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Nikolay M. NIKOLOV, Karl-Henning REHREN and Ivan T. TODOROV

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Avril 2005

IHES/P/05/13

Partial wave expansion and Wightman positivity in conformal field theory

Nikolay M. Nikolov[1,2*], Karl-Henning Rehren[2*], Ivan T. Todorov[1,2*]

[1] Institute for Nuclear Research and Nuclear Energy
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria

[2] Institut für Theoretische Physik, Universität Göttingen,
Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany

Abstract

A new method for computing exact conformal partial wave expansions is developed and applied to approach the problem of Hilbert space (Wightman) positivity in a non-perturbative four-dimensional quantum field theory model. The model is based on the assumption of global conformal invariance on compactified Minkowski space (GCI). Bilocal fields arising in the harmonic decomposition of the operator product expansion (OPE) prove to be a powerful instrument in exploring the field content. In particular, in the theory of a field $\mathcal{L}$ of dimension 4 which has the properties of a (gauge invariant) Lagrangian, the scalar field contribution to the 6-point function of the twist 2 bilocal field is analyzed with the aim to separate the free field part from the nontrivial part.

PACS 2003: 11.10.–z, 03.70.+k
MSC 2000: 81T10

* e-mail addresses:
mitov@inrne.bas.bg, nikolov@theorie.physik.uni-goe.de
rehren@theorie.physik.uni-goe.de
todorov@inrne.bas.bg, itodorov@theorie.physik.uni-goe.de

Contents

1 Introduction

Conformal partial wave expansions [DPPT76, DMPPT77] (an outgrow of conformal *operator product expansions* (OPE) [FGGP72]) provide a powerful method in conformal quantum field theory and continue to attract attention (see [LR93, HPR02, NST03, DO01, DO04, NST04, NT04]). We derive a new method for the complete partial wave expansion in closed form, and apply it to the positive definite 4-point functions of a pair of scalar fields, in order to approach the non-linear problem of Hilbert space positivity which is the central remaining problem when all linear properties of the correlation functions are satisfied.

Specifically, we study a model generated by a conformal field $\mathcal{L}$ of dimension 4 (in four space-time dimensions) which has the properties of a Lagrangian density [NST03]. Such a model appears to offer the best chance for constructing a nontrivial non-perturbative local quantum field theory.

To set the stage, we begin by a brief survey of relevant results of earlier work ([NT01, NST02, NST03]; for a recent review – see [NT04]).

Invariance under finite conformal transformations in Minkowski space $\mathbb{M}$, combined with Wightman axioms [SW64], has far reaching implications. Together with local commutativity it implies the *Huygens principle*: the commutator of any two local Bose fields vanishes for non-isotropic (non-lightlike) separations. Moreover, the Huygens principle is equivalent (assuming Wightman axioms) to *strong locality* stating that the commutator vanishes if multiplied by a sufficiently large power of the Lorentz distance square of the fields' arguments. The Huygens principle and energy positivity yield rationality of correlation functions ([NT01], Theorem 3.1), as well as meromorphy of (analytically continued) products of fields acting on the vacuum ([N05, Theorem 9.2]). These results allow one to extend a quantum field theory to *compactified* Minkowski space $\overline{\mathbb{M}}$.

We therefore call such a QFT *globally conformal invariant* (GCI). (It has been pointed out [LM75] that infinitesimal conformal covariance in general warrants global conformal symmetry (exponentiation to finite transformations) to hold *only on a covering space.* Calling a QFT globally conformal invariant without further qualification emphasizes that finite transformations are well-defined on $\overline{\mathbb{M}}$ itself.)

The compactified Minkowski space $\overline{\mathbb{M}}$ admits a convenient complex variable parametrization ([T86])[1]:

$$\overline{\mathbb{M}} = \left\{ z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \ z = \frac{\bar{z}}{\bar{z}^2} \right\} = \left\{ z = e^{i\zeta u}, \ \zeta \in \mathbb{R}, u \in \mathbb{S}^3 \right\} \qquad (z^2 := \sum_{\alpha=1}^{4} z_\alpha^2 = \mathbf{z}^2 + z_4^2) \quad (1.1)$$

(isomorphic to the $U(2)$ group manifold – *cf.* [U63]). The embedding $\mathbb{M} \hookrightarrow \overline{\mathbb{M}}$ amounts to a complex conformal transformation from Minkowski space coordinates x^μ, $\mu = 0, 1, 2, 3$ to z_α:

$$\mathbf{z} = \frac{\mathbf{x}}{\omega(x)}, \qquad z_4 = \frac{1 - x^2}{2\,\omega(x)}, \qquad 2\,\omega(x) = 1 + x^2 - 2\,i\,x^0 \qquad (x^2 := \mathbf{x}^2 - (x^0)^2).$$

Applying this conformal transformation, we obtain the so called *analytic* (*z-picture*) in which the correlation functions look essentially the same as in the x-picture. The free massless Minkowski space 2-point function $(2\pi)^{-2}\,(\mathbf{x}_{12}^2 - (x_{12}^0 - i\,\varepsilon)^2)^{-1}$ is turned into the distribution $\langle 0|\ \varphi(z_1)\varphi(z_2)\ |0\rangle = (z_{12}^2)_+^{-1}$ which is defined as the Taylor series in z_2 with coefficients in the polynomial algebra $\mathbb{C}\left[z_1, \frac{1}{z_1^2}\right]$. The same applies to the 2-point functions $\sim (z_{12}^2)_+^{-d}$ of scalar fields ϕ of dimension d.

The z-picture series expansions provide a convenient description of the fields as formal power series in z and negative powers of z^2 that is completely equivalent to the Wightman approach using distributions. Together with strong locality, this yields a concept of higher dimensional vertex algebras introduced in [N05], generalizing the chiral vertex algebra formalism of [K96].

For our purposes it is sufficient to treat the correlators as rational functions (rather than distributions) and we shall omit the subscript $+$.

Let $\phi(z)$ be a hermitean scalar field on $\overline{\mathbb{M}}$ of dimension $d \in \mathbb{N}$. The OPE of the product of two fields can be written down as a series of "bilocal fields" $V_\kappa(z_1, z_2)$ ([NST02, NST03, NST04, NT04]) with $d-1$ singular terms:

$$\begin{aligned} \phi(z_1)\,\phi(z_2) &= B_\phi (z_{12}^2)^{-d} \left\{ 1 + \sum_{\kappa=1}^{\infty} (z_{12}^2)^\kappa\, V_\kappa(z_1, z_2) \right\} \qquad (1.2) \\ &= B_\phi (z_{12}^2)^{-d} \left\{ 1 + \sum_{\kappa=1}^{d-1} (z_{12}^2)^\kappa\, V_\kappa(z_1, z_2) \right\} + :\phi(z_1)\,\phi(z_2): \end{aligned}$$

Here the *normal ordered product* $:\phi(z_1)\,\phi(z_2): \,=\, :\phi(z_2)\,\phi(z_1):$ is defined as the regular part of the sum in the first line of (1.2); in particular, it has a local field limit for $z_{12} \equiv z_1 - z_2 \to 0$.

Each bilocal field $V_\kappa(z_1, z_2)$ is defined in terms of its expansion in local symmetric traceless tensors $O_{2\kappa, L}(z)$ (L even for $V_\kappa(z_1, z_2) = V_\kappa(z_2, z_1)$) of fixed *twist* (= dimension minus rank) 2κ – *i.e.* of scale dimension $2\kappa + L$. This implies the orthogonality relation

$$\langle 0|\ V_\kappa(z_1, z_2)\, V_\lambda(z_3, z_4)\ |0\rangle = 0 \quad \text{for } \kappa \neq \lambda\,. \qquad (1.3)$$

The twist two field, $V_1(z_1, z_2)$, involves in its expansion an infinite series of conserved tensor fields, including the stress-energy tensor $T_{\alpha\beta}(z)$, and is harmonic in each argument:

$$\partial_{z_1}^2\, V_1(z_1, z_2) = 0 = \partial_{z_2}^2\, V_1(z_1, z_2), \qquad \partial_z^2 := \sum_{\alpha=1}^{4} \frac{\partial^2}{\partial\, z_\alpha^2}\,. \qquad (1.4)$$

[1]For the generalization to arbitrary space-time dimension D and further developments see, e.g., [NT04].

For $d = 2$ the sum of singular terms in (1.2) reduces to a single one, V_1, and it was proven (see Sect. 5) that V_1 (and ϕ) can be expressed in this case as a sum of normal products of (mutually commuting) free massless scalar fields.

It was then natural to turn attention to a GCI field $\mathcal{L}(z)$ of dimension $d = 4$ with the properties of a (gauge invariant) Lagrangian density [NST03, NT04]. Indeed, it is the lowest dimensional scalar field of $d > 2$ which can be interpreted as a local observable in a gauge field theory. It involves higher twist bilocal fields in the expansion (1.2) which do not obey the free field equation (1.4) and thus offer a chance for a nontrivial GCI theory of local observables.

The general GCI Wightman 4-point function $\langle 1234\rangle \equiv \langle 0|\ \mathcal{L}(z_1)\,\mathcal{L}(z_2)\ \mathcal{L}(z_3)\,\mathcal{L}(z_4)\ |0\rangle$ has the form [NST03, NST04]

$$\begin{aligned}\langle 1234\rangle &= \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle + \frac{B_{\mathcal{L}}^2 (z_{13}^2\, z_{24}^2)^2}{(z_{12}^2\, z_{23}^2\, z_{34}^2\, z_{14}^2)^3}\, P_5(s,t)\\ &= \langle 12\rangle\langle 34\rangle \left\{ 1 + s^4 + s^4\, t^{-4} + \frac{s}{t^3}\, P_5(s,t)\right\}\end{aligned} \tag{1.5}$$

where $\langle ij\rangle$ stands for the 2-point function $\langle 0|\ \mathcal{L}(z_i)\,\mathcal{L}(z_j)\ |0\rangle = B_{\mathcal{L}}\ (z_{ij}^2)^{-4}\ (B_{\mathcal{L}} > 0)$ and $P_5(s,t)$ is a crossing symmetric polynomial of overall degree five in the conformally invariant cross ratios

$$s = \frac{z_{12}^2\, z_{34}^2}{z_{13}^2\, z_{24}^2}\,, \qquad t = \frac{z_{14}^2\, z_{23}^2}{z_{13}^2\, z_{24}^2}\,. \tag{1.6}$$

The crossing symmetry ($s_{12}\, P_5(s,t) := t^5\, P_5\left(\frac{s}{t}, \frac{1}{t}\right) = P_5(s,t)$, $s_{13}\, P_5(s,t) := P_5(t,s) = P_5(s,t)$) reflects the local commutativity of the field $\mathcal{L}$.

The polynomial P_5 involves five free parameters and will be displayed in Section 2. (If we add to these the normalization B^2 of the product of 2-point functions and view two models differing by rescaling $\mathcal{L} \to \lambda\,\mathcal{L}$ of the basic field as equivalent we will end up with a five-dimensional projective space of 4-point functions described in terms of six homogeneous coordinates.)

We study in Section 2 the partial wave expansion of the 4-point function (1.5) of the scalar field $\mathcal{L}$, i.e., the decomposition

$$\langle 1234\rangle = \sum_{\kappa,L} \langle 1234\rangle_{\kappa L} \equiv \sum_{\kappa,L} \langle 0|\ \mathcal{L}(z_1)\,\mathcal{L}(z_2)\ \Pi_{\kappa L}\ \mathcal{L}(z_3)\mathcal{L}(z_4)\ |0\rangle \tag{1.7}$$

where $\Pi_{\kappa L}$ is the projection onto the irreducible positive-energy representation of the conformal group $SU(2,2)$ with $U(1) \times SU(2) \times SU(2)$ weight $(2\kappa + L, \frac{1}{2}L, \frac{1}{2}L)$, i.e., the representation by a rank L symmetric traceless tensor of scaling dimension $2\kappa + L$. Except for the vacuum contribution ($\kappa = 0$, $L = 0$) only positive even twists 2κ and nonnegative even spins L appear in the expansion of the product of two identical neutral scalar fields [NST03].

Each partial wave has the form

$$\langle 1234\rangle_{\kappa L} = \langle 12\rangle\langle 34\rangle \cdot B_{\kappa L} \cdot \beta_{\kappa L}(s,t) \tag{1.8}$$

where $\beta_{\kappa L}(s,t)$ are universal functions kinematically determined by the representation $\left(2\kappa + L,\ \frac{1}{2}\,L,\ \frac{1}{2}\,L\right)$. Only the coefficients $B_{\kappa L}$ are specific for the field $\mathcal{L}$ and characterize its couplings to twist 2κ spin L fields present in the theory. We shall compute all partial wave coefficients $B_{\kappa L}$ of the 4-point function (1.5) in closed form (Appendix B).

Wightman positivity of the 4-point function is equivalent to the positivity of the infinite series of coefficients $B_{\kappa L}$ (Section 3). It turns out that only two of the six structures entering (1.5) are separately positive. Nevertheless positivity is satisfied for the closure of a non-empty open set in the five-dimensional projective parameter space.

Of course, the discussion of positivity has to be extended beyond the 4-point level, where we have at this moment only partial control. For the case of a scalar field of scaling dimension 2, a complete classification has been obtained [NST02, N-un] by solving a moment problem involving the parameters of arbitrary n-point functions. In the case of two-dimensional CFT, a similar procedure implicitly extending to higher n-point functions by exploiting the presence of the stress-energy tensor in the operator product expansion of a neutral field with itself, has been demonstrated to yield the Friedan-Qiu-Shenker quantization of the central charge and the scaling dimensions [RS88].

