

**On the geometry of moduli spaces of holomorphic chains
over compact Riemann surfaces**

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ON THE GEOMETRY OF MODULI SPACES OF HOLOMORPHIC CHAINS OVER COMPACT RIEMANN SURFACES

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ABSTRACT. We study holomorphic $(n + 1)$ -chains $E_n \rightarrow E_{n-1} \rightarrow \dots \rightarrow E_0$ consisting of holomorphic vector bundles over a compact Riemann surface and homomorphisms between them. A notion of stability depending on n real parameters was introduced in [1] and moduli spaces were constructed in [22, 24]. In this paper we study the variation of the moduli spaces with respect to the stability parameters. In particular we characterize a parameter region where the moduli spaces are birationally equivalent. A detailed study is given for the case of 3-chains, generalizing that of 2-chains (triples) in [6]. Our work is motivated by the study of the topology of moduli spaces of Higgs bundles and their relation to representations of the fundamental group of the surface.

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1. INTRODUCTION

Let X be a compact Riemann surface of genus $g \geq 2$. A holomorphic $(n + 1)$ -chain over X is an object

$$E_n \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_1} E_0$$

consisting of holomorphic vector bundles E_j on X , $j = 0, \dots, n$, and homomorphisms $\phi_i: E_i \rightarrow E_{i-1}$, $i = 1, \dots, n$. The ranks and degrees of E_i define the type of the chain. A notion of stability for $(n + 1)$ -chains, depending on n real parameters α_i , has been introduced in [1]

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and moduli spaces have been constructed in [22, 24]. These objects generalize the holomorphic triples $E_1 \xrightarrow{\phi} E_0$ introduced in [10, 4]. The variation of the moduli spaces of holomorphic triples with respect to the stability parameter α has been studied in [6], where a birational characterization of the moduli space has been given. It turns out that the moduli space of α -stable triples is empty outside of an interval (α_m, α_M) where the bounds are determined by the type of the triple, and $\alpha_M = \infty$ if the ranks of E_0 and E_1 are equal. The main result in [6] is that for $\alpha \in (\alpha_m, \alpha_M)$ and $\alpha \geq 2g - 2$, the moduli space of α -stable triples is non-empty, smooth and irreducible.

In this paper we undertake a systematic study of holomorphic $(n+1)$ -chains for arbitrary n . We study the parameter region where the moduli spaces may be non-empty. This region is partitioned into chambers and we study the variations in the moduli space as we cross a wall. We show that the region is bounded by n hyperplanes, which play the role of α_m in the case of triples. The determination of other bounding hyperplanes — the analogues of α_M — is more difficult and is only done in some cases. However, we characterize a region where the moduli spaces of chains of a given type are birationally equivalent. It turns out that, similarly to the case of triples, the stability parameters α_i must satisfy $\alpha_i - \alpha_{i-1} \geq 2g - 2$. After developing the general theory, we study in more detail the case of 3-chains and finish the paper giving the birational characterization of the moduli spaces for some special values of the ranks.

Our main motivation to study this problem comes from the theory of Higgs bundles on a Riemann surface X and its relation to representations of the fundamental group of X . By results of Hitchin [15], Donaldson [8], Simpson [25], and Corlette [7], the moduli space of reductive representations of the fundamental group of X in a non-compact reductive Lie group G can be identified with the moduli space of polystable G -Higgs bundles. As shown by Hitchin, the L^2 -norm of the Higgs field with respect to the solution of the Hitchin equations defines a proper function on the moduli space of G -Higgs bundles, which in many cases is a perfect Morse-Bott function. Hence the study of the topology of the moduli space of Higgs bundles, such as Betti numbers, reduces to the study of the topology of the critical subvarieties of the Morse function. It turns out [25, 1] that for $G = \mathrm{GL}(n, \mathbb{C})$ and $G = \mathrm{U}(p, q)$ these critical subvarieties correspond precisely to moduli spaces of $(n+1)$ -chains for different values of n and for certain values of the stability parameters, namely $\alpha_i - \alpha_{i-1} = 2g - 2$ — exactly the extremes of the region where our results on the birationality of moduli spaces of chains apply.

It is indeed the irreducibility of the moduli space of triples proved in [6] that has allowed to count in [5] the number of connected components of the moduli space of representations of the fundamental group (and

its universal central extension) in $U(p, q)$. Also, the computation of the Betti numbers of the moduli spaces of triples $\mathcal{O} \xrightarrow{\phi} E$ with $\text{rk}(E) = 2$ by Thaddeus [27] has enabled Gothen [12, 13] to compute the Betti numbers of the moduli spaces of $SL(3, \mathbb{C})$ -Higgs bundles and $U(2, 1)$ -Higgs bundles, and the parabolic versions given in [11] and [20].

To carry out this programme, one has then to study the topology of the moduli spaces of $(n + 1)$ -chains for arbitrary n when $\alpha_i - \alpha_{i-1} = 2g - 2$. As it turns out in most of the cases studied so far, it is easier to understand the moduli space for some particular chamber, and then analyze the wall-crossings until we get to $\alpha_i - \alpha_{i-1} = 2g - 2$. In this paper we give the first steps in this direction beyond the $n = 1$ case. In fact, the examples we consider include the case $E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0$ with $\text{rk}(E_1) = \text{rk}(E_0) = 1$, whose Betti numbers, together with the Betti numbers of the moduli space of triples $E_1 \xrightarrow{\phi} E_0$ with $\text{rk}(E_0) = 1$ would give the Betti numbers of the moduli space of representations of the fundamental group in $U(n, 1)$ — the group of isometries of the n -dimensional hyperbolic space. This is indeed a computation that we plan to undertake in a future paper.

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2. DEFINITIONS AND BASIC FACTS

In this section, we will, on the one hand, review the formalism of holomorphic chains from [1] as well as existing results and, on the other hand, prove some substantial new results concerning the stability parameters for holomorphic chains. Recall that we do work exclusively on a smooth projective curve X of genus $g \geq 2$ defined over the complex numbers.

The concepts that we shall explain below are natural generalizations of notions and ideas from the theory of vector bundles on the curve X . Let us thus pause to recall the relevant concepts from the setting of vector bundles. If E is a vector bundle on X , we write $\text{rk}(E)$ for its *rank*, i.e., the dimension of the fiber vector spaces and denote by $\text{deg}(E)$ its *degree*. Using Chern classes, we find

$$c_1(E) = \text{deg}(E) \cdot [\text{pt}].$$

The *slope of E* is the quotient $\mu(E) := \text{deg}(E)/\text{rk}(E)$. A vector bundle is said to be *(semi)stable*, if for every non-zero proper subbundle $0 \subsetneq F \subsetneq E$ the inequality

$$\mu(F) \leq \mu(E)$$

is satisfied. The symbol “ \leq ” means that “ $<$ ” is used in the definition of “stable” and “ \leq ” in the definition of “semistable”. The semistable vector bundles of rank r and degree d are classified by an irreducible normal projective variety $\mathcal{U}(r, d)$ of dimension $r^2(g-1)+1$. It contains the smooth dense open subvariety $\mathcal{U}^s(r, d)$ that parameterizes stable vector bundles.

2.1. Definitions. A *holomorphic $(n+1)$ -chain* is a tuple $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$, consisting of vector bundles E_j on X , $j = 0, \dots, n$, and homomorphisms $\phi_i: E_i \rightarrow E_{i-1}$, $i = 1, \dots, n$. The tuple $t := (\text{rk}(E_j), j = 0, \dots, n; \text{deg}(E_j), j = 0, \dots, n)$ will be referred to as the *type of the chain* $(E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$. We will often write a chain in the form

$$C : E_n \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_1} E_0.$$

A *subchain of the holomorphic chain* $(E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$ is a tuple $C' := (F_j, j = 0, \dots, n)$ with F_j a subsheaf of E_j , $j = 0, \dots, n$, such that $\phi_i(F_i) \subseteq F_{i-1}$, $i = 1, \dots, n$. The subchains $(0, \dots, 0)$ and $(E_j, j = 0, \dots, n)$ are called the *trivial subchains*.

Remark 2.1. Note that a subchain $(F_j, j = 0, \dots, n)$ gives rise to the holomorphic chain $(F_j, j = 0, \dots, n; \phi_{i|F_i}: F_i \rightarrow F_{i-1}, i = 1, \dots, n)$.

Now, fix a tuple $\alpha = (\alpha_j, j = 0, \dots, n)$ of real numbers. For a holomorphic $(n+1)$ -chain $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$, we define the α -*degree* as

$$\text{deg}_\alpha(C) := \sum_{j=0}^n (\text{deg}(E_j) + \alpha_j \text{rk}(E_j))$$

and the α -*slope* as

$$\mu_\alpha(C) := \frac{\text{deg}_\alpha(C)}{\sum_{j=0}^n \text{rk}(E_j)}.$$

A holomorphic $(n + 1)$ -chain C is said to be α -(semi)stable, if the inequality

$$\mu_\alpha(F_j, j = 0, \dots, n; \phi_i|_{F_i}, i = 1, \dots, n) (\leq) \mu_\alpha(C)$$

is verified for any non-trivial subchain $C' = (F_j, j = 0, \dots, n)$ of C . Here, the convention for “ (\leq) ” is as before. Last but not least, we call a chain C α -polystable, if it may be written as a direct sum $C = C_1 \oplus \dots \oplus C_t$ where C_k is an α -stable holomorphic chain with $\mu_\alpha(C_k) = \mu_\alpha(C)$, $k = 1, \dots, t$. Since holomorphic chains form in a natural way an Abelian category, one easily derives the following result.

Proposition 2.2 (The Jordan-Hölder filtration). *Let C be an α -semi-stable holomorphic chain. Then, C possesses a (in general non-unique) so-called Jordan-Hölder filtration*

$$0 =: C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_m := C$$

by holomorphic subchains, such that $\mu_\alpha(C_i) = \mu_\alpha(C)$ and C_i/C_{i-1} is α -stable, $i = 1, \dots, m$. The so-called graduation

$$G := \text{gr}(C) := \bigoplus_{i=1}^m G_i, \quad G_i := C_i/C_{i-1}, \quad i = 1, \dots, m,$$

of C is then α -polystable. The equivalence class of $\text{gr}(C)$ does not depend on the Jordan-Hölder filtration of C .

Using the above proposition, we call two α -semistable holomorphic chains C and C' S -equivalent, if their graduations $\text{gr}(C)$ and $\text{gr}(C')$ are equivalent.

Remark 2.3. i) Suppose $C' = (F_j, j = 0, \dots, n)$ is a subchain of the holomorphic chain $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$. Let \overline{F}_j be the subbundle of E_j generated by F_j , $j = 0, \dots, n$. Then, $\overline{C}' := (\overline{F}_j, j = 0, \dots, n)$ is still a subchain with $\deg(F_j) \leq \deg(\overline{F}_j)$, $j = 0, \dots, n$. Thus, semistability has to be checked only against subchains composed of subbundles.

ii) Let $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$ be a holomorphic chain. A holomorphic chain $(Q_j, j = 0, \dots, n; \psi_i, i = 1, \dots, n)$ is called a *quotient chain of C* , if there exist surjective homomorphisms $\pi_j: E_j \rightarrow Q_j$, $j = 0, \dots, n$, such that $\psi_i \circ \pi_i = \pi_{i-1} \circ \phi_i$, $i = 1, \dots, n$. Note that $(\ker(\pi_j), j = 0, \dots, n)$ is then a holomorphic subchain of C and that we have the *trivial quotients* C and $(0, \dots, 0; 0, \dots, 0)$. Moreover, for any subchain $(F_j, j = 0, \dots, n)$, we obtain the induced quotient chain $(E_j/F_j, j = 0, \dots, n; \overline{\phi}_i, i = 1, \dots, n)$.

Standard arguments now show that a holomorphic chain C is α -(semi)stable, if and only if the inequality

$$\mu_\alpha(C) (\leq) \mu_\alpha(Q_j, j = 0, \dots, n; \psi_i, i = 1, \dots, n)$$

holds for any non-trivial quotient $(Q_j, j = 0, \dots, n; \psi_i, i = 1, \dots, n)$ of C .

iii) Let $\alpha = (\alpha_j, j = 0, \dots, n)$ be as above and $\beta \in \mathbb{R}$. Set $\alpha' := (\alpha_j + \beta, j = 0, \dots, n)$. Then, it is obvious that a holomorphic chain C is α -(semi)stable, if and only if it is α' -(semi)stable. Thus, we may assume that α_0 be zero. In particular, the semistability concept for holomorphic $(n + 1)$ -chains depends only on n rational parameters.

iv) If $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$ is a holomorphic chain, we get the *dual holomorphic chain* $C^\vee := (E'_j, j = 0, \dots, n; \phi'_i, i = 1, \dots, n)$ with $E'_j := E_{n-j}^\vee$, $j = 0, \dots, n$, and $\phi'_i := \phi_{n+1-i}^\vee$, $i = 1, \dots, n$. Then, C is $(\alpha_0, \dots, \alpha_{n+1})$ -(semi)stable, if and only if C^\vee is $(-\alpha_{n+1}, \dots, -\alpha_1)$ -(semi)stable.

2.2. Moduli spaces. Given a fixed type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$ and a fixed stability parameter $\alpha = (\alpha_j, j = 0, \dots, n)$ consisting of *rational* numbers, the S-equivalence classes of α -semistable holomorphic chains C of type t form a projective moduli scheme $\mathcal{M}_\alpha(t)$. The GIT construction for $\mathcal{M}_\alpha(t)$ is contained in [24] (see also [22] where the semistability condition appears in a different guise). The chamber structure of the parameter region (see Remark 2.13) will reveal that, for any parameter α , there is a rational parameter α' , such that the notion of α -(semi)stability for holomorphic chains of type t is equivalent to the notion of α' -(semi)stability, so that the moduli spaces do indeed exist for any stability parameter.

2.3. The parameter region. We need to study the moduli spaces $\mathcal{M}_\alpha(t)$ for fixed type t in dependence of the parameter α . As remarked in 2.3, iii), we may write $\alpha = (0, \alpha_1, \dots, \alpha_n)$. Thus, we may view the stability parameter α as an element of \mathbb{R}^n . The first interesting problem is to determine *a priori* a region $R(t) \subset \mathbb{R}^n$, such that the existence of an α -semistable holomorphic chain of the predetermined type t implies that α lies in the region $R(t)$. Here, we show that the semistability condition for certain natural subchains gives some halfspaces H_1, \dots, H_n , such that $R(t)$ lies in the intersection of those halfspaces. The next natural and important question is whether $R(t)$ is bounded or not. We will relate this question to the existence or non-existence of semistable chains in the category of k -vector spaces where $k = \mathbb{C}(X)$ is the function field of the curve X .

Let $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$ be a holomorphic $(n + 1)$ -chain. Define the *i -th standard subchain* to be $C_i := (E_0, \dots, E_i, 0, \dots, 0)$, $i = 0, \dots, n - 1$. A straightforward calculation gives the following result.

Proposition 2.4. *Suppose C is α -(semi)stable with $\alpha = (0, \alpha_1, \dots, \alpha_n)$. Then, the condition arising from the i -th standard subchain C_i is*

$$\begin{aligned} & (\alpha_1 r_1 + \dots + \alpha_i r_i)(r_{i+1} + \dots + r_n) - \\ & - (\alpha_{i+1} r_{i+1} + \dots + \alpha_n r_n)(r_0 + \dots + r_i) \\ (\leq) \quad & (r_0 + \dots + r_i)(d_{i+1} + \dots + d_n) - \\ & - (r_{i+1} + \dots + r_n)(d_0 + \dots + d_i), \quad i = 0, \dots, n - 1. \end{aligned}$$

Let h_i be the hyperplane determining the halfspace H_i from Proposition 2.4, i.e., h_i is defined by the equation

$$\begin{aligned} & (r_{i+1} + \dots + r_n)(r_1\alpha_1 + \dots + r_i\alpha_i) - \\ & - (r_0 + \dots + r_i)(r_{i+1}\alpha_{i+1} + \dots + r_n\alpha_n) \\ = & (r_0 + \dots + r_i)(d_{i+1} + \dots + d_n) - \\ & - (r_{i+1} + \dots + r_n)(d_0 + \dots + d_i), \quad i = 0, \dots, n. \end{aligned} \quad (2.1)$$

Example 2.5. In the case $n = 2$, we find the following two inequalities:

$$\begin{aligned} \alpha_1 r_0 r_1 + \alpha_2 r_0 r_2 & \geq (r_1 + r_2)d_0 - r_0 d_1 - r_0 d_2 \\ -\alpha_1 r_1 r_2 + \alpha_2 (r_0 + r_1)r_2 & \geq r_2 d_0 + r_2 d_1 - (r_0 + r_1)d_2. \end{aligned}$$

Note that these two inequalities bound α_2 from below. The region cut out by these inequalities is sketched in Figure 1.

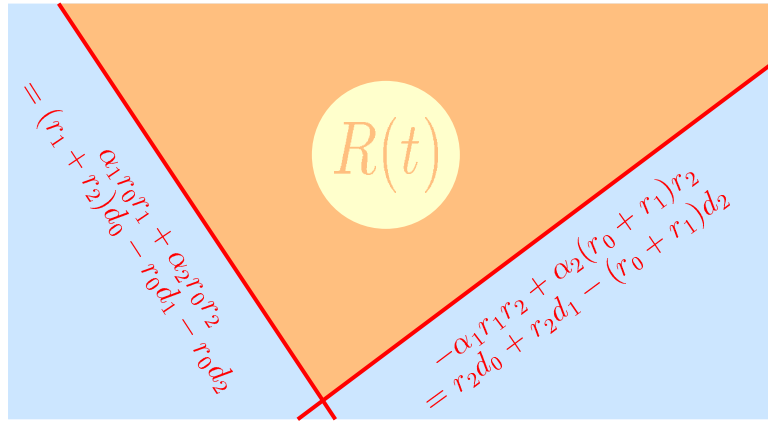


FIGURE 1. The parameter region for 3-chains.

Remark 2.6 (Degenerate holomorphic chains). Fix the type t and suppose we are given a stability parameter α and an α -semistable holomorphic chain $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$, such that, say, ϕ_{i_0+1} is zero. Then, $(0, \dots, 0, E_{i_0+1}, \dots, E_n, 0, \dots, 0, \phi_{i_0+2}, \dots, \phi_n)$ is both a subchain and a quotient chain. By Remark 2.3, ii), this implies that the inequality arising from the i_0 -th standard subchain must become an equality, that is,

$$\begin{aligned} & (\alpha_1 r_1 + \dots + \alpha_{i_0} r_{i_0})(r_{i_0+1} + \dots + r_n) - \\ & - (\alpha_{i_0+1} r_{i_0+1} + \dots + \alpha_n r_n)(r_0 + \dots + r_{i_0}) \\ = & (r_0 + \dots + r_{i_0})(d_{i_0+1} + \dots + d_n) - \\ & - (r_{i_0+1} + \dots + r_n)(d_0 + \dots + d_{i_0}). \end{aligned}$$

Therefore, the parameter α lies on the boundary of the region cut out by the inequalities in Proposition 2.4. Moreover, the moduli space for $(\alpha_j, j = 0, \dots, n)$ -semistable holomorphic chains of type t for which the inequality associated with the i_0 -th standard subchain becomes an equality can be easily seen to be a product of the moduli space of

$(\alpha_1, \dots, \alpha_{i_0})$ -semistable $(i_0 + 1)$ -chains of type $(r_j, j = 0, \dots, i_0; d_j, j = 0, \dots, i_0)$ (which, for $i_0 = 0$, is the moduli space of semistable vector bundles of rank r_0 and degree d_0) and the moduli space of $(\alpha_{i_0+1}, \dots, \alpha_n)$ -semistable $(n + i_0)$ -chains of type $(r_j, j = i_0 + 1, \dots, n; d_j, j = i_0 + 1, \dots, n)$ (see Section 2.6). The upshot is that, in our study, we may restrict to holomorphic chains in which all homomorphisms are non-trivial.

It seems to be a very difficult problem to determine the “exact” shape of the parameter region in general (see Section 5 for the case $n = 2$). Thus, one should be more modest and try to understand the behaviour along a half line in \mathbb{R}^n . Here, we may offer the following result.

Theorem 2.7. *Fix the type t . Choose rational stability parameters β and γ in \mathbb{Q}^n and set*

$$\alpha^\lambda := \beta + \lambda \cdot \gamma, \quad \lambda \in \mathbb{R}_{\geq 0}.$$

Then, there exists a value λ_∞ , such that for any $\lambda > \lambda_\infty$, a holomorphic chain $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$ is α^λ -(semi)stable, if and only if it satisfies the following conditions:

- (1) *For every subchain $(F_j, j = 0, \dots, n)$, the condition*

$$\frac{\gamma_1 \operatorname{rk}(F_1) + \dots + \gamma_n \operatorname{rk}(F_n)}{\operatorname{rk}(F_0) + \dots + \operatorname{rk}(F_n)} \leq \frac{\gamma_1 \operatorname{rk}(E_1) + \dots + \gamma_n \operatorname{rk}(E_n)}{\operatorname{rk}(E_0) + \dots + \operatorname{rk}(E_n)}$$

is verified.

- (2) *If we have equality above, then*

$$\mu_\beta(F_j, j = 0, \dots, n; \phi_{i|F_i}, i = 1, \dots, n) (\leq) \mu_\beta(C).$$

In order to appreciate the above statement, let us discuss linear chains over the field k . A *linear $(n + 1)$ -chain of type $r = (r_j, j = 0, \dots, n)$* is a tuple $V = (V_j, j = 0, \dots, n; f_i, i = 1, \dots, n)$ composed of k -vector spaces V_j with $\dim(V_j) = r_j, j = 0, \dots, n$, and linear maps $f_i: V_i \rightarrow V_{i-1}, i = 1, \dots, n$. As before, we may speak of *subchains, quotient chains, dual chains*, and so on. If we are given a tuple $\alpha = (\alpha_j, j = 0, \dots, n)$ of real numbers, we say that a chain $V = (V_j, j = 0, \dots, n; f_i, i = 1, \dots, n)$ is α -(semi)stable, if for every subchain $(W_j, j = 0, \dots, n)$, the condition

$$\frac{\alpha_0 \dim(W_0) + \dots + \alpha_n \dim(W_n)}{\dim(W_0) + \dots + \dim(W_n)} (\leq) \frac{\alpha_0 \dim(V_0) + \dots + \alpha_n \dim(V_n)}{\dim(V_0) + \dots + \dim(V_n)}$$

holds true. These concepts are special cases of King’s general formalism [17].

Remark 2.8. Any chain is 0-semistable.

Now, let C be a holomorphic chain of type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$. If η stands for the generic point of the curve X , the restriction C_η of C to the generic point is a linear chain of type $(r_j, j = 0, \dots, n)$ over $\mathbb{C}(X)$. Condition (1) in Theorem 2.7 just says that C_η is a γ -semistable linear chain. An immediate consequence is

Corollary 2.9. *Suppose there are no γ -semistable $\mathbb{C}(X)$ -linear chains, then there are no α^λ -semistable holomorphic chains of type t for $\lambda \gg 0$.*

Therefore, the intersection of the region of parameters α for which there do exist α -semistable holomorphic chains of type t intersected with the half line $\beta + \mathbb{R}_{\geq 0} \cdot \gamma$ is always bounded. Of course, one hopes that one can choose λ_∞ “uniformly”. We state the following.

Conjecture 2.10. *Suppose that 0 is the only parameter for which there exist α -semistable linear chains of type $(r_j, j = 0, \dots, n)$ over $\mathbb{C}(X)$. Then, for any type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$, there is a bounded region $R(t) \subset \mathbb{R}^n$, such that the existence of an α -semistable holomorphic chain of type t implies $\alpha \in R(t)$.*

We will prove this conjecture for $n = 2$ (see Theorem 5.5).

Sketch of proof of Theorem 2.7. For the proof, one has to place oneself in the more general setting of decorated tuples of vector bundles (see [24]). In our sketch, we will use the terminology of [24] without repeating it here. Fix non-negative integers a, b , and c . Then, we study tuples $(E_j, j = 0, \dots, n; \phi)$ where the E_j are vector bundles of rank r_j and degree $d_j, j = 0, \dots, n$, and

$$\phi: (E^{\otimes a})^{\oplus b} \longrightarrow \det(E)^{\otimes c}, \quad E := E_0 \oplus \cdots \oplus E_n,$$

is a non-trivial homomorphism. We have a natural equivalence relation on those objects which always identifies ϕ with $z \cdot \phi, z \in \mathbb{C}^*$. Note that $\mathbb{C} \oplus \bigoplus_{i=1}^n \text{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{r_{i-1}})$ is a direct summand of

$$\left(\left((\mathbb{C}^{r_0 + \cdots + r_n})^{\otimes a} \right)^{\oplus b} \right) \otimes \det(\mathbb{C}^{r_0 + \cdots + r_n})^{\otimes -c}$$

for appropriate non-negative integers a, b , and c . Thus, the formalism of holomorphic chains is contained in the formalism of decorated tuples of vector bundles. For a tuple $\alpha = (\alpha_0, \dots, \alpha_n)$ of rational numbers with $\sum_{i=0}^n \alpha_i r_i = 0$ and $\lambda \in \mathbb{Q}_{>0}$, a tuple $(E_j, j = 0, \dots, n; \phi)$ is called (α, λ) -*(semi)stable*, if

$$L(E^\bullet, a) - \sum_{j=0}^n r_j \cdot \left(\sum_{\nu=1}^s a_\nu \cdot \left(\sum_{j=0}^n \alpha_j \text{rk}(E_j^\nu) \right) \right) + \lambda \cdot \mu(E^\bullet, a; \phi) (\geq) 0$$

with

$$L(E^\bullet, a) := \sum_{\nu=1}^s a_\nu \left(\deg \left(\bigoplus_{j=0}^n E_j \right) \cdot \text{rk} \left(\bigoplus_{j=0}^n E_j^\nu \right) - \deg \left(\bigoplus_{j=0}^n E_j^\nu \right) \cdot \text{rk} \left(\bigoplus_{j=0}^n E_j \right) \right)$$

holds for every weighted filtration

$$(E^\bullet, a): \quad (0 \subsetneq (E_j^1, j = 0, \dots, n) \subsetneq \cdots \subsetneq (E_j^s, j = 0, \dots, n) \\ \subsetneq (E_j, j = 0, \dots, n), \quad a = (a_1, \dots, a_s))$$

of the “split” vector bundle $(E_j, j = 0, \dots, n)$. What we want to do is study the condition of $(\alpha^\lambda, \lambda)$ -semistability for large λ . Let V and V_j be the fibres of E and E_j , respectively, over the generic point and $\sigma \in \mathbb{P}((V^{\otimes a})^{\oplus b})$ the point defined by ϕ . Condition (1) says that σ must be semistable w.r.t. to the action of $\mathrm{SL}(V) \cap (\mathrm{GL}(V_0) \times \cdots \times \mathrm{GL}(V_n))$ and its linearization in $\mathcal{O}(1)$ modified by the character corresponding to γ . Set

$$\mu_\gamma(E^\bullet, a; \phi) := - \sum_{j=0}^n r_j \cdot \left(\sum_{\nu=1}^s a_\nu \cdot \left(\sum_{j=0}^n \gamma_j \mathrm{rk}(E_j^\nu) \right) \right) + \mu(E^\bullet, a; \phi).$$

If σ fails to be semistable, one applies the theory of the instability flag in order to produce a weighted filtration (E^\bullet, a) with $\mu_\gamma(E^\bullet, a; \phi) < 0$ and $L(E^\bullet, a)$ bounded from above by a constant which depends only on the type t , a , b , and c . The details for this may be easily adapted from our paper [23]. Moreover, the possible tuples a and $(\mathrm{rk}(\bigoplus_{j=0}^n E_j^\nu), \nu = 1, \dots, s)$ belong to a finite set whence

$$- \sum_{j=0}^n r_j \cdot \left(\sum_{\nu=1}^s a_\nu \cdot \left(\sum_{j=0}^n \beta_j \mathrm{rk}(E_j^\nu) \right) \right)$$

may also be bounded from above by a constant which depends only on the type t , a , b , and c . It is now clear that we may find λ_0 , such that for $\lambda > \lambda_0$ an $(\alpha^\lambda, \lambda)$ -semistable tuple satisfies

- (1) $\mu_\gamma(E^\bullet, a; \phi) \geq 0$ for every weighted filtration (E^\bullet, a) (which is equivalent to saying that σ is semistable).
- (2) If “=” holds, then

$$L(E^\bullet, a; \phi) - \sum_{j=0}^n r_j \cdot \left(\sum_{\nu=1}^s a_\nu \cdot \left(\sum_{j=0}^n \beta_j \mathrm{rk}(E_j^\nu) \right) \right) (\geq) 0.$$

The fact that Condition (1) and (2) are also sufficient will follow easily once one knows that the tuples $(E_j, j = 0, \dots, n; \phi)$ of type t , satisfying (1) and (2), live in a bounded family. This is again established along the lines of the corresponding result in [23]. \square

2.4. Walls and the chamber structure. In this section, we would like to subdivide \mathbb{R}^n into locally closed subsets, called chambers, such that the concept of α -(semi)stability is constant within each chamber.

