Universal enveloping algebras and some applications in physics

Xavier BEKAERT

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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Xavier Bekaert

Institut des Hautes Études Scientifiques
Le Bois-Marie, 35 route de Chartres
91440 Bures-sur-Yvette, France

bekaert@ihes.fr

Abstract

These notes are intended to provide a self-contained and pedagogical introduction to the universal enveloping algebras and some of their uses in mathematical physics. After reviewing their abstract definitions and properties, the focus is put on their relevance in Weyl calculus, in representation theory and their appearance as higher symmetries of physical systems.

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These lecture notes are written by a layman in abstract algebra and are aimed for other aliens to this vast and dry planet, therefore many basic definitions are reviewed. Indeed, physicists may be unfamiliar with the daily-life terminology of mathematicians and translation rules might prove to be useful in order to have access to the mathematical literature. Each definition is particularized to the finite-dimensional case to gain some intuition and make contact between the abstract definitions and familiar objects.

The lecture notes are divided into four sections. In the first section, several examples of associative algebras that will be used throughout the text are provided. Associative and Lie algebras are also compared in order to motivate the introduction of enveloping algebras. The Baker-Campbell-Haussdorff formula is presented since it is used in the second section where the definitions and main elementary results on universal enveloping algebras (such as the Poincaré-Birkhoff-Witt) are reviewed in details. Explicit formulas for the product are provided. In the third section, the Casimir operators are introduced as convenient generators of the center of the enveloping algebra. Eventually, in the fourth section the Coleman-Mandula theorem is reviewed and discussed on Lie algebraic grounds, leading to a rough conjecture on the appearance of enveloping algebras as physical higher symmetry algebras.[1]

1 Associative versus Lie algebras

An **algebra** $\mathcal{A}$ over a field $\mathbb{K}$ is a vector space over $\mathbb{K}$ endowed with a bilinear map $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ generally referred to as **multiplication** or **product**. The algebra is **associative** iff

$$x * (y * z) = (x * y) * z\,, \quad \forall x, y, z \in \mathcal{A}\,.$$

The algebra is **unital** if it possesses a **unit** element 1 such that

$$x * 1 = x = 1 * x\,.$$

[1]Slight additions to standard textbook material on the enveloping algebra topic are presented: An independent heuristic proof of Berezin's formula for the enveloping algebra composition law is provided, and a mean to evade the negative conclusions of the no-go theorems on S-matrix symmetries is discussed, while mentioning higher-spin algebras as a specific example.

Proposition 1. *Let $\mathcal{A}$ be a unital associative algebra, of finite dimension over the field $\mathbb{K}$. Then $\mathcal{A}$ is isomorphic to a subalgebra of the algebra $M(n\,;\,\mathbb{K})$ of $n \times n$ matrices for some non-negative integer $n \in \mathbb{N}$.*

The **center** of $\mathcal{A}$ is the subalgebra of elements that commute with all elements of $\mathcal{A}$ and is denoted by

$$\mathcal{Z}(\mathcal{A}) = \{z \in \mathcal{A} \,|\, z * a = a * z\,, \forall a \in \mathcal{A}\}\,.$$

The **centralizer** of a subset $S \subset \mathcal{A}$ is the subalgebra of elements that commute with all elements of S and is referred to as

$$\mathcal{C}_{\mathcal{A}}(S) = \{z \in \mathcal{A} \,|\, z * s = s * z\,, \forall s \in S\}\,.$$

If $\{e_i\}$ is a basis of $\mathcal{A}$, then the product $*$ is completely determined by the **structure constants** $f^i_{jk} \in \mathbb{K}$ defined by

$$e_j * e_k = e_i\, f^i_{jk}\,.$$

For (anti)commutative algebras, the associativity property is equivalent to the condition

$$f^i{}_{j[k} f^j{}_{l]m} = 0\,.$$

1.1 Representations of algebras

An **algebra homomorphism** from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\Phi(x * y) = \Phi(x) * \Phi(y)$ for all $x, y \in \mathcal{A}$.

A **linear representation** of the associative algebra $\mathcal{A}$ over the vector space V is an algebra homomorphism from $\mathcal{A}$ to the associative algebra $End(V)$ of endomorphisms (*i.e.* linear operators on V)

$$\mathcal{R}_V : \mathcal{A} \to End(V)\,.$$

The vector space V **carries the representation** and is called a **representation space** of $\mathcal{A}$ or, in more fancy terms, a (left) $\mathcal{A}$-**module**. The kernel of $\mathcal{R}_V$ is called the **annihilator** of the $\mathcal{A}$-module V and is denoted by $Ann(V)$. The image $\mathcal{R}_V\big(\mathcal{A}\big)$ of the algebra $\mathcal{A}$ in $End(V)$ is called the **realization** of the algebra $\mathcal{A}$ on the vector space V.