In the present case, we shall advance the analysis by assuming that the $d = 4$, $L = 2$ field appearing in the OPE of two $\mathcal{L}$'s coincides with the stress-energy tensor $T_{\alpha\beta}$. This allows us to relate, in Section 4, the amplitude c of the (unique up to a factor) 2-point function of T with three parameters appearing in the 4-point function (2.1) of $\mathcal{L}$. Similar relations, providing information about the stress-energy tensor from the OPE of other fields, have been known (and exploited for various purposes) before, e.g., [P96, AFP00, AEPS02].

We have to be aware of the possibility of the presence of free fields in the theory. Especially in the case when the OPE of $\mathcal{L}$ with itself contains a scalar field of dimension 2, we know ([NST02, N-un], see above) that the subtheory generated by the latter can be represented by the even Wick polynomials of a massless free field theory. In this situation, the general structure theory of subsystems and local extensions (e.g., [CC05, Sect. 5]) suggests that the local algebras of the full theory are contained in the tensor product of the local algebras of the massless free field theory and those of a decoupled theory

$$F(O) \subset A(O) \otimes \widehat{F}(O)\,. \tag{1.9}$$

A field theoretic interpretation of this formula would be a factorization with the general structure

$$\mathcal{L} = :\varphi^4: + \varphi\,\widehat{X} + \widehat{\mathcal{L}} \tag{1.10}$$

where the scalar fields $\widehat{X}$ of dimension 3 and $\widehat{\mathcal{L}}$ of dimension 4 are local fields associated with the decoupled theory. In particular, the OPE of $\widehat{\mathcal{L}}$ generates no scalar field of dimension 2. Without loss of generality, one may then assume from the outset that $\mathcal{L}$ generates no scalar field of dimension 2, thus effectively decoupling the free field from the theory.

Unfortunately, neither do we have in our approach sufficient control over the technical assumptions underlying (1.9), nor is there a precise derivation of (1.10) from (1.9). For this reason, we attempt to obtain more direct support for the structure (1.10) by studying the field content through iterated OPE's. This is the motivation for our analysis in Section 5, where we draw some first consequences of the analysis of the 6-point function (sketched in Appendix C) whose systematic investigation is postponed to future work.

Our analysis is based to a large extent on a systematic study of positivity of mixed 4-point functions involving two pairs of scalar fields. We demonstrate in particular, that only one among the four possible 6-point function structures of $V_1(z_1, z_2)$, that do not vanish for coinciding arguments, survives. The more interesting (but also more complicated) four additional 6-point structures will be studied in a separate publication.

Free field constructions, which are known to satisfy positivity, only fill a discrete (measure zero) subset of our positivity domain, thus leaving room for a nontrivial theory (Appendix A).

2 Partial wave expansion

The most general 4-point function of a neutral scalar field of scaling dimension $d = 4$ in four space-time dimensions satisfying GCI is given in the form (1.5) where the polynomial P_5 is parametrized by

$$P_5(s,t) = \sum_{\nu=0}^{2} a_\nu \cdot J_\nu(s,t) + st\,[\,b \cdot D(s,t) + b' \cdot Q(s,t)\,]. \tag{2.1}$$

Here the three structures $J_\nu(s,t)$ are given by

$$\begin{aligned} J_0(s,t) &= (t^2+t^3)+s^2(1+t^3)+s^3(1+t^2), \\ J_1(s,t) &= (t+t^4)-(t^2+t^3)+s(1+t^4)-2s(t+t^3)- \\ &\quad -s^2(1+t^3)-s^3(1+t^2)-2s^3t+s^4(1+t), \\ J_2(s,t) &= (1+t^5)-2(t+t^4)+(t^2+t^3)-2s(1+t^4)+s(t+t^3)+ \\ &\quad +s^2(1+t^3)+s^3(1+t^2)+s^3t-2s^4(1+t)+s^5. \end{aligned} \tag{2.2}$$

Locality dictates their symmetry under $[s_{13}f](s,t)=f(t,s)$ and $[s_{23}f](s,t)=s^5f(1/s,t/s)$. The origin of these structures [NST03] are the (harmonic) s-channel twist 2 contributions

$$\langle 0|\ V_1(z_1,z_2)V_1(z_3,z_4)\ |0\rangle = \frac{1}{z_{13}^2 z_{24}^2}\cdot\sum_{\nu=0}^{2} a_\nu\cdot j_\nu(s,t) \tag{2.3}$$

where V_1 is the twist 2 bilocal field introduced in (1.2), and

$$\begin{aligned} j_0 &= t^{-1}(1+t), \\ j_1 &= t^{-2}[(1-t-t^2+t^3)-s(1+t^2)], \\ j_2 &= t^{-3}[(1-2t+t^2+t^3-2t^4+t^5)+s(-2+t+t^3-2t^4)+s^2(1+t^3)], \end{aligned} \tag{2.4}$$

such that $j_\nu/z_{13}^2z_{24}^2$ have pole singularities in z_{ij}^2 ($i=1,2$; $j=3,4$) of degree $\nu+1$. The structures (2.2) then arise by symmetrization of the resulting contributions to the full 4-point function.

The last two terms in (2.1) contribute to twist 4 (and higher) partial waves: D and Q are degree two symmetric polynomials, D coinciding with the discriminant of the system (B.6) below:

$$\begin{aligned} D(s,t) &= 1-2t+t^2-2s(1+t)+s^2, \\ Q(s,t) &= t+s(1+t). \end{aligned} \tag{2.5}$$

The partial wave expansion of the 4-point function reads

$$\langle 1234\rangle = \langle 12\rangle\langle 34\rangle\cdot\sum_{\kappa\geq 0, L\geq 0} B_{\kappa L}\cdot\beta_{\kappa L}(s,t). \tag{2.6}$$

where the partial waves $\beta_{\kappa L}$ are fixed kinematically by the representation theory of the conformal group [DMPPT77]. In contrast, the coefficients $B_{\kappa L}$ contain dynamical information about the couplings of the fields in the model.

The partial wave of the identity operator is $\beta_{00}=1$, hence the leading term 1 in (1.5) corresponds to $B_{00}=1$. The expansion of the remaining terms is equivalent to

$$s^4+s^4t^{-4}+st^{-3}P_5(s,t)=\sum_{\kappa L} B_{\kappa L}\cdot\beta_{\kappa L}(s,t). \tag{2.7}$$

The coefficients $B_{\kappa L}$ of this expansion are explicitly computed by a method described in Appendix B, in terms of the parameters a_ν, b and b' specifying the truncated 4-point function $P_5(s,t)$, (2.1). To obtain them, it was essential to combine the presentation of the partial waves given in closed form by Dolan and Osborn in terms of "chiral variables" [DO01] with a new expansion formula presented in the Appendix. The result is that only even $L\geq 0$ occur, as expected, and

$$\tfrac{1}{2}\binom{2L}{L}B_{1L} = a_0+L(L+1)a_1+\tfrac{1}{4}(L-1)L(L+1)(L+2)a_2, \tag{2.8}$$

while for even $\kappa \geq 2$ (twist 4, 8, ...) we obtain

$$\begin{aligned}\frac{1}{2}\binom{2\kappa+2L-2}{\kappa+L-1}\binom{2\kappa-4}{\kappa-2}B_{\kappa L} &= (c^4_{\kappa+L}c^3_{\kappa-1} - c^3_{\kappa+L}c^4_{\kappa-1}) + a_0\cdot(2c^3_{\kappa+L}c^2_{\kappa-1} - 2c^2_{\kappa+L}c^3_{\kappa-1} + 2c^3_{\kappa-1}) - \\ &- a_1\cdot(c^3_{\kappa+L}c^2_{\kappa-1} + c^2_{\kappa+L}c^3_{\kappa-1} - c^3_{\kappa-1} - c^3_{\kappa+L}) + a_2\cdot c^3_{\kappa+L}c^3_{\kappa-1} \\ &+ b\cdot(c^2_{\kappa+L}c^2_{\kappa-1} + c^2_{\kappa+L} + c^2_{\kappa-1} - 2) - b'\cdot(2c^2_{\kappa-1} - 1), \end{aligned} \tag{2.9}$$

and for odd $\kappa \geq 3$ (twist 6, 10, ...)

$$\begin{aligned}\frac{1}{2}\binom{2\kappa+2L-2}{\kappa+L-1}\binom{2\kappa-4}{\kappa-2}B_{\kappa L} &= (c^4_{\kappa+L}c^3_{\kappa-1} - c^3_{\kappa+L}c^4_{\kappa-1}) + a_0\cdot(2c^3_{\kappa+L}c^2_{\kappa-1} - 2c^2_{\kappa+L}c^3_{\kappa-1} - 2c^3_{\kappa+L}) + \\ &+ a_1\cdot(c^3_{\kappa+L}c^2_{\kappa-1} + c^2_{\kappa+L}c^3_{\kappa-1} - c^3_{\kappa-1} - c^3_{\kappa+L}) - a_2\cdot c^3_{\kappa+L}c^3_{\kappa-1} - \\ &- b\cdot(c^2_{\kappa+L}c^2_{\kappa-1} + c^2_{\kappa+L} + c^2_{\kappa-1} - 2) + b'\cdot(2c^2_{\kappa+L} - 1), \end{aligned} \tag{2.10}$$

where c^p_ν vanish for $\nu < p$, and otherwise

$$c^p_\nu = \frac{1}{(p-1)!^2}\frac{(\nu+p-2)!}{(\nu-p)!} > 0. \tag{2.11}$$

Of special interest are the coefficients $B_{10} = 2a_0$ and $B_{20} = b'$, indicating the presence of a scalar field of dimension 2 and of a scalar field of dimension 4 (e.g., the field $\mathcal{L}$ itself), respectively, in the operator product expansion of $\mathcal{L}$ with itself, and $B_{12} = \frac{1}{3}a_0 + 2a_1 + 2a_2$ which includes the contribution of the stress-energy tensor $\Theta^{\mu\nu}$. As we shall see in Sect. 4, the value of B_{12} gives a lower bound on the amplitude of the 2-point function of $\Theta^{\mu\nu}$.

3 Wightman positivity

We want to study the constraints on the amplitudes, deriving from Wightman positivity. For the 2-point function, positivity requires $B_{\mathcal{L}} \geq 0$.

For the 4-point function, Wightman positivity has already been partially exploited by the maximal degree of singularities of the truncated 4-point function. This excludes partial waves corresponding to non-unitary representations of the conformal group. To get further insight, we note that each partial wave (1.8) amounts to the insertion of a projection $\Pi_{\kappa L}$ into the 4-point function. Because Hilbert space positivity implies positivity of the quadratic forms $\langle 0|\ \cdot \Pi_{\kappa L}\cdot\ |0\rangle$, each partial wave must separately satisfy Wightman positivity. Since all $\beta_{\kappa L}$ occur with positive coefficients in the partial wave expansion of the 4-point function of the massless scalar free field (which is certainly positive), we know that $\beta_{\kappa L}$ are separately positive. Hence Wightman positivity of the general 4-point function is equivalent to

$$B_{\kappa L} \geq 0 \qquad \forall \kappa, L. \tag{3.1}$$

We see from (2.8–2.10) that the disconnected part as well as the structure J_0 involve separately only positive partial wave coefficients, while J_1 and Q involve negative coefficients at even κ, while J_2 and D involve negative coefficients at odd κ. To have a positive 4-point function, all negative contributions of these structures must be dominated by positive contributions from other structures, giving relative bounds between the amplitudes a_ν and b, b'.

For $\kappa = 1$, (2.8) must be nonnegative for all L:

$$a_0 + L(L+1)a_1 + \tfrac{1}{4}(L-1)L(L+1)(L+2)a_2 \geq 0. \tag{3.2}$$

At $\kappa = 2$, all terms in (2.9) involving $c^p_{\kappa-1}$ $(p \geq 2)$ vanish, hence

$$\tfrac{1}{4}L(L+1)(L+2)(L+3)a_1 + L(L+3)b + b' \geq 0. \tag{3.3}$$

For $L = 0$ and for $L \to \infty$, these two conditions imply

$$a_\nu \geq 0 \quad (\nu = 0, 1, 2) \qquad \text{and} \qquad b' \geq 0. \tag{3.4}$$

Note that $2a_0$ and b' are the coefficients B_{10} and B_{20} of the partial waves due to scalar fields of dimension 2 and 4, respectively, in the operator product expansion.

For intermediate L, one obtains from $\kappa = 2$ a negative lower bound on b depending on the values of a_1 and b':

$$b \geq -\min_{L \text{ even}} \left\{ \frac{(L+1)(L+2)}{4} \cdot a_1 + \frac{1}{L(L+3)} \cdot b' \right\}. \tag{3.5}$$

If $a_1 = 0$, then $b \geq 0$. If $b' = 0$, then $b \geq -3a_1$.