We fix the type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$ and set $r := r_0 + \cdots + r_n$ and $d := d_0 + \cdots + d_n$. For a holomorphic chain $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$, the *total rank* is given by $r(C) := \mathrm{rk}(E_0) + \cdots + \mathrm{rk}(E_n)$ and the *total degree* by $d(C) := \mathrm{deg}(E_0) + \cdots + \mathrm{deg}(E_n)$. The idea is to first define hyperplanes which cut out parameters for which there might exist properly α -semistable (i.e., semistable but not stable) holomorphic $(n+1)$ -chains of type t . Suppose C is such a chain and

$C' = (F_j, j = 0, \dots, n)$ is a destabilizing subchain. Then, with α the stability parameter in question, we obtain the equation

$$\begin{aligned} & rd(C') - r(C')d \\ &= \alpha_1(r_1r(C') - \text{rk}(F_1)r) + \dots + \alpha_n(r_nr(C') - \text{rk}(F_n)r). \end{aligned}$$

Define

$$\mathcal{S} := \left\{ (s_0, \dots, s_n; e) \mid \begin{array}{l} 0 \leq s_j \leq r_j, \quad j = 0, \dots, n, \quad 0 < s < r, \\ s := s_0 + \dots + s_n, \quad e \in \mathbb{Z} \end{array} \right\}.$$

For an element $\sigma \in \mathcal{S}$, let

$$w_\sigma := \left\{ \alpha \in \mathbb{R}^n \mid \alpha_1(r_1s - s_1r) + \dots + \alpha_n(r_ns - s_nr) = re - sd \right\}$$

be the *wall defined by* σ . Note that we may have an empty wall $w_\sigma = \emptyset$ or an *improper wall* $w_\sigma = \mathbb{R}^n$. For an improper wall, we must have $r_js - rs_j = 0$, $j = 1, \dots, n$, and $re - sd = 0$. Set $r_{\text{red}} := r / \gcd(r, s) > 1$, because $s < r$. Then, r_{red} divides r_i , $i = 1, \dots, n$, and d .

Proposition 2.11. *If $\gcd(r_1, \dots, r_n, d) = 1$, then there do not exist any improper walls.*

The n -dimensional chambers are given as the connected components of

$$\mathbb{R}^n \setminus \bigcup_{\sigma \in \mathcal{S}: w_\sigma \neq \mathbb{R}^n} w_\sigma.$$

The $(n-1)$ -dimensional chambers are given as the connected components of

$$\bigcup_{\sigma \in \mathcal{S}: w_\sigma \neq \mathbb{R}^n} \left(w_\sigma \setminus \bigcup_{\tau \in \mathcal{S}: w_\sigma \not\subseteq w_\tau} w_\tau \right),$$

and so on. We label the j -dimensional chambers \mathcal{C}_k^j , $k \in J_j$, $j = 0, \dots, n$. Note that we have

$$\mathbb{R}^n = \bigsqcup_{j=0}^n \bigsqcup_{k \in J_j} \mathcal{C}_k^j. \quad (2.2)$$

Observe also that this chamber decomposition is locally finite, i.e., every bounded subset $R \subset \mathbb{R}^n$ intersects only finitely many chambers. By construction, we have the following property.

Proposition 2.12. i) *Let \mathcal{C} be any chamber and $\alpha_1, \alpha_2 \in \mathcal{C}$. Then, a holomorphic chain of type t is α_1 -(semi)stable, if and only if it is α_2 -(semi)stable.*

ii) *Let \mathcal{C}_1 be any chamber and \mathcal{C}_2 a chamber in the closure of \mathcal{C}_1 . Choose $\alpha_i \in \mathcal{C}_i$, $i = 1, 2$. Then, a holomorphic chain which is α_1 -semistable is also α_2 -semistable, and a holomorphic chain which is α_2 -stable is also α_1 -stable.*

Remark 2.13. By definition, any chamber contains elements of \mathbb{Q}^n , so that it suffices to consider rational stability parameters.

Finally, we note the following consequence of Proposition 2.11.

Corollary 2.14. *If $\gcd(r_1, \dots, r_n, d) = 1$, then, for a stability parameter α which lies in an n -dimensional chamber, the conditions of α -stability and α -semistability coincide.*

Proof. The definition of the walls shows that a stability parameter α for which there exists a properly α -semistable holomorphic chain of type t must lie on a wall. Since the assumption grants that no improper wall exists, we are done. \square

To conclude this paragraph, we remark that the chamber structure has been obtained by rough a priori considerations. Usually, one expects a coarser chamber structure based on a refined analysis of stability. This seems, unfortunately, rather involved and may be carried out only in more specialized situations. In general, we would expect only finitely many chambers, even if the region $R(t)$ of possible stability parameters were not bounded. This and other phenomena will be explained for $n = 2$ in Section 5.

2.5. Vortex equations and Hitchin–Kobayashi correspondence.

There are natural gauge-theoretic equations on a holomorphic chain

$$C: E_n \xrightarrow{\phi_n} E_{n-1} \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_1} E_0,$$

which we describe now. Define $\tau = (\tau_0, \dots, \tau_n) \in \mathbb{R}^{n+1}$ by

$$\tau_j = \mu_\alpha(C) - \alpha_j, \quad j = 0, \dots, n, \quad (2.3)$$

where we make the convention $\alpha_0 = 0$. Then α can be recovered from τ by

$$\alpha_j = \tau_0 - \tau_j, \quad j = 0, \dots, n. \quad (2.4)$$

The τ -vortex equations

$$\sqrt{-1}\Lambda F(E_j) + \phi_{j+1}\phi_{j+1}^* - \phi_j^*\phi_j = \tau_j \text{id}_{E_j}, \quad j = 0, \dots, n, \quad (2.5)$$

are equations for Hermitian metrics on E_0, \dots, E_n . Here, $F(E_j)$ is the curvature of the Hermitian connection on E_j , Λ is contraction with the Kähler form of a fixed metric on X such that $\text{vol}(X) = 2\pi$, and ϕ_j^* is the adjoint of ϕ_j . By convention $\phi_0 = \phi_{n+1} = 0$. One has the following.

Theorem 2.15 ([1, Theorem 3.4]). *A holomorphic chain C is α -polystable if and only if the τ -vortex equations have a solution, where α and τ are related by (2.3).*

2.6. Moduli spaces for parameters on and near the standard hyperplanes. A standard procedure to study moduli spaces is to start with known moduli spaces and create new ones out of them by “flip-type” operations. In our setting, we might try a kind of inductive procedure, by relating moduli spaces of holomorphic $(n + 1)$ -chains to moduli of “shorter” holomorphic chains. This is indeed possible for stability parameters in or near the hyperplanes where the inequalities in Proposition 2.4 become equalities.

Let h_i , $i = 0, \dots, n$, be the hyperplanes that were defined by the equations (2.1).

Proposition 2.16. *Let $C = (E_j, j = 0, \dots, n; \phi_i, i = 1, \dots, n)$ be a holomorphic $(n + 1)$ -chain.*

i) *Assume C to be α -semistable. If $\alpha \in h_{i_0+1}$, then C is S -equivalent to $(E_j, j = 0, \dots, n; \tilde{\phi}_i, i = 1, \dots, n)$ with $\tilde{\phi}_{i_0+1} = 0$ and $\tilde{\phi}_i = \phi_i$, for $i \neq i_0 + 1$. In particular, if C is α -polystable, then $\phi_{i_0+1} = 0$.*

ii) *The $(i_0 + 1)$ -chain $C' = (E_j, j = 0, \dots, i_0; \phi_i, i = 1, \dots, i_0)$ is β -semistable for $\beta = (\alpha_0, \dots, \alpha_{i_0})$, and the $(n - i_0)$ -chain $C'' = (E_{i_0+1+j}, j = 0, \dots, n - i_0 - 1; \phi_{i_0+1+i}, i = 1, \dots, n - i_0 - 1)$ is γ -semistable for $\gamma = (\alpha_{i_0+1}, \dots, \alpha_n)$. If, furthermore, C is α -polystable, then C' and C'' are β - and γ -polystable, respectively.*

iii) *If, conversely, the chains C' and C'' are β - and γ -semistable (polystable), then C is α -semistable (polystable).*

Proof. The arguments are essentially the same as for vector bundles, working in the Abelian category of holomorphic chains with the α -degree, the total rank, the α -slope, and the notion of α -semistability as the semistability concept.

If $\alpha \in h_{i_0+1}$, then $\tilde{C} := (E_0, \dots, E_{i_0}, 0, \dots, 0; \phi_1, \dots, \phi_{i_0}, 0, \dots, 0)$ is a subchain with $\mu_\alpha(\tilde{C}) = \mu_\alpha(C)$. The quotient chain $\overline{C} := C/\tilde{C}$ is $(0, \dots, 0, E_{i_0+1}, \dots, E_n; \phi_1, \dots, \phi_{i_0}, 0, \dots, 0)$, and $\mu_\alpha(C) = \mu_\alpha(\overline{C})$.

i) By definition of S -equivalence, C is S -equivalent to $\tilde{C} \oplus \overline{C}$, and, for α -polystable chains, S -equivalence is the same as equivalence.

ii) and iii) Standard arguments (parallel to those for semistable vector bundles) show that C is α -semistable, if and only if both \tilde{C} and \overline{C} are α -semistable. Now, note that C' is β -semistable, if and only if \tilde{C} is α -semistable and that C'' is γ -semistable, if and only if \overline{C} is α -semistable. This proves the assertions on semistability in ii) and iii). The corresponding claims about polystability are left as an exercise to the reader. \square

Remark 2.17. All the above observations regarding polystability may also be easily derived from the existence of solutions to the vortex equations (2.5) on the chain C .

Corollary 2.18. i) Let $\alpha \in h_{i_0+1}$. With the same notation as in the above proposition, we have that

$$\mathcal{M}_\alpha^{\text{red}}(t) \cong \mathcal{M}_\beta^{\text{red}}(t') \times \mathcal{M}_\gamma^{\text{red}}(t''),$$

where t' and t'' are the types of C' and C'' , respectively. Here, the superscript “red” refers to the induced reduced scheme structure.

ii) Let $\{\alpha\} = \bigcap_{i=1}^n h_i$. Then

$$\mathcal{M}_\alpha^{\text{red}}(t) \cong \mathcal{M}(r_1, d_1) \times \dots \times \mathcal{M}(r_n, d_n)$$

with $\mathcal{M}(r, d)$ the moduli space of semistable vector bundles of rank r and degree d .

Proposition 2.19. Fix the type t , and suppose $\alpha \in h_{i_0+1}$. Let \mathcal{C} be any chamber, such that $\alpha \in \overline{\mathcal{C}}$, and choose $\kappa \in \mathcal{C}$.

i) If C is κ -semistable, then the chain C' is β -semistable and the chain C'' is γ -semistable.

ii) If C' is β -stable and C'' is γ -stable, then, for any $\phi_{i_0+1}: E_{i_0+1} \rightarrow E_{i_0}$ different from zero, the resulting chain C is κ -stable.

Proof. i) This is a trivial continuity statement, observing the discussions in the proof of Proposition 2.16.

ii) We will demonstrate the following property:

There is an open subset $\mathcal{U} \subset \mathbb{R}^n$, containing α , such that, for $\kappa \in \mathcal{U} \cap (H_{i_0+1} \setminus h_{i_0+1})$, we have the following. If C' is β -stable and C'' is γ -stable, then, for any $\phi_{i_0+1}: E_{i_0+1} \rightarrow E_{i_0}$ different from zero, the resulting chain C is κ -stable.

In view of the general properties of the chamber decomposition (Proposition 2.12), this will imply the assertion of the proposition.

Define \mathcal{S} as in Section 2.4, and $\mathcal{S}^{\text{real}}$ as the set of elements of \mathcal{S} which come from a subchain of a holomorphic chain C of type t , such that C' is β -stable and C'' is γ -stable. Declare the finite set

$$\mathcal{R} := \left\{ (s_0, \dots, s_n) \mid \begin{array}{l} 0 \leq s_j \leq r_j, \quad j = 0, \dots, n, \\ (0 < s_0 + \dots + s_{i_0} < r_0 + \dots + r_{i_0}) \vee \\ \vee (0 < s_{i_0+1} + \dots + s_n < r_{i_0+1} + \dots + r_n) \end{array} \right\}.$$

For $s \in \mathcal{R}$, we define $\mathcal{S}_s^{\text{real}}$ as the set of elements $(s_0, \dots, s_n, e) \in \mathcal{S}^{\text{real}}$ with $(s_0, \dots, s_n) = s$, and set

$$e_0(s) := \max \{ e \mid (s, e) \in \mathcal{S}_s^{\text{real}} \}.$$

For each $s \in \mathcal{R}$, there is the function

$$\begin{aligned} \chi_s: \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (\kappa_1, \dots, \kappa_n) &\longmapsto \sum_{j=0}^n s_j \cdot \sum_{j=0}^n \kappa_j r_j - \sum_{j=0}^n r_j \cdot \sum_{j=0}^n \kappa_j s_j. \end{aligned}$$

Obviously, we can define open subsets \mathcal{U}_s by the condition $\chi_s(\kappa) > e_0(s)(r_0 + \cdots + r_n) - d(s_0 + \cdots + s_n)$ for any $\kappa \in \mathcal{U}_s$, $s \in \mathcal{R}$. We set

$$\mathcal{U} := \bigcap_{s \in \mathcal{R}} \mathcal{U}_s.$$

Let \tilde{C} be any proper subchain of C . We view C' as a subchain and C'' as a quotient chain of C . Suppose that both $\hat{C} := \tilde{C} \cap C'$ and the projection $\overline{\tilde{C}}$ of \tilde{C} to C'' are trivial subchains of C' and C'' , respectively. Because ϕ_{i_0+1} is non-trivial, this can happen only for $\tilde{C} = 0$, $\tilde{C} = C$, and $\tilde{C} = C'$. In the third case, $\mu_\kappa(\hat{C}) < \mu_\kappa(C)$ is equivalent to the assumption $\kappa \in H_{i_0+1} \setminus h_{i_0+1}$. Thus, we may assume that \hat{C} is a non-trivial subchain of C' or that $\overline{\tilde{C}}$ is a non-trivial subchain of C'' . But then, $\mu_\kappa(\hat{C}) < \mu_\kappa(C)$, for all $\kappa \in \mathcal{U}$, follows immediately from the definition of \mathcal{U} , because $(\text{rk}(F_0), \dots, \text{rk}(F_n)) \in \mathcal{R}$.

We still have to show that \mathcal{U} contains α . For given $s \in \mathcal{R}$, we may choose a chain C of type t , such that C' is β -stable and C'' is γ -stable and a subchain \tilde{C} of C with $(\text{rk}(F_0), \dots, \text{rk}(F_n)) = s$ and $\deg(F_0) + \cdots + \deg(F_n) = e_0(s)$. Then,

$$\begin{aligned} & \mu_\alpha(\tilde{C}) \\ = & \frac{\sum_{j=0}^n \deg(F_j) + \alpha_1 \text{rk}(F_1) + \cdots + \alpha_n \text{rk}(F_n)}{\text{rk}(F_0) + \cdots + \text{rk}(F_n)} \\ = & \frac{\sum_{j=0}^{i_0} \text{rk}(F_j) \cdot \mu_{(0, \alpha_1, \dots, \alpha_{i_0})}(\hat{C}) + \sum_{j=i_0+1}^n \text{rk}(F_j) \cdot \mu_{(\alpha_{i_0+1}, \dots, \alpha_n)}(\overline{\tilde{C}})}{\text{rk}(F_0) + \cdots + \text{rk}(F_n)} \\ < & \frac{\sum_{j=0}^{i_0} \text{rk}(F_j) \cdot \mu_{(0, \alpha_1, \dots, \alpha_{i_0})}(C') + \sum_{j=i_0+1}^n \text{rk}(F_j) \cdot \mu_{(\alpha_{i_0+1}, \dots, \alpha_n)}(C'')}{\text{rk}(F_0) + \cdots + \text{rk}(F_n)} \\ = & \frac{\sum_{j=0}^{i_0} \text{rk}(F_j) \cdot \mu_\alpha(C) + \sum_{j=i_0+1}^n \text{rk}(F_j) \cdot \mu_\alpha(C)}{\text{rk}(F_0) + \cdots + \text{rk}(F_n)} \\ = & \mu_\alpha(C). \end{aligned}$$

This inequality is equivalent to $\alpha \in \mathcal{U}_s$. \square

Remark 2.20. The same game can be played with a parameter α lying on several of the hyperplanes h_i . In particular, we can apply it to the parameter α defined in Corollary 2.18, ii).

3. EXTENSIONS AND DEFORMATIONS OF CHAINS

As a first step to study the variation of the moduli spaces of α -semistable holomorphic chains as the parameter α changes, we study in this section the deformation theory of holomorphic chains. As in the case of holomorphic triples, which is treated in [6, §3], the infinitesimal deformations of holomorphic chains are given by the hypercohomology groups of certain sheaf complexes associated to the holomorphic chains.

Throughout this section we fix a stability parameter $\alpha = (\alpha_j, j = 0, \dots, n)$ and two holomorphic chains C' and C'' , of types $t' = (r'_j, j = 0, \dots, n; d'_j, j = 0, \dots, n)$ and $t'' = (r''_j, j = 0, \dots, n; d''_j, j = 0, \dots, n)$, respectively, given by

$$C' : E'_n \xrightarrow{\phi'_n} E'_{n-1} \xrightarrow{\phi'_{n-1}} \dots \xrightarrow{\phi'_1} E'_0, \quad (3.1)$$

$$C'' : E''_n \xrightarrow{\phi''_n} E''_{n-1} \xrightarrow{\phi''_{n-1}} \dots \xrightarrow{\phi''_1} E''_0. \quad (3.2)$$

Given two vector bundles E and F over X , $\text{Hom}(E, F)$ and $\text{End}(E)$ denote the vector bundles of homomorphisms from E to F and of endomorphisms of E , respectively. The corresponding spaces of global sections are denoted $\text{Hom}_X(E, F)$ and $\text{End}_X(E)$, respectively.

3.1. Hypercohomology. In this subsection we analyze $\text{Ext}^1(C'', C')$ using the hypercohomology groups $\mathbb{H}^i(F^\bullet(C'', C'))$ of a 2-step complex of vector bundles

$$F^\bullet(C'', C') : F^0 \xrightarrow{b} F^1. \quad (3.3)$$

This complex has terms

$$F^0 = \bigoplus_{i=0}^n \text{Hom}(E''_i, E'_i), \quad F^1 = \bigoplus_{i=1}^n \text{Hom}(E''_i, E'_{i-1}),$$

and differential

$$b(\psi_0, \dots, \psi_n) = \sum_{i=1}^n b_i(\psi_{i-1}, \psi_i), \quad \text{for } \psi_i \in \text{Hom}(E''_i, E'_i),$$

where

$$b_i : \text{Hom}(E''_{i-1}, E'_{i-1}) \oplus \text{Hom}(E''_i, E'_i) \rightarrow \text{Hom}(E''_i, E'_{i-1}) \hookrightarrow F^1$$

is given by

$$b_i(\psi_{i-1}, \psi_i) = \psi_{i-1} \circ \phi''_i - \phi'_i \circ \psi_i.$$

Applying the cohomology functor to this complex of vector bundles, we obtain maps of vector spaces

$$d = H^p(b) : H^p(F^0) \longrightarrow H^p(F^1), \quad (3.4)$$

for $p = 0, 1$, where

$$H^p(F^0) = \bigoplus_{i=0}^n \text{Ext}_X^p(E''_i, E'_i), \quad H^p(F^1) = \bigoplus_{i=1}^n \text{Ext}_X^p(E''_i, E'_{i-1}),$$

and

$$d(\psi_0, \dots, \psi_n) = \sum_{i=1}^n d_i(\psi_{i-1}, \psi_i), \quad \text{for } \psi_i \in \text{Ext}_X^i(E''_i, E'_i).$$

Here,

$$d_i: \begin{array}{c} \text{Ext}_X^p(E''_{i-1}, E'_{i-1}) \\ \oplus \\ \text{Ext}_X^p(E''_i, E'_i) \end{array} \rightarrow \text{Ext}_X^p(E''_i, E'_{i-1}) \hookrightarrow H^p(F^1) \quad (3.5)$$

is given by

$$d_i(\psi_{i-1}, \psi_i) = \psi_{i-1} \circ \phi_i'' - \phi_i' \circ \psi_i,$$

where \circ is composition of maps for $p = 0$, and the Yoneda product for $p = 1$.

The following result generalizes [6, Proposition 3.1] from holomorphic triples to holomorphic chains of arbitrary length.

Proposition 3.1. *There are natural isomorphisms*

$$\begin{aligned} \text{Hom}(C'', C') &\cong \mathbb{H}^0(F^\bullet(C'', C')), \\ \text{Ext}^1(C'', C') &\cong \mathbb{H}^1(F^\bullet(C'', C')), \end{aligned}$$

and an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(F^\bullet(C'', C')) \longrightarrow H^0(F^0) \xrightarrow{d} H^0(F^1) \longrightarrow \mathbb{H}^1(F^\bullet(C'', C')) \\ &\longrightarrow H^1(F^0) \xrightarrow{d} H^1(F^1) \longrightarrow \mathbb{H}^2(F^\bullet(C'', C')) \longrightarrow 0. \end{aligned}$$

Proof. This follows from [14, Theorem 4.1 and Theorem 5.1], since a holomorphic chain is a holomorphic quiver bundle, for the quiver $n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow 0$. \square

Given two sheaves or vector bundles E and F , we define $h^i(E, F) = \dim(\text{Ext}_X^i(E, F))$ and $\chi(E, F) = h^0(E, F) - h^1(E, F)$. Similarly, for any pair of chains C' and C'' as before, we define

$$\begin{aligned} h^i(C'', C') &= \dim(\mathbb{H}^i(C'', C')), \\ \chi(C'', C') &= h^0(C'', C') - h^1(C'', C') + h^2(C'', C'). \end{aligned}$$

Recall that

$$r'_i = \text{rk}(E'_i), \quad d'_i = \text{deg}(E'_i), \quad r''_i = \text{rk}(E''_i), \quad d''_i = \text{deg}(E''_i).$$

Proposition 3.2.

$$\begin{aligned} \chi(C'', C') &= \sum_{i=0}^n \chi(E''_i, E'_i) - \sum_{i=1}^n \chi(E''_i, E'_{i-1}) \\ &= (1-g) \left(\sum_{i=0}^n r''_i r'_i - \sum_{i=1}^n r''_i r'_{i-1} \right) \\ &\quad + \sum_{i=0}^n (r''_i d'_i - r'_i d''_i) - \sum_{i=1}^n (r''_i d'_{i-1} - r'_{i-1} d''_i). \end{aligned}$$

Proof. The first equality is a consequence of the exact sequence in Proposition 3.1. The Riemann–Roch formula

$$\chi(E, F) = (1 - g) \operatorname{rk}(E) \operatorname{rk}(F) + \operatorname{rk}(E) \operatorname{deg}(F) - \operatorname{rk}(F) \operatorname{deg}(E),$$

for vector bundles E and F , implies now the second equality. \square

The previous proposition shows that $\chi(C'', C')$ only depends on the types t' and t'' of C' and C'' , respectively, so we may use the notation

$$\chi(t'', t') := \chi(C'', C').$$

Corollary 3.3. *For any extension $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ of holomorphic chains,*

$$\chi(C, C) = \chi(C', C') + \chi(C'', C'') + \chi(C'', C') + \chi(C', C'').$$

3.2. Vanishing of \mathbb{H}^0 and \mathbb{H}^2 . The following result is proved as in the case of semistable vector bundles, given the identification of $\mathbb{H}^0(F^\bullet(C'', C'))$ with $\operatorname{Hom}(C'', C')$ of Proposition 3.1.

Proposition 3.4. *Suppose that C' and C'' are α -semistable.*

- i) *If $\mu_\alpha(C') < \mu_\alpha(C'')$, then $\mathbb{H}^0(F^\bullet(C'', C')) = 0$.*
- ii) *If $\mu_\alpha(C') = \mu_\alpha(C'')$ and C'' is α -stable, then*

$$\mathbb{H}^0(F^\bullet(C'', C')) \cong \begin{cases} \mathbb{C} & \text{if } C' \cong C'', \\ 0 & \text{if } C' \not\cong C''. \end{cases}$$

\square

In the following result, $\{u_0, \dots, u_n\}$ is the standard basis of \mathbb{R}^{n+1} .