An **invariant subspace** is a vector subspace $W \subseteq V$ such that $\mathcal{R}_V(x)W \subseteq W$ for all $x \in \mathcal{A}$. In abstract jargon, it is also christened **submodule** or **left ideal**. An **irreducible** (or **simple**) module V has only two distinct submodules, $\{0\}$ and V itself.

1.2 Tensor algebras

The direct sum of all possible tensorial powers of V is

$$\bigotimes(V) \cong \bigoplus_{p=0}^{\infty} \otimes^p V \,,$$

where the first summand ($p = 0$) is the field $\mathbb{K}$ and the second ($p = 1$) is the space V itself. The vector space $\otimes(V)$ endowed with a unital associative algebra structure via the tensor product $\otimes$, is called the **tensor algebra** of V.

Let $\mathbb{K}$ be a field. The **free algebra** on n indeterminates X_1, ..., X_n (the construction works also for any countable set S of "indeterminates"), is the algebra spanned by all linear combinations

$$P(X_i) = \sum_{k=0} \Pi^{i_1 \ldots i_k} \, X_{i_1} \ldots X_{i_k} \,,$$

of formal products of the **generators** X_i, with coefficients $\Pi^{i_1 \cdots i_k} \in \mathbb{K}$. This algebra is denoted by $\mathbb{K} < X_i >$ and is said to be **freely generated** by the X's.

Proposition 2. *The free algebra on n indeterminates can be constructed as the tensor algebra of an n-dimensional vector space. More precisely, if the set $\{e_i\}$ is a basis of a vector space V over a field $\mathbb{K}$, then we have the following isomorphism of algebras*

$$\otimes(V) \cong \mathbb{K} < e_i > \ .$$

1.3 Presentation modulo relations

Because of the wide generality of the tensor algebra (that may be encoded in abstract terms in its "universality" property), many other interesting algebras are constructed by starting with the tensor algebra and then imposing certain relations on the generators, i.e. by constructing certain quotients of the tensor algebra.

A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ of an algebra $\mathcal{A}$ is a **left (or right) ideal** if $\mathcal{A} * \mathcal{I} \subseteq \mathcal{I}$ (or $\mathcal{I} * \mathcal{A} \subseteq \mathcal{I}$). Moreover, if $\mathcal{I}$ is both a left <u>and</u> a right ideal, then it is called **invariant subalgebra** or **(two-sided) ideal**. In such case, one may define the **quotient algebra** $\mathcal{A}/\mathcal{I}$ that is the algebra of equivalence classes $[a]$ defined by the equivalence relation $a \sim a + b$ where $b \in \mathcal{I}$.

Exercise 1: Check that the product of classes is well defined. ✍

For any representation $\mathcal{R}_V : \mathcal{A} \to End(V)$, the annihilator $Ann(V)$ is a (two-sided) ideal made of all elements of $\mathcal{A}$ which are represented by the operator zero in $End(V)$.

Proposition 3. *Let $\mathcal{R}_V : \mathcal{A} \to End(V)$ be a linear representation of the algebra $\mathcal{A}$ over the space V.*

Then the realization of the algebra $\mathcal{A}$ on the space V is isomorphic to the quotient of the algebra $\mathcal{A}$ by the annihilator of the $\mathcal{A}$-module V,

$$\mathcal{R}_V(\mathcal{A}) \cong \frac{\mathcal{A}}{Ann(V)}\,.$$

Let S be a set of generators and $R \subset \mathbb{K} < S >$ a subset of the free algebra generated by S. A **presentation** of an algebra $\mathcal{A}$ by **generators modulo relations** is the definition of $\mathcal{A}$ as the quotient of the free algebra $\mathbb{K} < S >$ by its smallest (two-sided) ideal containing all elements of R. In more pragmatic terms, this procedure puts all elements of R to zero in $\mathcal{A}$ and the relations $R = 0$ are thereby imposed on the corresponding product of generators in $\mathcal{A}$.

1.4 Symmetric and exterior algebras

The **symmetric algebra**[2] denoted by $\odot(V)$ (or also $\mathcal{S}(V)$ or $\vee(V)$) is the commutative associative algebra defined as the quotient of the tensor algebra $\otimes(V)$ by the smallest two-sided ideal containing all elements of the form $x \otimes y - y \otimes x$. It is graded by the order p of the contravariant tensors

$$\bigodot(V) = \bigoplus_{p=0}^{\infty} \odot^p(V)\,.$$

Another way to construct the symmetric algebra is by using the projector $\mathcal{S} : \otimes(V) \to \odot(V)$ on the symmetric part of a contravariant tensor. The quotient of the tensor algebra $\otimes(V)$ by the kernel of $\mathcal{S}$ is $\odot(V) \cong \otimes(V)/Ker(\mathcal{S})$.