Similarly, $\kappa = 3$ gives

$$\tfrac{1}{4}(L+1)(L+2)(L+3)(L+4)(2a_0 + a_1) + (L+2)(L+3)(2b' - 3b) - b' \geq 0. \tag{3.6}$$

These are upper bounds on b of which, in view of (3.4), $L = 0$ is the strongest one:

$$b \leq \tfrac{1}{3}(2a_0 + a_1) + \tfrac{11}{18}b'. \tag{3.7}$$

Further upper bounds arise on a_1, b' at even $\kappa \geq 4$, and on a_2, b at odd $\kappa \geq 5$. They are always relative to the respective remaining amplitudes and to the coefficient $B_{00} = 1$ of the disconnected terms $s^4 + (s/t)^4$ in (2.7). Since at large angular momentum and large twist, the disconnected contributions dominate all partial wave coefficients, the strongest bounds will arise from small values of L and κ, e.g.,

$$\begin{aligned} 216a_1 + 11b' &\leq 120 + 228a_0 + 180a_2 + 88b \qquad (\kappa = 4, L = 0), \\ 900a_2 + 90b &\leq 800 + 980a_0 + 880a_1 + 13b' \qquad (\kappa = 5, L = 0). \end{aligned} \tag{3.8}$$

All bounds (3.4–3.8) taken together, leave a closed admitted region in the space of the five amplitudes, with a nontrivial open interior. Wightman positivity for mixed vectors spanned by $\mathcal{L}(\cdot)\Omega$ and $\mathcal{L}(\cdot)\mathcal{L}(\cdot)\Omega$ gives rise to an upper bound on the square of the 3-point amplitude, relative to the coefficient $B_{20} = b'$, which we don't display. Clearly, if $B_{20} = 0$ then the 3-point function must vanish.

More decisive bounds are expected, however, beyond the 4-point level. Although these involve more and more new amplitude parameters, it is expected (e.g., from experience with two-dimensional models) that the resulting inequalities also put further constraints on the 4-point amplitudes alone.

4 Twist two contribution, the stress-energy tensor and the central charge

The importance of assuming the existence of a stress-energy tensor $\Theta_{\mu\nu}(x)$, $\mu, \nu = 0, 1, 2, 3$, in the axiomatic approach to conformal field theory has been recognized long ago ([MS72]; see also [M88]). It is a (conserved, symmetric, traceless rank 2) tensor field of *twist 2* (dimension 4), whose z-picture counterpart will be written in the form

$$T(z, v) := T_{\alpha\beta}(z)\, v^\alpha\, v^\beta\,, \qquad \text{so that} \quad (\partial_z \cdot \partial_v)\, T(z, v) = 0 = \partial_v^2\, T(z, v)\,, \tag{4.1}$$

($\alpha, \beta = 1, 2, 3, 4$); by definition it gives rise to the generators of infinitesimal conformal transformations. If a field T with such properties exists then it should appear in the OPE of the product of any local field ψ with its conjugate, ψ^*; in particular, in the product of any hermitean scalar field ϕ with itself. Thus, the partial wave coefficient $B_{12} = \frac{1}{3}\, a_0 + 2\, a_1 + 2\, a_2$ (2.8) should be strictly positive.

The normalization of the stress-energy tensor is fixed by the condition that its integrals give the generators of the conformal symmetry. While it is customary to use to this end the Minkowski energy-momentum components $P_\mu = \int \Theta_{\mu 0}(x) d^3x$, which can be defined as bilinear forms for rational correlation functions, it is technically simpler to introduce instead the *conformal Hamiltonian* given by the integral over the unit 3-sphere in the z-picture

$$H = \int_{\mathbb{S}^3} T(u,u)(du) \quad \text{for } z_\alpha = e^{2\pi i \zeta}\, u_\alpha\,, \qquad (du) = \delta(u^2-1)\, \frac{d^4\, u}{2\,\pi^2}\,. \tag{4.2}$$

The conformal dimension d of a local field $\phi(z)$ coincides with the minimal eigenvalue of the conformal Hamiltonian on the "1-particle space" Span $\{\phi(z)\ |0\rangle\}$; we have

$$[H, \phi(z)] = (d + z\cdot\partial_z)\,\phi(z) \qquad \text{implying} \quad (H-d)\,\phi(0)\ |0\rangle = 0 \tag{4.3}$$

(since $H\ |0\rangle = 0$).

The ratio of the mixed 3-point function to the 2-point function of the hermitean scalar field ϕ of dimension d is uniquely determined by conformal invariance, the normalization being fixed by (4.3) with H given by (4.2):

$$\langle 0|\ \phi(z_1)\, T(z_2,v)\, \phi(z_3)\ |0\rangle = d\cdot \langle 0|\ \phi(z_1)\, \phi(z_3)\ |0\rangle\, z_{13}^2\, w_3^{(1)}(z_1; z_2, v; z_3) \tag{4.4}$$

where

$$(z_{12}^2)^d\, \langle 0|\ \phi(z_1)\, \phi(z_2)\ |0\rangle = \langle d \mid d\rangle = B_\phi > 0 \tag{4.5}$$

and $w_3^{(1)} = \langle 0|\ \varphi \tilde{T} \varphi\ |0\rangle$ is the 3-point function of a (dimension 1) free scalar field φ with its stress-energy tensor (eq.(4.18)):

$$w_3^{(1)}(z_1; z_2, v; z_3) = \frac{2}{3\, z_{12}^2\, z_{23}^2} \left\{ (2X_{13}^2\cdot v)^2 - v^2\, \frac{z_{13}^2}{z_{12}^2\, z_{23}^2} \right\}, \qquad X_{13}^2 := \frac{z_{23}}{z_{23}^2} + \frac{z_{12}}{z_{12}^2}\,. \tag{4.6}$$

The ket vector $|d\rangle$ (and its dual bra vector), appearing in (4.5), is defined by the analytic continuation of $\phi(z)\ |0\rangle$ ($\langle 0|\ \phi(z)$):

$$|d\rangle := \phi(0)\ |0\rangle\,, \qquad \langle d| := \lim_{w\to 0} \left\{ (w^2)^{-d}\, \langle 0|\ \phi\left(\frac{w}{w^2}\right) \right\} = \lim_{z\to\infty} \left\{ (z^2)^d\, \langle 0|\ \phi(z) \right\},$$

so that Eq. (4.4) implies the relation

$$\langle d|\ T(z,v)\ |d\rangle = \frac{2d}{3}\, \langle d \mid d\rangle \left\{ 4\, \frac{(z\cdot v)^2}{(z^2)^3} - \frac{v^2}{(z^2)^2} \right\}. \tag{4.7}$$

This gives, in accord with (4.2) and (4.3),

$$\tfrac{1}{2}\langle d|\ T(u,u)\ |d\rangle = d\,\langle d \mid d\rangle = \int_{\mathbb{S}^3} \langle d|\ T(u,u)\ |d\rangle\,(du) = \langle d|\ H\ |d\rangle\,. \tag{4.8}$$

Conversely, the 3-point function (4.4), (4.6) can be restored from (4.7) by a suitable (complex) conformal transformation. The 3-point function for any other order of factors is obtained from (4.4) by using locality. Applying the result to $\phi = \mathcal{L}$ ($d=4$) and inserting the expansion (1.2) for the product of two $\mathcal{L}$'s we find

$$\langle 0|\ V_1(z_1,z_2)\, T(z_3,v)\ |0\rangle = 4\, w_3^{(1)}(z_1; z_2; z_3, v) := \frac{8}{3\, z_{13}^2\, z_{23}^2} \left\{ (2\, X_{12}^3\cdot v)^2 - v^2\, \frac{z_{12}^2}{z_{13}^2\, z_{23}^2} \right\} \tag{4.9}$$

where $X_{12}^3 = \frac{z_{13}}{z_{13}^2} - \frac{z_{23}}{z_{23}^2}$.

We shall now explore the assumption that the traceless conserved symmetric tensor of dimension 4 appearing in the OPE of $\mathcal{L}(z_1)\,\mathcal{L}(z_2)$ (or, equivalently in the expansion of $V_1(z_1,z_2)$) is proportional to the stress energy tensor T. (The relation between the 4-point function of a basic field and the 2-point function of the stress-energy tensor has been exploited before [P96] in the context of a conformally invariant $O(N)$ model in $2 < D < 4$ space-time dimensions.)

The relevant tensor in the OPE is isolated by applying the differential operator

$$\mathcal{D}_{12}(v) := \tfrac{1}{6}\,\{(v\cdot\partial_{z_1})^2 + (v\cdot\partial_{z_2})^2 - 4\,(v\cdot\partial_{z_1})\,(v\cdot\partial_{z_2}) + v^2\,(\partial_{z_1}\cdot\partial_{z_2})\} \tag{4.10}$$

to $V_1(z_1,z_2)$ and equating $z_1 = z_2$. (The resulting tensor of dimension 4 is symmetric, traceless and conserved due to the structure of $\mathcal{D}_{12}$ and due to the fact that $V_1(z_1,z_2)$ is harmonic.) We thus require that

$$T(z,v) = \gamma\cdot\mathcal{D}_{12}(v)V_1(z_1,z_2)|_{z_1=z_2=z} \tag{4.11}$$

with some factor of proportionality γ. We claim that $\gamma = \frac{4}{a_0+6\,a_1+6\,a_2}$.

Namely, applying the operator $\mathcal{D}_{34}(v)$ to the 4-point function $\langle 0|\ V_1(z_1,z_2)V_1(z_3,z_4)\ |0\rangle$ given by (2.3) and equating $z_3 = z_4$ we obtain

$$\mathcal{D}_{34}(v)\,\{\langle 0|\ V_1(z_1,z_2)\,V_1(z_3,z_4)\ |0\rangle\}|_{z_4=z_3} = (a_0 + 6\,a_1 + 6\,a_2)\cdot w_3^{(1)}(z_1;z_2;z_3,v)\,, \tag{4.12}$$

where $w_3^{(1)}$ is defined in (4.9). Comparing with the required normalization (4.9) of T, the factor of proportionality is determined.

Applying also the operator $\mathcal{D}_{12}(v_1)$ to (4.12) and equating $z_1 = z_2$, we finally arrive at

$$\langle 0|\ T(z_1,v_1)\,T(z_2,v_2)\ |0\rangle = c\cdot w_2^{(1)}(z_1,v_1;z_2,v_2) \tag{4.13}$$

where

$$w_2^{(1)}(z_1,v_1;z_2,v_2) = \frac{4}{3\,(z_{12}^2)^4}\,\{(2\,v_1\,r(z_{12})\,v_2)^2 - v_1^2\,v_2^2\}\,, \tag{4.14}$$

$$r(z) = \mathbb{I} - 2\,\frac{z\otimes z}{z^2} \qquad \text{i.e.} \quad r_{\alpha\beta}(z) = \delta_{\alpha\beta} - 2\,\frac{z_\alpha\,z_\beta}{(z^2)_+}\,, \tag{4.15}$$

and the coefficient is

$$c = \frac{16}{a_0 + 6\,a_1 + 6\,a_2}\,. \tag{4.16}$$

In fact, because T is a symmetric traceless and conserved rank 2 tensor field of dimension 4, its 2-point function is necessarily of the form (4.13), and the proportionality coefficient c can be viewed as a four-dimensional generalization of the central charge[2] of the Virasoro algebra.

In general, when the proportionality (4.11) fails, then also the resulting relation (4.16) between the amplitude of the 2-point function (4.13) and the coefficients of the 4-point function (2.3) fails. The contribution to the OPE of $\mathcal{L}\cdot\mathcal{L}$ orthogonal to the stress-energy tensor will add to the partial wave coefficient $B_{12} = (a_0 + 6a_1 + 6a_2)/3$, hence one has in general

$$a_0 + 6a_1 + 6a_2 \geq \frac{16}{c} \qquad \Leftrightarrow \qquad c \geq \frac{16}{a_0 + 6a_1 + 6a_2}. \tag{4.17}$$

The saturation of this bound, i.e. (4.16), is equivalent to our stress-energy tensor condition (4.11). This condition also excludes the presence of generalized free fields (see Appendix A).

[2]It corresponds to the "primary central charge" as discussed in the literature – see [A98] and references therein.

Let us now determine c for the stress-energy tensors of free fields. For the stress-energy tensor

$$\begin{aligned} T_\varphi(z,v) &:= D_{12}(v) :\varphi(z_1)\,\varphi(z_2):|_{z_1=z_2=z} \\ &= \tfrac{1}{3} :\{\varphi(z)\,(v\cdot\partial_z)^2\,\varphi(z) - 2\,[v\cdot\partial_z\,\varphi(z)]^2 + \tfrac{1}{2}\,v^2\,\partial_z\,\varphi(z)\cdot\partial_z\,\varphi(z)\}:, \end{aligned} \tag{4.18}$$

of a free massless scalar field φ (of dimension $d=1$, $\langle 0|\ \varphi(z_1)\varphi(z_2)\ |0\rangle = (z_{12}^2)^{-1}$), we find the 2-point function (4.14), i.e., $c_\varphi = 1$.