Proposition 3.5. *Let C' and C'' be two holomorphic chains.*

i) *Let $\mathcal{D} \subset \{1, \dots, n\}$ and for each $i \in \mathcal{D}$, let $\epsilon_i \geq 0$. Suppose that the following three conditions hold:*

- *C', C'' are α -semistable with $\mu_\alpha(C') = \mu_\alpha(C'')$.*
- *For all $i \in \{1, \dots, n\} \setminus \mathcal{D}$, $\alpha_i - \alpha_{i-1} > 2g - 2$.*
- *For all $i \in \mathcal{D}$, one of C', C'' is $(\alpha + \epsilon_i u_i)$ -stable and $\alpha_i - \alpha_{i-1} \geq 2g - 2$.*

Then $\mathbb{H}^2(F^\bullet(C'', C')) = 0$.

ii) *If C' and C'' are α -semistable with the same α -slope, and $\alpha_i - \alpha_{i-1} > 2g - 2$ for all $i = 1, \dots, n$, then $\mathbb{H}^2(F^\bullet(C'', C')) = 0$.*

iii) *If one of C', C'' is α -stable and the other one is α -semistable with the same α -slope, and $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all $i = 1, \dots, n$, then $\mathbb{H}^2(F^\bullet(C'', C')) = 0$.*

iv) *If for all $i = 1, \dots, n$, ϕ_i'' is injective or ϕ_i' is generically surjective, then $\mathbb{H}^2(F^\bullet(C'', C')) = 0$.*

Proof. From the exact sequence of Proposition 3.1, $\mathbb{H}^2(F^\bullet(C'', C')) = 0$ if and only if the map $d: H^1(F^0) \rightarrow H^1(F^1)$ (defined as in (3.4) for

$p = 1$) is surjective, that is, the maps d_i in (3.5) are surjective for all $i = 1, \dots, n$. Using Serre duality, d_i is surjective if and only if the map $P_i: \text{Hom}_X(E'_{i-1}, E''_i \otimes K) \rightarrow \text{Hom}_X(E''_{i-1}, E'_{i-1} \otimes K) \oplus \text{Hom}_X(E'_i, E''_i \otimes K)$ given by

$$P_i(\xi_i) = ((\phi''_i \circ \text{id}_K) \circ \xi_i, \xi_i \circ \phi'_i), \quad \text{for } \xi_i \in \text{Hom}_X(E'_{i-1}, E''_i \otimes K),$$

is injective. Let $i \in \{1, \dots, n\}$ and $\xi_i: E'_{i-1} \rightarrow E''_i \otimes K$ be a map in $\ker(P_i)$. Let

$$I_i = \text{im}(\xi_i) \subset E''_i \otimes K \quad \text{and} \quad N_{i-1} = \ker(\xi_i) \subset E'_{i-1}.$$

Then the fact that $\xi_i \in \ker(P_i)$ is equivalent to the fact that the maps

$$E'_{i-1} \xrightarrow{\xi_i} E''_i \otimes K \xrightarrow{\phi''_i \otimes \text{id}} E''_{i-1} \otimes K \quad \text{and} \quad E'_i \xrightarrow{\phi'_i} E'_{i-1} \xrightarrow{\xi_i} E''_i \otimes K \quad (3.6)$$

are both zero. The first map is zero if and only if

$$I_i \subset \ker(\phi''_i \otimes \text{id}), \quad (3.7)$$

i.e., $I_i \otimes K^* \subset \ker(\phi''_i)$, so the diagram

$$\begin{array}{ccc} I_i \otimes K^* & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ E''_i & \xrightarrow{\phi''_i} & E''_{i-1} \end{array}$$

commutes. In other words, there is a subchain $C''_{I_i} \hookrightarrow C''$ given by

$$\begin{array}{ccccccccccccccc} C''_{I_i}: & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & I_i \otimes K^* & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\ \downarrow & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ C'': & E''_n & \xrightarrow{\phi''_n} & \cdots & \xrightarrow{\phi''_{i+2}} & E''_{i+1} & \xrightarrow{\phi''_{i+1}} & E''_i & \xrightarrow{\phi''_i} & E''_{i-1} & \xrightarrow{\phi''_{i-1}} & \cdots & \xrightarrow{\phi''_1} & E''_0. \end{array}$$

Similarly, the second map in (3.6) is zero if and only if

$$\text{im}(\phi'_i) \subset N_{i-1}, \quad (3.8)$$

i.e., the diagram

$$\begin{array}{ccc} E'_i & \xrightarrow{\phi'_i} & N_{i-1} \\ \parallel & & \downarrow \\ E'_i & \xrightarrow{\phi'_i} & E'_{i-1} \end{array}$$

commutes, so we can define a subchain $C'_{N_{i-1}} \hookrightarrow C'$ by

$$\begin{array}{ccccccccccccccc} C'_{N_{i-1}}: & E'_n & \xrightarrow{\phi'_n} & \cdots & \xrightarrow{\phi'_{i+1}} & E'_i & \xrightarrow{\phi'_i} & N_{i-1} & \xrightarrow{\phi'_{i-1}|_{N_{i-1}}} & E'_{i-2} & \xrightarrow{\phi'_{i-2}} & \cdots & \xrightarrow{\phi'_1} & E'_0 \\ \downarrow & \parallel & & & & \parallel & & \downarrow & & \parallel & & & & \parallel \\ C': & E'_n & \xrightarrow{\phi'_n} & \cdots & \xrightarrow{\phi'_{i+1}} & E'_i & \xrightarrow{\phi'_i} & E'_{i-1} & \xrightarrow{\phi'_{i-1}} & E'_{i-2} & \xrightarrow{\phi'_{i-2}} & \cdots & \xrightarrow{\phi'_1} & E'_0. \end{array}$$

Let $k_{i-1} = \text{rk}(N_{i-1})$, $l_{i-1} = \text{deg}(N_{i-1})$, so

$$\mu_\alpha(C'_{N_{i-1}}) = \frac{\left(l_{i-1} + \sum_{j \neq i-1} d'_j\right) + \left(\alpha_{i-1} k_{i-1} + \sum_{j \neq i-1} \alpha_j r'_j\right)}{k_{i-1} + \sum_{j \neq i-1} r'_j}.$$

From the short exact exact sequence

$$0 \longrightarrow N_{i-1} \longrightarrow E'_{i-1} \longrightarrow I_i \longrightarrow 0,$$

we see that

$$\begin{aligned} \text{rk}(I_i \otimes K^*) &= \text{rk}(I_i) = r'_{i-1} - k_{i-1}, \\ \text{deg}(I_i \otimes K^*) &= \text{deg}(I_i) + \text{deg}(K^*) \text{rk}(I_i) \\ &= d'_{i-1} - l_{i-1} + (2 - 2g)(r'_{i-1} - k_{i-1}). \end{aligned}$$

Hence

$$\begin{aligned} \mu_\alpha(C''_{I_i}) &= \frac{\text{deg}(I_i \otimes K^*) + \alpha_i \text{rk}(I_i \otimes K^*)}{\text{rk}(I_i \otimes K^*)} \\ &= \frac{d'_{i-1} - l_{i-1} - k_{i-1}}{r'_{i-1} - k_{i-1}} + 2 - 2g + \alpha_i. \end{aligned}$$

Using these formulae for $\mu_\alpha(C'_{N_{i-1}})$ and $\mu_\alpha(C''_{I_i})$, we obtain

$$\begin{aligned} \left(k_{i-1} + \sum_{j \neq i-1} r'_j\right) \mu_\alpha(C'_{N_{i-1}}) + (r'_{i-1} - k_{i-1}) \mu_\alpha(C''_{I_i}) &= \\ \sum_{j=0}^n d'_j + (r'_{i-1} - k_{i-1})(2 - 2g) + (\alpha_{i-1} - \alpha_i) k_{i-1} + \left(\sum_{j \neq i-1} \alpha_j r'_j + \alpha_i r'_{i-1}\right). \end{aligned} \quad (3.9)$$

To prove Part i), suppose first that $i \in \{1, \dots, n\} \setminus \mathcal{D}$, so $\alpha_i - \alpha_{i-1} > 2g - 2$. Since C' and C'' are α -semistable with the same α -slope, $\mu_\alpha(C'_{N_{i-1}}) \leq \mu_\alpha(C')$, $\mu_\alpha(C''_{I_i}) \leq \mu_\alpha(C'')$ and $\mu_\alpha(C') = \mu_\alpha(C'')$, so (3.9) is smaller than or equal to

$$\begin{aligned} \left(k_{i-1} + \sum_{j \neq i-1} r'_j\right) \mu_\alpha(C') + (r'_{i-1} - k_{i-1}) \mu_\alpha(C'') &= \left(\sum_{j=0}^n r'_j\right) \mu_\alpha(C') \\ &= \sum_{j=0}^n d'_j + \sum_{j=0}^n \alpha_j r'_j. \end{aligned}$$

This is equivalent to the inequality

$$(\alpha_i - \alpha_{i-1})(r'_{i-1} - k_{i-1}) \leq (r'_{i-1} - k_{i-1})(2g - 2).$$

If $\xi_i \neq 0$, then $N_i \neq 0$, so $r'_{i-1} - k_{i-1} > 0$ and we see that

$$\alpha_i - \alpha_{i-1} \leq 2g - 2, \quad (3.10)$$

which contradicts $\alpha_i - \alpha_{i-1} > 2g - 2$. Therefore, $\xi_i = 0$. Thus, $\ker(P_i) = 0$, i.e., P_i is injective for all $i \in \{1, \dots, n\} \setminus \mathcal{D}$. If $i \in \mathcal{D}$,

the fact that C'' is α_i -stable, where $\alpha_i := \alpha + \epsilon_i u_i$, implies the strict inequality $\mu_{\alpha_i}(C''_{I_i}) < \mu_{\alpha_i}(C'')$ (note that C''_{I_i} is a proper subchain of C'' , since $r''_j \neq 0$ for all j , so $C''_{I_i} \neq C''$, and $\xi \neq 0$, so $C''_{I_i} \neq 0$). Defining $\lambda_i(C'') = r''_i / \sum_{j=0}^n r''_j$, this strict inequality can be written as

$$\mu_{\alpha}(C''_{I_i}) - \mu_{\alpha}(C'') < \epsilon_i \lambda_i(C'') - \epsilon_i = \epsilon_i (\lambda_i(C'') - 1),$$

so $\mu_{\alpha}(C''_{I_i}) < \mu_{\alpha}(C'')$, as $\epsilon_i \geq 0$. Hence, replacing the inequality $\mu_{\alpha}(C''_{I_i}) \leq \mu_{\alpha}(C'')$ by the strict inequality, we obtain a strict inequality in (3.10), which contradicts the hypothesis $\alpha_i - \alpha_{i-1} \geq 2g - 2$. Therefore, ξ_i must be zero, i.e., P_i is also injective when $i \in \mathcal{D}$. Thus, we conclude that $\mathbb{H}^2(F^{\bullet}(C'', C')) = 0$. This completes the proof of Part i).

Defining \mathcal{D} to be the empty set, we see that Part i) implies Part ii), and defining $\mathcal{D} = \{1, \dots, n\}$ and $\epsilon_i = 0$ for all $i = 1, \dots, n$, we see that Part i) implies Part iii).

To prove iv), note that (3.7) implies that, if ϕ''_i is injective, then $I_i = 0$, whereas (3.8) implies that, if ϕ''_i is generically surjective, then $N_{i-1} = E_{i-1}$. In both cases, we deduce that $\xi_i = 0$. Hence $\ker(P_i) = 0$, i.e., P_i is injective, for all $i = 1, \dots, n$. As explained above, this is equivalent to $\mathbb{H}^2(F^{\bullet}(C'', C')) = 0$. \square

Remark 3.6. Note that Part iv) of Proposition 3.5 generalizes and gives a direct proof of [4, Proposition 6.3] without dimensional reduction techniques.

The following is an immediate consequence of Proposition 3.5.

Corollary 3.7. *Suppose that C', C'' are α -semistable chains such that $\mu_{\alpha}(C') = \mu_{\alpha}(C'')$, where $\alpha_i - \alpha_{i-1} > 2g - 2$ for all $i = 1, \dots, n$. Then*

$$\dim \text{Ext}^1(C'', C') = h^0(C'', C') - \chi(C'', C').$$

The same holds if the conditions of Proposition 3.5, Part i), are satisfied for some subset $\mathcal{D} \subset \{1, \dots, n\}$ and $\epsilon_i \geq 0$ ($i \in \mathcal{D}$).

3.3. Deformation theory of chains. Let $\mathcal{M}_{\alpha}^s(t)$ be the moduli space of α -stable chains of type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$.

Theorem 3.8. *Let C be an α -stable chain of type t .*

i) *The Zariski tangent space at the point defined by C in the moduli space $\mathcal{M}_{\alpha}^s(t)$ is isomorphic to $\mathbb{H}^1(F^{\bullet}(C, C))$.*

ii) *If $\mathbb{H}^2(F^{\bullet}(C, C)) = 0$, then the moduli space $\mathcal{M}_{\alpha}^s(t)$ is smooth in a neighbourhood of the point defined by C .*

iii) *$\mathbb{H}^2(F^{\bullet}(C, C)) = 0$ if and only if the homomorphism in the corresponding exact sequence of Proposition 3.1,*

$$d: \bigoplus_{i=0}^n \text{Ext}_X^1(E_i, E_i) \longrightarrow \bigoplus_{i=1}^n \text{Ext}_X^1(E_i, E_{i-1}),$$

is surjective.

iv) At a smooth point $C \in \mathcal{M}_\alpha^s(t)$, the dimension of the moduli space $\mathcal{M}_\alpha^s(t)$ is

$$\begin{aligned} \dim \mathcal{M}_\alpha^s(t) &= h^1(C, C) = 1 - \chi(C, C) \\ &= (g-1) \left(\sum_{i=0}^n r_i^2 - \sum_{i=1}^n r_i r_{i-1} \right) \\ &\quad + \sum_{i=1}^n (r_i d_{i-1} - r_{i-1} d_i) + 1. \end{aligned}$$

v) If for each $i = 1, \dots, n$, $\phi_i: E_i \rightarrow E_{i-1}$ is injective or generically surjective, then $\mathbb{H}^2(F^\bullet(C, C)) = 0$ and therefore C defines a smooth point of $\mathcal{M}_\alpha^s(t)$.

vi) If $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all $i = 1, \dots, n$, then C defines a smooth point in the moduli space, and hence $\mathcal{M}_\alpha^s(t)$ is smooth.

Proof. All the results of this theorem follow immediately from the results of this section, as in the proof of [6, Theorem 3.8], noting that the space of infinitesimal deformations of C is $\mathbb{H}^1(F^\bullet(C, C))$. \square

Part iv) of Proposition 3.8 highlights the importance of the following.

Definition 3.9. The region $R_{2g-2} \subset \mathbb{R}^{n+1}$ is the set of points α such that $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all $i = 1, \dots, n$.

4. WALL-CROSSING

4.1. Flip loci. In this section we study the variations in the moduli spaces $\mathcal{M}_\alpha^s(t)$, for fixed type t and different values of α . We begin with a set-theoretic description of the differences between two spaces $\mathcal{M}_\alpha^s(t)$ and $\mathcal{M}_\beta^s(t)$ when α and β are separated by a wall w (as defined in Section 2.4). For the rest of this section we adopt the following notation: Let α_w be the point in the parameter space obtained by intersecting the line determined by α and β with the wall w . Set

$$\alpha_w^+ = \alpha_w + \epsilon, \quad \alpha_w^- = \alpha_w - \epsilon,$$

where $|\epsilon| > 0$ is small enough so that α_w^- is in the same chamber as α and α_w^+ is in the same chamber as β .

We define *flip loci* $\mathcal{S}_{\alpha_w^\pm} \subset \mathcal{M}_{\alpha_w^\pm}^s$ by the conditions that the points in $\mathcal{S}_{\alpha_w^+}$ represent chains which are α_w^+ -stable but α_w^- -unstable, while the points in $\mathcal{S}_{\alpha_w^-}$ represent chains which are α_w^- -stable but α_w^+ -unstable. The following is immediate.

Lemma 4.1. *In the above notation:*

$$\mathcal{M}_{\alpha_w^+}^s - \mathcal{S}_{\alpha_w^+} = \mathcal{M}_{\alpha_w}^s = \mathcal{M}_{\alpha_w^-}^s - \mathcal{S}_{\alpha_w^-}.$$

Remark 4.2. If the wall w is included in one of the bounding hyperplanes we have that either $\mathcal{S}_{\alpha_w^+} = \mathcal{M}_{\alpha_w^+}^s$, or $\mathcal{S}_{\alpha_w^-} = \mathcal{M}_{\alpha_w^-}^s$. The only interesting cases are thus those for which w is not a bounding wall.

A chain $C \in \mathcal{S}_{\alpha_w^\pm}$ must be strictly α_w -semistable. Hence in order to compute the codimension of $\mathcal{S}_{\alpha_w^\pm}$ in $\mathcal{M}_{\alpha_w^\pm}^s$ we have to “count” Jordan-Hölder filtrations of C . Let C be a strictly α_w -semistable chain. As we have seen in Section 2, there is a filtration of chains given by

$$0 = C_0 \subset C_1 \subset \cdots \subset C_m = C, \quad (4.1)$$

with $G_i = C_i/C_{i-1}$ α_w -stable and $\mu_{\alpha_w}(G_i) = \mu_{\alpha_w}(C)$ for $1 \leq i \leq m$. Let $\text{gr}(C) = \bigoplus_i G_i$ be the graduation of C .

Proposition 4.3. *Let w be a wall contained in the region R_{2g-2} . Let \mathcal{S} be a family of α_w -semistable chains C of type t , all of them pairwise non-isomorphic, and whose Jordan-Hölder filtration (4.1) has graduation $\text{gr}(C) = \bigoplus_{i=1}^m G_i$, with G_i of type t_i . Then*

$$\dim \mathcal{S} \leq - \sum_{i \leq j} \chi(t_j, t_i) - \frac{m(m-3)}{2}. \quad (4.2)$$

Proof. It is clear that

$$\dim \mathcal{S} \leq \sum_{1 \leq i \leq m} \dim \mathcal{M}_{\alpha_w}^s(t_i) + \sum_{1 \leq i < j \leq m} \dim \mathbb{P}(\text{Ext}^1(G_j, G_i)),$$

since $\mathbb{P}(\text{Ext}^1(G_j, G_i))$ parametrizes equivalence classes of extensions

$$0 \longrightarrow G_i \longrightarrow G \longrightarrow G_j \longrightarrow 0.$$

By Theorem 3.8, $\mathcal{M}_{\alpha_w}^s(t_i)$ is smooth and $\dim \mathcal{M}_{\alpha_w}^s(t_i) = 1 - \chi(t_i, t_i)$. From Corollary 3.7, we have that the dimension of $\text{Ext}^1(G_j, G_i)$ is given by $h^1(G_j, G_i) = -\chi(t_j, t_i)$. Here we are using the vanishing of $h^2(G_j, G_i)$ given by Proposition 3.5, and we have assumed that G_i and G_j are not isomorphic, and hence by Proposition 3.4 $h^0(G_j, G_i) = 0$, since otherwise we would have a subfamily of positive codimension in \mathcal{S} . The result follows now by adding up these dimensions. \square

In order to show that the flip loci $\mathcal{S}_{\alpha_w^\pm} \subset \mathcal{M}_{\alpha_w^\pm}^s$ have positive codimension we need to bound the values of $\chi(t_j, t_i)$ in (4.2). This is what we do next.

4.2. Bounds for χ .

Proposition 4.4. *Let C', C'' be two holomorphic chains, as in (3.1), (3.2), and b the differential of the complex $F^\bullet(C'', C')$, as in (3.3). If C' and C'' are α -polystable and $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all $i = 1, \dots, n$, then*

$$\mu(\ker(b)) \leq \mu_\alpha(C') - \mu_\alpha(C''), \quad (4.3)$$

$$\mu(\text{coker}(b)) \geq \mu_\alpha(C') - \mu_\alpha(C'') + 2g - 2. \quad (4.4)$$

Proof. We start constructing a holomorphic chain

$$\tilde{C}(C'', C'): F^{-1} \xrightarrow{c} F^0 \xrightarrow{b} F^1 \xrightarrow{a} F^2,$$

where F^0, F^1 and b are defined as in (3.3),

$$F^{-1} = \bigoplus_{i=1}^n \text{Hom}(E''_{i-1}, E'_i), \quad F^2 = \bigoplus_{i=2}^n \text{Hom}(E''_i, E'_{i-2}),$$

and

$$a(\theta_1, \dots, \theta_n) = \sum_{i=2}^n a_i(\theta_{i-1}, \theta_i), \quad c(\omega_1, \dots, \omega_n) = \sum_{i=0}^n c_i(\omega_i, \omega_{i+1}),$$

where $\theta_i \in \text{Hom}(E''_i, E'_{i-1})$, $\omega_i \in \text{Hom}(E''_{i-1}, E'_i)$, and

$$\begin{aligned} a_i: \text{Hom}(E''_{i-1}, E'_{i-2}) \oplus \text{Hom}(E''_i, E'_{i-1}) &\rightarrow \text{Hom}(E''_i, E'_{i-2}) \hookrightarrow F^2, \\ c_i: \text{Hom}(E''_{i-1}, E'_i) \oplus \text{Hom}(E''_i, E'_{i+1}) &\rightarrow \text{Hom}(E''_i, E'_i) \hookrightarrow F^0, \end{aligned}$$

are given by

$$a_i(\theta_{i-1}, \theta_i) = \theta_{i-1} \circ \phi''_i - \phi'_{i-1} \circ \theta_i, \quad c_i(\omega_i, \omega_{i+1}) = \omega_i \circ \phi''_i - \phi'_{i+1} \circ \omega_{i+1},$$

with $E'_{n+1} = 0 = E''_{-1}$ by convention. Note that the holomorphic chain $\tilde{C}(C'', C')$ is not in general a complex.

Suppose that C', C'' are α -polystable. Then by Theorem 2.15, there are Hermitian metrics on the vector bundles E'_i and E''_i satisfying the τ' - and τ'' -vortex equations

$$\begin{aligned} \sqrt{-1}\Lambda F(E'_i) + \phi'_{i+1}\phi'^*_{i+1} - \phi'^*_i\phi_i &= \tau'_i \text{id}_{E'_i}, \\ \sqrt{-1}\Lambda F(E''_i) + \phi''_{i+1}\phi''^*_{i+1} - \phi''^*_i\phi''_i &= \tau''_i \text{id}_{E''_i}, \end{aligned} \quad (4.5)$$

for $i = 0, \dots, n$, where $\tau', \tau'' \in \mathbb{R}^{n+1}$ are given by

$$\tau'_i = \mu_\alpha(C') - \alpha_i \quad \text{and} \quad \tau''_i = \mu_\alpha(C'') - \alpha_i. \quad (4.6)$$

Using these equations, we now show that the induced metrics on the bundles F^i , for $i = -1, 0, 1, 2$, which are the terms of the holomorphic chain $\tilde{C}(C', C'')$, satisfy the equations

$$\begin{aligned} \sqrt{-1}\Lambda F(F^0) + c \circ c^* - b^* \circ b &= (\mu_\alpha(C') - \mu_\alpha(C'')) \text{id}, \\ \sqrt{-1}\Lambda F(F^1) + b \circ b^* - a^* \circ a &= (\mu_\alpha(C') - \mu_\alpha(C'')) \text{id} \\ &+ \sum_{i=1}^n (\alpha_i - \alpha_{i-1}) \pi_i^1, \end{aligned} \quad (4.7)$$

where $\pi_i^1: F^1 \rightarrow \text{Hom}(E''_i, E'_{i-1})$ is the canonical projection. To prove this, let $\psi_i \in \text{Hom}(E''_i, E'_i)$ and $\zeta_i \in \text{Hom}(E''_i, E'_{i-1})$. The curvature $F(F^p)$ of the induced connection on F^p , for $p = 0, 1$, is the $(\text{End}(F^p))$ -valued 2-form given by

$$\begin{aligned} F(F^0)(\psi_i) &= F(E'_i) \wedge \psi_i - \psi_i \wedge F(E''_i), \\ F(F^1)(\zeta_i) &= F(E'_{i-1}) \wedge \zeta_i - \zeta_i \wedge F(E''_i), \end{aligned}$$

so the first terms in the left-hand sides of (4.7) are given by

$$\sqrt{-1}\Lambda F(F^0)(\psi_i) = \sqrt{-1}\Lambda F(E'_i) \circ \psi_i - \psi_i \circ \sqrt{-1}\Lambda F(E''_i), \quad (4.8)$$

$$\sqrt{-1}\Lambda F(F^1)(\zeta_i) = \sqrt{-1}\Lambda F(E'_{i-1}) \circ \zeta_i - \zeta_i \circ \sqrt{-1}\Lambda F(E''_i). \quad (4.9)$$

The remaining terms in the left-hand side of the first equation in (4.7) are

$$\begin{aligned} c \circ c^*(\psi_i) &= \psi_i \circ \phi_i''^* \phi_i'' + \phi_{i+1}' \phi_{i+1}''^* \circ \psi_i \\ &\quad - \phi_{i+1}''^* \circ \psi_i \circ \phi_{i+1}'' - \phi_i' \circ \psi_i \circ \phi_i''^*, \\ b^* \circ b(\psi_i) &= \psi_i \circ \phi_{i+1}'' \phi_{i+1}''^* + \phi_i''^* \phi_i' \circ \psi_i \\ &\quad - \phi_i' \circ \psi_i \circ \phi_i''^* - \phi_{i+1}''^* \circ \psi_i \circ \phi_{i+1}'', \end{aligned} \quad (4.10)$$

whereas the remaining terms in the left-hand side of the second equation in (4.7) are

$$\begin{aligned} b \circ b^*(\zeta_i) &= \zeta_i \circ \phi_i''^* \phi_i'' + \phi_i' \phi_i''^* \circ \zeta_i \\ &\quad - \phi_i''^* \circ \zeta_i \circ \phi_{i+1}' - \phi_{i-1}' \circ \zeta_i \circ \phi_i''^*, \\ a^* \circ a(\zeta_i) &= \zeta_i \circ \phi_{i+1}'' \phi_{i+1}''^* + \phi_{i-1}''^* \phi_{i-1}' \circ \zeta_i \\ &\quad - \phi_{i-1}' \circ \zeta_i \circ \phi_i''^* - \phi_i''^* \circ \zeta_i \circ \phi_{i+1}''. \end{aligned} \quad (4.11)$$

Using (4.8) and (4.10), and (4.9) and (4.11), together with (4.5), it follows immediately that the left-hand sides of (4.7) are

$$\sqrt{-1}\Lambda F(F^0) + c \circ c^* - b^* \circ b = \sum_{i=0}^n (\tau_i' - \tau_i'') \pi_i^0, \quad (4.12)$$

$$\sqrt{-1}\Lambda F(F^1) + b \circ b^* - a^* \circ a = \sum_{i=1}^n (\tau_{i-1}' - \tau_i'') \pi_i^1, \quad (4.13)$$

where $\pi_i^0: F^0 \rightarrow \text{Hom}(E''_i, E'_i)$ is the canonical projection. Now, it follows from (4.6) that the right-hand sides of (4.12) and (4.13) equal the right-hand sides of the first and the second equation in (4.7), respectively, so equations (4.7) are satisfied.

We can now use the equations (4.7) to obtain the inequalities (4.3) and (4.4). Let $G \subset \text{coker}(b)$ be the maximal vector subbundle of $\text{coker}(b)$. Note that

$$\text{rk}(\text{coker}(b)) = \text{rk}(G), \quad \deg(\text{coker}(b)) \geq \deg(G), \quad (4.14)$$

and, by standard results (see e.g. [28, 25, 3]),

$$\deg(\ker(b)) = \frac{1}{2\pi} \left(\int_X \text{tr}(\pi_0 \sqrt{-1}\Lambda F(F^0)) - \|\beta_0\|_{L^2}^2 \right), \quad (4.15)$$

$$\deg(G) = \frac{1}{2\pi} \left(\int_X \text{tr}(\pi_1 \sqrt{-1}\Lambda F(F^1)) + \|\beta_1\|_{L^2}^2 \right), \quad (4.16)$$

where

$$\pi_0: F_0 \longrightarrow F_0 \quad \text{and} \quad \pi_1: F^1 \longrightarrow F^1$$

are the orthogonal projection operators onto $\ker(b)$ and G , with respect to the induced Hermitian metrics on F^0 and F^1 (using the metric on F^1 , G is regarded as a (smooth) subbundle of F^1), and β_0 and β_1 are the corresponding second fundamental forms, i.e., the $(\text{End}(F^0))$ -valued and $(\text{End}(F^1))$ -valued $(0, 1)$ -forms

$$\beta_0 = \bar{\partial}_{F^0}(\pi_0), \quad \beta_1 = \bar{\partial}_{F^1}(\pi_1).$$

Let $a_\perp = a \circ \pi_1: F^1 \rightarrow F^2$ and $c_\perp = \pi_0 \circ c: F^{-1} \rightarrow F^0$. Since $a_\perp^* = \pi_1 \circ a^*$ and $c_\perp^* = c^* \circ \pi_0$,

$$\begin{aligned} \text{tr}(\pi_1 \circ a^* a) &= \text{tr}((\pi_1 a^*)(a \pi_1)) = \text{tr}(a_\perp^* a_\perp) = |a_\perp|^2, \\ \text{tr}(c c^* \circ \pi_0) &= \text{tr}((\pi_0 c)(c^* \pi_0)) = \text{tr}(c_\perp c_\perp^*) = |c_\perp|^2, \end{aligned} \quad (4.17)$$

(where $|\cdot|$ are the induced norms). By the definitions of π_0 and π_1 ,

$$b \circ \pi_0 = 0 \quad \text{and} \quad \pi_1 \circ b = 0. \quad (4.18)$$

Applying $\text{tr}(- \circ \pi_0)$ and $\text{tr}(\pi_1 \circ -)$ to (4.7), and using (4.17) and (4.18), we obtain

$$\text{tr}(\sqrt{-1} \Lambda F(F^0) \pi_0) + |c_\perp|^2 = (\mu_\alpha(C') - \mu_\alpha(C'')) \text{rk}(\ker(b)), \quad (4.19)$$

$$\begin{aligned} \text{tr}(\sqrt{-1} \Lambda F(F^1) \pi_1) - |a_\perp|^2 &= (\mu_\alpha(C') - \mu_\alpha(C'')) \text{rk}(\text{coker}(b)) \\ &\quad + \sum_{i=1}^n (\alpha_i - \alpha_{i-1}) \text{tr}(\pi_i^1 \pi_1), \end{aligned} \quad (4.20)$$

respectively (where $\text{tr}(\pi_1) = \text{rk}(\text{coker}(b))$ by (4.14)). Since $\text{tr}(\pi_i^1 \pi_1) \geq 0$, $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all i , and $\sum_{i=1}^n \pi_i^1 = \text{id}_{F^1}$, the last term in the right-hand side of (4.20) satisfies

$$\sum_{i=1}^n (\alpha_i - \alpha_{i-1}) \text{tr}(\pi_i^1 \pi_1) \geq (2g - 2) \sum_{i=1}^n \text{tr}(\pi_i^1 \pi_1) = (2g - 2) \text{rk}(\text{coker}(b)),$$

so (4.20) implies

$$\begin{aligned} \text{tr}(\sqrt{-1} \Lambda F(F^1) \pi_1) - |a_\perp|^2 \\ \geq (\mu_\alpha(C') - \mu_\alpha(C'')) + 2g - 2 \text{rk}(\text{coker}(b)). \end{aligned} \quad (4.21)$$

Integrating (4.19) and (4.21) over X , using (4.14), (4.15) and (4.16), and dividing by $\text{vol}(X) = 2\pi$, we obtain

$$\text{deg}(\ker(c)) + \frac{1}{2\pi} (\|\beta_0\|_{L^2}^2 + \|c_\perp\|_{L^2}^2) = (\mu_\alpha(C') - \mu_\alpha(C'')) \text{rk}(\ker(b)),$$

$$\begin{aligned} \text{deg}(\text{coker}(c)) - \frac{1}{2\pi} (\|\beta_1\|_{L^2}^2 + \|a_\perp\|_{L^2}^2) \\ \geq (\mu_\alpha(C') - \mu_\alpha(C'')) + 2g - 2 \text{rk}(\text{coker}(b)), \end{aligned}$$

respectively, which imply (4.3) and (4.4). \square

Proposition 4.5. *Let C' and C'' be non-zero holomorphic chains of types t' and t'' , respectively, and let $\alpha \in \mathbb{R}^{n+1}$. Suppose that the following conditions hold:*

- C' and C'' are α -polystable with $\mu_\alpha(C') = \mu_\alpha(C'')$,
- $\alpha_i - \alpha_{i-1} \geq 2g - 2$ for all $i = 1, \dots, n$,
- the map $b: F^0 \rightarrow F^1$ of (3.3) is not generically an isomorphism.

