in G$.

Define *Baer invariants* of crossed module (M,∂,G) as

$$B^{(k)}(M,\partial,F) = ker\{\partial_* : M/\gamma_{k+1}(G,M) \to F/\gamma_{k+1}(F)\}.$$

For any crossed module (M,∂,G) there exists the following exact sequence:

$$H_2(G) \to H_2(B(M,\partial,G)) \to M/\gamma_2(G,M) \to G/\gamma_2(G) \to H_1(Coker(\partial)) \to 0.$$

In particular, if $H_2(G) = 0$, for example G is free, then $B^{(1)}(M,\partial,G) = H_2(B(M,\partial,G))$.

Theorem 1. *Let (M,∂,F) be a crossed module with F free, $H_1(Coker(\partial))$ torsion free and $H_2(B(M,\partial,F)) = 0$. Then for all $n \geq 1$,*

$$B^{(n)}(M,\partial,F) = 0.$$

Proof. Suppose $B^{(k)}(M,\partial,F) = 0$ for some k. Lets prove the same for $k+1$. Consider the following short exact sequence of crossed modules:

$$(\gamma_k(F,M), \partial_k, \gamma_k(F)) \to (M,\partial,F) \to (M/\gamma_k(F,M), \partial_k^*, F/\gamma_k(F)).$$

It induces the following epimorphism, coming from the 5-lemma analog for crossed modules:

$$H_2(M/\gamma_k(F,M), \partial_k^*, F/\gamma_k(F)) \to (\gamma_k(F,M)/\gamma_{k+1}(F,M), \partial_k^*, \gamma_n(F)/\gamma_{n+1}(F)).$$

That is, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & ker(h_2) & \xrightarrow{h_2} & \xi H_2(M/\gamma_k(F,M), \partial_k^*, F/\gamma_k(F)) & \longrightarrow & H_2(F/\gamma_k(F)) \\
 & & {\scriptstyle q}\downarrow & & {\scriptstyle s}\downarrow & & \| \\
0 & \longrightarrow & ker(\partial_k^*) & \longrightarrow & \gamma_k(F,M)/\gamma_{k+1}(F,M) & \longrightarrow & \gamma_k(F)/\gamma_{k+1}(F),
\end{array}$$

where s is an epimorphism. Hence q is also an epimorphism.

Since we have the following exact sequence

$$0 \to ker(\partial_k^*) \to B^{(k+1)}(M,\partial,F) \to B^{(k)}(M,\partial,F),$$

it is enougth to prove that $ker(h_2) = 0$. But $ker(h_2)$ can be presented as

$$ker(h_2) = Coker\{H_3(F/\gamma_n(F)) \to H_3(B(M/\gamma_k(F,M), \partial_k, F/\gamma_k(F))\}.$$

Note that $ker(\partial_k) = B^{(k)}(M, \partial, F) = 0$, hence

$$B(M/\gamma_k(F,M), \partial_k, F/\gamma_k(F)) = K(Coker(\partial)/\gamma_k(Coker(\partial)), 1)$$

and

$$ker(h_2) = Coker\{H_3(F/\gamma_k(F)) \to H_3(Coker(\partial)/\gamma_k(Coker(\partial)))\}. \quad (2)$$

Since $H_1(Coker(\partial))$ is free Abelian, there exists a subgroup H in F, such that the restriction of the homomorphism ∂:

$$\partial|_H : H \to Coker(\partial)$$

induces isomorphism of abelianizations $H/\gamma_2(H) \simeq Coker(\partial)/\gamma_2(Coker(\partial))$. Observe that $H_2(Coker(\partial)) = 0$, since $H_2(B(M, \partial, F)$ maps epimorphically onto $H_2(Coker(\partial))$. Therefore, $\partial|_H$ induces isomorphisms of quotients $H/\gamma_n(H) \simeq Coker(\partial)/\gamma_n(Coker(\partial))$ for all $n \geq 1$. Hence we have the following commutative diagram

$$\begin{array}{ccc} H_*(H/\gamma_n(H)) & \xrightarrow{iso} & H_*(Coker(\partial)/\gamma_n(\partial)) \\ \downarrow & & \uparrow \\ H_*(H/H \cap \gamma_n(F)) & \longrightarrow & H_*(F/\gamma_n(F)), \end{array}$$

which implies that the natural map $H_*(F/\gamma_n(F)) \to H_*(G/\gamma_n(G))$ is an epimorphism for all $n \geq 1$ and therefore, $ker(h_2) = 0$ by (2). □