For a pair ψ, ψ^* of conjugate (2-component) Weyl spinor fields of dimension 3/2 and 2-point function

$$\langle 0|\ \psi(z_1)\,\psi^*(z_2)\ |0\rangle\,(\equiv \langle 0|\ \psi(z_1)\otimes\psi^*(z_2)\ |0\rangle) = \frac{2\,\not{z}_{12}^{+}}{(z_{12}^2)^2}\,,\quad \not{v}^{+} = \sum_{\alpha=1}^{4} Q_\alpha^{+}\,v_\alpha = \begin{pmatrix} v_4+iv_3 & iv_1+v_2 \\ iv_1-v_2 & v_4-iv_3 \end{pmatrix} \tag{4.19}$$

($Q_j = -\,i\,\sigma_j$ being the imaginary quaternion units, $Q_j^{+} = -\,Q_j$, $j=1,2,3$) the stress energy tensor $T_\psi(z,v)$ is given by

$$T_\psi(z,v) = -\tfrac{1}{2}\,(v\cdot\partial_{z_1})\,(v\cdot\partial_{z_2})\,V_1^{\psi}(z_1,z_2)|_{z_1=z_2=z} = \tfrac{1}{4} :\{v\cdot\partial_z\,\psi^*(z))\,\not{v}\,\psi(z) - \psi^*(z)\,\not{v}\,v\cdot\partial_z\,\psi(z)\}: \tag{4.20}$$

where the Weyl field contribution to the twist 2 bilocal field V_1,

$$V_1^{\psi}(z_1,z_2) = \tfrac{1}{4} :(\psi^*(z_1)\,\not{z}_{12}\,\psi(z_2) - \psi^*(z_2)\,\not{z}_{12}\,\psi(z_1)):, \tag{4.21}$$

reproduces the term proportional to $j_1(s,t)$ in the 4-point function (2.3)

$$\langle 0|\ V_1^{\psi}(z_1,z_2)\,V_1^{\psi}(z_3,z_4)\ |0\rangle = \frac{j_1(s,t)}{2\,z_{13}^2\,z_{24}^2}\,. \tag{4.22}$$

where $V_1(z_1,z_2)$ is the bilocal twist 2 field in the OPE of $\mathcal{L}$ with itself, defined as in eq. (1.2) with $\phi = \mathcal{L}$. Repeated application of (4.20) gives

$$\langle 0|\ V_1^{\psi}(z_1,z_2)\,T_\psi(z_3,v)\ |0\rangle = 3\,w_3^{(1)}(z_1;z_2;z_3,v) \tag{4.23}$$

and

$$\langle 0|\ T_\psi(z_1,v_1)\,T_\psi(z_2,v_2)\ |0\rangle = 3\,w_2^{(1)}(z_1,v_1;z_2,v_2) \qquad \text{i.e.}\quad c_\psi = 3\,. \tag{4.24}$$

The factor 3 in (4.23) as compared to 4 in (4.9) is correct because V_1^{ψ} arises from the dimension 3 field $\psi^*\psi$. The fact that T_ψ (4.20) is properly normalized (*i.e.* satisfies (4.8)) can also be read off the 3-point function

$$\langle 0|\ \psi(z_1)\,T_\psi(z_2,v)\,\psi^*(z_3)\ |0\rangle = \frac{1}{(z_{12}^2\,z_{23}^2)^2}\,\{(2v\cdot X_{13}^2)\,\not{z}_{12}^{+}\,\not{v}\,\not{z}_{23}^{+} - v^2\,\not{z}_{13}^{+}\}\,. \tag{4.25}$$

For the free electromagnetic field $F_{\alpha\beta}(z)$ with 2-point function

$$\langle 0|\ F_{\alpha_1\beta_1}(z_1)\,F_{\alpha_2\beta_2}(z_2)\ |0\rangle = 4(z_{12}^2)^{-2}\{r_{\alpha_1\alpha_2}(z_{12})\,r_{\beta_1\beta_2}(z_{12}) - r_{\alpha_1\beta_2}(z_{12})\,r_{\beta_1\alpha_2}(z_{12})\} \tag{4.26}$$

where $r(z)$ is defined in (4.16) (and satisfies $[r(z)]^2 = \mathbb{1}$, $r(z)\,z = -z$) and

$$\mathcal{L}^F(z) = -\tfrac{1}{4} :F_{\alpha\beta}(z)\,F_{\alpha\beta}(z): \tag{4.27}$$

we find (according to [NST03])

$$\mathcal{L}^F(z_1)\,\mathcal{L}^F(z_2) = \frac{48}{(z_{12}^2)^4}\,\{1 + z_{12}^2\,V_1^F(z_1,z_2) + O((z_{12}^2)^8)\} \tag{4.28}$$

with

$$V_1^F(z_1,z_2) \;=\; \tfrac{1}{24}\, z_{12}^2 \cdot r_{\alpha_1\alpha_2}(z_{12})\, r_{\beta_1\beta_2}(z_{12}) :F_{\alpha_1\beta_1}(z_1)\, F_{\alpha_2\beta_2}(z_2): =$$
$$= \tfrac{1}{24} :\{ z_{12}^2\, F_{\sigma\tau}(z_1)\, F_{\sigma\tau}(z_2) + 4\, z_{12}^{\alpha}\, F_{\alpha\sigma}(z_1)\, F_{\sigma\beta}(z_2)\, z_{12}^{\beta}\}: . \tag{4.29}$$

This implies

$$\langle 0|\; V_1^F(z_1,z_2)\, V_1^F(z_3,z_4)\; |0\rangle = \frac{2}{9}\, \frac{j_2(s,t)}{z_{13}^2\, z_{24}^2}\, . \tag{4.30}$$

Applying (4.12) to (4.30) for $a_0 = a_1 = 0$, $a_2 = \frac{2}{9}$, we find

$$\langle 0|\; V_1^F(z_1,z_2)\, T_F(z_3,v)\; |0\rangle = 3\, \mathcal{D}_{34}(v) \left\{ \frac{2}{9}\, \frac{j_2(s,t)}{z_{13}^2\, z_{24}^2} \right\} \Bigg|_{z_4=z_3} = 4\, w_3^{(1)}(z_1;z_2;z_3,v)\, , \tag{4.31}$$

and hence,

$$\langle 0|\; T_F(z_1,v_1)\, T_F(z_2,v_2)\; |0\rangle = 3\, \mathcal{D}_{01}(v_1)\, \langle 0|\; V_1^F(z_0,z_1)\, T_F(z_2,v_2)\; |0\rangle|_{z_0=z_1} = 12\, w_2^{(1)}(z_1,v_1;z_2,v_2)\, . \tag{4.32}$$

To summarize, the three basic free field theories reproduce the three structures in the general conformal invariant 3-point function of T (see [S88]), as well as the three contributions $j_a(s,t)$, $a = 0,1,2$, to the twist 2 part of the 4-point function of $\mathcal{L}$. For the "central charge" c defined in terms of the 2-point function (4.13) of T, we find the following values:

$$\langle 0|\; T_X(z_1,v_1)\, T_X(z_2,v_2)\; |0\rangle = c_X \cdot w_2^{(1)}(z_1,v_1;z_2,v_2)\, ,\;\; c_\varphi = 1\, ,\;\; c_\psi = 3\, ,\;\; c_F = 12\, . \tag{4.33}$$

(For a Dirac spinor, one gets twice the value for the Weyl spinor.)

Remark 4.1: In Minkowski space notation the generic 2-point function of the stress energy tensor $\Theta_{\mu\nu}(x)$, the counterpart of (4.33), has the form

$$\langle 0|\; \Theta_{\kappa\lambda}(x_1)\, \Theta_{\mu\nu}(x_2)\; |0\rangle = \frac{8\, c}{3\,(2\,\pi)^4}\, (x_{12}^2)_+^{-4} \left(r_{\kappa\mu}(x_{12})\, r_{\lambda\nu}(x_{12}) + r_{\kappa\nu}(x_{12})\, r_{\lambda\mu}(x_{12}) - \frac{1}{2}\, \eta_{\kappa\lambda}\, \eta_{\mu\nu} \right)$$

where

$$r_{\lambda\mu}(x_{12}) = \eta_{\lambda\mu} - 2\, \frac{(x_{12})_\lambda\, (x_{12})_\mu}{(x_{12}^2)_+}\, , \qquad (x_{12}^2)_+ = \mathbf{x}_{12}^2 - (x_{12}^0 - i\,\varepsilon)^2\, .$$

5 Positivity restrictions on the six-point function of V_1 and factoring the $d = 2$ contribution

The positivity conditions for the 4-point functions restrict the possible structures of the 6-point function. We shall demonstrate this studying the four structures of $\langle 0|\; V_1(z_1,z_2)\, V_1(z_3,z_4)\, V_1(z_5,z_6)\; |0\rangle$ displayed in Appendix C for which the local field $\phi(z) = \frac{1}{2}\, V_1(z,z)$ of dimension 2 has a non-vanishing contribution. Such structures can only appear if a_0 in (2.1) is non-zero as (noting that $j_\nu(0,1) = 2\,\delta_{\nu 0}$)

$$\phi(z) := \frac{1}{2}\, V_1(z,z) \quad \Rightarrow \quad \langle 0|\; \phi(z_1)\, \phi(z_2)\; |0\rangle = \frac{a_0}{2\,(z_{12}^2)^2}\, . \tag{5.1}$$

The 6-point functions F_1, $F_2^{(1)}$, $F_3^{(1)}$ and $F_3^{(2)}$ of Appendix C contribute to the GCI 4-point function

$$z_{34}^2\, \langle 0|\; V_1(z_1,z_2)\, \phi(z_3)\, \phi(z_4)\; |0\rangle = \frac{1}{z_{13}^2\, z_{24}^2} \sum_{\nu=0}^{2} A_\nu\, j_\nu(s,t) \;=\; \langle 0|\; \phi(z_1)\, \phi(z_2)\, V_1(z_3,z_4)\; |0\rangle\, z_{12}^2 \tag{5.2}$$

with certain (real) coefficients A_ν. We conclude from (5.2) that the 3-point function of ϕ is given by

$$\langle 0|\,\phi(z_1)\,\phi(z_2)\,\phi(z_3)\,|0\rangle = \frac{A_0}{z_{12}^2\, z_{23}^2\, z_{13}^2}\,. \tag{5.3}$$

In what follows we shall combine results of earlier work with implications of Hilbert space positivity on the mixed 4-point functions like (5.2), in order to restrict the number and the values of the parameters involved in the above 6-point function.

We first apply results of [NST02] and [N-un] concerning the subtheory generated by the field $\phi(z)$ which exploit the Wightman positivity condition for the set of *all* n-point functions of ϕ. One proves that $\phi(z)$ can be presented as a sum of a generalized free field and Wick squares of independent massless scalar fields. In the present situation, the assumed presence of a stress-energy tensor excludes the generalized free field.

More precisely, one shows (by solving a moments problem [N-un]) that $\phi(z)$ can be decomposed as:

$$\phi(z) \;=\; \sum_{k=1}^{N} \alpha_k\, \phi_k(z), \qquad \langle 0|\,\phi_j(z_1)\,\phi_k(z_2)\,|0\rangle \;=\; \frac{c_k}{2}\,\delta_{jk}\,(z_{12}^2)^{-2}, \tag{5.4}$$

where α_k are *all different*, and the absolute normalizations of the fields ϕ_k are fixed by the algebra[3]

$$\phi_j(z_1)\,\phi_k(z_2) \;=\; \delta_{jk}\left(\frac{c_k}{2(z_{12}^2)^2} + \frac{1}{z_{12}^2}\,V_1^{(k)}(z_1,z_2)\right) + :\phi_j(z_1)\,\phi_k(z_2):\,, \qquad \phi_k(z) \;=\; \frac{1}{2}\,V_1^{(k)}(z,z). \tag{5.5}$$

Alternatively, one may first define the amplitudes c_k in a normalization-independent way by the *homogeneous* relation

$$c_k\,(\langle 0|\,\phi_k(z_1)\,\phi_k(z_2)\,\phi_k(z_3)\,|0\rangle)^2 \;=\; 8\langle 0|\,\phi_k(z_1)\,\phi_k(z_2)\,|0\rangle\,\langle 0|\,\phi_k(z_2)\,\phi_k(z_3)\,|0\rangle\,\langle 0|\,\phi_k(z_1)\,\phi_k(z_3)\,|0\rangle$$

and then normalize ϕ_k as in (5.5).

Upon iteration of the OPE, arbitrary linear combinations of the fields ϕ_k can be generated, so the latter belong separately to the subtheory generated by ϕ.

Finally, the amplitudes c_k are proven to be positive integers (see Theorem 5.1 of [NST02]) and each ϕ_k can be represented, at the expense of extending the state space, as a sum of normal squares of commuting free massless fields $\{\varphi_{jk}(z) : 1 \le j \le c_k\}$,

$$\phi_k(z) \;=\; \frac{1}{2}\sum_{j=1}^{c_k} :\varphi_{jk}^2(z):\,, \qquad V_1^{(k)}(z_1,z_2) = :\varphi_k(z_1)\,\varphi_k(z_2):\,. \tag{5.6}$$

Using eqs. (5.1-5.6) one finds

$$2(z_{12}^2)^2\,\langle 0|\,\phi(z_1)\,\phi_k(z_2)\,|0\rangle \;=\; c_k\,\alpha_k, \qquad a_0 \;=\; \sum_{k=1}^{N} c_k\,\alpha_k^2\,, \qquad A_0 \;=\; \sum_{k=1}^{N} c_k\,\alpha_k^3 \tag{5.7}$$

and

$$z_{13}^2\, z_{24}^2\,\langle 0|\,V_1^{(j)}(z_1,z_2)\,V_1^{(k)}(z_3,z_4)\,|0\rangle \;=\; \delta_{jk}\,c_k\,j_0(s,t). \tag{5.8}$$

The integers c_k also determine the central charge c of (4.13) of the stress-energy tensor $T_\phi(z,v)$ of the subtheory generated by the field ϕ (or, equivalently, by ϕ_k, $k=1,\ldots,N$):

$$c_\phi \;=\; \sum_{k=1}^{N} c_k. \tag{5.9}$$

[3]This algebra is shown to hold on the vacuum state space of the subtheory generated by ϕ. By the Reeh-Schlieder theorem, it holds also on the full state space of the field $\mathcal{L}$.