Then $\chi(C'', C') \leq 1 - g$. In particular, if $g \geq 2$, then $\chi(C'', C') < 0$.

Proof. Let $F^\bullet(C'', C'): F^0 \xrightarrow{b} F^1$ be the complex (3.3). By Proposition 3.2,

$$\chi(C'', C') = (1 - g) (\text{rk}(F^0) - \text{rk}(F^1)) + \deg(F^0) - \deg(F^1).$$

Using

$$\begin{aligned} \deg(F^0) &= \deg(\ker(b)) + \deg(\text{im}(b)), \\ \deg(F^1) &= \deg(\text{im}(b)) + \deg(\text{coker}(b)), \\ \text{rk}(F^1) &= \text{rk}(\text{im}(b)) + \text{rk}(\text{coker}(b)), \end{aligned}$$

and the inequalities (4.3) and (4.4) with $\mu_\alpha(C') = \mu_\alpha(C'')$, we see that

$$\begin{aligned} \deg(F^0) - \deg(F^1) &\leq 2(1 - g) \text{rk}(\text{coker}(b)) \\ &= 2(1 - g) (\text{rk}(F^1) - \text{rk}(\text{im}(b))), \end{aligned}$$

so

$$\chi(C'', C') \leq (1 - g) (\text{rk}(F^0) + \text{rk}(F^1) - 2 \text{rk}(\text{im}(b))). \quad (4.22)$$

Note that

$$\text{rk}(F^0) + \text{rk}(F^1) - 2 \text{rk}(\text{im}(b)) \geq 0, \quad (4.23)$$

with equality in (4.23) if and only if $\text{rk}(F^0) = \text{rk}(\text{im}(b)) = \text{rk}(F^1)$. But b is not generically an isomorphism, so the equality in (4.23) does not hold, i.e., $\text{rk}(F^0) + \text{rk}(F^1) - 2 \text{rk}(\text{im}(b)) \geq 1$. Therefore, (4.22) implies $\chi(C'', C') \leq 1 - g$. \square

4.3. The birationality region. Proposition 4.5 motivates the definition of a region $\tilde{R}(t) \subset \mathbb{R}^{n+1}$ as follows. First, we recall from §2.3 that a *linear chain* V (over \mathbb{C}) is a chain in the category of complex vector spaces, i.e., a diagram of complex vector spaces V_i and linear maps f_i which compose as follows.

$$V: V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} V_0. \quad (4.24)$$

Note that this is simply a holomorphic chain when X is a point. The *dimension vector* of V is the $(n + 1)$ -tuple of integers $r = (r_j, j = 0, \dots, n)$, with $r_j = \dim V_j$. Given two linear chains V', V'' , we define a 2-step complex of vector spaces over \mathbb{C} ,

$$F^\bullet(V'', V'): F^0 \xrightarrow{b} F^1, \quad (4.25)$$

exactly as in (3.3), where now X is a point, so b is simply a linear map. Note also that the α -slope $\mu_\alpha(C)$ of a holomorphic chain C only depends on the type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$ of C , so we may use the notation

$$\mu_\alpha(t) := \frac{\sum_{i=0}^n (d_i + \alpha_i r_i)}{\sum_{i=0}^n r_i}.$$

Definition 4.6. Fix a type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$. The region $\tilde{R}(t) \subset \mathbb{R}^{n+1}$ is the set of points α such that for all types $t' = (r'_j, j = 0, \dots, n; d'_j, j = 0, \dots, n)$ and $t'' = (r''_j, j = 0, \dots, n; d''_j, j = 0, \dots, n)$, with $t' + t'' = t$ and $\mu_\alpha(t') = \mu_\alpha(t'')$, and for all linear chains V' and V'' with dimension vectors $r' = (r'_j, j = 0, \dots, n)$ and $r'' = (r''_j, j = 0, \dots, n)$, respectively, the map b of (4.25) is not an isomorphism.

Note that the region $\tilde{R}(t)$ is generally non-connected. Furthermore, the definition of $\tilde{R}(t)$ does not involve the geometry of X but only linear algebra.

The following is an immediate consequence of Proposition 4.5.

Theorem 4.7. *Let C' and C'' be non-zero holomorphic chains of types t' and t'' , respectively, and let $\alpha \in \mathbb{R}^{n+1}$. Suppose that the following conditions hold:*

- C' and C'' are α -polystable with $\mu_\alpha(C') = \mu_\alpha(C'')$,
- $\alpha \in \tilde{R}(t) \cap R_{2g-2}$.

Then $\chi(C'', C') \leq 1 - g$. In particular, if $g \geq 2$, then $\chi(C'', C') < 0$.

Because of this theorem, it becomes an important problem to characterize the *birationality region* $\tilde{R}(t)$ of Definition 4.6. The rest of this subsection is devoted to the determination of $\tilde{R}(t)$.

Definition 4.8. (1) Fix a dimension vector $r = (r_j, j = 0, \dots, n)$.

Let $\mathcal{V}(r)$ be the set of pairs (r', r'') of dimension vectors $r' = (r'_j, j = 0, \dots, n)$ and $r'' = (r''_j, j = 0, \dots, n)$ with $r' + r'' = r$, such that there exist linear chains V' and V'' of dimension vectors r' and r'' , respectively, for which the differential b of the complex (4.25), corresponding to V' and V'' , is an isomorphism.

(2) Given a type $t = (r, d)$, let $\mathcal{T}(t)$ be the set of pairs (t', t'') of types $t' = (r'_j, j = 0, \dots, n; d'_j, j = 0, \dots, n)$ and $t'' = (r''_j, j = 0, \dots, n; d''_j, j = 0, \dots, n)$, with $t' + t'' = t$, such that $(r', r'') \in \mathcal{V}(r)$, where $r' = (r'_j, j = 0, \dots, n)$ and $r'' = (r''_j, j = 0, \dots, n)$.

(3) Given $(t', t'') \in \mathcal{T}(t)$, let $\mathcal{B}(t', t'') \subset \mathbb{R}^{n+1}$ be the set of points α on the hyperplane

$$\mu_\alpha(t') = \mu_\alpha(t'').$$

Proposition 4.9. $\widetilde{R}(t) = \mathbb{R}^{n+1} \setminus \mathcal{B}(t)$, where the ‘boundary’ is

$$\mathcal{B}(t) = \bigcup_{(t', t) \in \mathcal{T}(t)} \mathcal{B}(t', t').$$

Proof. This follows automatically from Definitions 4.6 and 4.8. \square

Thus, to determine the region $\widetilde{R}(t)$ we need to find the set $\mathcal{V}(r)$. This can be done by using the following results. Note first that the Euler characteristic of two linear chains V' and V'' is

$$\chi(V'', V') := \dim \operatorname{Hom}(V'', V') - \dim \operatorname{Ext}^1(V'', V').$$

The following lemma can be compared with Propositions 3.1 and 3.2.

Lemma 4.10. *Let V' and V'' be two linear chains with dimension vectors $r' = (r'_j, j = 0, \dots, n)$ and $r'' = (r''_j, j = 0, \dots, n)$, respectively. Let b be the map in (4.25). Then there is a canonical exact sequence*

$$0 \rightarrow \operatorname{Hom}(V'', V') \rightarrow F^0 \xrightarrow{b} F^1 \rightarrow \operatorname{Ext}^1(V'', V') \rightarrow 0. \quad (4.26)$$

Hence, the Euler characteristic is given by

$$\chi(V'', V') = \dim(F^0) - \dim(F^1) = \sum_{i=0}^n r''_i r'_i - \sum_{i=1}^n r''_i r'_{i-1}. \quad (4.27)$$

Proof. The exact sequence (4.26) is obtained as in Proposition 3.1 when X is point, whereas equation (4.27) follows immediately from this. \square

By Lemma 4.10, $\chi(V'', V')$ only depends on the dimension vectors r' and r'' of V' and V'' , respectively, so we may use the notation

$$\chi(r'', r') := \chi(V'', V').$$

Proposition 4.11. *Let V', V'' be linear chains, given by*

$$V' = \bigoplus_{j=1}^{s'} V'_j, \quad V'' = \bigoplus_{k=1}^{s''} V''_k,$$

where V'_j, V''_k are linear chains. Let

$$F^\bullet(V'', V'): F^0 \xrightarrow{b} F^1$$

be the 2-step complex corresponding to V' and V'' , as in (4.25), and

$$F^\bullet(V''_k, V'_j): F_{jk}^0 \xrightarrow{b_{jk}} F_{jk}^1, \quad \text{for } 1 \leq j \leq s', 1 \leq k \leq s'',$$

the 2-step complexes corresponding to the subchains V''_k and V'_j . The following conditions are equivalent:

- i) The map b is an isomorphism.
- ii) $\operatorname{Hom}(V'', V') = 0 = \operatorname{Ext}^1(V'', V')$.
- iii) The maps b_{jk} are isomorphisms, for all $1 \leq j \leq s', 1 \leq k \leq s''$.
- iv) $\operatorname{Hom}(V''_k, V'_j) = 0 = \operatorname{Ext}^1(V''_k, V'_j)$, for all $1 \leq j \leq s', 1 \leq k \leq s''$.

Proof. The equivalence ii) \Leftrightarrow iv) comes from the obvious isomorphisms

$$\mathrm{Hom}(V'', V') = \bigoplus_{\substack{1 \leq j \leq s' \\ 1 \leq k \leq s''}} \mathrm{Hom}(V_k'', V_j'), \quad \mathrm{Ext}^1(V'', V') = \bigoplus_{\substack{1 \leq j \leq s' \\ 1 \leq k \leq s''}} \mathrm{Ext}^1(V_k'', V_j'),$$

whereas i) \Leftrightarrow ii) and iii) \Leftrightarrow iv) follow from Lemma 4.10. \square

Definition 4.12. Given integers $0 \leq p \leq q \leq n$, the linear chain $\delta_{[p,q]}$ is defined by the diagram

$$\delta_{[p,q]}: \quad 0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathbb{C} = \mathbb{C} = \cdots = \mathbb{C} = \mathbb{C} \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

$q \qquad \qquad \qquad p$

Thus, $\delta_{[p,q]}$ is given by (4.24), where $V_i = \mathbb{C}$ if $p \leq i \leq q$ and $V_i = 0$ otherwise, whereas $f_i = \mathrm{id}: \mathbb{C} \rightarrow \mathbb{C}$ if $p < i \leq q$ and $f_i = 0$ otherwise. The dimension vector of $\delta_{[p,q]}$ is denoted by $r_{[p,q]}$.

Proposition 4.13. i) *All linear chains are direct sums of indecomposable ones.*

ii) *The linear chains $\delta_{[p,q]}$ are indecomposable and any indecomposable linear chain is isomorphic to such a $\delta_{[p,q]}$.*

Proof. Part i) is well-known. Part ii) can be found, e.g., in [9]. \square

Thus, the linear chains V' and V'' can be written as

$$V' \cong \bigoplus_{j=1}^{s'} \delta_{[p'_j, q'_j]}, \quad V'' \cong \bigoplus_{k=1}^{s''} \delta_{[p''_k, q''_k]}, \quad (4.28)$$

for sets of pairs of integers (p'_j, q'_j) and (p''_k, q''_k) , for $1 \leq j \leq s'$ and $1 \leq k \leq s''$, satisfying $0 \leq p'_j \leq q'_j \leq n$ and $0 \leq p''_k \leq q''_k \leq n$, for all j and k . Now, Proposition 4.9 reduces the problem of finding $\tilde{R}(t)$ to the problem of finding $\mathcal{T}(t)$ or equivalently (by Definition 4.8), $\mathcal{V}(r)$, whereas Propositions 4.11 and 4.13 reduce this problem to finding when there are no homomorphisms and extensions between the indecomposable linear chains $\delta_{[p',q']}$ and $\delta_{[p'',q']}$. This is in Proposition 4.14 below.

Given two integers p and q , let $[p, q] = \{p, p+1, \dots, q-1, q\}$ if $p \leq q$, and $[p, q] = \emptyset$ otherwise. Note that, by (4.27), for all pairs of integers (p', q') and (p'', q'') , with $0 \leq p' \leq q' \leq n$ and $0 \leq p'' \leq q'' \leq n$, we have $\chi(\delta_{[p'',q'']}, \delta_{[p',q']}) = \#([p', q'] \cap [p'', q'']) - \#([p'+1, q'+1] \cap [p'', q''])$, where, given any set S , $\#S$ denotes its cardinal.

Proposition 4.14. *Given pairs of integers (p', q') and (p'', q'') , with $0 \leq p' \leq q' \leq n$ and $0 \leq p'' \leq q'' \leq n$, the following conditions are equivalent:*

i) $\mathrm{Hom}(\delta_{[p'',q'']}, \delta_{[p',q']}) = 0 = \mathrm{Ext}^1(\delta_{[p'',q'']}, \delta_{[p',q']})$.

ii) *At least one of the following inequalities is satisfied*

$$p'' > p', \quad q'' > q', \quad p' > q'', \quad p'' > q', \quad (4.29)$$

and, furthermore,

$$\#([p', q'] \cap [p'', q'']) = \#([p' + 1, q' + 1] \cap [p'', q'']). \quad (4.30)$$

Proof. We first show that $\text{Hom}(\delta_{[p'', q'']}, \delta_{[p', q']}) = 0$ if and only if one of the inequalities (4.29) is satisfied. Note that the only way that the following two diagrams can commute is that the maps f and g are zero.

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & 0 \\ f \downarrow & & \downarrow \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & \mathbb{C}. \end{array}$$

Hence, all the maps f_i and g_i in the following diagrams are zero, provided they commute, so that they define morphisms $\delta_{[p'', q'']} \rightarrow \delta_{[p', q']}$.

$$\begin{array}{ccccccc} \cdots & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\ & & f_{p''+2} \downarrow & & f_{p''+1} \downarrow & & f_{p''} \downarrow & & \downarrow & & \downarrow \\ \text{(Case } p'' > p') & & \cdots & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \cdots \end{array}$$

$$\begin{array}{ccccccc} \cdots & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \cdots \\ & & \downarrow & & g_{q'} \downarrow & & g_{q'-1} \downarrow & & g_{q'-2} \downarrow & & \downarrow \\ \text{(Case } q'' > q') & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \cdots \end{array}$$

Thus, if $p'' > p'$ or $q'' > q'$, then $\text{Hom}(\delta_{[p'', q'']}, \delta_{[p', q']}) = 0$. If $p' > q''$ or $p'' > q'$, then $\text{Hom}(\delta_{[p'', q'']}, \delta_{[p', q']}) = 0$ as well, because the set $[p', q'] \cap [p'', q'']$ is empty. Conversely, if none of the inequalities (4.29) holds, i.e., if $q' \geq q'' \geq p' \geq p''$ (so $[p', q'] \cap [p'', q''] \neq \emptyset$), then the following commutative diagram shows that $\text{Hom}(\delta_{[p'', q'']}, \delta_{[p', q']}) = \mathbb{C}$.

$$\begin{array}{cccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \cdots \\ & & \downarrow & & \downarrow & & \parallel & & \parallel & & & & \parallel & & \downarrow & & \downarrow & & \\ \cdots & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \mathbb{C} & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Finally, if $\text{Hom}(\delta_{[p'', q'']}, \delta_{[p', q']}) = 0$, then $\text{Ext}^1(\delta_{[p'', q'']}, \delta_{[p', q']}) = 0$ if and only if $\chi(\delta_{[p'', q'']}, \delta_{[p', q']}) = 0$, which is equivalent to (4.30), by the observation before this proposition. \square

Remark 4.15. As observed in the proof of Proposition 4.14, given pairs of integers (p', q') and (p'', q'') with $1 \leq p' \leq q' \leq n$ and $1 \leq p'' \leq q'' \leq n$, the fact that at least one of the inequalities (4.29) is satisfied is equivalent to the fact that the following does not hold:

$$q' \geq q'' \geq p' \geq p''. \quad (4.31)$$

The regions $\mathcal{V}(r)$ and $\tilde{R}(t)$ can now be obtained by applying Propositions 4.11 and 4.14 to all possible direct sums (4.28) with dimension vectors r' and r'' such that $r' + r'' = r$. Thus, we have proved the following.

Theorem 4.16. *Fix a dimension vector $r = (r_j, j = 0, \dots, n)$. A pair of dimension vectors (r', r'') , with $r' + r'' = r$, belongs to $\mathcal{V}(r)$ if and only if there are decompositions*

$$r' = r_{[p'_1, q'_1]} + \dots + r_{[p'_{s'}, q'_{s'}]} \quad \text{and} \quad r'' = r_{[p''_1, q''_1]} + \dots + r_{[p''_{s''}, q''_{s''}]},$$

for two sequences of pairs of integers

$$((p'_i, q'_i); i = 1, \dots, s') \quad \text{and} \quad ((p''_j, q''_j); j = 1, \dots, s''),$$

with $0 \leq p'_i \leq q'_i \leq n$ and $0 \leq p''_j \leq q''_j \leq n$, such that the following conditions hold for all $i = 1, \dots, s'$ and $j = 1, \dots, s''$. At least one of the following inequalities is satisfied

$$p''_j > p'_i, \quad q''_j > q'_i, \quad p'_i > q''_j, \quad p''_j > q'_i, \quad (4.32)$$

and, furthermore,

$$\#([p'_i, q'_i] \cap [p''_j, q''_j]) = \#([p'_i + 1, q'_i + 1] \cap [p''_j, q''_j]). \quad (4.33)$$

Example 4.17. As an application of Theorem 4.16, here we obtain the birationality region when $n = 1$. Note that in this case the $(n+1)$ -holomorphic chains are actually the holomorphic triples studied in [6] and that in that paper, although not explicitly defined, the birationality region was completely determined. First, we list the only indecomposable linear 2-chains δ', δ'' satisfying $\text{Hom}(\delta'', \delta') = 0 = \text{Ext}(\delta'', \delta')$:

- (1) $\delta' = \delta_{[1,1]}$, $\delta'' = \delta_{[0,0]}$;
- (2) $\delta' = \delta_{[0,1]}$, $\delta'' = \delta_{[1,1]}$;
- (3) $\delta' = \delta_{[0,0]}$, $\delta'' = \delta_{[0,1]}$.

Applying Theorem 4.16 to a dimension vector $r = (r_0, r_1)$, it follows that $\mathcal{V}(r)$ is the set of pairs (r', r'') , formed by pairs of integers $r' = (r'_0, r'_1)$ and $r'' = (r''_0, r''_1)$ with $r' + r'' = r$, satisfying exactly one of the following conditions:

- (1) $r'_0 = r''_1 = 0$;
- (2) $r''_0 = 0$, $r''_1 \neq 0$ and $r'_0 = r'_1$;
- (3) $r'_0 \neq 0$, $r'_1 = 0$ and $r''_0 = r''_1$.

These conditions also follow from of [6, Lemma 4.5]. Fix now a type $t = (r_0, r_1; d_0, d_1)$. Using the previous description of $\mathcal{V}(r)$, together with Definition 4.8(3) and Proposition 4.9, we immediately see that

$$\tilde{R}(t) \cap \{(0, \alpha) \in \mathbb{R}^2\} = \{0\} \times (\mathbb{R} \setminus \{\alpha_m(t), \alpha_M(t)\}),$$

where

$$\alpha_m(t) := \frac{d_0}{r_0} - \frac{d_1}{r_1},$$

$$\alpha_M(t) := \frac{2r_0}{|r_0 - r_1|} \alpha_m(t) = \left(1 + \frac{r_0 + r_1}{|r_0 - r_1|}\right) \left(\frac{d_0}{r_0} - \frac{d_1}{r_1}\right).$$

Here we set $\alpha_M(t) = +\infty$ when $r_0 = r_1$, by convention. Hence, a connected component of $\tilde{R}(t) \cap \{(0, \alpha) \in \mathbb{R}^2\}$ is given by the open interval $\alpha_m(t) < \alpha < \alpha_M(t)$. We have thus recovered [6, Lemma 4.6].

In the previous example we have seen that, when $n = 1$, the parameter region $R(t)$ is in fact a component of the birationality region $\tilde{R}(t)$. For $n > 1$, this is not generally true, as we will see in §6. The following proposition clarifies the relation between the parameter region $R(t)$ and the birationality region $\tilde{R}(t)$. Recall that $R(t)$ is bounded by the hyperplanes h_i defined in §2.3 for $0 \leq i < n$ and that, by Proposition 4.9, $\tilde{R}(t)$ is the complementary of the union $\mathcal{B}(t)$ of the hyperplanes $\mathcal{B}(t', t'')$ of Definition 4.8.

Proposition 4.18. *Fix a type $t = (r_j, j = 0, \dots, n; d_j, j = 0, \dots, n)$. The hyperplanes h_i are contained in the boundary $\mathcal{B}(t)$ of \tilde{R} , for all $0 \leq i < n$.*

Proof. Given a linear chain V and an integer $k \geq 0$, let kV be the direct sum of k copies of V if $k > 0$ or 0 if $k = 0$. Observe now that $\text{Hom}(\delta_{[j,j]}, \delta_{[k,k]}) = 0 = \text{Ext}^1(\delta_{[i,i]}, \delta_{[j,j]})$ for all $0 \leq j < k \leq n$ (this corresponds to case (1) in Example 4.17), so

$$\text{Hom}\left(\bigoplus_{0 \leq j \leq i} r_j \delta_{[j,j]}, \bigoplus_{i < j \leq n} r_j \delta_{[j,j]}\right) = 0 = \text{Ext}^1\left(\bigoplus_{0 \leq j \leq i} r_j \delta_{[j,j]}, \bigoplus_{i < j \leq n} r_j \delta_{[j,j]}\right).$$

for all $0 \leq i < n$. Since the linear chains

$$\bigoplus_{0 \leq j \leq i} r_j \delta_{[j,j]} \quad \text{and} \quad \bigoplus_{i < j \leq n} r_j \delta_{[j,j]}$$

have dimension vectors

$$r_{(\leq i)} := (r_0, \dots, r_i, 0, \dots, 0) \quad \text{and} \quad r_{(> i)} := (0, \dots, 0, r_{i+1}, \dots, r_n),$$

respectively, it follows that $(r_{(> i)}, r_{(\leq i)}) \in \mathcal{V}(r)$, so $(t_{(> i)}, t_{(\leq i)}) \in \mathcal{T}(t)$, where

$$t_{(\leq i)} := (r_{(\leq i)}, d_{(\leq i)}) \quad \text{and} \quad t_{(> i)} := (r_{(> i)}, d_{(> i)}),$$

with

$$d_{(\leq i)} := (d_0, \dots, d_i, 0, \dots, 0) \quad \text{and} \quad d_{(> i)} := (0, \dots, 0, d_{i+1}, \dots, d_n).$$

By definition, this implies that $\mathcal{B}(t_{(> i)}, t_{(\leq i)})$ is contained in the boundary $\mathcal{B}(t)$. But $\mathcal{B}(t_{(> i)}, t_{(\leq i)})$ is the set of points α on the hyperplane

$$\mu_\alpha(t_{(> i)}) = \mu_\alpha(t_{(\leq i)})$$

Writing explicitly this equation, we see that $\mathcal{B}(t_{(> i)}, t_{(\leq i)})$ is in fact the hyperplane h_i defined in §2.3. \square

4.4. Birationality of moduli spaces. Let α_w, α_w^+ and α_w^- be defined as in §4.1, where now $|\epsilon| > 0$ is small enough so that α_w^- and α_w^+ are in the same connected component of $\tilde{R}(t)$.

Proposition 4.19. *Let w be a wall contained in the region $\tilde{R}(t) \cap R_{2g-2}$. Let \mathcal{S} be a family of α_w -semistable chains of type t all of which are pairwise non-isomorphic, and such that \mathcal{S} maps generically one-to-one in an open set in the moduli space $\mathcal{M}_{\alpha_w^\pm}$. Then the codimension of the strictly semistable locus (which we assume non-empty) in \mathcal{S} is at least $g - 1$.*

Proof. The codimension of the strictly semistable locus is at least

$$\min \left\{ - \sum_{j < i} \chi(t_j, t_i) + \frac{m(m-3) + 2}{2} \right\}, \quad (4.34)$$

where the minimum is taken over all the numerically possible types t_i and m that may occur for a Jordan-Hölder filtration of a strictly α_w -semistable change of type t . This follows from subtracting (4.2) to the dimension of the moduli space of α_w^\pm -stable chains of type t which is $1 - \chi(t, t)$ (recall that under the hypotheses of the proposition $h^2(C, C)$ vanishes), and using that $\chi(t, t) = \sum_{i,j} \chi(t_i, t_j)$ by Corollary 3.3.

Now, from Theorem 4.7, we have that $-\chi(t_j, t_i) \geq g - 1$. Hence, the codimension is at least

$$\min \left\{ \frac{m(m-1)}{2}(g-1) + \frac{m(m-3) + 2}{2} \right\}.$$

Clearly, the minimum is attained when $m = 2$ giving the result. \square

From Proposition 4.19 we immediately obtain the following.

Theorem 4.20. *Let w be a wall contained in the region $\tilde{R}(t) \cap R_{2g-2}$. Then $\mathcal{M}_{\alpha_w^+}^s$ and $\mathcal{M}_{\alpha_w^-}^s$ are birationally equivalent. Moreover, if in addition, $\gcd(r_1, \dots, r_n, d) = 1$ α -stability coincides with α -semistability, by Corollary 2.14, and hence $\mathcal{M}_{\alpha_w^+}$ and $\mathcal{M}_{\alpha_w^-}$ are birationally equivalent.*

Remark 4.21. We emphasize that the boundary $\mathcal{B}(t)$ of the birational region $\tilde{R}(t)$ contains in general more hyperplanes than just the h_i of Proposition 4.18. However, in the examples considered in §6 we will be able to bound the dimensions of the flip loci when the parameter α crosses the hyperplanes $\mathcal{B}(t', t'')$ which are inside the parameter region $R(t)$. A natural question is whether this can always be done in the analysis of moduli spaces $\mathcal{M}_\alpha^s(t)$ of holomorphic $(n+1)$ -chains for all possible types t and all the hyperplanes $\mathcal{B}(t', t'')$ which are inside $R(t)$.

5. PARAMETER REGIONS FOR SEMISTABLE 3-CHAINS

In this section, we study the region of possible stability parameters for holomorphic 3-chains. Among other things, we will prove Conjecture 2.10 for $n = 2$. To this end, we first study linear 3-chains over the field $k = \mathbb{C}(X)$. A general assumption will be $r_j > 0$, $j = 0, 1, 2$.

5.1. 3-Chains of k -vector spaces. Recall that we have established in Theorem 2.7 a connection between the semistability of a holomorphic chain and the semistability of the induced linear chain over $k := \mathbb{C}(X)$. Therefore, we now study linear 3-chains over k .