[2]The **exterior algebra** $\wedge(V)$ of antisymmetric forms is constructed analogously to the symmetric algebra $\vee(V)$ as a quotient of the tensor algebra $\otimes(V)$ by the relations $x \otimes y + y \otimes x$.

Let $\mathbb{K}$ be a field. The **polynomial algebra** on n indeterminates X_1, ..., X_n is the algebra spanned by all linear combinations over $\mathbb{K}$ of products of the commuting variables X_i. This algebra is denoted by $\mathbb{K}[X_i]$.

Proposition 4 (Polynomial and symmetric algebra $\cong$). *Let $\{e_i\}$ be a basis of an n-dimensional space V over the field $\mathbb{K}$. Then we have the following isomorphism of associative commutative algebras*

$$\odot(V) \cong \mathbb{K}[e_i]\,.$$

Moreover, $\odot^r(V)$ is isomorphic to the space of homogeneous polynomials of order r in the variables e_i and its dimension is equal to $C^r_{n+r-1} = \frac{(n+r-1)!}{(n-1)!\,r\,!}$.

Exercise 2: Verify that the correspondence is a bijection (hint: an arbitrary element p of $\odot^r(V)$ decomposes as $p = p^{i_1\cdots i_r} e_{i_1} \odot \ldots \odot e_{i_r}$, where the components $p^{i_1\cdots i_r}$ are completely symmetric) and a commutative algebra homomorphism (hint: the product in $\odot(V)$ reads in components $(p \odot q)^{i_1\cdots i_{r+s}} = p^{(i_1\cdots i_r} q^{i_{r+1}\cdots i_{r+s})}$, where the curved bracket denotes complete symmetrization with strength one). ✍

1.5 Lie algebras

A **Lie algebra** is an algebra $\mathfrak{g}$ whose product $[\;]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is called **(Lie) bracket** and obeys the following properties:

1. $[X,Y] = -\,[Y,X]$ for all $X,Y\in\mathfrak{g}$ (**antisymmetry**)
2. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for all $X,Y,Z\in\mathfrak{g}$ (**Jacobi identity**).

A Lie algebra $\mathfrak{c}$ is said to be **Abelian** if its Lie bracket vanishes identically, $[\,\mathfrak{c}\,,\mathfrak{c}\,] = 0$. The **central extension** of an arbitrary Lie algebra $\mathfrak{g}$ by an Abelian Lie algebra $\mathfrak{c}$ is the Lie algebra that is the direct sum $\mathfrak{g}\oplus\mathfrak{c}$ endowed with the Lie bracket defined by $[\,\mathfrak{g}\,,\mathfrak{c}\,] = 0$.

Proposition 5. *A Lie algebra $\mathfrak{g}$ is associative if and only if*

$$\big[X,[Y,Z]\big] = 0\,,\quad \textit{for any}\quad X,Y,Z\in\mathfrak{g}$$

if and only if $\mathfrak{g}$ is a central extension of an Abelian Lie algebra.

Exercise 3: Prove Proposition 5 by using the Jacobi identity and the definition of associativity and central extension. ✍

Example: The **Heisenberg algebra** $\mathfrak{h}_n$ is the Lie algebra of dimension $2n+1$ that is algebraically generated by the generators X^i, P_j $(i, j = 1, 2, \ldots, n)$ and C which are subject to the following Lie bracket relations

$$[P_j, X^i] = C\delta^i_j\,, \quad [C, X^i] = 0\,, \quad [C, P_j] = 0\,,$$

hence C is a central element. The Heisenberg algebra is an associative Lie algebra (it is actually the elementary building block of any finite-dimensional non-Abelian associative Lie algebra).

From Proposition 5, one knows that, generally speaking, a Lie algebra is *not* associative. Nevertheless, there is a canonical procedure to construct a Lie algebra out of an associative one.

Proposition 6. *Any associative algebra $\mathcal{A} \cong (V\,, *)$ may be turned into a Lie algebra $\mathfrak{g} \cong (V\,, [\,]\,)$ that has the same underlying vector space V, but whose multiplication operation $[\ ,\ ]$ is given by the* ***commutator bracket***

$$[x, y] := x * y - y * x\,.$$

It seems there is no name for this widely spread construction but I propose to call the obtained Lie algebra $\mathfrak{g}$ "**commutator algebra** associated to $\mathcal{A}$" and to denote it by $[\mathcal{A}\,]$.

A natural question arises: Does the converse of Proposition 6 hold? More accurately: Is it possible to canonically construct an associative algebra out of any given Lie algebra? The answer is positive and "universal": the enveloping algebra does the job, which explains its usefulness.