Since $V_1(z_1,z_2)$ and $V_1^{(k)}(z_1,z_2)$ are built only of twist 2 contributions we have, in view of (5.5):

$$\langle 0|\, V_1(z_1,z_2)\,\phi_j(z_3)\,\phi_k(z_4)\,|0\rangle \;=\; \delta_{jk}\,(z_{34}^2)^{-1}\,\langle 0|\, V_1(z_1,z_2)\,V_1^{(k)}(z_3,z_4)\,|0\rangle. \tag{5.10}$$

By Proposition 4.3 of [NT01] the power of, say, $(z_{23}^2)^{-1}$ appearing in the 4-point function $\langle 0|\ \mathcal{L}(z_1)\ \mathcal{L}(z_2)\ \phi_k(z_3)\ \phi_k(z_4)\ |0\rangle$ does not exceed two. It follows from Eq. (5.10) and the property that the singularities of $\langle 0|\ V_1(z_1,z_2)\ \phi_k(z_3)\ \phi_k(z_4)\ |0\rangle$ do not exceed those of $\langle 0|\ \mathcal{L}(z_1)\,\mathcal{L}(z_2)\,\phi_k(z_3)\,\phi_k(z_4)\ |0\rangle$, that A_2 in (5.2) *vanishes*, and that we can write

$$\langle 0|\, V_1(z_1,z_2)\,V_1^{(k)}(z_3,z_4)\,|0\rangle \;=\; \frac{c_k\,\alpha_k\,j_0(s,t)+c_k\,A_{1k}\,j_1(s,t)}{z_{13}^2\,z_{24}^2} \;=\langle 0|\, V_1^{(k)}(z_1,z_2)\,V_1(z_3,z_4)\,|0\rangle \tag{5.11}$$

for some real constants A_{1k} ($k=1,\ldots,N$). The coefficients in front of $j_0(s,t)$ in (5.11) are fixed by (5.1) and (5.4) and (5.7) in the limits $z_1\to z_2$ and $z_3\to z_4$. The constants A_{1k} are related to A_1 of (5.2) by

$$A_1 \;=\; \sum_{k\,=\,1}^{N} c_k\,\alpha_k^2\,A_{1k}. \tag{5.12}$$

On the other hand, we observe that for a general bilocal field $W(z_1,z_2)$ which is conformal invariant of weight (κ,κ) the field

$$C_{12}^{(\kappa)}\,W(z_1,z_2) \;:=\; \left\{\tfrac{1}{2}z_{12}^2\,r_{\alpha\beta}(z_{12})\partial_{z_1\alpha}\partial_{z_2\beta}+\kappa\,z_{12}\cdot(\partial_{z_1}-\partial_{z_2})\right\}W(z_1,z_2) \tag{5.13}$$

is again a bilocal field of weight (κ,κ). This follows by the fact that $C_{12}^{(\kappa)}$ is constructed as a conformal (2-point) Casimir operator on the space of weight (κ,κ) bilocal fields. In particular, for $\kappa=1$, the conformal partial waves β_{1L} are eigenfunctions of $C_{12}:=z_{13}^2z_{24}^2\circ C_{12}^{(1)}\circ(z_{13}^2z_{24}^2)^{-1}$ of eigenvalue $L(L+1)$:

$$(C_{12}-\lambda)\,\beta_{1L}(s,t) \;=\; 0, \quad \lambda:=L(L+1)\,. \tag{5.14}$$

(The operator C_{12} has been introduced in [DO04] in order to reduce the problem of finding partial waves to an eigenvalue problem.) Accordingly, as one verifies directly,

$$C_{12}^{(1)}\;\frac{j_0(s,t)}{z_{13}^2\,z_{24}^2}\;=\;\frac{j_1(s,t)}{z_{13}^2\,z_{24}^2},\qquad C_{12}^{(1)}\;\frac{j_1(s,t)}{z_{13}^2\,z_{24}^2}\;=\;2\,\frac{j_1(s,t)+2j_2(s,t)}{z_{13}^2\,z_{24}^2}. \tag{5.15}$$

Therefore, if we introduce the bilocal field $V_1^{\perp}(z_1,z_2)$ by

$$V_1(z_1,z_2)=V_1^{\perp}(z_1,z_2)\;+\;\sum_{k\,=\,1}^{N}\Big(\alpha_k\,V_1^{(k)}(z_1,z_2)\;+\;A_{1k}\,C_{12}^{(1)}\,V_1^{(k)}(z_1,z_2)\Big) \tag{5.16}$$

then it follows that

$$\langle 0|\, V_1^{\perp}(z_1,z_2)\,V_1^{(k)}(z_3,z_4)\,|0\rangle \;=\; 0. \tag{5.17}$$

We summarize this discussion in

Proposition 5.1: *The bilocal field $V_1^{\perp}(z_1,z_2)$ is orthogonal to any product of ϕ's:*

$$\langle 0|\, V_1^{\perp}(z_1,z_2)\,\phi(z_3)\,\ldots\,\phi(z_n)\,|0\rangle \;=\; 0 \qquad (n=3,4,\ldots). \tag{5.18}$$

In other words, the bilinear form defined by $V_1^{\perp}$ vanishes on the cyclic subspace $\Pi_\phi\mathcal{H}$ generated by polynomials in ϕ acting on the vacuum:

$$\Pi_\phi\,V_1^{\perp}(z_1,z_2)\,\Pi_\phi \;=\; 0. \tag{5.19}$$

Proof. Π_ϕ is the projection on the vertex subalgebra generated by $\phi(z)$: working in this purely algebraic approach we do not encounter any domain problem. Then we note that $\Pi_\phi\, V_1^\perp(z_1,z_2)\,\Pi_\phi$ is strongly bilocal with respect to $\phi(z)$. Thus, if we prove that it vanishes acting on the vacuum, then (5.19) will follow from the Reeh-Schlieder theorem. To this end we observe that $\Pi_\phi\, V_1^\perp(z_1,z_2)\,\Omega$ is a vector containing only twist 2 contributions so that it is contained in the linear span of the 2-particle spaces $V_1^{(k)}(z_1,z_2)\,\Omega$. But $V_1^\perp(z_1,z_2)\,\Omega$ has zero projection on the latter space because of (5.17), which completes the proof.

We interpret Proposition 5.1 as a "decoupling" of the subtheory of massless free fields or their Wick products. The presence of the free bilocal fields $V_1^{(k)}$ in the OPE of $\mathcal{L}$ with itself suggests the presence of the free massless scalar fields φ as factors in $\mathcal{L}$, while the orthogonal contribution $V_1^\perp$ is due to factors which decouple from the free scalars. This indicates a structure of the general form (1.10)

$$\mathcal{L} = :\varphi^4: + \varphi\widehat{X} + \widehat{\mathcal{L}} \tag{5.20}$$

where $\widehat{X}$ and $\widehat{\mathcal{L}}$ are scalar fields whose OPE generates no scalar of dimension 2. $\widehat{\mathcal{L}}$ would then be the candidate for a nontrivial field of dimension 4. Since Prop. 5.1 does not suffice to draw such a conclusion, we try to get further insight into the structure of $\mathcal{L}$ by considering its mixed 4-point functions $\langle 0|\ \phi_i(z_1)\,\mathcal{L}(z_2)\,\mathcal{L}(z_3)\,\phi_i(z_4)\ |0\rangle$ and the OPE

$$\phi(z_1)\mathcal{L}(z_2) = \frac{1}{(z_{12}^2)^2}\left\{\widetilde{\phi}(z_2) + V_{02}(z_1,z_2) + O(z_{12}^2)\right\} \tag{5.21}$$

where ϕ is any real scalar field of dimension 2 (which can be either one of the basic fields ϕ_i or some linear combination such as (5.4)). In particular, we wish to derive and exploit further implications of Wightman positivity for these 4-point functions.

The local field in (5.21)

$$\widetilde{\phi}(z) = \lim_{\varepsilon\to 0}\left\{(\varepsilon^2)^2\phi(z+\varepsilon)\mathcal{L}(z)\right\} \tag{5.22}$$

is another scalar field of dimension 2, hence again a linear combination of the ϕ_i (which may vanish). V_{02} is a hypothetical bilocal field of dimension $(0,2)$ with the properties that $V_{02}(z,z)=0$ but $V_{02}(z_1,z_2)\neq 0$ for $z_{12}^2=0$, $z_{12}\neq 0$. We shall now show that such a field must vanish.

The expansion of V_{02} in local fields involves only *twist 2* tensor fields $O_L(z;z_{12}) := O_{\alpha_1\cdots\alpha_L}(z)z_{12}^{\alpha_1}\cdots z_{12}^{\alpha_L}$ with $L>0$. The 2-point function of O_L satisfies the conservation law $\partial_z\cdot\partial_v\langle 0|\ O_L(z;v)O_L(z_2;v_2)\ |0\rangle = 0$, hence by Wightman positivity and the Reeh-Schlieder theorem, the conservation law holds for the field itself. On the other hand, the unique conformally invariant 3-point function $\langle 0|\ A(z_1)B(z_2)O_L(z_3;v)\ |0\rangle$ satisfies the conservation law only if the scalar fields A and B have equal dimension. Therefore, O_L cannot contribute to the OPE (5.21) of ϕ with $\mathcal{L}$.

More generally, the above argument allows to prove the following complement to Proposition 4.3 of [NT01].

Proposition 5.2: *Let $A(z)$ and $B(z)$ be two scalar fields of different (positive integer) dimensions d_A and d_B such that $d_A+d_B =: 2m$ is even. Then the bilocal twist 2 field $V_{AB}(z_1,z_2)$ defined on the light-like surface $z_{12}^2=0$ by $V_{AB}(z_1,z_2)=\lim_{z_{12}^2\to 0}\{(z_{12}^2)^{m-1}A(z_1)B(z_2)\}$ only involves a scalar field of dimension 2 in its expansion in local fields.*

In the case at hand, we have $m=3$ and $V_{\phi\mathcal{L}}(z_1,z_2)=\widetilde{\phi}(z_2)$. Applying this result to the 4-point function of ϕ_i and $\mathcal{L}$, we find that $A_{1i}=0=A_1$ (in particular, the subtraction involving the Casimir operator in

(5.16) is absent), and

$$\langle 0|\,\phi_i(z_1)\mathcal{L}(z_2)\mathcal{L}(z_3)\phi_i(z_4)\,|0\rangle = B_{\mathcal{L}}\left\{\frac{c_i}{2(z_{14}^2)^2(z_{23}^2)^4}+\right.$$
$$\left.+\,\frac{\tilde b_i}{(z_{23}^2)^2}\left(\frac{1}{(z_{12}^2z_{34}^2)^2}+(2\leftrightarrow 3)\right)+\frac{\tilde c_i}{z_{12}^2z_{13}^2(z_{23}^2)^2z_{24}^2z_{34}^2}+\frac{\alpha_i c_i}{z_{14}^2(z_{23}^2)^3}\left(\frac{1}{z_{12}^2z_{34}^2}+(2\leftrightarrow 3)\right)\right\}$$
$$= \frac{B_{\mathcal{L}}}{(z_{12}^2)^2(z_{23}^2)^2(z_{34}^2)^2}\left\{\frac{c_i}{2}\;s^2/t^2+\tilde b_i\;(1+s^2)+\tilde c_i\;s+\alpha_i c_i\;(s/t+s^2/t)\right\} \tag{5.23}$$

with new undetermined amplitudes $\tilde b_i$, $\tilde c_i$, while the other amplitudes have been determined using eqs. (5.4), (5.5) and (5.11). Here, the leading constant term in the braces (with coefficient $\tilde b_i$) represents the *complete* twist 2 partial wave $\beta_{10}^{\delta=2}=1$ (see eq. (B.8)). The next-to-leading terms s and s/t can be expanded into partial waves of twist 4 while all other terms are of higher twist. Performing the exact partial wave expansion by the method explained in Appendix B, one finds

$$\begin{aligned}
B_{1L} &= \delta_{L0}\,\tilde b_i,\\
\binom{2L+2}{L}B_{2L} &= \alpha_i c_i\cdot\binom{(L+1)(L+2)}{2}+\tilde c_i\cdot(-1)^L,\\
\binom{2\kappa+2L-2}{\kappa+L}\binom{2\kappa-4}{\kappa-1}B_{\kappa L} &= 2c_i\cdot\left(\binom{\kappa-1}{2}\binom{\kappa+L+1}{4}-\binom{\kappa}{4}\binom{\kappa+L}{2}\right)-\\
-\alpha_i c_i\cdot(-1)^\kappa\left(\binom{\kappa+L}{2}+(-1)^L\binom{\kappa-1}{2}\right) &+ \tilde b_i\cdot(-1)^L\left((\kappa+L+1)(\kappa+L-2)-\kappa(\kappa-3)\right)\qquad(\kappa\geq 3).
\end{aligned} \tag{5.24}$$

From the twist 2 and twist 4 coefficients we see that Wightman positivity requires $\tilde b_i$ and $\alpha_i c_i$ to be nonnegative, while we already know that c_i are positive and $\alpha_i\neq 0$. Hence we conclude

$$\tilde b_i\;(\equiv B_{10})\geq 0,\qquad \alpha_i>0\quad\text{for}\quad i=1,\dots N. \tag{5.25}$$

It then follows that A_0 in (5.2) and (5.3) is also positive. Further nontrivial bounds among the amplitudes arise at small values $\kappa=2,3,4$, $L=0,1$:

$$-\alpha_i\,c_i\leq\tilde c_i\leq 3\alpha_i\,c_i,\qquad (-3+\tfrac{3}{2}\alpha_i)\,c_i\leq\tilde b_i\leq(1+\tfrac{1}{2}\alpha_i)\,c_i,\qquad \alpha_i\leq 4\,. \tag{5.26}$$

When α_i takes the minimal value 0, then $\tilde c_i=0$ is fixed; when α_i takes the maximal value 4, then $\tilde b_i=3$ is fixed. All these bounds are clearly consistent with the structure (5.20), and we expect that considerations of this kind should ultimately allow to prove it.