Theorem 5.1. i) *Suppose (r_0, r_1, r_2) satisfies $r_0 > r_1 \neq r_2$. If $V = (V_0, V_1, V_2; f_1, f_2)$ is an (α_1, α_2) -semistable linear 3-chain of type (r_0, r_1, r_2) , then*

$$(\alpha_1, \alpha_2) = (0, 0).$$

ii) *Assume that $r_0 < r_1 > r_2$ and that $V = (V_0, V_1, V_2; f_1, f_2)$ is an (α_1, α_2) -semistable 3-chain of type (r_0, r_1, r_2) , then $(\alpha_1, \alpha_2) = 0$ or $r_0 = r_2$, $f_1 \circ f_2$ is an isomorphism, and*

$$(\alpha_1, \alpha_2) = \lambda \cdot (1, 2)$$

for some $\lambda \in \mathbb{R}_{\geq 0}$. If V is a chain, such that $r_0 = r_2$ and $f_1 \circ f_2$ is an isomorphism, then V is $(1, 2)$ -semistable.

iii) *If, in (r_0, r_1, r_2) , $r_0 \neq r_1 = r_2$ and if $V = (V_0, V_1, V_2; f_1, f_2)$ is an (α_1, α_2) -semistable 3-chain of type (r_0, r_1, r_2) , then f_2 is an isomorphism and*

$$(\alpha_1, \alpha_2) = \lambda \cdot (-1, 1)$$

for some $\lambda \in \mathbb{R}_{\geq 0}$. Conversely, if V is a chain in which f_2 is an isomorphism, then V is $(-1, 1)$ -semistable.

iv) *Assume $r_0 = r_1 = r_2$. If $V = (V_0, V_1, V_2; f_1, f_2)$ is an (α_1, α_2) -semistable linear 3-chain of type (r_0, r_1, r_2) , then either $(\alpha_1, \alpha_2) = (0, 0)$ or f_2 is an isomorphism and (α_1, α_2) is a non-negative multiple of $(-1, 1)$ or f_1 is an isomorphism and $(\alpha_1, \alpha_2) = \lambda \cdot (2, 1)$ for some non-negative number λ or both f_1 and f_2 are isomorphisms, $\alpha_1 \leq 2\alpha_2$, and $\alpha_1 + \alpha_2 \geq 0$. Conversely, a chain in which f_1 and f_2 are both isomorphisms is (α_1, α_2) -semistable for all (α_1, α_2) with $\alpha_1 \leq 2\alpha_2$ and $\alpha_1 + \alpha_2 \geq 0$, a chain in which f_2 is an isomorphism is $(-1, 1)$ -semistable, and a chain in which f_1 is an isomorphism is $(2, 1)$ -semistable.*

Proof of Theorem 5.1. We will check the condition of semistability on several non-trivial subobjects. The first one is $(V_0, 0, 0)$. Semistability yields

$$\alpha_1 r_1 + \alpha_2 r_2 \geq 0. \quad (5.1)$$

From the subobject $(V_0, V_1, 0)$, we get the condition

$$-\alpha_1 r_1 + \alpha_2 (r_0 + r_1) \geq 0. \quad (5.2)$$

i) Since we assume $r_0 > r_1$, $(\text{im}(f_1), V_1, V_2)$ is a non-trivial subchain. If $r'_0 < r_0$ is the dimension of $\text{im}(f_1)$, this subobject yields $(r_0 - r'_0)(\alpha_1 r_1 + \alpha_2 r_2) \leq 0$. We infer

$$\alpha_1 r_1 + \alpha_2 r_2 \leq 0. \quad (5.3)$$

Observe that (5.1) and (5.3) give

$$\alpha_1 r_1 = -\alpha_2 r_2. \quad (5.4)$$

Now, assume $r_2 > r_1$. Then, $(0, 0, \ker(f_2))$ is a non-trivial subchain. We see

$$-\alpha_1 r_1 + \alpha_2(r_0 + r_1) \leq 0. \quad (5.5)$$

Together with (5.2), we conclude $\alpha_1 r_1 = \alpha_2(r_0 + r_1)$. Invoking (5.4), we find $\alpha_2 = 0$ and derive the assertion of the proposition in the case $r_0 > r_1 < r_2$.

Finally, we have to consider the case $r_2 < r_1$. In this case, we use the condition of semistability for the subchain $(V_0, \text{im}(f_2), V_2)$. This condition is $\alpha_2(r_1 - r'_1)r_2 \leq \alpha_1(r_1 - r'_1)(r_0 + r_2)$, $r'_1 := \dim(\text{im}(f_2))$. Since $r'_1 \leq r_2 < r_1$, this gives

$$\alpha_1(r_0 + r_2) - \alpha_2 r_2 \geq 0. \quad (5.6)$$

Together with (5.1), this yields $\alpha_1 \geq 0$. On the other hand, (5.2) shows that $\alpha_2 \geq 0$. By (5.3), we have $(\alpha_1, \alpha_2) = (0, 0)$ as desired.

ii) If $r_0 < r_1 > r_2$, then there is the subchain $(0, \ker(f_1), 0)$ which gives the estimate

$$\alpha_1(r_0 + r_2) \leq \alpha_2 r_2. \quad (5.7)$$

Inequality (5.6) is also true, so that

$$\alpha_1(r_0 + r_2) = \alpha_2 r_2.$$

Note that $\alpha_2 \geq 0$, by (5.1) and (5.2), so that also $\alpha_1 \geq 0$. One can now check that a chain V is $(0, \alpha_1, \alpha_2)$ -semistable, if and only if every subchain (W_0, W_1, W_2) with $\dim(W_j) = r'_j$, $j = 0, 1, 2$, satisfies

$$r_0 r'_2 \leq r'_0 r_2. \quad (5.8)$$

If $f_1 \circ f_2$ were not injective, then this condition would be violated by $(0, f_2(\ker(f_1 \circ f_2)), \ker(f_1 \circ f_2))$. Thus, $f_1 \circ f_2$ is injective and $r_0 \geq r_2$. Since the dual chain V^\vee is $(0, \alpha_2 - \alpha_1, \alpha_2)$ -semistable (compare Remark 2.3), we must also have $r_2 \geq r_0$ and, consequently, $r_0 = r_2$. If we assume conversely that $r_0 = r_2$ and that $f_1 \circ f_2$ is an isomorphism, then $r'_2 \leq r'_0$ for every subchain, and (5.8) is verified.

iii) First, assume that $r_0 > r_1$. Then, (5.3) is still satisfied. Thus, by (5.4) and $r_1 = r_2$, $\alpha_1 = -\alpha_2$. Since $\alpha_2 \geq 0$, we see that (α_1, α_2) is a non-negative multiple of $(-1, 1)$ and, if it is a positive multiple, the semistability condition becomes

$$\dim(W_2) \leq \dim(W_1) \quad (5.9)$$

for all subchains $(0, W_1, W_2)$. This would be violated by $(0, 0, \ker(f_2))$, if f_2 were not injective. Conversely, if f_2 is injective, then (5.9) will obviously be satisfied for all subchains.

For $r_0 < r_1$, $W_1 := \ker(f_1)$ is a non-trivial subspace of V_1 of dimension, say, r'_1 . Choose a subspace W_2 of V_2 of dimension r'_1 which maps to W_1 under f_1 . The subchain $(0, W_1, W_2)$ yields

$$(\alpha_1 + \alpha_2)(r_0 + 2r_1) \leq 2(\alpha_1 + \alpha_2)r_1,$$

i.e., (5.3) is again verified, and we may conclude as before.

iv). A chain V is $(0, \alpha_1, \alpha_2)$ -semistable if and only if it is α' -semistable for $\alpha' = (\alpha'_0, \alpha'_1, \alpha'_2)$ with $\alpha'_0 := -(\alpha_1 + \alpha_2)/3$, $\alpha'_1 := (2\alpha_1 - \alpha_2)/3$, and $\alpha'_2 := (-\alpha_1 + 2\alpha_2)/3$. Note that we have $\alpha'_0 + \alpha'_1 + \alpha'_2 = 0$. We will require $\alpha' \neq 0$ in the following. For $\alpha'_0 = 0$, i.e., $(\alpha_1, \alpha_2) = \lambda \cdot (-1, 1)$ for some $\lambda \in \mathbb{R}_{>0}$, α' -semistability is given by (5.9) for all subchains. As we have discussed before, this is equivalent to f_2 being an isomorphism. For $\alpha'_2 = 0$, that is $(\alpha_1, \alpha_2) = \lambda \cdot (2, 1)$, $\lambda > 0$, we find the condition

$$\dim(W_0) \leq \dim(W_1) \quad (5.10)$$

for all subchains. This is equivalent to f_1 being an isomorphism. In the case that $\alpha'_1 = 0$, i.e., $(\alpha_1, \alpha_2) = \lambda \cdot (1, 2)$, $\lambda > 0$, the condition of α' -semistability becomes

$$\dim(W_2) \leq \dim(W_0) \quad (5.11)$$

for all subchains. This is equivalent to the fact that $f_1 \circ f_2$ is an isomorphism, so that both f_1 and f_2 must be isomorphisms.

Finally, we treat the case $\alpha'_j \neq 0$, $j = 0, 1, 2$. The conditions $\alpha_1 + \alpha_2 \geq 0$ and $\alpha_1 \leq 2\alpha_2$ result from (5.1) and (5.2), keeping in mind $r_0 = r_1 = r_2$. If f_1 were not surjective, then (5.3) would hold and, thus, $\alpha_1 = -\alpha_2$ and $\alpha'_0 = 0$, a contradiction. Similarly, we derive that f_2 must be an isomorphism. To conclude, assume that f_1 and f_2 are isomorphisms, $\alpha_1 \leq 2\alpha_2$, and $\alpha_1 + \alpha_2 \geq 0$. Let (W_0, W_1, W_2) be a subchain with $\dim(W_j) = r'_j$, $j = 0, 1, 2$. Then, $r'_0 \geq r'_1 \geq r'_2$, and we may estimate as follows:

$$\begin{aligned} & 3(\alpha'_0 r'_0 + \alpha'_1 r'_1 + \alpha'_2 r'_2) \\ &= -(\alpha_1 + \alpha_2)r'_0 + (2\alpha_1 - \alpha_2)r'_1 + (-\alpha_1 + 2\alpha_2)r'_2 \\ &\leq -(\alpha_1 + \alpha_2)r'_1 + (2\alpha_1 - \alpha_2)r'_1 + (-\alpha_1 + 2\alpha_2)r'_2 \\ &= (\alpha_1 - 2\alpha_2)r'_1 + (-\alpha_1 + 2\alpha_2)r'_2 \\ &\leq (\alpha_1 - 2\alpha_2)r'_2 + (-\alpha_1 + 2\alpha_2)r'_2 = 0. \end{aligned}$$

This shows that V is a semistable chain. \square

Remark 5.2. Observe that the proof shows that there never exists any stable chain.

5.2. Parameter regions for rank maximal 3-chains. A holomorphic chain will be called *rank maximal*, if all the homomorphisms, that is, ϕ_1 , ϕ_2 , and, in case $r_1 \geq \max\{r_0, r_1\}$, also $\phi_1 \circ \phi_2$, have generically maximal rank, i.e., are either injective or generically surjective. Note that being rank maximal is an open property, so that describing the moduli spaces for rank maximal chains will provide birational models for some components of the moduli space of all chains. (Those components will be smooth in stable points by Theorem 3.8, v.) For rank maximal chains, one can use the test objects analogous to those used in the proof of Proposition 5.1 to find inequalities which limit the parameters for which semistable rank maximal 3-chains might occur. The

advantage is that, for rank maximal chains, one obtains more explicit bounds on the parameters. We state the following more precise result.

Proposition 5.3. *Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ be a holomorphic 3-chain of type $t = (r_0, r_1, r_2; d_0, d_1, d_2)$. Define the inequalities*

$$\alpha_1 r_1 + \alpha_2 r_2 \geq \frac{r_1 + r_2}{r_0} \cdot d_0 - d_1 - d_2 \quad (5.12)$$

$$-\alpha_1 r_1 + \alpha_2 (r_0 + r_1) \geq d_0 + d_1 - \frac{r_0 + r_1}{r_2} \cdot d_2. \quad (5.13)$$

and, if applicable, the following additional inequalities:

(1) *If $r_0 > r_1$ and ϕ_1 is injective,*

$$\alpha_1 r_1 + \alpha_2 r_2 \leq \frac{2r_1 + r_2}{r_0 - r_1} \cdot d_0 - \frac{2r_0 + r_2}{r_0 - r_1} \cdot d_1 - d_2. \quad (5.14)$$

(2) *If $r_1 < r_2$ and ϕ_2 is generically surjective,*

$$-\alpha_1 r_1 + \alpha_2 (r_0 + r_1) \leq d_0 + \frac{r_0 + 2r_2}{r_2 - r_1} \cdot d_1 - \frac{r_0 + 2r_1}{r_2 - r_1} \cdot d_2. \quad (5.15)$$

(3) *If $r_1 > r_2$ and ϕ_2 is injective,*

$$-\alpha_1 (r_0 + r_2) + \alpha_2 r_2 \leq -d_0 + \frac{r_0 + 2r_2}{r_1 - r_2} \cdot d_1 - \frac{r_0 + 2r_1}{r_1 - r_2} \cdot d_2. \quad (5.16)$$

(4) *If $r_0 < r_1$ and ϕ_1 is generically surjective,*

$$\alpha_1 (r_0 + r_2) - \alpha_2 r_2 \leq \frac{2r_1 + r_2}{r_1 - r_0} \cdot d_0 - \frac{2r_0 + r_2}{r_1 - r_0} \cdot d_1 + d_2. \quad (5.17)$$

(5) *$r_0 < r_1 > r_2$, $r_0 < r_2$, and $\phi_1 \circ \phi_2$ is generically surjective,*

$$\begin{aligned} & \alpha_1 (r_0 - r_1 + r_2) + \alpha_2 (r_0 + r_1 - r_2) \\ & \leq \frac{-r_0 + r_1 + 3r_2}{r_2 - r_0} \cdot d_0 + 2d_1 - \frac{3r_0 + r_1 - r_2}{r_2 - r_0} \cdot d_2. \end{aligned} \quad (5.18)$$

Let $R(t)$ be the region described by all applicable inequalities. If C is $(0, \alpha_1, \alpha_2)$ -semistable, then

$$(\alpha_1, \alpha_2) \in R(t).$$

Proof. The inequalities (5.12) and (5.13) have already been given in Example 2.5. Inequality (5.14) follows from the subchain $(\text{im}(\phi_1) \cong E_1, E_1, E_2)$. For (5.15), one uses the subchain $(0, 0, \ker(\phi_2))$ and the inequality $\deg(\ker(\phi_2)) = d_2 - \deg(\text{im}(\phi_2)) \geq d_2 - d_1$. The subchain $(E_0, \text{im}(\phi_2) \cong E_2, E_2)$ provides us with (5.16). Inequality (5.17) is derived from the test object $(0, \ker(\phi_1), 0)$ and the fact $\deg(\ker(\phi_1)) \geq d_1 - d_0$. Last but not least, the subchain $(0, \phi_2(\ker(\phi_1 \circ \phi_2)) \cong \ker(\phi_1 \circ \phi_2), \ker(\phi_1 \circ \phi_2))$ and the estimate $\deg(\ker(\phi_1 \circ \phi_2)) \geq d_2 - d_0$ yield (5.18). \square

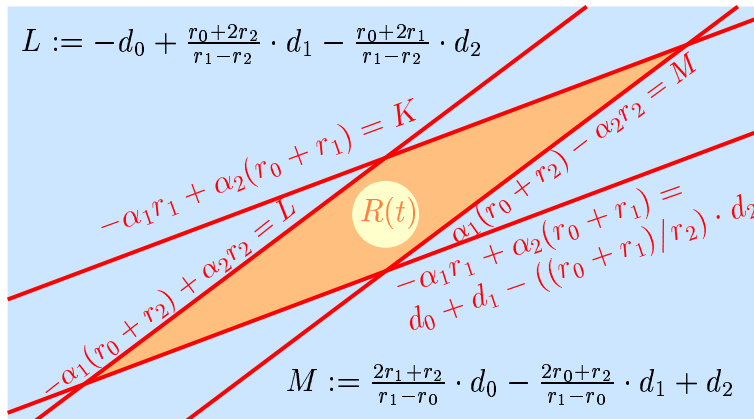


FIGURE 2. The parameter region for rank maximal 3-chains with $r_0 < r_1 > r_2$ and $r_0 < r_2$.

Example 5.4. Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ be a holomorphic 3-chain of type $t = (r_0, r_1, r_2; d_0, d_1, d_2)$, such that $r_0 < r_1 > r_2$, $r_0 < r_2$, and $\phi_1 \circ \phi_2$ is generically surjective. Then, obviously, ϕ_2 must be injective and ϕ_1 generically surjective. If we add (5.16) and (5.18), we find an estimate

$$- \alpha_1 r_1 + \alpha_2 (r_0 + r_1) \leq K = K(t). \tag{5.19}$$

Note that

$$\frac{r_1}{r_0 + r_1} < 1 < \frac{r_0 + r_2}{r_2}.$$

It follows that the inequalities (5.13), (5.19), (5.16), and (5.17) bound a region R in the shape of a parallelogram (see Figure 2). Thus, the possible stability parameters for rank maximal 3-chains of type $t = (r_0, r_1, r_2; d_0, d_1, d_2)$ with $r_0 < r_1 > r_2$ and $r_0 < r_2$ live in the bounded region R . (Note that the other inequalities may still “cut away” some pieces from R .)

5.3. Bounded parameter regions.

We now demonstrate

Theorem 5.5. Conjecture 2.10 holds true for $n = 2$.

Proof. Fix the type $t = (r_0, r_1, r_2; d_0, d_1, d_2)$. The cases we have to consider are a) $r_0 > r_1 < r_2$, b) $r_0 > r_1 > r_2$, and c) $r_0 < r_1 > r_2$ and $r_0 < r_2$. We would like to adapt the strategy in the proof of Proposition 5.1. However, if a test object contains a kernel or an image of a map, then the semistability condition contains the degree of that kernel or image which we do not know in general. So, we have to modify some arguments.

Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ be an α -semistable 3-chain. From the Inequalities (5.12) and (5.13) which always hold, we derive the estimate

$$\alpha_2 \geq K_0 := \frac{d_0}{r_0} - \frac{d_2}{r_2}.$$

The case $r_0 > r_1$. We will give a bound on α_1 under the condition that ϕ_1 is *not* injective. Suppose the rank of $\ker(\phi_1)$ is r'_1 and its degree is d'_1 . Then, the semistability condition for the test object $(0, \ker(\phi_1), 0)$ reads

$$\begin{aligned} & (r_0 + r_1 + r_2)d'_1 + \alpha_1 r'_1(r_0 + r_1 + r_2) \\ & \leq r'_1(d_0 + d_1 + d_2) + r'_1(\alpha_1 r_1 + \alpha_2 r_2). \end{aligned} \quad (5.20)$$

We invoke the condition arising from the subchain $(\text{im}(\phi_1), E_1, E_2)$, i.e.,

$$\begin{aligned} & (r_0 + r_1 + r_2)(d_1 + d_2) + (\alpha_1 r_1 + \alpha_2 r_2)(r_0 + r_1 + r_2) \\ & + (r_0 + r_1 + r_2)(d_1 - d'_1) \\ \leq & (r_1 + r_2)(d_0 + d_1 + d_2) + (r_1 + r_2)(\alpha_1 r_1 + \alpha_2 r_2) \\ & + (r_1 - r'_1)(d_0 + d_1 + d_2) + (r_1 - r'_1)(\alpha_1 r_1 + \alpha_2 r_2). \end{aligned} \quad (5.21)$$

Now, add (5.20) and (5.21) in order to find

$$-\alpha_1(r_0 + r_1 + r_2)r'_1 \geq K_1 + (\alpha_1 r_1 + \alpha_2 r_2)(r_0 - r_1).$$

Here, $K_1 := -(2r_1 + r_2)d_0 + (2r_0 + r_2)d_1 + (r_0 - r_1)d_2$. Together with (5.12), we find

$$-\alpha_1 r'_1 \geq K_2 := -\frac{r_1}{r_0} \cdot d_0 + d_1,$$

i.e.,

$$\alpha_1 \leq K_3 := -\min \left\{ r'_1 = 1, \dots, r_1 \mid \frac{K_2}{r'_1} \right\}.$$

If, on the other hand, ϕ_1 is injective, then we have (5.14). Now, $-r_2 \cdot (5.13) + (r_0 + r_1) \cdot (5.14)$ provides the estimate

$$\alpha_1 \leq K_4 := \frac{2}{r_0 - r_1} \left(d_0 - \frac{r_0}{r_1} \cdot d_1 \right).$$

The case $r_0 > r_1 < r_2$. First, we will derive a bound on α_2 under the condition that ϕ_2 is not generically surjective. Note that the latter is equivalent to the fact that ϕ_2^\vee is not injective. Thus, we may look at the dual holomorphic 3-chain C^\vee which is $(-\alpha_2, -\alpha_1, 0)$ -semistable, by Remark 2.3, iv), and thus $(0, \alpha_2 - \alpha_1, \alpha_2)$ -semistable, by Remark 2.3, iii). Our previous computations may now be applied to find a bound

$$\alpha_2 - \alpha_1 \leq K_5 := \frac{2}{r_2 - r_1} \left(\frac{r_2}{r_1} \cdot d_1 - d_2 \right).$$

and, thus,

$$\alpha_2 \leq K_5 + \alpha_1 \leq K_5 + K_4. \quad (5.22)$$

If ϕ_2 is generically surjective, we have (5.15). Since we have already bounded α_1 from above, Inequality (5.15) provides an upper bound for α_2 . All in all, we have found lower and upper bounds for both α_1 and α_2 under the assumption a) $r_0 > r_1 < r_2$.

The case $r_1 > r_2$. We abbreviate $r := r_0 + r_1 + r_2$, $\mu := (d_0 + d_1 + d_2)/r$, and $\mu_K := d'_2/r'_2$, $r'_2 := \text{rk}(\ker(\phi_2))$ and $d'_2 := \text{deg}(\ker(\phi_2))$. From the subchain $(0, 0, \ker(\phi_2))$, we get the condition

$$\mu_K + \alpha_2 \leq \mu + \frac{\alpha_1 r_1 + \alpha_2 r_2}{r},$$

i.e.,

$$\alpha_2 - \frac{\alpha_1 r_1 + \alpha_2 r_2}{r} \leq \mu - \mu_K. \quad (5.23)$$

Next, we check the semistability condition for the subchain $(E_0, \text{im}(\phi_2), E_2)$. We find

$$\frac{d_0 + 2d_2 - r'_2 \mu_K}{r_0 + 2r_2 - r'_2} + \frac{\alpha_1(r_2 - r'_2) + \alpha_2 r_2}{r_0 + 2r_2 - r'_2} \leq \mu + \frac{\alpha_1 r_1 + \alpha_2 r_2}{r}.$$

This may be rewritten as

$$-r'_2 \mu_K + \alpha_1(r_2 - r'_2) + \alpha_2 r_2 \leq -r'_2 \mu + \frac{r_0 + 2r_2 - r'_2}{r}(\alpha_1 r_1 + \alpha_2 r_2) + K_6,$$

$K_6 := \mu(r_0 + 2r_2) - (d_0 + 2d_2)$, that is,

$$\mu - \mu_K \leq \frac{r_0 + 2r_2 - r'_2}{r'_2 r}(\alpha_1 r_1 + \alpha_2 r_2) + \frac{\alpha_1(r'_2 - r_2) - \alpha_2 r_2}{r'_2} + \frac{K_6}{r'_2}. \quad (5.24)$$

We combine (5.23) and (5.24) and multiply by $r'_2 r$:

$$\begin{aligned} & \alpha_2 r'_2 r - (\alpha_1 r_1 + \alpha_2 r_2) r'_2 \\ & \leq (r_0 + 2r_2 - r'_2)(\alpha_1 r_1 + \alpha_2 r_2) + \alpha_1(r'_2 - r_2)r - \alpha_2 r_2 r + r K_6. \end{aligned}$$

We conclude

$$\alpha_2(r'_2 r + r_2 r - r_2(r_0 + 2r_2)) \leq \alpha_1(r'_2 r - r_2 r + (r_0 + 2r_2)r_1) + r K_6. \quad (5.25)$$

Observe $r - r_0 - 2r_2 = r_1 - r_2 > 0$, whence the coefficient of α_2 is positive. Also,

$$(r_0 + 2r_2)r_1 - r_2 r = (r_0 + r_2)(r_1 - r_2) > 0,$$

so that the coefficient of α_1 is positive. (The value of $r K_6$ is $(r_2 - r_1)d_0 + (r_0 + 2r_2)d_1 - (r_0 + 2r_1)d_2$ (compare (5.16)).)

The case $r_0 > r_1 > r_2$. Recall that we have already bounded α_1 from above. Therefore, if ϕ_2 is not injective, then (5.25) provides an upper bound for α_2 , too. If, on the other hand, ϕ_2 is injective, Inequality (5.16) holds. This inequality also provides an upper bound of α_2 in terms of α_1 and constants depending only on the type t . Again, α_1 and α_2 are bounded both from above and below, and we are done for case b).

The case $r_0 < r_1 > r_2$; $r_0 < r_2$. By (5.25), there is the inequality

$$\alpha_2 c_2 \leq \alpha_1 c_1 + K_7,$$

with positive constants c_1 and c_2 , if ϕ_2 is not injective. If ϕ_2 is injective, we have Inequality (5.16). If ϕ_1 is not generically surjective, then (5.25) for the dual chain yields

$$\alpha_2(r'_0 r + r_0 r - r_0(2r_0 + r_2)) \leq (\alpha_2 - \alpha_1)(r'_0 r - r_0 r + (2r_0 + r_2)r_1) + rK_8,$$

$r'_0 := \text{rk}(\ker(\phi_1^\vee))$, i.e.,

$$\begin{aligned} & \alpha_1(r'_0 r - r_0 r + (2r_0 + r_2)r_1) \\ &= \alpha_1(r'_0 r + (r_0 + r_2)(r_1 - r_0)) \\ &\leq \alpha_2 r_2(r_1 - r_0) + rK_8. \end{aligned} \quad (5.26)$$

Set $c_3 := (r'_0 r + (r_0 + r_2)(r_1 - r_0))$ and $c_4 := r_2(r_1 - r_0)$. One easily checks

$$\frac{c_3}{c_4} > \frac{r_0 + r_2}{r_2} > \frac{c_1}{c_2}. \quad (5.27)$$

Hence, if ϕ_1 is not generically surjective, then (5.12), (5.25), and (5.26) bound a triangular region. The same goes for (5.12), (5.16), and (5.26) (cf. Figure 3). In our argument, we may therefore assume that ϕ_1 is generically surjective, so that we have Inequality (5.17). By (5.27), (5.12), (5.17), and (5.25) also bound a triangular region. Thus, we may also assume that ϕ_2 is injective. If $\phi_1 \circ \phi_2$ is generically surjective, we have Example 5.4. The final case to consider is the one in which ϕ_1 and ϕ_2 have generically the maximal possible rank and $\phi_1 \circ \phi_2$ has a cokernel of positive rank. We use again the abbreviations $d := d_0 + d_1 + d_2$, $r := r_0 + r_1 + r_2$, and $\mu := d/r$.

Claim. There is a constant $K_9 = K_9(t)$, such that

$$\mu - \mu(\ker(\phi_1 \circ \phi_2)) \leq K_9.$$

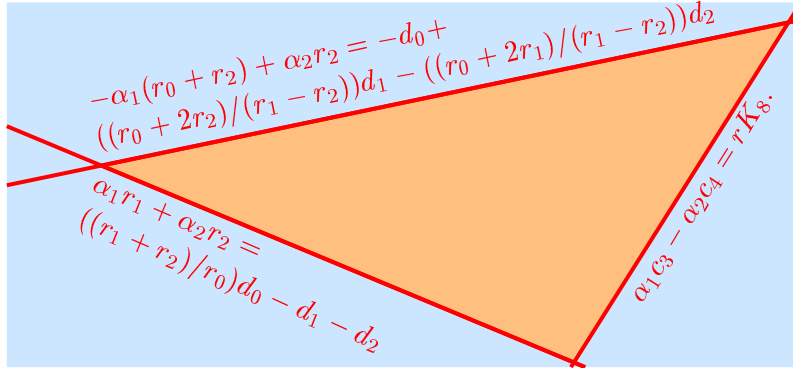


FIGURE 3. The triangle bounded by (5.12), (5.16), and (5.26).

Assume, for the moment, this claim. Note that (5.16) and (5.17) yield

$$\frac{\alpha_1(r_0 + r_2)}{r_2} - K_{10} \leq \alpha_2 \leq \frac{\alpha_1(r_0 + r_2)}{r_2} + K_{11}.$$

For a subchain $C' = (F_0, F_1, F_2)$ with $r'_j := \text{rk}(F_j)$, $j = 0, 1, 2$, $r' := r'_0 + r'_1 + r'_2$, $d' := \text{deg}(F_0) + \text{deg}(F_1) + \text{deg}(F_2)$, and $\mu_{C'} := d'/r'$, we then get from the condition of $(0, \alpha_1, \alpha_2)$ -semistability

$$\alpha_1(r_0 r'_2 - r'_0 r_2) \leq r' r_2 (\mu - \mu_{C'}) + r'_2 r_2 K_{10} + r' r_2 \frac{K_{11}}{r}. \quad (5.28)$$

If we apply this to the subchain $(0, \phi_2(\ker(\phi_1 \circ \phi_2)) \cong \ker(\phi_1 \circ \phi_2), \ker(\phi_1 \circ \phi_2))$, we find a bound

$$\alpha_1 \leq K_{12} = K_{12}(t),$$

using the above claim. This proves the boundedness of the parameter region.