The commutator algebra is very useful for defining a **linear representation of a Lie algebra** $\mathfrak{g}$ acting on V (or **Lie algebra module**) as a Lie algebra homomorphism

$$R_V : \mathfrak{g} \to \mathfrak{gl}(V)\,,$$

where V is the representation space and $\mathfrak{gl}(V) := [End(V)]$. Given two $\mathfrak{g}$-modules V and W, one can represent $\mathfrak{g}$ also on the **tensor product** $V \otimes W$ by

$$\Big((R_V \otimes R_W)(X)\Big)(v \otimes w) := \big(R_V(X)v\big) \otimes w \,+\, v \otimes \big(R_W(X)w\big)\,,$$

for all $X \in \mathfrak{g}$, $v \in V$ and $w \in W$. The latter expression may be rewritten as

$$R_V \otimes R_W = R_V \otimes 1_W \,+\, 1_V \otimes R_W\,.$$

1.6 Baker-Campbell-Haussdorff formula

Baker-Campbell-Hausdorff (BCH) type relations arise naturally when one describes Lie group elements by exponentials of Lie algebra elements. The most crucial consequence of the BCH formula is that it shows that the local structure of the Lie group G (the multiplication law for elements near the identity) is completely determined by its Lie algebra $\mathfrak{g}$ (where the Lie bracket is the multiplication law).

Theorem 1 (Baker-Campbell-Haussdorff formula). *For any two elements $X, Y \in \mathfrak{g}$ with $\| X \|$ and $\| Y \|$ "sufficiently small",*

$$\log(e^X e^Y) = Y + \int_0^1 dt\ \Omega\big(e^{t\, ad_X} e^{ad_Y}\big)(X) = X + \int_0^1 dt\ \Psi\big(e^{ad_X} e^{t\, ad_Y}\big)(Y)$$

where ad is the ***adjoint representation*** *defined by $ad_X(Y) = [X, Y]$ and Ω is the function*

$$\Omega(z) = \frac{\log z}{z-1}$$

and $\Psi(z) = z\Omega(z)$.

Though entirely explicit, the genuine computation of the group multiplication law from the Lie brackets is out of reach in most cases. Nevertheless, at first order it is possible to get more illuminating expressions.

The **left and right shift** are the respective actions of the group G on itself defined by $\lambda(g) : h \mapsto gh$ and $\rho(g) : h \mapsto hg$. For all elements X, Y of a Lie algebra $\mathfrak{g}$, one may consider the expansion of $\log(e^X e^Y)$ either in X or in Y:

$$\log(e^X e^Y) \;=\; Y + \mathcal{L}_Y(X) + \mathcal{O}_Y\big(\, \| X \|^2 \,\big) \;=\; X + \mathcal{R}_X(Y) + \mathcal{O}_X\big(\, \| Y \|^2 \,\big)$$

where $\mathcal{O}_Y\big(\, \| X \|^2 \,\big)$ and $\mathcal{O}_X\big(\, \| Y \|^2 \,\big)$ denote complicated functions of X and Y obeying

$$\frac{\mathcal{O}_Y\big(\, \| X \|^2 \,\big)}{\| X \|} \overset{\| X \| \to 0}{\longrightarrow} 0\,, \qquad \frac{\mathcal{O}_X\big(\, \| Y \|^2 \,\big)}{\| Y \|} \overset{\| Y \| \to 0}{\longrightarrow} 0\,.$$

The Lie algebra endomorphisms $\mathcal{L}_Y$ and $\mathcal{R}_X$ are respectively called the **left and right shift operators** since they correspond to the infinitesimal action of the Lie group element.

Corollary 1. *The left and right shift operators are respectively given by*

$$\mathcal{L}_Y = \mathcal{B}\,(\,ad_Y\,)\,, \qquad \mathcal{R}_X = \mathcal{B}\,(-ad_X\,)\,,$$

where $\mathcal{B}$ is the ***generating function of the Bernoulli numbers***

$$\mathcal{B}(z) = \frac{z}{\exp z - 1} = \sum_{m=0}^{\infty} \frac{1}{m!}\, b_m\, z^m$$

the coefficients of the Taylor series of which are the Bernoulli numbers b_m.

Even more explicitly, for a finite-dimensional Lie algebra with structure constants defined by

$$[T_j, T_k] = f^i{}_{jk}\, T_i$$

the left shift matrix reads

$$\mathcal{L}^j{}_i\,(Y) = \sum_{m=0}^{\infty} \frac{b_m}{m!}\, f^j_{k_1\ell_1}\, f^{\ell_1}_{k_2\ell_2}\, \ldots\, f^{\ell_{m-2}}_{k_{m-1}\ell_{m-1}}\, f^{\ell_{m-1}}_{k_m i}\, Y^{k_1} Y^{k_2} \ldots Y^{k_{m-1}} Y^{k_m}\,.$$

Exercise 4: Prove Corollary 1 by expanding the BCH formula of Theorem 1 at first order (hint: $\Omega\big(e^{t\,ad_X} e^{ad_Y}\big)(X) = \Omega\big(e^{ad_Y}\big)(X) + \mathcal{O}'_Y\big(\parallel X \parallel^2\,\big)$) ✍

2 Universal enveloping algebras

Mathematically speaking, the construction of the universal enveloping algebra of a Lie algebra is useful in order to pass from a non-associative structure to a more familiar (associative) algebra over the same field while preserving the representation theory.