Assuming for the moment (5.20) to be correct, and turning to $\widehat{\mathcal{L}}$ instead of $\mathcal{L}$, we may return to our discussion of Section 3 and assume $a_0=0$ in (2.1), which means that a scalar field of dimension 2 does *not* occur in the OPE of $\mathcal{L}$ with itself. This strengthens the bounds on the other parameters. Let us assume in addition the absence of a dimension 4 scalar in the OPE, i.e., $b'=0$ (in particular, the 3-point function of $\mathcal{L}$ vanishes, which is a characteristic property for the Lagrangian of a pure gauge field [NST03]). Then (3.5) and (3.6) reduce to the positivity restrictions for the remaining parameters

$$a_1\geq 0\,,\;a_2\geq 0\quad(a_1+a_2>0),\qquad -3\,a_1\leq b\leq\frac{1}{3}\,a_1\,. \tag{5.27}$$

The case $a_1=0$ (hence $b=0$) is expected to reduce to a direct sum of free Maxwell fields. It is thus interesting to study implications of positivity at 6- (and higher) point level for $a_1>0$.

6 Conclusion

Our central result is an exhaustive study of the consequences of Wightman positivity at the 4-point level, summarized in Sect. 3, of a GCI scalar field $\mathcal{L}$ of scaling dimension 4 in four space-time dimensions. It is based on an explicit computation of all partial wave coefficients $B_{\kappa L}$, presented in Appendix B. It turns out that the behaviour at asymptotically large spin L gives rise to non-trivial constraints, which were not accessible with recursive techniques yielding only the coefficients of a finite number of "leading" partial waves.

We analyze in Sect. 4 the consequences of the existence and uniqueness of a stress-energy tensor T whose appropriate moments generate the infinitesimal conformal transformations. Defining the *central charge* c by the normalization of the 2-point function (4.13) of T, we establish a lower bound (4.17) for c in terms of the parameters of the twist 2 contributions to the 4-point function of $\mathcal{L}$. This bound is saturated if the stress-energy tensor is precisely the conserved rank 2 tensor appearing in the OPE of the bilocal field V_1 defined by (1.2) and (1.4).

Section 5 exploits the results of previous work to analyze the consequences of the presence of a non-zero scalar field $\phi(z) = \frac{1}{2}V_1(z,z)$ of dimension 2. We prove in this case that V_1 admits an orthogonal decomposition of the form

$$V_1 = V_1^{(\phi)} + V_1^{\perp} \tag{6.1}$$

where $V_1^{(\phi)}$ belongs to the algebra generated by the scalar dimension 2 field ϕ, and $V_1^{\perp}$ has vanishing correlation functions with any number of fields ϕ. We interpret this decomposition, together with further results in Section 5, as a strong evidence in favour of a factorization of the theory according to (1.9), (1.10), allowing one to split off the free field content.

The positivity condition for the 4-point function $\langle 0|\, \phi(z_1)\mathcal{L}(z_2)\mathcal{L}(z_3)\phi(z_4)\, |0\rangle$ strongly restricts the parameters of the 6-point function of V_1 (discussed in Appendix C), and of its 4-point function.

These results are the first steps towards a more systematic analysis of Wightman positivity at the 6-point level, and to the identification of a nontrivial field of dimension 4.

Acknowledgments

I.T. would like to thank Yassen Stanev, and K.-H.R. would like to thank Sebastiano Carpi for helpful discussions at an early stage of this work. A major part of this work was done while N.N. and I.T. were visiting the Institut für Theoretische Physik der Universität Göttingen as an Alexander von Humboldt research fellow and a recipient of an AvH Research Award, respectively. A first draft was written (and corrected) during the stay of I.T. at l'Institut des Hautes Études Scientifiques, Bures-sur-Yvette (and at the Theory division of CERN). The authors thank these institutions for hospitality and support. The work of N.N. and I.T. was supported in part by the Research Training Network within Framework Programme 5 of the European Commission under contract HPRN-CT-2002-00325 and by the Bulgarian National Council for Scientific Research under contract PH-1406.

Appendix

A Free field constructions

In order to see which of the possible values of the parameters are with certainty realized in Wightman positive theories, we have considered various scalar fields $\mathcal{L}_I$ of dimension 4 which can be constructed as Wick products of free fields. These models give us admissible points $\vec{\boldsymbol{\alpha}}_I$ in the five-dimensional space of relative amplitudes $\vec{\boldsymbol{\alpha}} \equiv (a_0, a_1, a_2, b, b')$ (i.e., ignoring the absolute normalizations $B_{\mathcal{L}}$).

Before we display these values below, we observe that by taking the sum $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ of any two commuting fields (defined on the tensor product of their individual Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$), the absolute amplitudes $B_{\mathcal{L}}$ and $B_{\mathcal{L}} \cdot \vec{\alpha}$ are additive, so that the vector $\vec{\alpha}$ for the sum equals $\lambda^2\vec{\alpha}_1 + (1-\lambda)^2\vec{\alpha}_2$ where $\lambda = B_1/(B_1+B_2)$. For sums of more than two commuting fields, one finds the general "subconvex" composition law

$$\vec{\alpha} = \sum \lambda_I^2 \vec{\alpha}_I \qquad \text{with} \quad \lambda_I \geq 0, \sum \lambda_I = 1. \tag{A.1}$$

By taking subconvex combinations of the points $\vec{\alpha}_I$ realized by canonical free field models, one obtains a subset with a non-empty open interior in the five-dimensional parameter space defined by (3.1). In particular, taking sums of copies of identical fields $\mathcal{L}$, one can obtain any multiple $\mu \cdot \vec{\alpha}$ with $0 \leq \mu \leq 1$. We observe, however, that none of the free field models has $b < 0$, although 4-point positivity does not exclude this possibility.

Note also, that the assumption (4.11) of uniqueness of the stress-energy tensor introduced in Sect. 4 (requiring the $\kappa = 1, L = 2$ contribution to the operator product expansion to be a multiple of the stress-energy tensor, and consequently $c = \frac{16}{a_0+6a_1+6a_2}$) is not stable under taking arbitrary weighted sums, because the OPE contains the stress-energy tensors T_I with relative weight λ_I. This assumption is therefore consistent only with a parameter subset of measure zero.

A.1 Canonical free fields

The scalar free field φ in four space-time dimensions has dimension 1, so one obvious candidate is $\mathcal{L} = :\varphi^4:$. More generally, scalar fields of dimension 4 can be obtained as Wick products of any four components φ_a of a scalar free field multiplet.

The canonical vector field has dimension 3, so there is no way of constructing a scalar of dimension 4 with it. The canonical Dirac spinor has dimension $\frac{3}{2}$. With this, one can construct the scalar Yukawa-type Wick product $\varphi:\bar{\psi}\psi:$ of dimension 4.

Finally, there is the gauge model $\mathcal{L} = \frac{1}{4}:F_{\mu\nu}F^{\mu\nu}:$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is a vector field of dimension 1. But A_μ is not itself conformal (the conformally invariant 2-point function of a vector field of dimension 1 is a gradient, so the associated field strength tensor would vanish). Although A_μ has an indefinite 2-point function $(-\eta_{\mu\nu}\cdot(2\pi)^{-2}(x^2)^{-1}$ in the Feynman gauge), $F_{\mu\nu}$ is well-known to be a conformally covariant and Wightman positive free tensor field.

The computations of the 4-point functions are straight forward. We just give the result in the form of a table, displaying the value for the amplitude $B_{\mathcal{L}}$ of the 2-point function and the values of the relative amplitudes $\vec{\alpha} = (a_0, a_1, a_2, b, b')$ according to (2.1): Our convention for the normalization of fields in the z-picture is such that it includes, apart from the conformal transformation factor, an additional explicit factor of 2π so that, e.g., the scalar field with canonical 2-point function $(2\pi)^{-2}(x_{12}^2)^{-1}$ in x-space has the 2-point function $(z_{12}^2)^{-1}$ in z-space.

$\mathcal{L}$	$B_{\mathcal{L}}$	a_0	a_1	a_2	b	b'
$:\varphi^4:$	24	16	-	-	36	216
$:\varphi_1^3:\varphi_2$	6	10	-	-	18	54
$:\varphi_1^2::\varphi_2^2:$	4	8	-	-	18	76
$:\varphi_1^2:\varphi_2\varphi_3$	2	6	-	-	10	42
$\varphi_1\varphi_2\varphi_3\varphi_4$	1	4	-	-	6	24
$\varphi:\bar{\psi}\psi:$	16	1	$\frac{1}{4}$	-	$\frac{1}{4}$	-
$\varphi\,(\bar{\psi}_1\psi_2 + h.c.)$	32	1	$\frac{1}{8}$	-	$\frac{1}{8}$	-
$\frac{1}{4}:F_{\mu\nu}F^{\mu\nu}:$	48	-	-	$\frac{2}{9}$	-	-
$\frac{1}{4}F_{1\mu\nu}F_2^{\mu\nu}$	24	-	-	$\frac{1}{9}$	-	-

Clearly, all entries in this table satisfy the bounds given in Section 3.

A.2 Generalized free fields

Further free field constructions are possible involving generalized free fields with integer (scalars) or half-integer (spinors) scaling dimensions above the unitarity bound. Their Wightman functions are computed from 2-point functions of the generalized free fields by the same procedure (i.e., Wick's Theorem) as for Wick products of free fields.

Again, we need not consider vector fields, because their dimension above the unitarity bound of 3 is too big to contribute to a scalar of dimension 4.

Generalized fields do not arise from a Lagrangian, and consequently there is no associated canonical stress-energy tensor as a Wightman field. Note, however, that generalized free fields do possess a stress-energy tensor which is more singular than a Wightman field [DR03]: it has infinite vacuum fluctuations, hence formally $c = \infty$; in particular, this stress-energy tensor does not contribute to the operator product expansion.

Moreover, generalized free fields do not possess a canonical normalization. We are thus free to fix their normalizations by choosing the 2-point functions in x-space to be $(2\pi)^{-2}(x^2)^{-d}$ for scalars and $(2\pi)^{-2} i\not\partial(x^2)^{\frac{1}{2}-d}$ for spinors. As before, we absorb the factor 2π in the passage to the z-space fields.

Denoting by $A \equiv \varphi$, B, C, and D the scalar fields of dimension 1, 2, 3, 4, and by ψ and χ spinors of dimension $\frac{3}{2}$ and $\frac{5}{2}$, respectively, we have computed the following table.

$\mathcal{L}$	$B_{\mathcal{L}}$	a_0	a_1	a_2	b	b'
D	1	-	-	-	-	-
AC	1	1	-	-	-	-
$:B^2:$	2	-	-	-	4	8
B_1B_2	1	-	-	-	2	4
$:A^2:B$	2	4	-	-	2	8
A_1A_2B	1	2	-	-	2	6
$\bar{\chi}\psi + h.c.$	64	-	$\frac{1}{16}$	-	-	-

Also the entries of this table satisfy the bounds given in Sect. 3, because generalized free fields with dimension above the unitarity bounds satisfy Wightman positivity. On the other hand, there is no stress-energy tensor (with finite c) in all these models.

A.3 Subcanonical free fields

We have also considered free field constructions with $\mathcal{L}$ a Wick product involving subcanonical generalized free fields, i.e., generalized free tensor fields whose dimensions are below the unitarity bound such that these fields for themselves violate Wightman positivity. But we cannot exclude a priori the possibility that their Wick products might define a positive definite subtheory.

With these constructions, there are three possible scenarios how positivity might fail.

First, the operator product expansion of $\mathcal{L}$ with itself may contain fields whose scaling dimensions are below the unitarity bound. This occurs with Wick products involving subcanonical Fermi fields of dimension $\frac{1}{2}$, or subcanonical vector fields of dimension 1, such as $\mathcal{L} = j^\mu A_\mu$ where $j^\mu = :\bar{\psi}\gamma^\mu\psi:$ is a Dirac current and A_μ a subcanonical free conformal vector field of dimension 1. The operator product expansion of $\mathcal{L}\mathcal{L}$ contains the rank 2 symmetric tensor $:A_\mu A_\nu:-$ trace, whose dimension 2 is below the unitarity bound. In these cases, there contribute to the 4-point function partial waves which are more singular than the polynomial structures J_ν, D, Q of Section 2, and which were already excluded by the previous analysis [NST03]. We have not pursued these cases any further.

Second, the operator product expansion of $\mathcal{L}$ with itself may contain only fields with allowed quantum numbers, but a failure of Wightman positivity at the 4-point level could become manifest through some negative partial wave coefficient. Remarkably, we did not find examples for this scenario.