In order to establish the claim, we look at the dual chain $C^\vee = (\tilde{E}_0, \tilde{E}_1, \tilde{E}_2; \tilde{\phi}_1, \tilde{\phi}_2)$. If $\phi_1 \circ \phi_2$ is not generically surjective, then $\tilde{\phi}_1 \circ \tilde{\phi}_2$ is not injective. Set

$$\tilde{K} := \tilde{\phi}_2(\ker(\tilde{\phi}_1 \circ \tilde{\phi}_2)) \cong \ker(\tilde{\phi}_1 \circ \tilde{\phi}_2),$$

because $\tilde{\phi}_2$ is injective. If $d_K := \text{deg}(\ker(\phi_1 \circ \phi_2))$, $r_K := \text{rk}(\ker(\phi_1 \circ \phi_2))$, and $\mu_K := d_K/r_K$, then

$$\mu(\tilde{K}) = \frac{-d_0 + d_2 - r_K \mu_K}{r_0 - r_2 + r_K}.$$

Since C^\vee is $(0, \alpha_2 - \alpha_1, \alpha_2)$ -semistable, the subchain $(0, \tilde{K}, 0)$ gives the estimate

$$\mu(\tilde{K}) + \alpha_2 - \alpha_1 \leq -\mu + \frac{-\alpha_1 r_1 + \alpha_2 (r_0 + r_1)}{r}.$$

This may be rewritten as

$$\begin{aligned} r_K(\mu - \mu_K) &\leq (r_2 - r_0)\mu + d_0 - d_2 + \\ &\quad + \frac{r_0 - r_2 + r_K}{r} (\alpha_1(r_0 + r_2) - \alpha_2 r_2) \\ &\leq (r_2 - r_0)\mu + d_0 - d_2 + \\ &\quad + \frac{r_0 - r_2 + r_K}{r} \left(\frac{2r_1 + r_2}{r_1 - r_0} \cdot d_0 - \frac{2r_0 + r_2}{r_1 - r_0} \cdot d_1 + d_2 \right). \end{aligned}$$

For the second estimate, we have used the fact that $r_K \geq r_2 - r_0$ and (5.17). The above inequality clearly settles the claim. \square

5.4. Unbounded parameter regions and the finiteness of the number of chambers. If the region $R(t)$ of possible parameters for semistable chains of type t is bounded, then the local finiteness of the chamber decomposition (2.2) implies that there are only finitely many chambers. In this section, we will show that also if the parameter region is unbounded, there are only finitely many chambers. This is closely related to the question whether the set of isomorphism classes vector bundles F for which there exist a parameter α , an α -semistable 3-chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , and an index $j_0 \in \{0, 1, 2\}$ with $F \cong E_{j_0}$ is bounded. We will first give some additional bounds for the possible parameter regions, then prove the above boundedness statement, and, last but not least, derive the finiteness of the number of chambers.

Proposition 5.6. *i) Assume $r_0 \neq r_1 = r_2$. Then, there exist a constant $K_{12} = K_{12}(t)$ and a bounded region $R_0(t)$, such that, for any $(0, \alpha_1, \alpha_2)$ -semistable 3-chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , one has either that ϕ_2 is not injective and $(\alpha_1, \alpha_2) \in R_0(t)$ or ϕ_2 is injective and $(\alpha_1, \alpha_2) \in R_1(t)$. Here, $R_1(t)$ is the region bounded by (5.12), (5.13), and*

$$r_1(\alpha_1 + \alpha_2) = \alpha_1 r_1 + \alpha_2 r_2 \leq K_{12}. \quad (5.29)$$

ii) Suppose $r_0 = r_1 = r_2$. Then, there are constants K_{13} and K_{14} , depending only on the type t and a bounded region $R_0(t)$ with the following property: Given any $(0, \alpha_1, \alpha_2)$ -semistable 3-chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , then either a) neither ϕ_1 nor ϕ_2 is injective and $(\alpha_1, \alpha_2) \in R_0(t)$, or b) ϕ_1 is not injective but ϕ_2 is and $(\alpha_1, \alpha_2) \in R_1(t)$, or c) ϕ_1 is injective, ϕ_2 isn't, and $(\alpha_1, \alpha_2) \in R_2(t)$, or d) both ϕ_1 and ϕ_2 are injective and $(\alpha_1, \alpha_2) \in R_3(t)$. Here, $R_1(t)$ is the region limited by the inequalities (5.12), (5.13), and

$$\alpha_1 r_1 + \alpha_2 r_2 \leq K_{13},$$

$R_2(t)$ is the region confined by the restrictions (5.12), (5.13), and

$$-\alpha_1 r_1 + \alpha_2(r_0 + r_1) \leq K_{14},$$

and $R_3(t)$ is bounded by (5.12) and (5.13) (see Figure 1).

iii) Assume $r_0 < r_1 > r_2$, $r_0 = r_2$. Then, there is a bounded region $R_0(t)$, such that, for any $(0, \alpha_1, \alpha_2)$ -semistable 3-chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , C fails to be rank maximal and $(\alpha_1, \alpha_2) \in R_0(t)$ or C is rank maximal and (α_1, α_2) lies in the region $R_1(t)$ bounded by (5.12), (5.16), and (5.17).

Proof. i) We first treat the case that ϕ_2 is not injective. If ϕ_2 fails to be injective, then (5.25) holds true and yields $\alpha_2 \leq \alpha_1 + K_6/r'_2$. In the case $r_0 > r_1$, we have an upper bound on α_1 , from the proof of Theorem 5.5. Thus, α_2 is also bounded from above. As we have seen before, α_2 is always bounded from below. Finally, (5.12) provides a

lower bound for α_1 . If $r_0 < r_1$, then the arguments used in the proof of the case $r_0 < r_1 > r_2$ in Theorem 5.5¹ show that the parameter (α_1, α_2) lives in a bounded triangular region.

Now, we assume that ϕ_2 is injective. If, in the case $r_0 > r_1$, ϕ_1 is injective, then we are done by (5.14). We set $K := \ker(\phi_1 \circ \phi_2)$, and define $r'_2 := \text{rk}(\ker(\phi_1 \circ \phi_2))$, $\mu_K := \mu(K)$, and $r := r_0 + r_1 + r_2$. The semistability condition for the subchain $(0, K \cong \phi_2(K), K)$ reads

$$\frac{r_0}{2r} \cdot (\alpha_1 + \alpha_2) \leq \mu - \mu_K. \quad (5.30)$$

The semistability condition for the subchain $(\text{im}(\phi_1 \circ \phi_2), E_2 \cong \phi_2(E_2), E_2)$ may be written in the form

$$-(\alpha_1 r_1 + \alpha_2 r_2) \underbrace{\left(1 - \frac{3r_2 - r'_2}{r}\right)}_{=:c} \geq r'_2(\mu - \mu_K) + 3d_2 - 3r_2\mu. \quad (5.31)$$

Combining (5.30), (5.31), and (5.12) proves the claim, provided c is non negative. Observe $rc = r_0 - r_2 + r'_2 = r_0 - r_1 + r'_2$. Therefore, c is positive, if $r_0 > r_1$. If $r_0 < r_1$, we use the fact $r'_2 \geq r_2 - r_0 = r_1 - r_0$ to conclude.

ii) If neither ϕ_1 nor ϕ_2 is injective, then (5.25) and (5.26) hold true, so that (α_1, α_2) is an element of the triangular region bounded by (5.12), (5.25), and (5.26). If ϕ_2 is injective but ϕ_1 is not, one may use the arguments from the proof of i). Likewise, one gets the result in the case where ϕ_1 is injective but ϕ_2 is not, by looking at the dual chain.

iii) If $\phi_1 \circ \phi_2$ is not rank maximal, one may use the same arguments as in the proof of Theorem 5.5, Case $r_0 < r_1 > r_2$, $r_0 < r_2$, to see that (α_1, α_2) belongs to a bounded subset of \mathbb{R}^2 . \square

Theorem 5.7. *Fix the type t . Then, the set of isomorphism classes of vector bundles F for which there exist a parameter α , an α -semistable 3-chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , and an index $j_0 \in \{0, 1, 2\}$ with $F \cong E_{j_0}$ is bounded.*

Proof. The assertion is well known for one fixed parameter α . By the local finiteness of the chamber decomposition (2.2), the theorem is also clear, if α is allowed to move in a bounded region. Thus, we are left with the cases of Proposition 5.6.

Case i). Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ be 3-chain of type t which is semistable w.r.t. some parameter α . Again, we may exclude a bounded region, so that we may assume that ϕ_2 is generically an isomorphism. First, we look at a subbundle F of E_0 . The semistability condition for $(F, 0, 0)$ gives

$$\mu(F) \leq \frac{d_0 + d_1 + d_2}{r_0 + r_1 + r_2} + \frac{\alpha_1 r_1 + \alpha_2 r_2}{r} \stackrel{(5.29)}{\leq} \frac{d_0 + d_1 + d_2}{r_0 + r_1 + r_2} + \frac{K_{12}}{r} =: K_{15}.$$

¹More precisely, one checks that either (5.12), (5.25), and (5.26) or (5.12), (5.17), and (5.25) apply, because ϕ_2 is not injective.

Since K_{15} depends only on the type, this implies that E_0 moves in a bounded family. Next, let F be any subbundle of E_2 that is contained in the kernel of $\phi_1 \circ \phi_2$. We look at the subchain $(0, F \cong \phi_2(F), F)$. This gives

$$\mu(F) \leq \frac{d_0 + d_1 + d_2}{r_0 + r_1 + r_2} + \left(\frac{2r_1}{r} - \frac{1}{2} \right) \cdot (\alpha_1 + \alpha_2) \leq K_{16},$$

by (5.12). Now, let F be an arbitrary subbundle of E_2 . Then, we find the extension

$$\begin{array}{ccccccc} 0 & \longrightarrow & F' := F \cap \ker(\phi_1 \circ \phi_2) & \longrightarrow & F & \longrightarrow & \\ & & \longrightarrow & & F'' := (\phi_1 \circ \phi_2)(F) & \longrightarrow & 0. \end{array}$$

Thus,

$$\mu(F) = \frac{\text{rk}(F')\mu(F') + \text{rk}(F'')\mu(F'')}{\text{rk}(F)} \leq \max\{K_{15}, K_{16}\}.$$

This shows that E_2 lives in a bounded family, too. Finally, E_1 is given as an extension

$$0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow T \longrightarrow 0,$$

with T a torsion sheaf of length $d_1 - d_2$. It follows easily that E_1 also belongs to a bounded family.

Case ii). If ϕ_2 is injective, but ϕ_1 is not, then we may argue as in Case i). Similarly, we obtain the result in the case that ϕ_1 is injective, but ϕ_2 is not. Finally, we look at the case where both ϕ_1 and ϕ_2 are injective. For any subbundle F of E_2 , we look at the subchain $(F \cong (\phi_1 \circ \phi_2)(F), F \cong \phi_2(F), F)$. This yields the condition

$$\mu(F) \leq \mu.$$

This proves that E_2 is a member of a bounded family. For E_0 and E_1 , we find the analogous result by looking at the extensions

$$0 \longrightarrow E_2 \longrightarrow E_{0,1} \longrightarrow T_{0,1} \longrightarrow 0,$$

with $T_{0,1}$ a torsion sheaf of length $d_{0,1} - d_2$.

Case iii). If C is a rank maximal chain, then we get Condition (5.28) for every subchain $C' = (F_0, F_1, F_2)$ of C . Now, suppose that F is a subbundle of E_2 and look at the subchain $(F \cong (\phi_1 \circ \phi_2)(F), F \cong \phi_2(F), F)$. Then, (5.28) yields the estimate

$$\mu(F) \leq \mu + \frac{1}{3}K_{10} + \frac{1}{r}K_{11} =: K_{17},$$

so that the family of possible E_2 's is bounded. Since E_0 is an extension of a torsion sheaf of length $d_0 - d_2$ by E_2 , the family of possible E_0 's is bounded, too. Next, if F is a subbundle of E_1 which is contained in the kernel of ϕ_1 , then the condition for the subchain $(0, F, 0)$ gives

$$\mu(F) \leq \mu + \frac{-\alpha_1(r_0 + r_2) + \alpha_2 r_2}{r} \stackrel{(5.16)}{\leq} K_{18}.$$

An arbitrary subbundle F of E_1 is written as an extension

$$0 \longrightarrow F' := F \cap \ker(\phi_1) \longrightarrow F \longrightarrow F'' := \phi_1(F) \longrightarrow 0.$$

We infer

$$\mu(F) = \frac{\text{rk}(F')\mu(F') + \text{rk}(F'')\mu(F'')}{\text{rk}(F)} \leq \max\{K_{17}, K_{18}\}$$

and settle the theorem. \square

Corollary 5.8. *Fix the type t . Then, there are only finitely many “effective” chambers.*

Proof. By Theorem 5.7, there is a constant $d_\infty = d_\infty(t)$, such that, for any $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, any α -semistable holomorphic chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , $\alpha := (0, \alpha_1, \alpha_2)$, any index $j_0 \in \{0, 1, 2\}$, and any subbundle $F \subseteq E_{j_0}$, one has $\deg(F) \leq d_\infty$. One easily derives the following assertion.

Lemma 5.9. *Fix a constant L . Then, there is an integer $d_l = d_l(t, L)$, such that for any $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, any α -semistable holomorphic chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ of type t , $\alpha := (0, \alpha_1, \alpha_2)$, and any subchain $C' = (F_0, F_1, F_2)$, one has*

$$\frac{\deg(F_0) + \deg(F_1) + \deg(F_2)}{r_0 + r_1 + r_2} < L,$$

whenever there exists an index $j_0 \in \{0, 1, 2\}$ with $\deg(F_{j_0}) < d_l$.

We go again through the cases of Proposition 5.6.

Case i). As usual, we may assume that $\phi_2 \neq 0$ for the chains we are dealing with. Let $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ be such a chain and $C' = (F_0, F_1, F_2)$ a subchain. Note that (5.29) provides an upper bound for $\alpha_1 + \alpha_2$ and that “ $-r_2(5.13) + (r_0 + r_1)(5.29)$ ” gives an upper bound for α_1 . Since $\text{rk}(F_1) \geq \text{rk}(F_2)$, it is easy to find a constant $K_{19} = K_{19}(t)$ with

$$\begin{aligned} & \mu(F_0 \oplus F_1 \oplus F_2) + \frac{\alpha_1 \text{rk}(F_1) + \alpha_2 \text{rk}(F_2)}{\text{rk}(F_0 \oplus F_1 \oplus F_2)} \\ &= \mu(F_0 \oplus F_1 \oplus F_2) + \frac{\alpha_1(\text{rk}(F_1) - \text{rk}(F_2))}{\text{rk}(F_0 \oplus F_1 \oplus F_2)} + \frac{(\alpha_1 + \alpha_2) \text{rk}(F_2)}{\text{rk}(F_0 \oplus F_1 \oplus F_2)} \\ &\leq \mu(F_0 \oplus F_1 \oplus F_2) + K_{19}. \end{aligned}$$

On the other hand, by (5.12),

$$\frac{d_0 + d_1 + d_2}{r_0 + r_1 + r_2} + \frac{\alpha_1 r_1 + \alpha_2 r_2}{r_0 + r_1 + r_2} \geq K_{20}.$$

With Lemma 5.9, we see that, if there is one index $j_0 \in \{0, 1, 2\}$ with $\mu(F_{j_0}) < d_l(t, K_{20} - K_{19})$, then the condition

$$\mu(F_0 \oplus F_1 \oplus F_2) + \frac{\alpha_1 \text{rk}(F_1) + \alpha_2 \text{rk}(F_2)}{\text{rk}(F_0 \oplus F_1 \oplus F_2)} < \frac{d_0 + d_1 + d_2}{r_0 + r_1 + r_2} + \frac{\alpha_1 r_1 + \alpha_2 r_2}{r_0 + r_1 + r_2}$$

is satisfied. We, therefore, define

$$\mathcal{S}_{\text{eff}} := \left\{ (s_0, s_1, s_2; e) \mid \begin{array}{l} 0 \leq s_j \leq r_j, \quad j = 0, 1, 2, \\ 0 < s_0 + s_1 + s_2 < r_0 + r_1 + r_2, \\ 3d_l(t, K_{20} - K_{19}) \leq e \leq 3d_\infty(t) \end{array} \right\}.$$

This is a finite set. As in Section 2.4, we derive a decomposition of \mathbb{R}^2 into a (now finite) set of chambers. Together with the chamber decomposition of $R_0(t)$, we thus obtain a decomposition of \mathbb{R}^2 into a finite set of locally closed chambers, such that the property of Proposition 2.12 remains true.

Case ii). The arguments of Case i) apply to such chains where one map fails to be a generic isomorphism. Hence, we are reduced to study chains $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ where both ϕ_1 and ϕ_2 are injective. Note that, by (5.12) and (5.13), there is constant K_{21} with

$$\alpha_1 \leq 2\alpha_2 + K_{21}.$$

We may use a simple modification of the argument given at the end of the proof of Theorem 5.1 to see that there is a constant K_{21} which depends only on the type t , such that

$$\mu(F_0 \oplus F_1 \oplus F_2) + \frac{\alpha_1 \text{rk}(F_1) + \alpha_2 \text{rk}(F_2)}{\text{rk}(F_0 \oplus F_1 \oplus F_2)} \leq \mu(F_0 \oplus F_1 \oplus F_2) + K_{22}.$$

The rest of the proof proceeds as before.

Case iii). We look only at chains $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ where $\phi_1 \circ \phi_2$ is generically an isomorphism. Note that (5.16) and the lower bound for α_2 provide us with a lower bound for α_1 . Using (5.28) and $\text{rk}(F_2) \leq \text{rk}(F_0)$ for every subchain $C' = (F_0, F_1, F_2)$ of C , we find a constant K_{23} with

$$\alpha r_2(r'_2 - r'_0) = \alpha_1(r_0 r'_2 - r'_0 r_2) \leq K_{23}.$$

For an appropriate choice of L , (5.28) will be satisfied with “ $<$ ”, if there is $j_0 \in \{0, 1, 2\}$ with $\mu(F_{j_0}) < d_l(t, L)$. Hence, we may continue as before. \square

Example 5.10. In Figure 4, we sketch the shape of the parameter region together with its chamber structure away from some bounded region in the cases $r_0 \neq r_1 = r_2$ and $r_0 < r_1 > r_2$, $r_0 = r_2$. We have also marked an “extremal chamber” \mathcal{C}_∞ . The hope is that one can understand the corresponding “extremal moduli space” sufficiently well to start the investigation of other moduli spaces via birational transformations, using the results from Section 4.

5.5. Concluding remarks. i) For any type t , we have found a region $R(t) \subset \mathbb{R}^2$, such that the existence of an $(0, \alpha_1, \alpha_2)$ -semistable chain of type t implies $(\alpha_1, \alpha_2) \in R(t)$. (Although the bounds we have found as well all the constants appearing are given by complicated expressions,

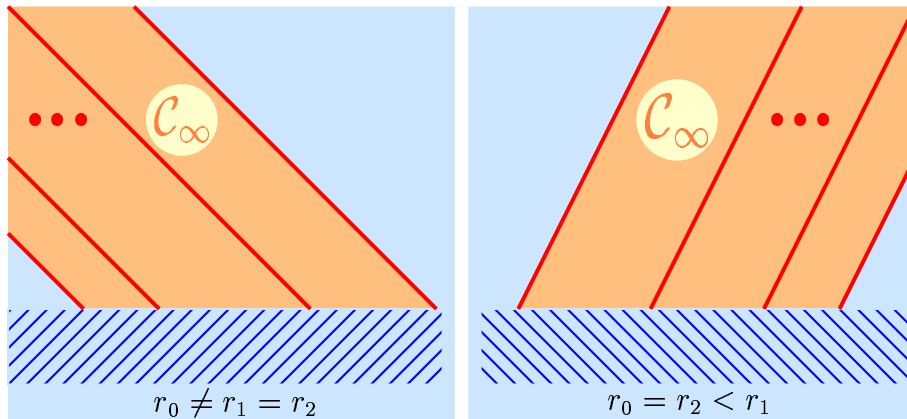


FIGURE 4. Unbounded parameter regions for 3-chains.

they can be explicitly determined.) If we want that $R(t)$ has a non-empty interior, we find non-trivial restrictions on the type. E.g., for $r_0 > r_1 < r_2$, we have

$$\frac{d_0}{r_0} - \frac{d_1}{r_2} \leq \alpha_2 \leq \frac{2}{r_0 - r_1} \left(d_0 - \frac{r_0}{r_1} d_1 \right) + \frac{2}{r_2 - r_1} \left(\frac{r_2}{r_1} d_1 - d_2 \right).$$

ii) For augmented or decorated vector bundles, i.e., vector bundles together with a section in the vector bundle associated by means of a homogeneous representation $\rho: \mathrm{GL}_r(\mathbb{C}) \rightarrow \mathrm{GL}(V)$, the connection between the behaviour of the semistability concept for large parameters and the invariant theory in V has been understood in general in [23]. To our knowledge, we investigate here for the first time the analogous question for a reductive group other than $\mathrm{GL}_r(\mathbb{C})$ (namely, $\mathrm{GL}_{r_0}(\mathbb{C}) \times \mathrm{GL}_{r_1}(\mathbb{C}) \times \mathrm{GL}_{r_2}(\mathbb{C})$). Our arguments are valid only for the special situation we are looking at, but in view of Theorem 5.1, the relationship between the shape of the region of possible stability parameters and the invariant theory in $\mathrm{Hom}(\mathbb{C}^{r_2}, \mathbb{C}^{r_1}) \oplus \mathrm{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_0})$ is clearly perceptible. Thus, we get some feeling why Conjecture 2.10 and some more general properties should be true.

6. EXTREMAL MODULI SPACES FOR 3-CHAINS

This section serves as an illustration of the geometry of moduli spaces for 3-chains and its relation to other problems. We will study a few specific types in which we have inserted “ones”. This condition is used to grant that the chains we will consider are all rank maximal, so that we have a good picture of the a priori parameter region $R(t)$ and may exhibit a two-dimensional chamber \mathcal{C}_∞ which yields the “asymptotic moduli spaces”. We may expect that these moduli spaces are in a certain way the easiest and are related to other well-known moduli spaces such as moduli spaces of semistable vector bundles. On the other hand, we have laid in previous chapters the foundations for studying

other, birationally equivalent moduli spaces via the “flip-technology”. Although we discuss only very special types, it becomes clear how one may in general relate the moduli spaces in the extremal chamber \mathcal{C}_∞ to the moduli spaces on the nearby boundaries that one may usually understand quite easily (compare Propositions 6.4 and 6.5 below). This should give the reader sufficient material to attack any special case she or he is interested in.

6.1. Generalities on moduli spaces for type $(m, 1, n; d_0, d_1, d_2)$. In the next two sections, we will describe the moduli spaces $\mathcal{M}_\alpha(t)$ for the type $t = (m, 1, n; d_0, d_1, d_2)$ with respect to the stability parameters α which lie in a certain “extremal” two-dimensional chamber. To this end, we first recall the results concerning the parameter region and the moduli spaces that we have already obtained.

Let \mathcal{C} be a two dimensional chamber. All the chains that we will have to consider will be automatically rank maximal in the sense of Section 5.2, because of Remark 2.6. Therefore, we only have to look at the relevant inequalities from Proposition 5.3 that bound the parameter region $R(t)$. Let us remind the reader what these inequalities are and how they are obtained.

Inequality I. Obviously, $(\text{im}(\phi_1), E_1, E_2)$ is a subchain. Note that $\text{im}(\phi_1)$ is isomorphic to E_1 , so that it has degree d_1 and rank 1. The condition of α -semistability for this subchain reads

$$(m-1)\alpha_1 + (m-1)n\alpha_2 \leq A_I := (n+2)d_0 - (2m+n)d_1 + (1-m)d_2.$$

Inequality II. The condition of α -semistability for the subchain $(E_0, 0, 0)$ produces the inequality

$$m\alpha_1 + mn\alpha_2 \geq A_{II} := (n+1)d_0 - md_1 - md_2.$$

Inequality III. Here, one checks α -semistability for the subchain $(E_0, E_1, 0)$. This gives

$$-n\alpha_1 + (m+1)n\alpha_2 \geq A_{III} := nd_0 + nd_1 - (m+1)d_2.$$

Inequality IV. This inequality only applies, if $n > 1$. One uses the subchain $(0, 0, \ker(\phi_2))$. Clearly, $\ker(\phi_2)$ has rank $n-1$ and degree $\deg(E_2) - \deg(\text{im}(\phi_2))$. Since $\text{im}(\phi_2)$ is a non-trivial subsheaf of the line bundle E_1 , $\deg(\text{im}(\phi_2)) \leq d_1$, i.e., $\deg(\ker(\phi_2)) \geq d_2 - d_1$. Thus, the condition of α -semistability for the given subchain implies the necessary condition

$$-(n-1)\alpha_1 + (m+1)(n-1)\alpha_2 \leq A_{IV} := (n-1)d_0 + (m+2n)d_1 - (m+2)d_2.$$

Remark 6.1. i) If $n > 1$, then the Inequalities I-IV bound a parallelogram. One checks that potential destabilizing objects are of the form $(F_0, 0, 0)$, $(0, 0, F_2)$, and (F_0, E_1, F_2) (we do not have to consider subchains of the form $(F_0, 0, F_2)$, because the condition of α -(semi)stability for such a subchain follows from those for the subchains $(0, 0, F_2)$ and

$(F_0, 0, 0)$). One checks that the corresponding one-dimensional walls are parallel to one of the sides of the parallelogram.

ii) For $n = 1$, the inequalities I-III bound an “open parallelogram”. We claim that all the one-dimensional walls are defined by an equation of the form $\alpha_1 + \alpha_2 = c$. Therefore, the parameter region with its chamber structure looks as depicted in Figure 5. In fact, if $C := (E_0, E_1, E_2; \phi_1, \phi_2)$ is an α -semistable but not α -stable chain, then we find an α -destabilizing subchain (F_0, F_1, F_2) . Setting $C' := (F_0, F_1, F_2; \phi_{1|F_1}, \phi_{2|F_2})$ and $C'' := C/C'$, the chain $\tilde{C} := C' \oplus C''$ is still α -semistable. If we assume that α lies in the *interior* of $R(t)$, then none of the homomorphisms in \tilde{C} must be zero. If we replace C by \tilde{C} , then we easily see that the wall containing α is defined via a subchain $(0, 0, F_2)$. From this, one immediately arrives at our claim.

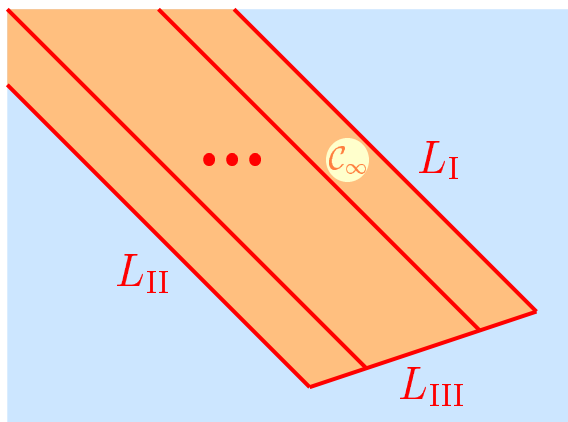


FIGURE 5. The chamber structure for 3-chains of type $(m, 1, 1; d_0, d_1, d_2)$.

Finally, we also add the following useful observation.

Lemma 6.2. i) *Suppose $n = 1$. Then, $R(t)$ has a non-empty interior if and only*

$$d_1 < \frac{d_0}{m},$$

that is, the subsheaf $\text{im}(\phi_1)$ does not destabilize E_0 .

ii) *If $n > 1$, then the parameter region has a non-empty interior if and only if the condition in i) holds and additionally*

$$(m + 1)(nd_1 - d_2) - md_2 > 0.$$

Proof. i) The interior of $R(t)$ is obviously non-empty if and only if $m \cdot A_I > (m - 1) \cdot A_{II}$ in the above notation. A few simplifications lead to the assertion.

ii) The second condition arises from evaluating the inequality $n \cdot A_{IV} > (n - 1) \cdot A_{III}$. \square

Remark 6.3. i) We note that the inequality in Lemma 6.2 implies that $d_1 < (d_0 - d_1)/(m - 1)$.

ii) It will later be interesting to know that, in Lemma 6.2, we may choose d_1 and d_2 in such a way that the inequality holds and $(nd_1 - d_2)$ is a prescribed value (which might also be negative).