Theorem 2. *Let (M, ∂, F) be a crossed module with F free, $H_1(Coker(\partial))$ free Abelian and $H_2(B(M, \partial, F)) = 0$ then $ker(\partial) = \gamma_\omega(F, M)$.*

Proof. It is clear that

$$\frac{ker(\partial)}{ker(\partial) \cap \gamma_n(F, M)} \subseteq ker\{M/\gamma_n(F, M) \to F/\gamma_n(F)\},\ n \geq 1. \quad (3)$$

It follows from Theorem 1 that $B^{(n)}(M, \partial, F) = 0$, i.e. the kernel (3) is trivial and $ker(\partial) \in \gamma_n(F, M)$ for all $n \geq 1$.

From the other hand, $\partial(\gamma_n(F, M)) \subseteq \gamma_n(F)$. Therefore $\partial(\gamma_\omega(F, M)) \subseteq \gamma_\omega(F) = 1$. Hence $\gamma_\omega(F, M) \subseteq ker(\partial)$. □

Analogically can be proved the following

Proposition 1. *Let (M, ∂, F) be a crossed module with F free, $H_1(Coker(\partial))$ free Abelian and $H_2(B(M, \partial, F))$ torsion. Then $B^{(n)}(M, \partial, F)$ are torsion groups for all $n \geq 1$.*

Proof. Assuming $B^{(k)}(M, \partial, F) \otimes \mathbb{Q} = 0$, it is enougth to show that the group

$$ker(h_2) = Coker\{H_3(F/\gamma_n(F)) \to H_3(B(M/\gamma_k(F, M), \partial_k, F/\gamma_k(F))\}$$

is a torsion group.

It is well-known that for any Abelian group A,

$$H_1K(A, 2) = 0, H_2K(A, 2) = A, H_3K(A, 2) = 0,$$

therefore, the spectral sequence

$$H_p(Coker(\partial)/\gamma_k(Coker(\partial)), H_qK(B^{(k)}(M, \partial, F)) \to H_{p+q}B(M/\gamma_k(F, M), \partial_k, F/\gamma_k(F))$$

gives that there exists a monomorphism

$$B = ker\{H_3B(M/\gamma_k(F,M), \partial_k, F/\gamma_k(F)) \to H_3(Coker(\partial)/\gamma_k(Coker(\partial)))\} \to H_1(Coker(\partial), B^{(k)}(M,\partial,F))$$

and hence $B \otimes \mathbb{Q} = 0$. It follows from the proof of Theorem 1, that there is an epimorphism $B \to ker(h_2)$. Therefore, $ker(h_2) \otimes \mathbb{Q} = 0$ and thus we have proved the needed statement. □

2. Applications of faithfulness

Recall that, given a group G, a right $\mathbb{Z}[G]$-module M is said to be *faithful* if $Ann(M) := \{\alpha \in \mathbb{Z}[G] \mid M.\alpha = 0\} = 0$ and we say that G *acts faithfully on* M if $G \cap (1 + Ann(M)) = 1$. The following is shown in [M]:

Theorem 3. *For a given projective crossed module $\partial : M \to F$, with F be a free group, either $ker(\partial) = 0$ or $coker(\partial)$ acts faithfully on $ker(\partial)$.*

J. Ratcliffe proved that for any CW-complex X, which is dominated by a 2-complex, its standard crossed module $\delta : \pi_2(X, X^{(1)}) \to \pi_1(X^{(1)})$ is projective [R]. Connecting these two results, we have the following