The third and most intriguing scenario is encountered [S-un, work in progress] with the scalar Wick square $\mathcal{L} = :B_\mu B^\mu:$ of a subcanonical vector field of dimension 2. Here the 4-point function is given by the relative amplitudes $\vec{\boldsymbol{\alpha}} = (a_0, a_1, a_2, b, b') = (0, 0, \frac{1}{4}; \frac{1}{4}, \frac{1}{2})$. Remarkably, all partial wave coefficients are positive, thus Wightman positivity is satisfied at the 4-point level. Moreover, the field $S_{\mu\nu}$ given by the traceless Wick square of the vector field, which contributes to the operator product expansion of $\mathcal{L}$ with itself, has all the properties of a stress-energy tensor. But a careful analysis by separating singular contributions in successive operator product expansions of $\mathcal{L}$ shows that positivity-violating fields such as the bilocal field $:B_\mu(x)B_\nu(y):$ arise within the OPE of six $\mathcal{L}$'s, so Wightman positivity must fail not later than at the 12-point level. In particular, this model illustrates that positivity at the 4-point level is certainly not a sufficient condition for Wightman positivity to all orders.

To conclude, we did not discover any free field model involving subcanonical conformal fields, which would satisfy positivity to all orders. The scenario of the free Maxwell tensor being positive to all orders although it involves a positivity-violating gauge field, does not repeat itself with subcanonical conformal fields.

B Partial wave expansion of scalar four-point functions

We describe a method to obtain the partial wave expansion for any globally conformal invariant scalar 4-point function. We concentrate, however, on 4-point functions of the form

$$\langle 0|\, A(z_1)B(z_2)B(z_3)A(z_4)\,|0\rangle = \frac{f(s,t)}{(z_{14}^2)^{d_A}(z_{23}^2)^{d_B}} \equiv \frac{(s/t)^{d_A} f(s,t)}{(z_{12}^2)^{d_A}(z_{23}^2)^{d_B-d_A}(z_{34}^2)^{d_A}} \tag{B.1}$$

which are relevant for the issue of Wightman positivity. We have used this method in order to compute the partial wave coefficients (2.8–2.10) of the 4-point function (1.5) of the field $\mathcal{L}$, and to obtain the positivity conditions (5.25) for the mixed 4-point function (5.23).

Let $d = d_A$ and $\delta = d_B - d_A$. Then the partial wave expansion has the form

$$\langle 0|\, A(z_1)B(z_2)B(z_3)A(z_4)\,|0\rangle = \frac{1}{(z_{12}^2)^d(z_{23}^2)^\delta(z_{34}^2)^d} \sum B_{\kappa L} \cdot \beta^\delta_{\kappa L}(s,t)\,, \tag{B.2}$$

i.e.,

$$(s/t)^d \cdot f(s,t) = \sum B_{\kappa L} \cdot \beta^\delta_{\kappa L}(s,t)\,. \tag{B.3}$$

The partial waves $\beta^\delta_{\kappa L}$ are represented [DO01] as

$$\beta^\delta_{\kappa L}(s,t) = \frac{uv}{u-v}\,\Big(G^\delta_{\kappa+L-\delta/2}(u)\cdot G^\delta_{\kappa-1-\delta/2}(v) - (u \leftrightarrow v)\Big) \tag{B.4}$$

or equivalently

$$(s/t)^\delta\;\beta^\delta_{\kappa L}(s,t) = \frac{uv}{u-v}\,\Big(G^{-\delta}_{\kappa+L+\delta/2}(u)\cdot G^{-\delta}_{\kappa-1+\delta/2}(v) - (u \leftrightarrow v)\Big) \tag{B.5}$$

where u and v are the "chiral variables"

$$s = uv\,, \qquad t = (1-u)(1-v)\,, \tag{B.6}$$

and

$$G^\delta_\nu(z) = z^\nu \cdot F(\nu,\nu;2\nu+\delta;z). \tag{B.7}$$

(For odd δ, only odd twists arise, corresponding to half-integer values of κ, thus ν is always an integer.) These representations of the scalar twist $2\kappa = 0, 1, 2$ partial waves (which are present only if $\delta = \pm 2\kappa$) involve singular hypergeometric functions $F(\cdot,\cdot;2\kappa-2;\cdot)$; but the singularity disappears upon antisymmetrization $u\leftrightarrow v$. In fact, these partial waves are particularly simple:

$$\beta^{+2\kappa}_{\kappa 0} = 1\,, \qquad \beta^{-2\kappa}_{\kappa 0} = (s/t)^{2\kappa}\,. \tag{B.8}$$

Remark B.1: Different authors [DO01, LR93] use different normalizations of the partial waves. Our normalization of $\beta^\delta_{\kappa L}$ is such that the leading term in s equals $s^{\kappa-\delta/2}(1-t)^L\cdot(1+O(1-t))$.

Inserting (B.4), the partial wave expansion of a given 4-point function (B.1) is equivalent to the expansion

$$\frac{u-v}{uv}(s/t)^d\cdot f(s,t) = \sum B_{\kappa L}\Big(G^\delta_{\kappa+L-\delta/2}(u)\cdot G^\delta_{\kappa-1-\delta/2}(v) - (u\leftrightarrow v)\Big). \tag{B.9}$$

For GCI 4-point functions, the function $f(s,t)$ is rational, in fact, a finite linear combination of terms of the form $s^r t^q$, where $r>-d$ except for a possible distinguished term $(s/t)^{-d}$ which can be present only if $\delta\geq 0$. The latter corresponds separately to the scalar partial wave $\beta^\delta_{\kappa=\delta/2,L=0}$ (= the vacuum contribution if $A=B$). It is then possible to represent the left-hand side of (B.9) (without the distinguished term) as a finite linear combination of terms of the form

$$(\text{nonnegative power of } u \text{ or } \tfrac{u}{1-u})\cdot(\text{nonnegative power of } v \text{ or } \tfrac{v}{1-v}) - (u\leftrightarrow v)\,. \tag{B.10}$$

Remark B.2: The 4-point structures J_ν, D and Q of Sect. 2 simplify considerably when represented in this way in terms of the chiral variables u and v. E.g., the twist 2 contributions j_ν of eq. (2.4), after multiplication by $(u-v)$, become simply $[(-1)^\nu u^{\nu+1} + (\frac{u}{1-u})^{\nu+1}] - (u\leftrightarrow v)$.

Each term in (B.10) has to be expanded into products $(G^\delta_\mu(u)G^\delta_\nu(v) - (u\leftrightarrow v))$. This is achieved with the help of the expansion formulae

$$z^p = \sum_{\nu\in p+\mathbb{N}_0} \frac{(-1)^{\nu-p}}{(\nu-p)!}\frac{(p+\alpha)_{\nu-p}(p+\beta)_{\nu-p}}{(\nu+p+\gamma-1)_{\nu-p}}\cdot z^\nu F(\nu+\alpha,\nu+\beta;2\nu+\gamma;z), \tag{B.11}$$

$$\left(\frac{z}{1-z}\right)^p = (1-z)^\alpha \sum_{\nu\in p+\mathbb{N}_0} \frac{1}{(\nu-p)!}\frac{(p+\alpha)_{\nu-p}(p+\gamma-\beta)_{\nu-p}}{(\nu+p+\gamma-1)_{\nu-p}}\cdot z^\nu F(\nu+\alpha,\nu+\beta;2\nu+\gamma;z) \tag{B.12}$$

which are valid if $2p+\gamma>0$. We shall prove these expansions below.

If $\delta\geq 0$, we may choose[4] $\alpha=\beta=0$, $\gamma=\delta$ and apply (B.11), (B.12) to both factors in (B.10). Thus, the left-hand side of (B.9) (without the distinguished term) is represented as a series of the form $\sum_{\mu\nu}X_{\mu\nu}(G^\delta_\mu(u)G^\delta_\nu(v) - (u\leftrightarrow v))$, with $\mu\neq\nu$ running over all nonnegative integers. It remains to relabel $\mu=\kappa+L-\delta/2$ and $\nu=\kappa-1-\delta/2$ if $\mu>\nu$, and $\mu=\kappa-1-\delta/2$ and $\nu=\kappa+L-\delta/2$ if $\mu<\nu$. Comparing with (B.9), one obtains the partial wave coefficients

$$B_{\kappa L} = X_{\kappa+L-\delta/2,\kappa-1-\delta/2} - X_{\kappa-1-\delta/2,\kappa+L-\delta/2} \qquad (2\kappa>\delta)\,. \tag{B.13}$$

If $\delta<0$, the expansions (B.11) and (B.12) for small p involve singular hypergeometric functions with $2\nu+\delta<0$. To avoid this problem, one proceeds in the same way by using the alternative representation (B.12) instead of (B.4). Equivalently, one may use the the "reversed" 4-point function $\langle 0|\,BAAB\,|0\rangle$, which has the same partial wave coefficients as $\langle 0|\,ABBA\,|0\rangle$.

Remark B.3: If $A=B$, then locality implies the symmetry of (B.1) under the permutation $(z_1\leftrightarrow z_2)$. It follows that (B.9) is antisymmetric under $u\mapsto -u/(1-u)$ and $v\mapsto -v/(1-v)$, and consequently the

[4] Other instances of these formulae are relevant for the partial wave expansion of mixed 4-point functions involving scalar fields of different dimensions.

symmetry $G^0_\nu(z) = (-1)^\nu\, G^0_\nu(-\frac{z}{1-z})$ cancels all terms with odd L, thus reproducing the well-known fact that the operator product expansion of a field with itself produces only fields of even spin.

Suitably adapting the parameters α, β, γ, the method works also for the general mixed 4-point function.

It remains to prove (B.11) and (B.12). The latter is equivalent to the former by the functional identity for hypergeometric functions under $z \mapsto -z/(1-z)$. By shifting both the summation index and the parameters α, β, γ by p, (B.11) is equivalent to the identity ($\gamma > 0$)

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\alpha)_n(\beta)_n}{(\gamma+n-1)_n} \cdot z^n F(n+\alpha, n+\beta; 2n+\gamma; z). \tag{B.14}$$

To prove this identity, we insert the expansion for the hypergeometric functions and collect the coefficients of z^k, giving

$$\frac{(\alpha)_k(\beta)_k}{k!} \sum_{n=0}^{k} (-1)^n \cdot (\gamma+2n-1) \binom{k}{n} \cdot \frac{1}{(\gamma+n-1)_{k+1}}.$$

This sum trivially equals 1 if $k=0$. For $k>0$, we insert the elementary identity $(\gamma+2n-1) = (\gamma+k+n-1) \cdot \frac{n}{k} + (\gamma+n-1) \cdot \frac{k-n}{k}$, giving

$$\frac{(\alpha)_k(\beta)_k}{k!} \sum_{n=0}^{k} (-1)^n \cdot \left[\binom{k-1}{n-1} \frac{1}{(\gamma+n-1)_k} + \binom{k-1}{n} \frac{1}{(\gamma+n)_k} \right] = 0.$$

This proves (B.14), hence (B.11) and (B.12).

C Six-point twist two contributions

We shall sum up results about the 6-point function

$$F(1,2;3,4;5,6) := \langle 0|\, V_1(z_1,z_2)\, V_1(z_3,z_4)\, V_1(z_5,z_6)\, |0\rangle \tag{C.1}$$

needed in Section 5. The function F appears as the leading contribution for $z^2_{2k-1,2k} \to 0$, $k=1,2,3$ to the truncated 6-point function w^t_6 of $\mathcal{L}(z)$

$$w^t_6(z_1,\ldots,z_6) = B^3 (z^2_{12}\, z^2_{34}\, z^2_{56})^{-3} \{ F(1,2;3,4;5,6) + O(|z^2_{12}| + |z^2_{34}| + |z^2_{56}|) \} \tag{C.2}$$

and is characterized by the following properties: (i) GCI; (ii) F is a rational function of z^2_{ij} ($i<j$) with not higher than third degree poles and no singularities in $z^2_{2k-1,2k}$, $k=1,2,3$; (iii) it is invariant under the 48 element subgroup $G_{8,6}$ of the permutation group $\mathcal{S}_6$ of $\{z_1,\ldots,z_6\}$ generated by the substitutions $s_{2k-1,2k}$, $k=1,2,3$ and the products $s_{13}\, s_{24}$, $s_{35}\, s_{46}$ which exchange pairs of variables ($(z_1,z_2) \rightleftarrows (z_3,z_4)$ etc.); (iv) F is harmonic in each z_i. We shall denote the linear manifold of all functions satisfying (i)–(iv) by $\mathcal{H}_6$.

It turns out that the space $\mathcal{H}_6$ is eight-dimensional. A basis in it, – *i.e.* a set of eight linearly independent rational functions satisfying the above conditions, will be displayed elsewhere. Here we shall briefly describe the method of deriving the general expression for F and shall write down the subset of four linearly independent functions for which $V_1(z,z) \neq 0$, implying the existence of a (hermitian) scalar field ϕ of dimension 2. Of the remaining four we shall display the one with lowest order of singularity (2) in z^2_{ij}.

We start with an observation which singles out GCI n-point functions with $n \leq 6$ in four space-time dimensions.

Consider the commutative algebra $\mathcal{A}^{(n)}$ of $n(n-1)$ generators, ρ_{ij} and ρ^{-1}_{ij} for $1 \leq i < j \leq n$. Let $\mathcal{A}^{(n,\mathcal{D})}$ be the algebra of scalar n-point functions in D-dimensional (compactified) space-time $\overline{\mathbb{M}}$ spanned by monomials of the type $\prod_{1\le i<j\le n} (z_{ij}^2)^{\mu_{ij}}$ with $\mu_{ij} \in \mathbb{Z}$, $z_{ij} = z_i - z_j$. For $n \ge 3$ the homomorphism $J_D : \mathcal{A}^{(n)} \to \mathcal{A}^{(n,\mathcal{D})}$ such that $\rho_{ij} \to z_{ij}^2$ is an isomorphism for $(3 \le) n \le D+1$. Otherwise, for $n > \mathcal{D}+1$, there are $\binom{n-D}{2}$ relations coming from the vanishing of Gram determinants of inner products of linearly dependent vectors. Thus, for space-time dimension $D = 4$ and $n \le 5$ the map J_D is an isomorphism. We shall now demonstrate that this allows to construct a complete set of *GCI 6-point functions* treating $\rho_{ij} = z_{ij}^2$ as independent, in spite of the fact that there is a (single, degree five) relation among them in this case.