The rather abstract definition is as follows (fortunately a more concrete definition is presented in Theorem 2): A **universal enveloping algebra** $\mathcal{U}$ of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a unital associative algebra $\mathcal{U}$ over $\mathbb{K}$, together with a homomorphism $i : \mathfrak{g} \to [\mathcal{U}]$ of Lie algebras, such that if $\mathcal{A}$ is another associative algebra over $\mathbb{K}$ and $\phi : \mathfrak{g} \to [\mathcal{A}]$ is another Lie algebra homomorphism, then there exists a unique homomorphism $\Psi : \mathcal{U} \to \mathcal{A}$ of associative algebras inducing a Lie algebra homomorphism $\psi : [\mathcal{U}] \to [\mathcal{A}]$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & [\mathcal{U}] \\ \phi \searrow & & \swarrow \psi \\ & [\mathcal{A}] & \end{array}\,.$$

The universal property above ensures that all universal enveloping algebras of $\mathfrak{g}$ are isomorphic; this justifies the standard notation $\mathcal{U}(\mathfrak{g})$.

Theorem 2 (Canonical universal enveloping algebra). *Any Lie algebra $\mathfrak{g}$ has a universal enveloping algebra:*
Let $\otimes(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$ and let $\mathcal{I}$ be the smallest two-sided ideal of $\otimes(\mathfrak{g})$ containing all elements of the form

$$x \otimes y - y \otimes x - [x, y]\,,$$

for $x, y \in \mathfrak{g}$. Then the quotient $\otimes(\mathfrak{g})/\mathcal{I}$ is a universal enveloping algebra of $\mathfrak{g}$, i.e.

$$\mathcal{U}(\mathfrak{g}) \cong \otimes(\mathfrak{g})/\mathcal{I}\,.$$

Example: If $\mathfrak{g}$ is Abelian, then this construction gives the symmetric algebra $\odot(\mathfrak{g})$.

In the finite-dimensional case, combining Theorem 2 with Proposition 2 leads to a construction more familiar to physicists.

Corollary 2. *Let $\{T_i\}$ be a basis of the finite-dimensional Lie algebra $\mathfrak{g}$ over the field $\mathbb{K}$.*
The universal algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to a quotient of the free algebra $\mathbb{K} < T_i >$ and the elements of $\mathcal{U}(\mathfrak{g})$ may be thought as formal polynomials $P(T_i)$ in the generators of $\mathfrak{g}$. More precisely, the universal algebra $\mathcal{U}(\mathfrak{g})$ may be presented by the generators T_i modulo the "commutation relations" of the Lie algebra $\mathfrak{g}$

$$T_j T_k - T_k T_j = [T_j, T_k]\,.$$

Example: The **Weyl algebra A_n** is the universal enveloping algebra of the Heinsenberg algebra $\mathfrak{h}_n$, that is $A_n \cong \mathcal{U}(\mathfrak{h}_n)$. The Weyl algebra is isomorphic to the algebra of operators polynomial in the positions and momenta (*i.e.* textbook quantum mechanics) of which only the associative algebra structure is retained (in other words, A_n does not know about Hilbert space, hermiticity properties, *etc*).

By definition a representation $R_V : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism. The universal property implies the existence of an *associative algebra* homomorphism $\mathcal{R}_V : \mathcal{U}(\mathfrak{g}) \to End(V)$ which is nothing else than a representation of $\mathcal{U}(\mathfrak{g})$ acting on V. Since the map $\mathcal{R}_V$ is unique, the representations of the Lie algebra $\mathfrak{g}$ and of the associative algebra $\mathcal{U}(\mathfrak{g})$ are in one-to-one correspondence. The realization of the universal enveloping algebra in $End(V)$, *i.e.* the image $\mathcal{R}_V\big(\mathcal{U}(\mathfrak{g})\big)$, will be called here the **enveloping algebra** of the realization of $\mathfrak{g}$ on the module V, and denoted by $Env(\mathfrak{g}\,;V)$.

2.1 Poincaré-Birkhoff-Witt theorem

Theorem 3 (Poincaré-Birkhoff-Witt). *Consider any countable basis* $\{T_i\}$ *of the Lie algebra* $\mathfrak{g}$ *with elements* T_i *indexed by a subset* $I \subseteq \mathbb{N}_0$.

The set of all lexicographically ordered monomials of $\mathcal{U}(\mathfrak{g})$

$$T_{i_1}T_{i_2}\dots T_{i_k} \qquad (i_1 \leq i_2 \leq \dots \leq i_{k-1} \leq i_k\,;\ k \in \mathbb{N})$$

is a basis of $\mathcal{U}(\mathfrak{g})$.