Corollary 1. *For a given CW-complex X, which is dominated by a 2-complex, either $\pi_2(X) = 0$, or $\pi_1(X)$ acts faithfully on $\pi_2(X)$.*

Lemma 1. *Let G be a semi-direct product $G = R \rtimes F$, where F is free and R its normal subgroup, viewed as an F-group via conjugation. Then $\gamma_\omega(G) = 1$.*

Proof. For any $m \geq 1$, every element $f \in \gamma_m(G)$ can be written uniquely as a product $f = gh, g \in [R,_{m-1} F], h \in \gamma_m(F)$. Since F is residually nilpotent, $\gamma_\omega(G) \subseteq R$. Therefore, for any $m \geq 1$, $\gamma_\omega(G) \subseteq [R,_m F]$. But the intersection of $[R,_m F]$ is trivial in F due to the residual nilpotence of F. □

For a given group G, denote by $\delta_n(G)$ the n-th term of the derived series of G, $\delta_\omega = \cap_n \delta_n(G)$, $\delta_{\omega+1}(G) = [\delta_\omega(G), \delta_\omega(G)]$. For a given complex K denote by K^+ its plus-construction complex.

Theorem 4. *Let K be a 2-dimensional complex with K^+ aspherical. Then the following conditions are equivalent: (i) $\mathfrak{L}^1(K)$ is residually solvable; (ii) K is aspherical.*

Proof. The implication (ii) implies (i) obviously. Assume that $\pi_2(K) = 0$, then $\pi_2(K, K^{(1)})$ is a normal subgroup in the free group $\pi_1(K^{(1)})$. Then the proposition (i) follows from Lemma 1. The statement (ii) clearly implies (i) for any complex K.

Now assume that the group $\mathfrak{L}^1(K)$ is residually solvable. It is shown in [G] that for a given 2-complex L the following conditions are equivalent: (i) L^+ is aspherical; (ii) $H_2(\mathcal{P}(\pi_1(L))) = 0$ and L is a $\mathcal{P}(\pi_1(L))$-Cockcroft complex. Suppose K is non-aspherical. Thus, $\mathcal{P}(\pi_1(K))$ is nontrivial and we can find an element $x \in \delta_n(\pi_1(K))$ for all $n \geq 1$. Note that for any crossed module $M \to P$,

$$(p-1) \circ m \in \delta_n(M \rtimes P), p \in \delta_n(P), m \in M,$$

due to the identification $(x-1) \circ m = [m, x]$. Therefore,

$$(x-1) \circ m \in \delta_n(\pi_2(K, K^{(1)}) \rtimes \pi_1(K^{(1)})),\ m \in \pi_2(K), n \geq 1.$$

The residual solvability of $\mathcal{L}^1(K)$ thus implies that $x-1$ annihilates whole $\pi_2(K)$, but this contradicts to the faithfulness of the action of $\pi_1(K)$ on $\pi_2(K)$. Therefore, $\pi_2(Y)=0$. $\square$

If for a given 2-complex K, its homology $H_1(K)$ is torsion-free and $H_2(K)=0$, then $\pi_1(K)$ is a so-called E-group in the sense of Strebel [S]. It follows from [S] and [BD] that in this case the condition (ii) from the cited [G] holds and therefore, we have

Corollary 2. *Let K be a 2-complex with $H_1(K)$ torsion free and $H_2(K)=0$. Then the following conditions are equivalent: (i) $\mathfrak{L}^1(K)$ is residually solvable; (ii) K is aspherical.*

Note that for any crossed module $\partial : M \to P$ with a free P, group M is residually nilpotent, since it is a central extension of a free group. Since P is residually solvable, the term $\delta_{2\omega}$ of the group $M \rtimes P$ is trivial. Therefore, the stabilization of the derived series $\delta_\omega(M \rtimes P)=\delta_{\omega+1}(M \rtimes P)$ implies residual solvability of $M \rtimes P$. Thus we have the following

Remark 1. *In Theorem 4 the condition (i) is equivalent to the stabilization of the transfinite derived series of $\mathcal{L}^1(K)$:*

$$\delta_\omega(\mathcal{L}^1(K))=\delta_{\omega+1}(\mathcal{L}^1(K)).$$

2.1. **Transfinite computations.** The above Remark shows that the obstructions to the asphericity of certain 2-complexes lies in the difference between δ_ω and $\delta_{\omega+1}$ of some groups. Here we shall show that for some important cases of the Whitehead asphericity conjecture, such kind of obstructions can be found in the difference between δ_ω and $\delta_{\omega+1}$ of certain finitely generated groups.