First, let us analyze the moduli spaces for parameters which do lie on the boundary of the parameter region. If the stability parameter α lies on one of the boundaries determined by Inequality II or III, then we have described the moduli space in Corollary 2.18. Thus, one of the remaining cases is the one when $(m - 1)\alpha_1 + (m - 1)n\alpha_2 = (n + 2)d_0 - (2m + n)d_1 + (1 - m)d_2$. We assume that the remaining Inequalities II, III, and IV are strict, so that both $\phi_2 \neq 0$ and $\phi_1 \neq 0$. Suppose $(E_0, E_1, E_2; \phi_1, \phi_2)$ is α -semistable. By the definition of the boundaries, the subchain $(\text{im}(\phi_1), E_1, E_2)$ becomes destabilizing. Standard arguments now show that

- $(\text{im}(\phi_1) \cong E_1, E_1, E_2)$ is an α -semistable holomorphic 3-chain of type $t' := (m, 1, 1; d_0, d_1, d_2)$,
- $Q_0 := E_0/\text{im}(\phi_1)$ is a semistable vector bundle of degree $d_0 - d_1$ and rank $m - 1$, and
- $(E_0, E_1, E_2; \phi_1, \phi_2)$ is S-equivalent to the chain $(\text{im}(\phi_1) \oplus Q_0, E_1, E_2; \phi_1, \phi_2)$.

We infer.

Proposition 6.4. *The natural morphism*

$$\begin{aligned} \sigma_1: \mathcal{M}_\alpha(t) &\longrightarrow \mathcal{M}_\alpha(t') \times \mathcal{U}(n - 1, d_0 - d_1) \\ [E_0, E_1, E_2; \phi_1, \phi_2] &\longmapsto ([\text{im}(\phi_1), E_1, E_2; \phi_1, \phi_2], [E_0/\text{im}(\phi_1)]) \end{aligned}$$

is bijective, and there is also the inverse morphism

$$\begin{aligned} \tau_1: \mathcal{M}_\alpha(t') \times \mathcal{U}(n - 1, d_0 - d_1) &\longrightarrow \mathcal{M}_\alpha(t) \\ ([E_1, E_1, E_2; \phi_1, \phi_2], [Q_0]) &\longmapsto [E_1 \oplus Q_0, E_1, E_2; \phi_1, \phi_2]. \end{aligned}$$

The other case to consider is the one when $-(n - 1)\alpha_1 + (m + 1)(n - 1)\alpha_2 = (n - 1)d_0 + (m + 2n)d_1 - (m + 2)d_2$. We may assume that the remaining Inequalities I, II, and III are strict, i.e., $\phi_2 \neq 0$ and $\phi_1 \neq 0$.

Proposition 6.5. *There are the bijective morphisms*

$$\begin{aligned} \sigma_{\text{IV}}: \mathcal{M}_\alpha(t) &\longrightarrow \mathcal{U}(n - 1, d_2 - d_1) \times \mathcal{M}_\alpha(m, 1, 1; d_0, d_1, d_1) \\ [E_0, E_1, E_2; \phi_1, \phi_2] &\longmapsto ([\ker(\phi_2)], [E_0, E_1, E_1 \cong E_2/\ker(\phi_2); \phi_1, \bar{\phi}_2]) \end{aligned}$$

and

$$\begin{aligned} \tau_{\text{IV}}: \mathcal{U}(n - 1, d_2 - d_1) \times \mathcal{M}_\alpha(m, 1, 1; d_0, d_1, d_1) &\longrightarrow \mathcal{M}_\alpha(t) \\ ([K_2], [E_0, E_1, E_1; \phi_1, \phi_2]) &\longmapsto [E_0, E_1, E_1 \oplus K_2; \phi_1, \phi_2]. \end{aligned}$$

6.2. Extremal moduli spaces for type $(m, 1, 1; d_0, d_1, d_2)$. Now, suppose $\alpha = (0, \alpha_1, \alpha_2)$ is such that (α_1, α_2) lies in the chamber \mathcal{C}_∞ (i.e., the chamber in the interior of $R(t)$ which is adjacent to the line L_I on which Inequality I becomes an equality; see Figure 5). Let $\alpha^M = (0, \alpha_1^M, \alpha_2^M)$ be such that (α_1^M, α_2^M) is an element of $\overline{\mathcal{C}}_\infty \cap L_I$. Note that $\alpha_1 + \alpha_2 < \alpha_1^M + \alpha_2^M$, so that $\mu_\alpha(t) < \mu_{\alpha^M}(t)$. Let $(E_0, E_1, E_2; \phi_1, \phi_2)$ be an α -semistable holomorphic chain of type $t = (m, 1, 1; d_0, d_1, d_2)$. We note the following properties.

Proposition 6.6. i) *The vector bundle E_0 does not possess a subbundle of slope $(d_0 - d_1)/(m - 1)$ or higher and is given by a non-split extension*

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow Q_0 \longrightarrow 0$$

of a semistable vector bundle Q_0 of degree $d_0 - d_1$ and rank $m - 1$ by E_1 .

ii) *In i), we have*

$$\dim_{\mathbb{C}}(\text{Ext}^1(Q_0, E_1)) = d_0 - md_1 + (m - 1)(g - 1).$$

Proof. i) By Proposition 2.12, we know that $(E_0, E_1, E_2; \phi_1, \phi_2)$ is also α^M -semistable. By the results stated before Proposition 6.4, this implies that $\text{im}(\phi_1) \cong E_1$ is a subbundle of E_0 and that $Q_0 := E_0/\text{im}(\phi_1)$ is a semistable vector bundle of degree $d_0 - d_1$ and rank $m - 1$. Observe

$$\frac{d_0 - d_1}{m - 1} = \mu(Q_0) = \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2). \quad (6.1)$$

For a subbundle F_0 of E_0 , we thus obtain

$$\mu(F_0) \leq \mu_\alpha(E_0, E_1, E_2; \phi_1, \phi_2) < \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2) = \frac{d_0 - d_1}{m - 1}.$$

This also implies that the extension is non-split.

ii) Recall from Lemma 6.2 that $d_1 < \mu(E_0)$, whence $\mu(Q_0) > \mu(E_0)$. Since Q_0 is semistable, this implies

$$H^0(Q_0^\vee \otimes E_1) = \text{Hom}(Q_0, E_1) = \{0\}.$$

Since $\text{Ext}^1(Q_0, E_1) = H^1(Q_0^\vee \otimes E_1)$, the given formula is a consequence of the Riemann-Roch theorem. \square

There is also a partial converse to Proposition 6.6.

Proposition 6.7. *Let $\alpha = (0, \alpha_1, \alpha_2)$ be a stability parameter with $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$. Then, a holomorphic chain $C = (E_0, E_1, E_2; \phi_1, \phi_2)$ with E_1 a line bundle of degree d_1 , D an effective divisor of degree $d_1 - d_2$, Q_0 a **stable** vector bundle of degree $d_0 - d_1$ and rank $m - 1$,*

$$0 \longrightarrow E_1 \xrightarrow{\phi_1} E_0 \longrightarrow Q_0 \longrightarrow 0$$

a non-split extension, $E_2 := E_1(-D)$, and $\phi_2: E_2 \subseteq E_1$ is α -stable.

Proof. For any non-trivial subbundle $F_0 \subsetneq E_0$, we have to check the stability condition for the subchain $(F_0, 0, 0)$, and, if $E_1 \subseteq F_0$, also for the subchains $(F_0, E_1, 0)$ and (F_0, E_1, E_2) .

In the following, let $\alpha^M = (0, \alpha_1^M, \alpha_2^M)$ be such that (α_1^M, α_2^M) is a point on L_I which lies in the interior of the region depicted in Figure 1. Let F_0 be a subbundle of E_0 . If $F_0 \cap E_1 = \{0\}$, we find

$$\mu(F_0) < \mu(Q_0) = \frac{d_0 - d_1}{m - 1} = \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2),$$

because the extension is non-split. Otherwise, $E_1 \subseteq F_0$, and we have the exact sequence

$$0 \longrightarrow E_1 \longrightarrow F_0 \longrightarrow F_0/E_1 \longrightarrow 0 \quad (6.2)$$

where

$$\mu(F_0/E_1) \leq \mu(Q_0) = \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2).$$

On the other hand,

$$\mu(E_1) = d_1 \stackrel{\text{Lemma 6.2}}{<} \mu(E_0) < \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2).$$

These two facts imply again

$$\mu(F_0) < \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2).$$

This strict inequality still holds for all (α_1, α_2) in the interior of the parameter region that are close enough to (α_1^M, α_2^M) . By definition of the chamber decomposition, it must then hold for all $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$.

Next, let us look at a subchain of the type $(F_0, E_1, 0)$, F_0 a subbundle of E_0 . In this case, F_0 must contain $\text{im}(\phi_1)$, so that we have again the extension (6.2). We claim that

$$\mu_{\alpha^M}(\text{im}(\phi_1), E_1, 0; \phi_1, 0) < \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2).$$

Since (α_1^M, α_2^M) lies above the line L_{III} , we have

$$\mu_{\alpha^M}(E_0, E_1, 0; \phi_1, 0) < \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2). \quad (6.3)$$

The subchain $\bar{C} = (E_0, E_1, 0; \phi_1, 0)$ is an extension of the chain $\bar{C}'' := (Q_0, 0, 0; 0, 0)$ by the chain $\bar{C}' := (\text{im}(\phi_1), E_1, 0; \phi_1, 0)$. Hence,

$$\mu_{\alpha^M}(\bar{C}) = \frac{\text{rk}(\bar{C}')\mu_{\alpha^M}(\bar{C}') + \text{rk}(\bar{C}'')\mu_{\alpha^M}(\bar{C}'')}{\text{rk}(\bar{C})}.$$

Since $\mu_{\alpha^M}(\bar{C}'') = \mu_{\alpha^M}(C)$, our contention follows from (6.3). By the stability of Q_0 , we also have

$$\mu(F_0/E_1) \leq \mu(Q_0) = \mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2).$$

Therefore,

$$\begin{aligned}
& \mu_{\alpha^M}(F_0, E_1, 0; \phi_1, 0) \\
= & \frac{d_1 + d_1 + \deg(F_0/E_1) + \alpha_1^M}{1 + \text{rk}(F_0)} \\
= & \frac{2\mu_{\alpha^M}(\text{im}(\phi_1), E_1, 0; \phi_1, 0) + (\text{rk}(F_0) - 1)\mu(F_0/E_1)}{1 + \text{rk}(F_0)} \\
< & \frac{2\mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2) + (\text{rk}(F_0) - 1)\mu_{\alpha^M}(E_0, E_1, E_2; \phi_1, \phi_2)}{1 + \text{rk}(F_0)} \\
= & \mu_{\alpha^M}(C).
\end{aligned}$$

Again, we see that the same inequality holds for stability parameters $\alpha = (0, \alpha_1, \alpha_2)$ with $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$.

For proper subchains (F_0, E_1, E_2) , we obtain the chain $(\text{im}(\phi_1), E_1, E_2; \phi_1, \phi_2)$ with

$$\mu_{\alpha^M}(\text{im}(\phi_1), E_1, E_2; \phi_1, \phi_2) = \mu_{\alpha^M}(C)$$

and the proper subbundle F_0/E_1 of Q_0 for which we have

$$\mu(F_0/E_1) < \mu(Q_0),$$

by the stability of Q_0 . This enables us to conclude as before. \square

Recall the following.

- For $(0, \alpha_1, \alpha_2)$ with $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$, a holomorphic chain $(E_0, E_1, E_2; \phi_1, \phi_2)$ of type t is $(0, \alpha_1, \alpha_2)$ -semistable if and only if it is $(0, \alpha_1, \alpha_2)$ -stable. (This follows from Proposition 2.11.)
- For $(\alpha_1^i, \alpha_2^i) \in \mathcal{C}_\infty$, $i = 1, 2$, a holomorphic chain $(E_0, E_1, E_2; \phi_1, \phi_2)$ of type t is $(0, \alpha_1^1, \alpha_2^1)$ -semistable if and only if it is $(0, \alpha_1^2, \alpha_2^2)$ -semistable.

We may now describe the moduli spaces which belong to a stability parameter in the chamber \mathcal{C}_∞ .

Corollary 6.8. *The moduli space $\mathcal{M}_\alpha(t)$ for $\alpha = (0, \alpha_1, \alpha_2)$ and $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$ is a connected smooth projective variety of dimension*

$$d_0 - (m - 1)d_1 - d_2 + (m - 1)m(g - 1) + g.$$

It is birationally equivalent to a \mathbb{P}^N -bundle over the product $\mathcal{J}^{d_1} \times X^{(d_1-d_2)} \times \mathcal{U}^s(m-1, d_0-d_1)$ of the Jacobian of degree d_1 line bundles, the (d_1-d_2) -fold symmetric product of the curve, and the moduli space of stable vector bundles of rank $(m-1)$ and degree (d_0-d_1) , $N := d_0 - md_1 + (m-1)(g-1) - 1$.

Proof. The only thing that we have to prove is the irreducibility. The smoothness results from the fact that all α -semistable 3-chains of type t are α -stable and Theorem 3.8, v). The assertions about the dimension and the birational model are evident from Propositions 6.6 and 6.7.

(Note that Theorem 3.8 also gives the dimension.) It suffices to exhibit an irreducible parameter space for all α -semistable objects. The product $\mathcal{J}^{d_1} \times X^{(d_1-d_2)}$ parameterizes the pairs $(\phi_2: E_1(-D) \subseteq E_1)$. Moreover, it is well known that one can construct an irreducible variety \mathcal{A} and a family $Q_{\mathcal{A}}$ on $\mathcal{A} \times X$ which contains any semistable vector bundle of rank $(m-1)$ and degree (d_0-d_1) . Using the theory of universal extensions [18], we may construct an affine bundle \mathcal{B} over $\mathcal{J}^{d_1} \times \mathcal{A}$ and a vector bundle $E_{\mathcal{B}}$ that consists of all vector bundles which are extensions of a vector bundle Q corresponding to a point $a \in \mathcal{A}$ by a line bundle of degree d_1 . Thus, $\mathcal{D} := X^{(d_1-d_2)} \times \mathcal{B}$ is an irreducible variety which carries a universal family of chains, such that any α -semistable chain belongs to that family. Since α -semistability is an open condition, there is an open subvariety \mathcal{D}^0 that parameterizes exactly the α -semistable chains. The irreducible variety \mathcal{D}^0 surjects onto the moduli space $\mathcal{M}_{\alpha}(t)$. \square

Proposition 6.9. *Let $\alpha^i = (0, \alpha_1^i, \alpha_2^i)$, $i = 1, 2$, be two stability parameters, such that $(\alpha_1^i, \alpha_2^i) \in R(t) \cap R_{2g-2}$, $i = 1, 2$, that do not lie on any wall. Then, $\mathcal{M}_{\alpha^1}(t)$ and $\mathcal{M}_{\alpha^2}(t)$ are birationally equivalent smooth projective varieties of dimension $d_0 - (m-1)d_1 - d_2 + (m-1)m(g-1) + g$ or empty.*

In particular, if $R_{2g-2} \cap \text{Interior}(R(t))$ is non-empty, then these varieties are birationally equivalent to a \mathbb{P}^N -bundle over the product $\mathcal{J}^{d_1} \times X^{(d_1-d_2)} \times \mathcal{U}^s(m-1, d_0-d_1)$, $N := d_0 - md_1 + (m-1)(g-1) - 1$.

Proof. If α does not lie on any wall, then the notions of α -stability and α -semistability are equivalent, by Corollary 2.14. Therefore, Theorem 3.8 grants the smoothness of the moduli spaces and determines the dimension. We will first study the region $\text{Interior}(R(t)) \cap \widetilde{R}(t)$.

Let $\alpha^0 = (0, \alpha_1^0, \alpha_2^0)$ be a parameter where (α_1^0, α_2^0) lies on a wall in the interior of $R(t)$ and $C' := (E'_0, E'_1, E'_2; \phi'_1, \phi'_2)$ and $C'' := (E''_0, E''_1, E''_2; \phi''_1, \phi''_2)$ two α^0 -semistable 3-chains of type t' and t'' , respectively, such that $t = t' + t''$ and the map b defined in (3.3) is an isomorphism. Since $\widetilde{C} = C' \oplus C''$ will be an α^0 -semistable chain and α^0 lies in the interior of $R(t)$, no map in \widetilde{C} must be zero, by Remark 2.6. We write $t' = (r'_0, r'_1, r'_2; d'_0, d'_1, d'_2)$ and $t'' = (r''_0, r''_1, r''_2; d''_0, d''_1, d''_2)$. Since we require the maps in \widetilde{C} to be non-zero, there are two possibilities: a) $(r'_0, r'_1, r'_2) = (m', 1, 1)$ and $(r''_0, r''_1, r''_2) = (m'', 0, 0)$ or b) $(r'_0, r'_1, r'_2) = (m', 0, 0)$ and $(r''_0, r''_1, r''_2) = (m'', 1, 1)$. For a general point $x \in X$, we let V' and V'' be the restrictions of C' and C'' , respectively, to $\{x\}$. These are \mathbb{C} -linear chains. With (4.27), we compute

$$\chi(V'', V') = m'm''$$

in Case a). If b is an isomorphism, we must have $\chi(V'', V') = 0$, by Lemma 4.10. Since m' and m'' are both non-zero, this is impossible.

In Case b), we compute $\chi(V'', V') = m'm'' - m'$. For this quantity to become zero, we must have $m'' = 1$ and $m' = m - 1$. Thus, V' has rank-type $(m - 1, 0, 0)$ and V'' has rank type $(1, 1, 1)$. Note that these types are *not excluded by Theorem 4.16*. It remains to compute the dimension of the resulting flip loci. Thus, let $C' = (E'_0, 0, 0; 0, 0)$ be an α^0 -semistable chain of type $(m - 1, 0, 0; d'_0, 0, 0)$ (which means that E'_0 is a semistable vector bundle of rank $m - 1$ and degree d'_0) and $C'' = (E''_0, E_1, E_2; \phi''_1, \phi_2)$ an α^0 -semistable holomorphic chain of type $(1, 1, 1; d''_0, d_1, d_2)$. We have to compute $\dim_{\mathbb{C}}(\text{Ext}^1(C'', C'))$. Note that $\text{Hom}(C'', C') = \{0\}$ and that $\mathbb{H}^2(C'', C') = \{0\}$, by Proposition 3.5. Therefore, Proposition 3.2 gives

$$\dim_{\mathbb{C}}(\text{Ext}^1(C'', C')) = (m - 1)(d''_0 - d_1).$$

Thus, the space of isomorphy classes of α^0 -semistable chains C which are non-split extensions of a chain C'' by a chain C' as above has dimension

$$(m - 1)^2(g - 1) + g + md''_0 + (1 - m)d_1 - d_2. \quad (6.4)$$

By assumption, we have

$$\mu_{\alpha^0}(C'') = \mu_{\alpha^0}(C')$$

that is

$$(m - 1)(\alpha_1^0 + \alpha_2^0) = 3d_0 - (m + 2)d''_0 + (1 - m)d_1 + (1 - m)d_2. \quad (6.5)$$

Since (α_1^0, α_2^0) is supposed to lie in the interior of the region $R(t)$, we find

$$\frac{m}{m - 1}(3d_0 - (m + 2)d''_0 + (1 - m)d_1 + (1 - m)d_2) > 2d_0 - md_1 - md_2$$

which amounts to

$$md''_0 < d_0.$$

Plugging this information into (6.4) and using the dimension formula Theorem 3.8, iv) (which applies for the same reasons as before), we see that the locus of chains C which are extensions of the type we have considered has codimension at least g in the relevant moduli spaces, whence we may forget about these extensions.

The arguments given in Proposition 4.19 deal with the remaining cases (in which the map b cannot be an isomorphism). \square

Remark 6.10. The minimal possible value for d''_0 is d_1 . In that case, (6.5) describes exactly the line L_1 .

6.3. Extremal moduli spaces for type $(m, 1, n; d_0, d_1, d_2)$ where $n > 1$. Under this assumption, the chamber structure looks like the one depicted in Figure 6 (compare Remark 6.1). In this section, we set out to describe the moduli spaces for stability parameters $\alpha = (0, \alpha_1, \alpha_2)$, such that $(\alpha_1, \alpha_2) \in \mathcal{C} := \mathcal{C}_\infty$. It turns out that the geometry of these moduli spaces is closely related to the geometry of Brill-Noether loci.

As before, we first describe the necessary conditions that have to be fulfilled by α -semistable objects. To this end, we fix stability parameters $\alpha^{M_I} = (0, \alpha_1^{M_I}, \alpha_2^{M_I})$ and $\alpha^{M_{IV}} = (0, \alpha_1^{M_{IV}}, \alpha_2^{M_{IV}})$. The assumption is that $(\alpha_1^{M_I}, \alpha_2^{M_I})$ is contained in the relative interior of $\overline{\mathcal{C}} \cap L_I$ and that $(\alpha_1^{M_{IV}}, \alpha_2^{M_{IV}})$ is contained in the relative interior of $\overline{\mathcal{C}} \cap L_{IV}$.

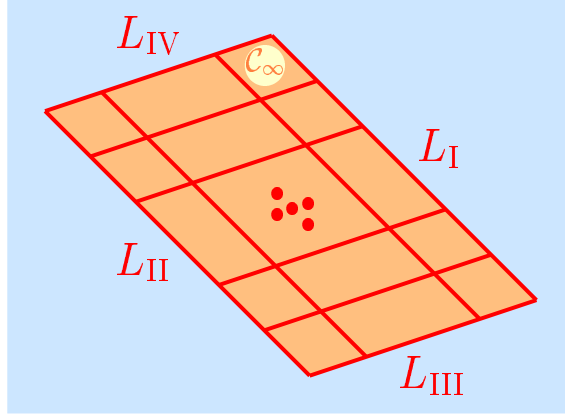


FIGURE 6. The chamber structure for 3-chains of type $(m, 1, n; d_0, d_1, d_2)$.

Proposition 6.11. i) *The vector bundle E_0 does not possess a subbundle of slope $(d_0 - d_1)/(m - 1)$ or higher and is given by a non-split extension*

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow Q_0 \longrightarrow 0$$

of a semistable vector bundle Q_0 of degree $d_0 - d_1$ and rank $m - 1$ by E_1 .

ii) *In i), we have*

$$\dim_{\mathbb{C}}(\text{Ext}^1(Q_0, E_1)) = d_0 - md_1 + (m - 1)(g - 1).$$

iii) *Set $K_2 := \ker(\phi_2)$. Then, K_2 is a semistable vector bundle of rank $n - 1$ and degree $d_2 - d_1$. The vector bundle E_2 is given as a non-split extension*

$$0 \longrightarrow K_2 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow 0.$$

Proof. Part i) and ii) are checked in the same fashion as their counterparts in Proposition 6.6 (i.e., by using that an α -semistable chain is also α^{M_I} -semistable).

In order to establish iii), we use that an α -semistable chain is $\alpha^{M_{IV}}$ -semistable, too. Then, the assertion becomes a straightforward consequence of Proposition 6.5. \square

Remark 6.12. We emphasize that we do not have a formula for computing $\dim(\text{Ext}^1(E_1, K_2)) = h^0(B_2)$, $B_2 := K_2^\vee \otimes E_1 \otimes \omega_X$, in Part iii) of the above proposition. The degree of B_2 is $b_2 := nd_1 - d_2 + (n - 1)(2g - 2)$.

In Remark 6.3, we have already stressed that b_2 can take on any prescribed value. The understanding of our moduli spaces thus rests on our understanding of the spaces of global sections of the semistable vector bundle B_2 . This gives an interesting link between our moduli problem and Brill-Noether theory that we shall exploit below.

In the next step, we will demonstrate a partial converse to Proposition 6.11.

Proposition 6.13. *Assume $\alpha = (0, \alpha_1, \alpha_2)$ is a stability parameter, such that (α_1, α_2) belongs to the chamber \mathcal{C}_∞ . Suppose that $C := (E_0, E_1, E_2; \phi_1, \phi_2)$ is a holomorphic 3-chain of type t , such that E_1 is a line bundle of degree d_1 , Q_0 a **stable** vector bundle of degree $d_0 - d_1$ and rank $m - 1$,*

$$0 \longrightarrow E_1 \xrightarrow{\phi_1} E_0 \longrightarrow Q_0 \longrightarrow 0$$

*a non-split extension, K_2 a **stable** vector bundle of rank $n - 1$ and degree $d_2 - d_1$, and*

$$0 \longrightarrow K_2 \longrightarrow E_2 \xrightarrow{\phi_2} E_1 \longrightarrow 0$$

a non-split extension. Then, C is α -stable.

Proof. First, we look at a subchain $(0, 0, F_2)$ where F_2 is a subbundle of K_2 . The definition of the wall L_{IV} implies that, for a stability parameter $\tilde{\alpha} = (0, \tilde{\alpha}_1, \tilde{\alpha}_2)$, such that $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ lies in the interior of the parameter region $R(t)$, one has

$$\frac{d_2 - d_1}{n - 1} + \tilde{\alpha}_2 < \mu_{\tilde{\alpha}}(C).$$

Since K_2 is, by assumption, a stable vector bundle, we also have

$$\mu(F_2) + \tilde{\alpha}_2 \leq \frac{d_2 - d_1}{n - 1} + \tilde{\alpha}_2 < \mu_{\tilde{\alpha}}(C),$$

so that the chain $(0, 0, F_2)$ is not destabilizing for any stability parameter as in the proposition.

With the methods of the proof of Proposition 6.7, one also checks that no subchain of the form $(F_0, 0, 0)$ is destabilizing. This also implies that subchains of the form $(F_0, 0, F_2)$ aren't destabilizing either.

The remaining case to study is that of a subchain of the shape (F_0, E_1, F_2) . We begin with the following construction. We have the subchain $C' := (0, 0, K_2; 0, 0)$ of C and form the quotient chain $\tilde{C} := C/C'$. Note that $\tilde{C} = (E_0, E_1, E_1; \phi_1, \tilde{\phi}_2)$. This 3-chain possesses the subchain $C'' := (E_1 \cong \text{im}(\phi_1), E_1, E_1; \phi_1, \tilde{\phi}_2)$. Let $C''' := \tilde{C}/C''$. The chain C''' is given as $(Q_0, 0, 0; 0, 0)$. Let $\alpha^M := (0, \alpha_1^M, \alpha_2^M)$ be the stability parameter that is characterized by the condition that (α_1^M, α_2^M) is the point of intersection of the lines L_I and L_{IV} . By construction, we have

$$\mu_{\alpha^M}(C') = \mu_{\alpha^M}(C'') = \mu_{\alpha^M}(C''') = \mu_{\alpha^M}(t). \quad (6.6)$$

The chains C' and C''' are α - and α^M -stable, because the vector bundles Q_0 and K_2 are stable. We claim that the chain C'' is α^M -stable, too. For this, we have to check the two subchains $C_1 := (E_1, E_1, 0)$ and $C_2 := (E_1, 0, 0)$. The condition

$$\mu_{\alpha^M}(C_i) < \mu_{\alpha^M}(t) = \mu_{\alpha^M}(C''), \quad i = 1, 2,$$

follows by applying a trick similar to the one used in the proof of Proposition 6.7, because C_1 and C_2 may also be viewed as subchains of C , and the fact that (α_1^M, α_2^M) lies in the interior of the intersection of the two half spaces defined by the Inequalities II and III (see Figure 1) implies that the two subchains $(E_0, 0, 0)$ and $(E_0, E_1, 0)$ do not destabilize C .

Now, we return to a subchain $\bar{C} = (F_0, E_1, F_2)$ of C . We write the vector bundle F_2 as the extension

$$0 \longrightarrow \tilde{K}_2 := F_2 \cap K_2 \longrightarrow F_2 \longrightarrow Q_2 \longrightarrow 0,$$

and the vector bundle F_0 as the extension

$$0 \longrightarrow E_1 \longrightarrow F_0 \longrightarrow \tilde{Q}_0 \longrightarrow 0.$$

With these constructions, we find the subchain $\bar{C}' := (0, 0, \tilde{K}_2)$ of C' , the subchain $\bar{C}'' := (E_1, E_1, Q_2)$ of C'' , and the subchain $\bar{C}''' := (\tilde{Q}_0, 0, 0)$ of C''' .