Remarks:

☞ By the Poincaré-Birkhoff-Witt (PBW) theorem, the map i (in the abstract definition) is injective. Usually $\mathfrak{g}$ is identified with $i(\mathfrak{g})$ since i is the canonical embedding.

☞ The PBW theorem is useful in representation theory because it determines a proper basis of Verma modules that are roughly spaces of the form $\mathcal{U}(\mathfrak{g}_-)|\,h.w.\rangle$, where $\mathfrak{g}_-$ is the space of lowering operators and $|\,h.w.\rangle$ is a highest weight state (see F. Dolan's lectures for more comments).

Corollary 3. *Let* $\{T_i\}$ *be a countable basis of the Lie algebra* $\mathfrak{g}$. *The set of all distinct Weyl-ordered homogeneous polynomials*

$$T_{i_1}T_{i_2}\dots T_{i_k} + \textit{all permutations}\,,$$

is a basis of $\mathcal{U}(\mathfrak{g})$. *Therefore, the universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ *and the symmetric algebra* $\odot(\mathfrak{g})$ *are isomorphic <u>as vector spaces</u>.*

Exercise 5: Prove Corollary 3 by showing that the change of basis passing from lexicographic to Weyl ordering is triangular. ✍

In order to illustrate Corollary 3, let us introduce a variable t_i for each generator T_i of $\mathfrak{g}$. Using the isomorphism of Proposition 4, we identify the symmetric algebra $\odot(\mathfrak{g})$ with the polynomial algebra $\mathbb{K}[t_i]$. It is convenient to use the set of all distinct completely symmetric products of the variables t_i as a basis of $\mathbb{K}[t_i]$. In such case any polynomial reads

$$P(t_i) = \sum_{k=0} \Pi^{i_1\dots i_k}\, t_{i_1}\dots t_{i_k}$$

where the coefficients $\Pi^{i_1 \ldots i_k} \in \mathbb{K}$ are symmetric over all contravariant indices. Consequently, to any polynomial $P(t_i) \in \mathbb{K}[t_i]$ corresponds a unique Weyl-ordered polynomial $P_W(T_i) \in \mathbb{K} < T_i >$, given by

$$P_W(T_i) = \sum_{k=0} \Pi^{i_1 \ldots i_k} \, T_{i_1} \ldots T_{i_k} \,,$$

and conversely. A Weyl-ordered polynomial in the generators is called a **Weyl polynomial**. The **symbol** of the Weyl polynomial $P_W(T_i) \in \mathbb{K} < T_i >$ is simply defined as the polynomial $P_W(t_i) \in \mathbb{K}[t_i]$ obtained by replacing each generator with its counterpart t_i.

Remarks

☞ The symbol of a Weyl polynomial is not necessarily written in Weyl-ordered form since it is an ordinary polynomial in the polynomial algebra $\mathbb{K}[t_i]$ where the variables t_i commute.

☞ For complex algebras, a useful – though somewhat formal – alternative bijection is given by the "double Fourier transform" formula

$$P_W(T_i) = \int d^n p \; \widetilde{P}(p^j) \;\; \exp(i \, p^i T_i) \,,$$

where $\widetilde{P}$ is the formal[3] Fourier transform of the polynomial P

$$\widetilde{P}(p^j) = \frac{1}{(2\pi)^n} \int d^n t \;\, P(t_i) \;\; \exp(-i \, p^i t_i) \,.$$

By construction, for any (not necessarily written in Weyl-ordered form) polynomial $P(t_i)$ the resulting polynomial $P_W(T_i)$ is Weyl-ordered because the power series expansion of the exponential does the job.

2.2 Weyl map and star product

The bijection between symbols and Weyl polynomials is extremely straightforward at the vector space level (it is even a mere exchange of arguments in

[3] Actually, the Fourier transform of a polynomial should require some care because it is neither a polynomial nor a power series in x but a distribution. I will not enter in these issues and consider this as a formal definition of P_W which, in any case, is not problematic because one plays with polynomials only (at the beginning and the end of the procedure).

the Weyl-ordered bases). However, the non-commutativity introduces some complications since in general the product of two Weyl-ordered polynomials is not properly ordered any more. Indeed, cumbersome reordering of the generators T_i should be performed before getting a Weyl polynomial. Moreover, the latter does not correspond to the naive (commutative) product of symbols. In order to take into account the reordering performed in $\mathcal{U}(\mathfrak{g})$ one should look for its counterpart in $\mathbb{K}[t_i]$, *i.e.* deform its commutative product.