The following theorem is due to Gutierrez and Ratcliffe.

Theorem 5. [GR] *If K is a connected 2-dimensional CW-complex, and K_1 and K_2 are subcomplexes such that $K=K_1\cup K_2$ and $K_1\cap K_2$ is the 1-skeleton K^1 of K, then there is an exact sequence of $\mathbb{Z}[\pi_1(K)]$-modules*

$$0\to i_1\pi_2(K_1)\oplus i_2\pi_2(K_2)\xrightarrow{\alpha}\pi_2(K)\to\frac{R\cap S}{[R,S]}\to 0,$$

where α is induced by inclusion, R is the kernel of $\pi_1(K^1)\to\pi_1(K_1)$, S is the kernel of $\pi_1(K^1)\to\pi_1(K_1)$ and the action of $\pi_1(K)\simeq\pi_1(K^1)/RS$ on $\frac{R\cap S}{[R,S]}$ is induced by conjugation.

This theorem gives a link between asphericity and pure group theoretical problems. Theorems 3 and 5 imply that in the case K_1 and K_2 aspherical, either $R\cap S=[R,S]$ or $\pi_1(K^1)/RS$ acts faithfully on $\frac{R\cap S}{[R,S]}$.

For a given group G denote

$$\delta_{(\omega)}(G)=\bigcap_n[\delta_\omega(G),\delta_n(G)].$$

Obviously, we have

$$\delta_\omega(G)\supseteq\delta_{(\omega)}(G)\supseteq\delta_{\omega+1}(G).$$

Lemma 2. *Let $g\in\delta_{\omega+1}(F/RS)$, then*

$$\frac{R\cap S}{[R,S]}\circ(g-1)\subseteq\delta_{(\omega)}(F/[R,\,S]).$$

Proof. The element $g \in \delta_{\omega+1}(F/RS)$ can be written as

$$g = \prod_{j=1}^{k} [x_{2j-1}, x_{2j}],$$

for some k, where $x_j \in \delta_\omega(F/RS)$. Let $r[R,\, S] \in \frac{R\cap S}{[R,S]}$, $r \in R \cap S$. Then

$$r[R,\, S] \circ (g-1) = [\prod_{j=1}^{k} [f_{2j-1}, f_{2j}], r^{-1}],$$

where $f_l RS = x_l, l = 1, \ldots, 2k$. We have

$$r[R,\, S] \circ (g-1) \in \langle [[f_j, r], f_l],\ j, l = 1, \ldots, 2k\rangle^F. \tag{4}$$

But $f_l RS \in \delta_\omega(F/RS)$, hence $[f_j, r] \in \delta_\omega(F/[R,\, S])$. The statement then immediately follows from the fact that $f_l RS$ from (4) also lies in $\delta_\omega(F/RS)$. □

Remark 2. *In Theorem 4 the condition (i) is equivalent to* $\delta_{(\omega)}(\mathcal{L}^1(K)) = 1$.

Lemma 3. *Let F be a free group, R and S its normal subgroups such that F/RS is torsion free and*
(i) $\delta_\alpha(F/R) = \delta_\alpha(F/S) = 1,\ \alpha \geq \omega,$
(ii) $\delta_\omega(\frac{F}{[R\cap S, RS]}) = \delta_{\omega+1}(\frac{F}{[R\cap S, RS]})$.
Then $\delta_\omega(F/RS) = 1$.