Exploiting the α^M -stability of C' , C'' , and C''' and the fact that one of the subchains \bar{C}' , \bar{C}'' , and \bar{C}''' will be a proper one, we find

$$\begin{aligned} & \mu_{\alpha^M}(\bar{C}) \\ &= \frac{1}{\text{rk}(\bar{C})} \left(\text{rk}(\bar{C}') \mu_{\alpha^M}(\bar{C}') + \text{rk}(\bar{C}'') \mu_{\alpha^M}(\bar{C}'') + \text{rk}(\bar{C}''') \mu_{\alpha^M}(\bar{C}''') \right) \\ &< \frac{1}{\text{rk}(\bar{C})} \left(\text{rk}(\bar{C}') \mu_{\alpha^M}(C') + \text{rk}(C'') \mu_{\alpha^M}(C'') + \text{rk}(C''') \mu_{\alpha^M}(C''') \right) \\ &= \mu_{\alpha^M}(t). \end{aligned}$$

From this inequality, it is clear that $\mu_{\alpha}(\bar{C}) < \mu_{\alpha}(t)$ holds for all α , such that (α_1, α_2) is sufficiently close to (α_1^M, α_2^M) . By the definition of the walls, the same must be true for all α , such that (α_1, α_2) lies in the chamber \mathcal{C}_{∞} . \square

Next, we will include some observations regarding the relationship with Brill-Noether theory. (A survey on Brill-Noether theory which was also very helpful to the authors is [21].) Let $\alpha = (0, \alpha_1, \alpha_2)$ be a stability parameter, such that $(\alpha_1, \alpha_2) \in \mathcal{C}_{\infty}$. Note that a holomorphic 3-chain C of type t is α -stable, if and only if it is α -semistable. By Theorem 3.8, the moduli space $\mathcal{M}_{\alpha}(t)$ is smooth, and, if non-empty, it

has dimension

$$\underbrace{(m-1)^2(g-1) + (d_0 - md_1) + (m-1)(g-1) + g}_{=:f} + \underbrace{((n-1)^2(g-1) + 1)}_{=:h} + \underbrace{(nd_1 - d_2 + (n-1)(g-1) - 1)}_{=:c}.$$

Recall the notation from Remark 6.12. In that notation, we define the morphism

$$\begin{aligned} \Phi: \mathcal{M}_\alpha(t) &\longrightarrow \mathcal{U}(n-1, b_2) \\ C = (E_0, E_1, E_2; \phi_1, \phi_2) &\longmapsto B_2. \end{aligned}$$

Call a vector bundle E on X *special*, if both $h^0(E) \neq 0$ and $h^1(E) \neq 0$. Using the map Φ and the dimension formula for $\mathcal{M}_\alpha(t)$, we find the following result.

Theorem 6.14 (Laumon [19]). *Let $r > 0$ and l be integers. Then, the generic vector bundle E of rank r and degree l on X is non-special.*

Proof. By Serre duality, we may assume that $l \leq r(g-1)$. For these values, a vector bundle E of rank r and degree l is special, if and only if $h^0(E) > 0$. Now, we may pick m, n, d_0, d_1, d_2 in such a way that $r = n-1$, $l = b_2 = b_2(g, n, d_1, d_2)$ (see Remark 6.12), and $d_0 > md_1$. Our considerations, in particular Proposition 6.13, imply that the intersection of the image of Φ with $\mathcal{U}^s(r, l)$ consists exactly of the set of isomorphism classes of special stable vector bundles of rank $r = n-1$ and degree $l = b_2$. One easily verifies that the dimension of any fiber of Φ over a special stable vector bundle is at least f . Therefore, the image has dimension at most $h + c$. Now, h is the dimension of $\mathcal{U}^s(r, l)$ and c is negative. This proves that the generic stable vector bundle is non-special. Since the generic vector bundle is stable, we are done. \square

Remark 6.15. The above result is Corollary 1.7 in [19]. Of course, our proof is neither easier nor more natural than the one of Laumon, but it is a nice illustration of the strength of our results. Moreover, the morphism Φ relates the geometry of the moduli space $\mathcal{M}_\alpha(t)$ to the geometry of the Brill-Noether locus inside $\mathcal{U}(n-1, b_2)$. This enables us to derive fundamental properties of our moduli spaces from the basic results in Brill-Noether theory.

Following Laumon, we let $W_{X,r}^l$ be the closed substack of vector bundles that do possess global sections inside the stack $\text{Bun}_{X,r}^l$ of all vector bundles of rank r and degree l on X .

Theorem 6.16. i) *The stack $W_{X,r}^l$ is irreducible.*

ii) *For every r and every $l > 0$, there exist stable vector bundles of rank r and degree l with global sections.*

Proof. i) This is [19], Corollary 5.2.

ii) For $l > n(g-1)$, the existence of global sections follows from the theorem of Riemann-Roch. In the remaining range $0 < l \leq n(g-1)$, Sundaram [26] shows that there exist stable vector bundles of rank r and degree l with global sections. (Of course, for $l = 0$, there are no stable vector bundles with global sections.) \square

Remark 6.17. If $l \leq r(g-1)$, then the generic vector bundle in $W_{X,r}^l$ has precisely one global section. This follows from [19], Lemma 2.6.

Proposition 6.18. *Assume that the type $t = (m, 1, n; d_0, d_1, d_2)$ is such that $b_2 > 0$ (see Remark 6.12) and $d_0 > md_1$ and that $\alpha = (0, \alpha_1, \alpha_2)$ is a stability parameter with $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$. Then, the moduli space $\mathcal{M}_\alpha(t)$ is a smooth connected projective variety of dimension*

$$(g-1)(m^2 + 1 + n^2 - m - n) + (d_0 - md_1) + (nd_1 - d_2) + 1.$$

Proof. Proposition 6.13 and Theorem 6.16 grant that $\mathcal{M}_\alpha(t)$ is non-empty. By Theorem 3.8, the moduli space is smooth of the indicated dimension. It remains to check that it is also connected. Since the semistable vector bundles form an open substack of the stack of all vector bundles, the stack of semistable vector bundles of rank r and degree l with global sections is still irreducible, by Theorem 6.16, i). Using this and the techniques from the proof of Corollary 6.8, one easily constructs a connected parameter space for α -stable chains of type t . \square

Remark 6.19. i) We remind the reader that the conditions $b_2 \geq 0$ and $d_0 > md_1$ are both necessary for the non-emptiness of the moduli spaces. The first one, because a semistable vector bundle of negative degree never possesses global sections, and the second one, because otherwise the interior of the parameter region is empty, by Remark 6.2. Observe that the existence problem for $b_2 = 0$ has been left open.

ii) Let $l > 0$ and d be integers and $\mathcal{W}_{X,r}^l \subset \mathcal{U}^s(r, l)$ the locus of stable vector bundles with global sections. Under the assumption that $0 < l \leq r(g-1)$, Hoffmann [16, Example 5.11] has recently shown that $\mathcal{W}_{X,r}^l$ is birationally equivalent to $\mathbb{P}^s \times \mathcal{J}^0(X)$. Thus, the morphism Φ introduced before will help in a more detailed investigation of the geometry of the moduli spaces of holomorphic chains.

To conclude this example, we will again determine a region of parameters, such that all moduli spaces associated to parameters in that region will be birationally equivalent.

Proposition 6.20. *Let $\alpha^i = (0, \alpha_1^i, \alpha_2^i)$, $i = 1, 2$, be two stability parameters, such that $(\alpha_1^i, \alpha_2^i) \in R(t) \cap R_{2g-2}$, $i = 1, 2$, that do not lie on any wall. Then, $\mathcal{M}_{\alpha^1}(t)$ and $\mathcal{M}_{\alpha^2}(t)$ are birationally equivalent smooth projective varieties which either have dimension*

$$(g-1)(m^2 + 1 + n^2 - m - n) + (d_0 - md_1) + (nd_1 - d_2) + 1$$

or are empty.

Proof. We use the same discussion as at the beginning of the proof of Proposition 6.9. Note that we may always replace C' by an α^0 -stable subchain, i.e., we can directly require C' to be α^0 -stable.

In the respective notation we distinguish the two cases a) $(r'_0, r'_1, r'_2) = (m', 1, n')$ with $m'n' > 0$ and $(r''_0, r''_1, r''_2) = (m'', 0, n'')$ with either $m'' \neq 0$ or $n'' \neq 0$ and b) $(r'_0, r'_1, r'_2) = (m', 0, n')$ with either $m' \neq 0$ or $n' \neq 0$ and $(r''_0, r''_1, r''_2) = (m'', 1, n'')$ with $m''n'' > 0$.

In Case a), the \mathbb{C} -linear chain $\delta_{[0,2]}$ will appear in the decomposition of V'

into indecomposable chains. We easily see $\text{Hom}(\delta_{[0,0]}, \delta_{[0,2]}) \cong \mathbb{C}$. Thus, $\delta_{[0,0]}$ cannot occur in the decomposition of V'' , so that $V'' = \delta_{[2,2]}^{\oplus n''}$. The formula for $\chi(V'', V')$ yields the value $n'n'' - n''$. We want this to be zero, so that we conclude $n' = 1$ and $n'' = n - 1$. The rank type of V' is thus $(m, 1, 1)$ and the rank type of V'' is $(0, 0, n - 1)$. This is again a possibility which is allowed by Theorem 4.16.

Let $C' = (E_0, E_1, E'_2; \phi_1, \phi'_2)$ be an α^0 -stable chain of type $(d_0, d_1, d'_2; m, 1, 1)$ and $C'' = (0, 0, E''_2; 0, 0)$ an α^0 -semistable holomorphic chain of type $(0, 0, d''_2; 0, 0, n - 1)$. We have to evaluate the dimension of the locus of chains C which might be obtained as non-split extensions of a chain such as C'' by a chain such as C' . By the dimension formula Theorem 3.8, iv), the moduli space for α^0 -stable chains of type $(d_0, d_1, d'_2; m, 1, 1)$ has dimension $d_0 - (m - 1)d_1 - d_2 + (m - 1)m(g - 1) + g$. The moduli space of chains of type $(0, 0, d''_2; 0, 0, n - 1)$ agrees with the moduli space of semistable vector bundles of rank $n - 1$ and degree d''_2 and has dimension $(n - 1)^2(g - 1)$. Again, we find $\text{Hom}(C'', C') = \{0\} = \mathbb{H}^2(C'', C')$ (Proposition 3.5, iv), so that Proposition 3.2 gives $\dim_{\mathbb{C}}(\text{Ext}^1(C'', C')) = (n - 1)(d_1 - d'_2)$. All in all, the dimension of the locus we wish to describe is

$$(g - 1)(m^2 + 1 + n^2 - m - 2n + 1) + (d_0 - md_1) + n(d_1 - d''_2) + 1. \quad (6.7)$$

We have $\mu_{\alpha^0}(C') = \mu_{\alpha^0}(C)$ which gives

$$-(n - 1)\alpha_1^0 + (m + 1)(n - 1)\alpha_2^0 = (n - 1)(d_0 + d_1) - (m + 2)d_2 + (m + n + 1)d'_2. \quad (6.8)$$

Since α^0 is supposed to lie in the interior of the parameter region $R(t)$, we must have

$$(n - 1)(d_0 + d_1) - (m + 2)d_2 + (m + n + 1)d'_2 > \frac{n - 1}{n}(nd_0 + nd_1 - (m + 1)d_2)$$

that is

$$nd'_2 > d_2.$$

Together with Formula (6.7), this shows that the codimension of the flip locus in question is at least $(n - 1)(g - 1) + 1$. It, therefore, may be neglected.

Case b) follows immediately from Case a) by passing to the *dual chains* (see Remark 2.3, iv). The remaining flip loci are covered by Proposition 4.19. \square

Remark 6.21. In (6.8), the maximal value of d_2' is d_1 . Plugging d_1 into that formula gives the equation for the line L_{IV} . Likewise, one obtains the equation for the line L_I (this is hidden in the argument with the dual chains).

6.4. Moduli spaces for type $(1, m, 1; d_0, d_1, d_2)$. In the examples above, we have merely used vector bundle techniques in order to analyze the asymptotic moduli spaces. Here, we will discuss an elementary example where we make use of extensions *in the category of holomorphic chains* in order to gather interesting information on the geometry of the asymptotic moduli spaces.

Let us recall the relevant inequalities from Proposition 5.3 that bound the parameter region $R(t)$.

Inequality I. Using the test object $(E_0, 0, 0)$, one arrives at the inequality

$$m\alpha_1 + \alpha_2 \geq A_I := (m+1)d_0 - d_1 - d_2.$$

Inequality II. Testing α -semistability with the subchain $(E_0, E_1, 0)$ yields the inequality

$$-m\alpha_1 + (m+1)\alpha_2 \geq A_{II} := d_0 + d_1 - (m+1)d_2.$$

Inequality III. The subchain $(E_0, E_2 \cong \text{im}(\phi_2), E_2)$ provides us with the inequality

$$-2\alpha_1 + \alpha_2 \leq A_{III} := -d_0 + \frac{3}{m-1}d_1 - \frac{2m+1}{m-1}d_2.$$

Inequality IV. Finally, one finds the inequality

$$-2\alpha_1 + \alpha_2 \geq A_{IV} := -\frac{2m+1}{m-1}d_0 + \frac{3}{m-1}d_1 - d_2$$

with the subchain $(0, \ker(\phi_1), 0)$.

Remark 6.22. i) If we wish that the interior of the parameter region $R(t)$ becomes non-empty, then we must choose the numerical data in such a way that $A_{III} > A_{IV}$ holds true. This condition simply amounts to $d_0 > d_2$.

ii) Inequality III and IV bound a strip in the (α_1, α_2) -plane. The remaining inequalities provide the lower bounds for the parameter region $R(t)$. Figure 7 illustrates the shape of the resulting domain.

iii) Let us examine when the map $\phi_1 \circ \phi_2: E_2 \rightarrow E_0$ is non-zero. If $\phi_1 \circ \phi_2 \equiv 0$, then we obtain the subchain $(0, \ker(\phi_1), E_2)$. Noting that $\deg(\ker(\phi_1)) \geq d_1 - d_0$, the condition of α -semistability applied to the given subchain yields the inequality

$$(m-2)\alpha_1 + 2\alpha_2 \leq 2(m+1)d_0 - 2d_1 - 2d_2. \quad (6.9)$$

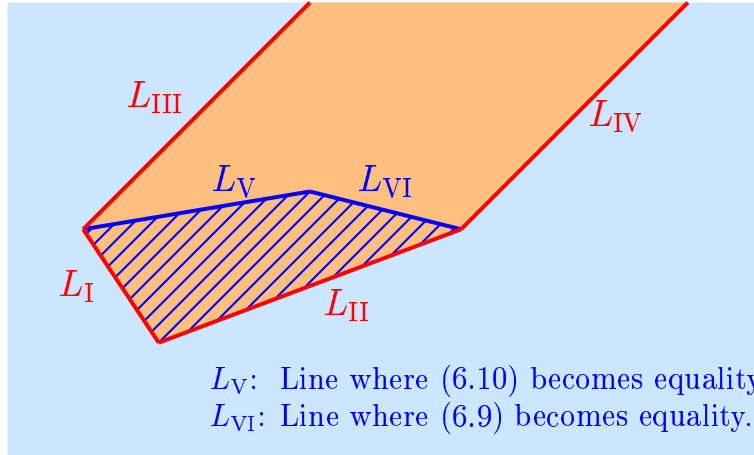


FIGURE 7. The parameter region for type $(r_0, r_1, r_2; d_0, d_1, d_2)$.

Alternatively, one may use the subchain $(0, E_2 \cong \phi_2(E_2), E_2)$. This leads to the inequality

$$(-m + 2)\alpha_1 + m\alpha_2 \leq 2d_0 + 2d_1 - 2(m + 1)d_2. \quad (6.10)$$

The reader may check that Inequality I-IV together with either Inequality (6.9) or Inequality (6.10) cuts out a bounded region in the (α_1, α_2) -plane. Therefore, if the converse to either (6.9) or (6.10) holds, then $\phi_1 \circ \phi_2$ is non-trivial (whence a generic isomorphism).

In Figure 4, we have already sketched the parameter region $R(t)$ away from some bounded subregion and selected a chamber \mathcal{C}_∞ . It is formally defined to be the (unique) unbounded two-dimensional chamber whose closure intersects the line L_{III} where Inequality III becomes an equality. This definition involves that the converse to both (6.9) and (6.10) is verified for the elements of \mathcal{C}_∞ .

Remark 6.23. In this example, one might also declare the unbounded two-dimensional chamber whose closure intersects the line L_{IV} where Inequality IV becomes equality to be the chamber \mathcal{C}_∞ . The reader may verify that techniques analogous to those presented in the following lead to a description of the relevant moduli spaces.

Proposition 6.24. *Let $\alpha = (0, \alpha_1, \alpha_2)$ be a stability parameter, such that $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$, and let $C := (E_0, E_1, E_2; \phi_1, \phi_2)$ be an α -semistable holomorphic chain of type $(1, m, 1; d_0, d_1, d_2)$. Then, C is a non-split extension of the holomorphic chain*

$$C'' := C/C' : 0 \longrightarrow Q_1 := E_1/\phi_2(E_2) \longrightarrow 0$$

by the chain

$$C' : E_2 \xrightarrow{\phi_2} E_2 \cong \text{im}(\phi_2) \xrightarrow{\cdot D} E_2(D) \cong E_0.$$

Here, D is an effective divisor of degree $d_0 - d_2$, and Q_1 is a semistable vector bundle of rank $m - 1$ and degree $d_1 - d_2$.

Furthermore,

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(C'', C')) = (m-1)(d_0 - d_2).$$

Proof. Everything apart from the formula for $\dim_{\mathbb{C}}(\mathrm{Ext}^1(C'', C'))$ follows from the definition of the line L_{III} and the adjacency of \mathcal{C}_{∞} to that line.

First, note that Proposition 3.5, iv), grants that $\mathbb{H}^2(C'', C') = \{0\}$. Next, we obviously have $\mathrm{Hom}(C'', C') = \{0\}$. Therefore, we find $\dim_{\mathbb{C}}(\mathrm{Ext}^1(C'', C')) = -\chi(C'', C')$. Finally, we use Proposition 3.2 to compute $\chi(C'', C')$. \square

Again, we prove a partial converse to Proposition 6.24.

Proposition 6.25. *Assume that $\alpha = (0, \alpha_1, \alpha_2)$ is a stability parameter with $(\alpha_1, \alpha_2) \in \mathcal{C}_{\infty}$. Let E_2 be a line bundle of degree d_2 , D an effective divisor of degree $d_0 - d_2$, and Q_1 a **stable** vector bundle of rank $m-1$ and degree $d_1 - d_2$. Set $E_0 := E_2(D)$ and define the chain*

$$C'' : 0 \longrightarrow Q_1 \longrightarrow 0$$

as well as the chain

$$C' : E_2 \xrightarrow{\mathrm{id}_{E_2}} E_2 \xrightarrow{\cdot D} E_0.$$

Any non-split extension C of the chain C'' by the chain C' is an α -stable holomorphic 3-chain of type $t = (1, m, 1; d_0, d_1, d_2)$.

Proof. We fix a stability parameter $\alpha^M = (0, \alpha_1^M, \alpha_2^M)$, such that (α_1^M, α_2^M) lies on the line L_{III} , in the interior of the region described by the remaining Inequalities I, II, and IV, and in the closure of \mathcal{C}_{∞} . Let C be a chain as in the statement of the proposition. We start with the investigation of some special subchains. First, we recall that the subchain $C_1 := (E_0, 0, 0)$ is neither α - nor α^M -destabilizing, because (α_1, α_2) and (α_1^M, α_2^M) both lie in the interior of the region from Figure 1. Next, we look at the subchain $C_2 := (E_0, E_2, E_2)$. The definition of the line L_{III} and the fact that (α_1, α_2) lies below that line imply that C_2 does not α -destabilize C either. (However, we have $\mu_{\alpha^M}(C_2) = \mu_{\alpha^M}(C)$.) As the third subchain, we define $C_3 := (E_0, E_2, 0)$ and we ask whether the inequality

$$\frac{d_0 + d_2 + \alpha_1}{2} < \frac{d_0 + d_1 + d_2 + m\alpha_1 + \alpha_2}{m+2}$$

is verified, that is, if

$$(m-2)\alpha_1 + 2\alpha_2 > md_0 - 2d_1 + md_2$$

holds true. As we have pointed out after Remark 6.22, the definition of the chamber \mathcal{C}_{∞} involves the inequality

$$(m-2)\alpha_1 + 2\alpha_2 > 2(m+1)d_0 - 2d_1 - 2d_2.$$

One checks that the necessary condition $d_0 > d_2$ implies

$$2(m+1)d_0 - 2d_1 - 2d_2 > md_0 - 2d_1 + md_2,$$

so that we may conclude that C_3 isn't an α -destabilizing subchain for C . The same conclusion applies with respect to α^M .

Now, let $\overline{C} = (F_0, F_1, F_2)$ be any subchain of C . Then, we write the vector bundle F_1 as the extension

$$0 \longrightarrow E_2 \cap F_1 \longrightarrow F_1 \longrightarrow \tilde{Q}_1 \longrightarrow 0.$$

First, we suppose that F_0 and F_2 are both trivial. Then, $E_2 \cap F_1 = \{0\}$, so that we may view F_1 as a subsheaf of Q_1 . It is a proper subsheaf. Otherwise, we would have $E_1 = E_2 \oplus Q_1$. But, since F_1 obviously agrees with $\ker(\phi_1)$, this implies $C = C' \oplus C''$, contradicting our assumption. Therefore, since Q_1 is stable,

$$\mu_{\alpha^M}(\overline{C}) = \mu(F_1) + \alpha_1^M < \mu(Q_1) + \alpha_1^M = \mu_{\alpha^M}(C') = \mu_{\alpha^M}(C).$$

In the remaining cases, we may write the chain \overline{C} as an extension of the chain $(0, \tilde{Q}_1, 0; 0, 0)$ by one of the subchains C_1 , C_2 , or C_3 . Since Q_1 is a stable bundle, we find

$$\mu(\tilde{Q}_1) + \alpha_1^M \leq \mu(Q_1) + \alpha_1^M = \mu_{\alpha^M}(C').$$

Using the results on the subchains C_1 , C_2 , and C_3 (and the fact that \tilde{Q}_1 will be a proper subbundle of Q_1 in the case of the chain C_2), one checks that

$$\mu_{\alpha^M}(\overline{C}) < \mu_{\alpha^M}(C)$$

is verified. As in the proof of Proposition 6.7, we easily derive the assertion of the proposition. \square

Our discussions yield the following information on the moduli spaces which belong to a stability parameter in the chamber \mathcal{C}_∞ .

Corollary 6.26. *The moduli space $\mathcal{M}_\alpha(t)$ for $\alpha = (0, \alpha_1, \alpha_2)$ and $(\alpha_1, \alpha_2) \in \mathcal{C}_\infty$ is a connected smooth projective variety of dimension*

$$(m-1)^2(g-1) + g + m(d_0 - d_2).$$

It is birationally equivalent to a \mathbb{P}^N -bundle over the product $\mathcal{J}^{d_2} \times X^{(d_0-d_2)} \times \mathcal{U}^s(m-1, d_1-d_2)$ of the Jacobian of degree d_0 line bundles, the (d_0-d_2) -fold symmetric product of the curve, and the moduli space of stable vector bundles of rank $(m-1)$ and degree (d_1-d_2) , $N := (m-1)(d_0-d_2) - 1$.

Proof. The smoothness follows again from the fact that all α -semistable 3-chains of type t are α -stable and Theorem 3.8, v). Proposition 6.24 and 6.25 establish the assertions about the dimension and the birational geometry. The irreducibility of the moduli space may be proved along the same lines as Corollary 6.8. \square

Remark 6.27. The reader may try to find the result by just using extension techniques for vector bundles. That approach doesn't seem to work properly.

Proposition 6.28. *Let $\alpha^i = (0, \alpha_1^i, \alpha_2^i)$, $i = 1, 2$, be two stability parameters, such that $(\alpha_1^i, \alpha_2^i) \in R(t) \cap R_{2g-2}$, $i = 1, 2$, that do not lie on any wall. Then, $\mathcal{M}_{\alpha^1}(t)$ and $\mathcal{M}_{\alpha^2}(t)$ are birationally equivalent smooth projective varieties which either have dimension*

$$(m-1)^2(g-1) + g + m(d_0 - d_2).$$

or are empty.

Proof. We use the set-up described at the beginning of the proof of Proposition 6.9. This time, there are the cases a) $(r'_0, r'_1, r'_2) = (0, m', 1)$ and $(r''_0, r''_1, r''_2) = (1, m'', 0)$ with $m'm'' > 0$, b) $(r'_0, r'_1, r'_2) = (1, m', 0)$ and $(r''_0, r''_1, r''_2) = (0, m'', 1)$ with $m'm'' > 0$, c) $(r'_0, r'_1, r'_2) = (1, m', 1)$ and $(r''_0, r''_1, r''_2) = (0, m'', 0)$ with $m'm'' > 0$, and d) $(r'_0, r'_1, r'_2) = (0, m', 0)$ and $(r''_0, r''_1, r''_2) = (1, m'', 1)$ with $m'm'' > 0$.

Recall that we would like to first determine the cases when the map b from (3.3) may be an isomorphism. This requires $\chi(V'', V') = 0$ and $\text{Hom}(V'', V') = \{0\}$ (Lemma 4.10). In Case a), we have $\chi(V'', V') = m'm''$. This is never zero, so that this case needs not be considered. In Case b), we have $\chi(V'', V') = m'm'' - m$. This is non-zero except for the case $m = 4$ and $m' = 2 = m''$. In that case, the decomposition of both V' and V'' contains the linear chain $\delta_{[1,1]}$, whence $\text{Hom}(V'', V') \neq \{0\}$, and that excludes this case, too.

In Case c), we have $\chi(V'', V') = 0$ and $\text{Hom}(V'', V') = \{0\}$, if and only if $m' = 1$ and $m'' = m - 1$. The α^0 -semistable chains C' of type $(1, 1, 1; d_0, d'_1, d_2)$ have a $(g + d_0 - d_2)$ -dimensional moduli space whereas the α^0 -semistable chains of type $(0, m - 1, 0; d_0, d'_1, d_2)$ have an $((m - 1)^2(g - 1) + 1)$ -dimensional moduli space. For these chains, we compute that $\text{Hom}(C'', C') = \{0\}$, $\mathbb{H}^2(C'', C') = \{0\}$, by Proposition 3.5, and that $\dim_{\mathbb{C}}(\text{Ext}^1(C'', C')) = (m - 1)(d_0 - d'_1)$, by Proposition 3.2. The dimension of the locus of chains C that may be written as an extension of a chain C'' as above by a chain C' as above thus has dimension

$$(m-1)^2(g-1) + g + (d_0 - d_2) + (m-1)(d_0 - d'_1).$$

We obviously have $d_2 \leq d'_1 \leq d_0$. If $d'_1 = d_2$, then α^0 has to lie on the line L_{III} , and we exclude that. Thus, we see that the flip locus under investigation has proper codimension.

Case d) reduces immediately to Case c), by passing to the dual chains (Remark 2.3, iv). Together with Proposition 4.19, these considerations imply our contention. \square

Remark 6.29. Again, we emphasize that the walls $\mathcal{B}(t, t'')$ from Definition 4.8 comprise the walls L_{III} and L_{IV} .

6.5. Concluding remarks regarding the parameter region and the birationality region. In Definition 4.8, a certain region $\widetilde{R}(t)$ in the plane \mathbb{R}^2 has been defined, so that non-empty moduli spaces belonging to parameters in the same connected component of the region $R_{2g-2} \cap \widetilde{R}(t)$ are birationally equivalent. In the proofs of Proposition 6.9, 6.20, and 6.28, we have seen that the definition of $\widetilde{R}(t)$ is not optimal. Indeed, there are many walls in $\mathcal{B}(t', t'')$, such that the moduli spaces in two open chambers adjacent to such a wall are still birationally equivalent. It would be interesting to know, if the relevant computations may be performed in greater generality in order to arrive at better results (cf. Remark 4.21).

Another interesting feature, at which we pointed in Remark 6.10, 6.21, and 6.29, is that the walls that bound the parameter region belong to those defined in Proposition 2.4 and, more generally, to those of the form $\mathcal{B}(t', t'')$. This observation might help to find the a priori region for parameters with non-empty moduli spaces in more general situations, i.e., for chains of greater length or eventually more general quivers as considered in [2, 24].

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