The associative (but in general non-commutative) **algebra of symbols** $\star(\mathfrak{g})$ is defined as the space $\mathbb{K}[t_i]$ of polynomials endowed with a product $\star$ such that the **Weyl map** between the space of symbols onto the universal enveloping algebra

$$W : \star(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g}) : P \mapsto W[P] := P_W$$

is an associative algebra isomorphism:

$$W[P \star Q] \,=\, W[P]\; W[Q]\,, \qquad \forall\, P, Q \in \mathbb{K}[t_i]\,.$$

The product in $\mathcal{U}(\mathfrak{g})$ is determined from the Lie algebra bracket, hence the right-hand-side is known. The crucial point is that the polynomial in the generators $W[P]\,W[Q]$ should be Weyl-ordered in order to determine the left-hand-side and get the symbol $P \star Q$. This procedure uniquely defines the product $\star$. More explicit formulas of the latter are presented in the next subsection.

Remark ☞ Since the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is unique, up to isomorphisms, it is clear that the algebra $\star(\mathfrak{g})$ of symbols is simply another realization of the same algebra. Nevertheless, it is quite useful from conceptual ($\hookleftarrow$ deformation quantization) and computational ($\hookleftarrow$ Weyl calculus) points of view. The $\star$-product is indeed a bidifferential operator that may be expanded as a formal power series in the structure constants.

2.3 Weyl calculus

The "composition formula" of the universal enveloping algebra has been written explicitly first by Berezin. It has been subject to many generalizations afterwards by other mathematicians. The explicit formulae for the star product are the cornerstone of the Weyl calculus and found direct application in physics for field theories on non-commutative space-times with constant or linear commutation relations.

On the one hand, the Berezin formula implements the left multiplication by a symbol P (*i.e.* the endomorphism $P\star$) as a differential operator. On the other hand, the BCH product is an explicit formula for the $\star$-product acting as a bidifferential operator. The main ideas of their proofs is to use the "double Fourier transform" formula of Subsection 2.1 and to interpret the exponentials of Lie algebra elements (in the Fourier transformations) as Lie group elements.

Theorem 4 (Berezin's formula). *Let $\mathfrak{g}$ be a complex Lie algebra of dimension $n \in \mathbb{N}$ with structure constants defined by*

$$[T_j, T_k] = i\, f^i{}_{jk}\, T_i \qquad (i, j, k = 1, 2, \ldots, n)\,.$$

The left multiplication by a given symbol $P(t_i)$ is the differential operator

$$P(t_i) \star = P_W\Big(t_j\, \mathcal{L}^j{}_i(\frac{\partial}{\partial t})\Big)\,,$$

where $\mathcal{L}^j{}_i(Y)$ are the components of the left shift matrix $\mathcal{L}_Y \in End(\mathfrak{g})$.

In Appendix A, a proof of Theorem 4 is presented, which uses only the corollary 1.

The Berezin formula is of interest because it shows that it is sufficient to know the BCH formula at first order in order to determine the left $\star$-multiplication. Nevertheless, if the BCH formula is known at all orders, then the following formula is much more convenient (because it does not involve Weyl symmetrization of one factor and it puts both factors on equal footing).

Theorem 5 (Baker-Campbell-Haussdorff's product). *The product of two polynomial symbols of $\star(\mathfrak{g})$ is a bidifferential operator acting as*

$$P(t) \star\, Q(t) = \exp\Big(t_i\, m^i\big(\frac{\partial}{\partial u}\,,\,\frac{\partial}{\partial v}\big)\Big) P(u)\, Q(v)\Big|_{u=v=t}$$

where the functions m^i are the components of

$$m(X, Y) = \log(e^X e^Y) - X - Y\,,$$

in some basis $\{T_i\}$.

The proof is a straightforward application of the BCH formula together with a couple of Fourier transformations and integration by parts. It is given in Appendix B.

Example: Theorem 5 applied to the Lie brackets of the Heisenberg algebra $\mathfrak{h}_n$ leads to the well-known **Moyal product** in the Weyl algebra A_n (see S. Cnockaert's lectures for more details).

3 Casimir operators

3.1 Intertwiners

In representation theory, it is useful to find maps between modules which commute with the action of the algebra. Let $\mathcal{A}$ be an algebra, let R_V and R_W be two representations of $\mathcal{A}$ acting respectively on the spaces V an W. A linear map $\phi : V \to W$ is called a **intertwiner of representations** if

$$\phi \circ R_V(x) = R_W(x) \circ \phi \quad \forall x \in \mathcal{A}.$$

If $V = W$ then an intertwiner $\phi : V \to V$ is called a **self-intertwiner**.