Proof. Recall from [M1] that if F a noncyclic free group and $\{1\} \neq S \subseteq R$ are its normal subgroups and F/R does not contain nontrivial finite normal subgroups, then $S/[R,S]$ is a faithful $\mathbb{Z}[F/R]$-module. Therefore the module $\frac{R\cap S}{[R\cap S, RS]}$ is a faithful $\mathbb{Z}[F/RS]$-module.

Consider the following exact sequence of groups:

$$1 \to \frac{R\cap S}{[R\cap S,\ RS]} \to \frac{F}{[R\cap S, RS]} \to F/R\cap S\ (\subseteq F/R \times F/S). \tag{5}$$

From (5), and condition (i) it follows that $\delta_{\alpha+1}(\frac{F}{[R\cap S,\, RS]}) = 1$, therefore, due to (ii), we have

$$\delta_\omega(\frac{F}{[R\cap S,\ RS]}) = 1. \tag{6}$$

Now let $g \in \delta_\omega(F/RS)$. Then, using the fact that $\frac{R\cap S}{[R\cap S, RS]} \circ (g-1) \subseteq \delta_\omega(\frac{F}{[R\cap S,\, RS]})$ and (6), we have

$$\frac{R\cap S}{[R\cap S, RS]} \circ (g-1) = 0.$$

But F/RS act faithfully on $\frac{R\cap S}{[R\cap S,\, RS]}$, therefore $g = 1$. Finally, we have $\delta_\omega(F/RS) = 1$. □

Analogically we have the following

Lemma 4. *Let F be a free group, R and S its normal subgroups such that the group F/RS is torsion free and*
(i) $\delta_\omega(F/R) = \delta_\omega(F/S) = 1$;
(ii) $\delta_{(\omega)}(\frac{F}{[R\cap S, RS]}) = \delta_{\omega+1}(\frac{F}{[R\cap S, RS]})$,
then $\delta_{\omega+1}(F/RS) = 1$.

Proof. Group $F/R\cap S$ is a subgroup of a direct product $F/R\times F/S$, and from the condition (i) it follows that the group $F/R\cap S$ is residually solvable, i.e. $\delta_\omega(F/R\cap S)=1$. The short exact sequence

$$1\to \frac{R\cap S}{[R\cap S,RS]}\to \frac{F}{[R\cap S,RS]}\to F/R\cap S\to 1$$

gives that $\delta_{\omega+1}(\frac{F}{[R\cap S,RS]})=1$, and due to (ii), we have

$$\delta_{(\omega)}(\frac{F}{[R\cap S,RS]})=1. \tag{7}$$

By Corollary 1 we have that F/RS acts faithfully on $\frac{R\cap S}{[R\cap S,\,RS]}$. Let $g\in\delta_{\omega+1}(F/RS)$. Then, by (7) and Lemma 2, we have

$$\frac{R\cap S}{[R,S]}\circ(g-1)=0,$$

and $g=1$. Therefore, $\delta_{\omega+1}(F/RS)=1$. □

Theorem 6. *Let K be a subcomplex of a contractible 2-dimensional complex and $K=K_1\cup K_2,\ K_1\cap K_2=K^1$. Suppose that*
(i) $\delta_\omega(\pi_1(K_1))=\delta_\omega(\pi_1(K_2))=1$,
(ii) $\delta_{(\omega)}(\pi_1(K^1)/[R,\,S])=\delta_{\omega+1}(\pi_1(K^1)/[R,\,S])$.
Then $\pi_2(K)=0$.