We first note that $\mathcal{A}^{(n)}$ (as well as $\mathcal{A}^{(n,\mathcal{D})}$) admits a $\mathbb{Z}^n$ grading $\mathcal{A}^{(n)} = \bigoplus_{d_1,\dots,d_n} \mathcal{A}^{(n)}_{d_1\dots d_n}$ defined by

$$\text{weight}\left(\prod_{1\le i<j\le n} \rho_{ij}^{\mu_{ij}}\right) = (d_1,\dots,d_n) \in \mathbb{Z}^n \quad \text{for} \quad d_k = -\sum_{i=1}^{k-1} \mu_{ik} - \sum_{j=k+1}^{n} \mu_{kj}\,, \tag{C.3}$$

the allowed weights satisfying

$$\sum_{k=1}^{n} d_k \left(= -2 \sum_{i<j} \mu_{ij} \right) \in 2\,\mathbb{Z}\,. \tag{C.4}$$

It is a crucial observation that the GCI n-point function $\langle 0|\, \phi_1(z_1)\dots\phi_n(z_n)\, |0\rangle$ where $\phi_i(z)$ is a conformal scalar field of dimension d_i is an n-homogeneous element of $\mathcal{A}^{(n,\mathcal{D})}$ of weight $(d_1,\dots,d_n)$. This follows from the fact that under a conformal transformation $z \to g(z)$ the monomial in the left hand side of (C.3) transforms as

$$\prod_{1\le i<j\le n} [(g(z_i) - g(z_j))^2]^{\mu_{ij}} = \prod_{k=1}^{n} [\omega(g,z_k)]^{d_k} \prod_{1\le i<j\le n} (z_{ij}^2)^{\mu_{ij}}\,, \tag{C.5}$$

where $\omega(g,z)$ is a quadratic polynomial in z. As one can send any point to infinity by an appropriate (complex) conformal transformation, every $(n+1)$-homogeneous monomial of weight $(d_0, d_1, \dots, d_n)$

$$\prod_{0\le i<j\le n} \rho_{ij}^{\mu_{ij}} \in \mathcal{A}^{(n+1)}_{d_0 d_1\dots d_n} \tag{C.6}$$

is uniquely determined by the factor

$$\prod_{1\le i<j\le n} \rho_{ij}^{\mu_{ij}} \in \mathcal{A}^{(n)}_{\delta_1\dots\delta_n} \quad \text{where} \quad \delta_1 + \dots + \delta_n = d_1 + \dots + d_n - d_0\,, \tag{C.7}$$

independent of the variable z_0. Indeed, given $(\delta_1,\dots,\delta_n)$ satisfying (C.7), one can restore (C.6) by setting $\mu_{0j} = \delta_j - d_j$. In particular, the set of GCI 6-point functions is determined by the set of homogeneous (rational) 5-point functions for which the variables ρ_{ij} are independent.

Our next task is to find a basis of 6-homogeneous harmonic functions of weight $(1,1,1,1,1,1)$ constraint by condition (ii) on their singularities. This is made easy by the fact that the Laplace operator acting on $\mathcal{A}^{(6,4)}_{111111}$ acquires a simple form in terms of the variables ρ_{ij}: if we set $\rho = \{\rho_{k\ell}, 1 \le k < \ell \le 6\}$, $F(1,2,3,4;5,6) = f(\rho)$ (or, using the above homomorphism $J_\mathcal{D}$ for $\mathcal{D} = 4$, $F = J_4\, f$) we find

$$\partial_{z_1}^2\, F(1,2;3,4;5,6) = -\mathcal{D}_1\, f(\rho) = 0\,, \quad \mathcal{D}_1 := \sum_{1<i<j\le 6} \rho_{ij}\, \frac{\partial^2}{\partial\rho_{1i}\,\partial\rho_{1j}}\,. \tag{C.8}$$

Here we have used the Euler equation $\sum_{k=2}^{6} \rho_{1k}\, \frac{\partial}{\partial\rho_{1k}}\, f = -f$.

Lemma C.1: *If* $f = \sum_{\mu} C(\mu) \prod_{1\le i<j\le 6} \rho_{ij}^{\mu_{ij}}$, *($\mu \in \mathbb{Z}^{15}$ being a multiindex), is a solution of (C.8) then*

$$C(\mu) = 0 \textit{ if there exist } 1 < k < \ell \textit{ such that } \mu_{1k} < 0 \textit{ and } \mu_{1\ell} < 0 \,. \tag{C.9}$$

To *prove* the lemma one chooses the minimal (*i.e.* the largest in absolute value, negative) μ_{1k} and $\mu_{1\ell}$ appearing in a single term in the expansion of f and then concludes that the derivative $\rho_{k\ell} \frac{\partial^2}{\partial\rho_{1k}\,\partial\rho_{1\ell}}$ of this monomial cannot be cancelled by any other term in the sum.

After these preliminaries, writing down a basis in $\mathcal{H}_6$, *i.e.* a complete set of linearly independent rational functions satisfying conditions (i)–(iv), becomes a (more or less) routine technical problem.

We shall present the solution in increasing order of singularity starting with first order poles. According to Lemma B.1 and to the symmetry assumption (iii) there are eight possible singular configurations (for a given order of singularity n), obtained from $(\rho_{16}\,\rho_{23}\,\rho_{45})^{-n}$ by transposing independently $1 \rightleftarrows 2$, $3 \rightleftarrows 4$, $5 \rightleftarrows 6$. Each such *elementary contribution* (with a fixed pole structure) is symmetric under the trihedral subgroup $\mathcal{D}_3$ ($\sim \mathcal{S}_3$) of $G_{8,6}$ generated by the (order three) cyclic permutation $(1\,2\,3\,4\,5\,6) \to (5\,6\,1\,2\,3\,4)$ and the reflection $(1\,2\,3\,4\,5\,6) \to (6\,5\,4\,3\,2\,1)$. Thus, the unique vector in $\mathcal{H}_6$ with only first order pole singularities is

$$F_1(1,2;3,4;5,6) = S(\mathbb{Z}_2^3)(\rho_{16}\,\rho_{23}\,\rho_{45})^{-1} \tag{C.10}$$

where $S(\mathbb{Z}_2^3)$ is symmetrization with respect to the 8-element abelian group $\mathbb{Z}_2^3$ generated by the transpositions s_{12}, s_{34} and s_{56}:

$$\begin{aligned} S(\mathbb{Z}_2^3)(\rho_{16}\,\rho_{23}\,\rho_{45})^{-1} &= (\rho_{16}\,\rho_{23}\,\rho_{45})^{-1} + (\rho_{13}\,\rho_{26}\,\rho_{45})^{-1} + (\rho_{16}\,\rho_{24}\,\rho_{35})^{-1} + (\rho_{15}\,\rho_{23}\,\rho_{46})^{-1} \\ &+ (\rho_{15}\,\rho_{24}\,\rho_{36})^{-1} + (\rho_{13}\,\rho_{25}\,\rho_{46})^{-1} + (\rho_{14}\,\rho_{26}\,\rho_{35})^{-1} + (\rho_{14}\,\rho_{25}\,\rho_{36})^{-1} \,. \end{aligned} \tag{C.11}$$

It is reproduced by the normal product of a free massless scalar field φ with itself: $V_1^{\varphi}(z_1, z_2) = :\varphi(z_1)\,\varphi(z_2):$.

There are two linearly independent functions in $\mathcal{H}_6$ with poles of order not exceeding two:

$$F_2^{(i)}(1,2;3,4;5,6) = S(\mathbb{Z}_2^3)\, W_2^{(i)}(12;34;56) \qquad i = 1,2 \tag{C.12}$$

where

$$\begin{aligned} W_2^{(1)}(12;34;56) &= \rho_{16}^{-2}\,\rho_{23}^{-2}\,\rho_{45}^{-2}\, S(\mathbb{Z}_3)\,\{\rho_{16}(\rho_{25}\,\rho_{34} - \rho_{24}\,\rho_{35}) \\ &= \frac{\rho_{16}(\rho_{25}\,\rho_{34} - \rho_{24}\,\rho_{35}) + \rho_{23}(\rho_{14}\,\rho_{56} - \rho_{16}\,\rho_{45}) + \rho_{45}(\rho_{12}\,\rho_{36} - \rho_{13}\,\rho_{26})}{\rho_{16}^2\,\rho_{23}^2\,\rho_{45}^2} \,, \end{aligned} \tag{C.13}$$

$$\begin{aligned} W_2^{(2)}(12;34;56) &= \frac{1}{\rho_{16}\,\rho_{23}\,\rho_{45}} + \frac{\rho_{12}\,\rho_{34}\,\rho_{56} - \rho_{14}\,\rho_{25}\,\rho_{36}}{\rho_{16}^2\,\rho_{23}^2\,\rho_{45}^2} \\ &+ S(\mathbb{Z}_3)\,\frac{\rho_{23}(\rho_{14}\,\rho_{56} - \rho_{15}\,\rho_{46}) - \rho_{13}\,\rho_{24}\,\rho_{56} + \rho_{14}\,\rho_{26}\,\rho_{35}}{\rho_{16}^2\,\rho_{23}^2\,\rho_{45}^2} \,, \end{aligned} \tag{C.14}$$

where $S(\mathbb{Z}_3)$ stands for symmetrization with respect to the 3-element cyclic group generated by the permutation $(1\,2\,3\,4\,5\,6) \to (5\,6\,1\,2\,3\,4)$. The first of them is characterized by the fact that it has non-zero limit for $z_{2k-1} = z_{2k}$, $k = 1,2,3$; for instance,

$$W_2^{(1)}(12;33;44) = \frac{\rho_{12}\,\rho_{34} - \rho_{13}\,\rho_{24} - 2\,\rho_{14}\,\rho_{23}}{\rho_{14}^2\,\rho_{23}^2\,\rho_{34}} \,. \tag{C.15}$$

The function $F_2^{(2)}$ is reproduced by the bilocal field V_1^{ψ} (4.21) (the composite of the free Weyl field) (*cf.* [NT04]):

$$\langle 0|\, V_1^{\psi}(z_1,z_2)\, V_1^{\psi}(z_3,z_4)\, V_1^{\psi}(z_5,z_6)\, |0\rangle = \frac{1}{8}\, F_2^{(2)}(1,2;3,4;5,6) \,. \tag{C.16}$$

There are two more independent elements of $\mathcal{H}_6$, both with third degree poles, also having non-zero limit for $z_{2k-1} = z_{2k}$, $k = 1, 2, 3$. Setting again $F_3^{(i)} = S(\mathbb{Z}_2^3)\, W_3^{(i)}$, $i = 1, 2$, we find

$$W_3^{(1)}(12; 34; 56) = S(\mathbb{Z}_3) \left\{ \frac{(\rho_{14}\,\rho_{56} - \rho_{15}\,\rho_{46})^2}{\rho_{23}\,\rho_{45}^3\,\rho_{16}^3} - \frac{\rho_{15}\,\rho_{46}}{\rho_{23}\,\rho_{45}^2\,\rho_{16}^2} \right\}, \tag{C.17}$$

$$\begin{aligned} W_3^{(2)}(12; 34; 56) &= 3\, \frac{\rho_{12}\,\rho_{34}\,\rho_{56}}{\rho_{23}^2\,\rho_{45}^2\,\rho_{16}^2} \\ &+ S(\mathbb{Z}_3) \left\{ \frac{\rho_{14}\,\rho_{26}\,\rho_{35} - 2\,\rho_{13}\,\rho_{24}\,\rho_{56}}{\rho_{23}^2\,\rho_{45}^2\,\rho_{16}^2} - 2\, \frac{\rho_{12}\,\rho_{14}\,\rho_{36}\,\rho_{56} + \rho_{13}\,\rho_{15}\,\rho_{26}\,\rho_{46}}{\rho_{23}^2\,\rho_{45}^2\,\rho_{16}^3} \right\} \\ &+ 2\, S(D_3)\, \frac{\rho_{13}\,\rho_{14}\,\rho_{26}\,\rho_{56}}{\rho_{23}^2\,\rho_{45}^2\,\rho_{16}^2}, \end{aligned} \tag{C.18}$$

where $S(D_3)$ stands for symmetrization with respect to the trihedral group $\mathcal{D}_3$ described before Eq. (C.10). None of these two structures can be reproduced by free fields.

The remaining three structures (which will be studied elsewhere) all have third order poles and vanish for coinciding arguments. One of them is reproduced by the 6-point function of V_1^F (4.29).

We note that the 4-point functions $\rho_{13}\,\rho_{24}\,\rho_{34}\,F_n^{(i)}(1, 2; 3, 3; 4, 4)$ for $(n, i) = (2, 1), (3, 1), (3, 2)$ are, as expected, linear combinations of $j_\nu(s, t)$. For instance, $\frac{1}{4}\,\rho_{13}\,\rho_{24}\,\rho_{34}\,F_2^{(1)}(1, 2; 3, 3; 4, 4) = -\,3\,j_0(s, t) - j_1(s, t)$.

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