Proof. By Adams' Theorem [A], $\pi_2(K_1)=\pi_2(K_2)=0$. Therefore,

$$\pi_2(K)\simeq\frac{R\cap S}{[R,S]}$$

by Theorem 5. Suppose that $\pi_2(K)\neq 0$. Then by Corollary 1 the action of $\pi_1(K)\simeq\pi_1(K^1)/RS$ on $\frac{R\cap S}{[R,S]}$ is faithful. By Lemma 2, we have

$$\frac{R\cap S}{[R,S]}\circ(g-1)\subseteq\delta_{(\omega)}(\pi_1(K^1)/[R,\,S]),\ g\in\delta_{\omega+1}(K). \tag{8}$$

Due to the exact sequence

$$1\to\frac{R\cap S}{[R,S]}\to\pi_1(K^1)/[R,\,S]\to\pi_1(K^1)/R\cap S\ (\subseteq\pi_1(K^1)/R\times\pi_1(K^1)/S),$$

we have $\delta_{\omega+1}(\pi_1(K^1)/[R,\,S])=1$. By condition (ii) and (8), we have $\delta_{\omega+1}(\pi_1(K^1)/RS)=1$ and complex K is aspherical by Adams' Theorem. □

Theorem 7. *Let K be a subcomplex of a contractible 2-dimensional complex and $K=K_1\cup\cdots\cup K_m,\ K_1\cap\cdots\cap K_m=K^1$. Suppose*

$$\delta_\omega(K_i)=1,\ 1\le i\le m, \tag{9}$$

and for all $i=1,\dots m-1$

$$\delta_\omega(\frac{\pi_1(K^1)}{[Q_1\dots Q_i,Q_{i+1}]})=\delta_{\omega+1}(\frac{\pi_1(K^1)}{[Q_1\dots Q_i,Q_{i+1}]}), \tag{10}$$

$$\delta_\omega(\frac{\pi_1(K^1)}{[Q_1\dots Q_i\cap Q_{i+1},Q_1\dots Q_{i+1}]})=\delta_{\omega+1}(\frac{\pi_1(K^1)}{[Q_1\dots Q_i\cap Q_{i+1},Q_1\dots Q_{i+1}]}), \tag{11}$$

where $Q_i=ker\{\pi_1(K^1)\to\pi_1(K_i)\},\ i=1,\dots,m$. Then $\pi_2(K)=0$.

Proof. The proof is by induction on m. The case $m = 1$ is exactly the Adams' Theorem [A]. Suppose that the Theorem is proved for a given $m - 1$, $m \geq 2$.

It is clear that conditions (9), (10) and (11) imply asphericity of $K_1 \cup \cdots \cup K_{m-1}$. Denote $R = ker\{\pi_1(K^1) \to \pi_1(K_1 \cup \cdots \cup K_{m-1})\}$, $S = Q_m$. Apply Theorem 6 to this case. We have that the complex $K_1 \cup \cdots \cup K_m$ is aspherical and therefore its group is torsion free. Now we can apply Lemma 3 (the condition (i) of Lemma 3 follows from (9) and the assumption of induction, but the condition (ii) follows from (11)). We get $\delta_\omega(\pi_1(K^1)/RS) = 1$ and the induction is finished. □

Conditions (9) take place in certain important cases of the Whitehead asphericity conjecture, for example, in the case of LOT (label oriented tree) presentations. Let $\mathcal{T}$ be a tree with vertices $\mathcal{X}$ and edges $\mathcal{E}$. Let $\phi : \mathcal{E} \to \mathcal{X}$ be some function. For a given pair $(\mathcal{F}, \phi)$ we can correspond the following presentation, called LOT-presentation:

$$\langle \mathcal{X} \mid i(e)\phi(e)t^{-1}(e)\phi(e)^{-1},\ e \in \mathcal{E} \rangle,$$

where $i(e)$ and $t(e)$ are the initial and final vertices of a giben edge e respectively.

It is easy to see that the standard 2-complex, associated with a LOT-presentation is a subcomplex of a 2-dimensional contractible complex. Clearly, we have a condition (9) for a one-relator group, where the relator is taken from the set of relators of a given LOT-presentation. Hence the obstructions to the asphericity of such presentations lie in the difference between δ_ω and $\delta_{\omega+1}$ of some finitely generated groups, associated with a given LOT-presentation. Note that modulo Andrews-Curtis conjecture about balanced presentations of a trivial group, the Whitehead asphericity conjecture for finite complexes is equivalent to the asphericity of LOT-presenations [Ho].

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