Theorem 6 (Schur's lemma). *Let $\phi \in End(V)$ be a self-intertwiner of a finite-dimensional irreducible complex module V. Then $\phi = \lambda I$, for some $\lambda \in \mathbb{C}$. The space of self-intertwiners of a finite-dimensional irreducible complex module is isomorphic to $\mathbb{C}$.*

Corollary 4. *Let R be a finite-dimensional irreducible representation of a complex algebra $\mathcal{A}$. If z is in the center of $\mathcal{A}$, then $R(z) = \lambda I$, with $\lambda \in \mathbb{C}$.*

Proof: The central elements $z \in \mathcal{Z}(\mathcal{A})$ are represented as self-intertwiners $\phi := R(z) \in End(V)$ because $R : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism. □

Lie algebra representations are in one-to-one correspondence with their universal enveloping algebra representations. It is therefore useful to know the basis elements of the center of $\mathcal{U}(\mathfrak{g})$ since they act as multiple of the identity on irreducible complex representation spaces and their corresponding complex values may characterize the irreducible module V.

3.2 Quadratic and higher order Casimir operator

Lemma 1. *The center $\mathcal{Z}\big(\mathcal{U}(\mathfrak{g})\big)$ of the universal enveloping algebra is equal to the centralizer $\mathcal{C}_{\mathcal{U}(\mathfrak{g})}(\mathfrak{g})$ of the (canonically embedded) Lie algebra $\mathfrak{g}$.*

Proof: Of course, any element of the center of the universal enveloping algebra must commute with all elements of any subset of $\mathcal{U}(\mathfrak{g})$, *e.g.* the canonically embedded Lie algebra $\mathfrak{g}$. Moreover, by the PBW theorem any element that commutes with a basis of $\mathfrak{g}$ commutes with any basis of $\mathcal{U}(\mathfrak{g})$. □

Theorem 7 (Gel'fand). *Let $\{T_i\}$ be any basis of a finite-dimensional Lie algebra $\mathfrak{g}$ with Lie brackets $[T_j, T_k] = f^i{}_{jk}\, T_i$.*

A Weyl-ordered polynomial $P_W(T)$ belongs to the center $\mathcal{Z}\big(\mathcal{U}(\mathfrak{g})\big)$ of the universal enveloping algebra if and only if the corresponding symmetric tensor $\Pi \in \odot(\mathfrak{g})$ is invariant under the adjoint action. More explicitly, the Weyl polynomial $P_W(T_i) = \sum \Pi^{i_1 \dots i_n}\, T_{i_1} \dots T_{i_n}$ (with completely symmetric contravariant components) commutes with all elements of $\mathcal{U}(\mathfrak{g})$ iff

$$0 = f^{i_1}{}_{k\ell}\, \Pi^{\ell i_2 \dots i_n} + \ldots + f^{i_n}{}_{k\ell}\, \Pi^{i_1 \dots i_{n-1} \ell}\,.$$

Exercise 6: Prove Gel'fand's Theorem by only verifying that $P_W(T)$ commutes with any basis element T_k of $\mathfrak{g}$. ✍

The **Casimir operators** of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ are a distinguished basis of the center $\mathcal{Z}\big(\mathcal{U}(\mathfrak{g})\big)$ of the universal enveloping algebra made of homogeneous polynomials

$$\mathcal{C}_k = d^{i_1 \dots i_k} T_{i_1} \dots T_{i_k}$$

with $d^{i_1 \dots i_k}$ suitable symmetric invariant tensors of the adjoint representation. The degree k of this homogeneous polynomial is called the **order** of $\mathcal{C}_k$.

Example: The inverse matrix of the Killing form κ^{ij} defines the **quadratic Casimir operator** equal to

$$\mathcal{C}_2 := \kappa^{ij} T_i T_j\,.$$

The quadratic Casimir $\mathcal{C}_2$ operator belongs to the center of $\mathcal{U}(\mathfrak{g})$ since the Killing form κ_{ij} is invariant under the adjoint action.

Theorem 8. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of rank r.*

The center $\mathcal{Z}\big(\mathcal{U}(\mathfrak{g})\big)$ of the universal enveloping algebra is isomorphic to the polynomial algebra $\mathbb{K}[\mathcal{C}^{(i)}]$ over the base field in r variables $\mathcal{C}^{(i)}$ ($i = 1, 2, \dots, r$). Therefore, the number of algebraically independent Casimir operators is equal to the rank.

For applications in physics, the Casimir operators are sometimes more useful than the r Dynkin labels (of which they are non-linear functions) because they more often correspond to physical quantities (such as the square of the momentum or of the Pauli-Lubanski vector).

4 Symmetries of the S-matrix

4.1 Rudiments of scattering theory

A proper beginning to address the symmetries of the S-matrix is a brief review of some fundamental definitions of scattering theory.

To start up with first quantization, Wigner showed that the rules of quantum mechanics, combined with the principle of special relativity, imply that the classification of all possible wave equations $\mathrm{K}|\,\psi\rangle = 0$ describing the evolution of the states $|\,\psi\rangle \in \mathcal{H}$ of a free relativistic particle moving in the Minkowski space $\mathbb{R}^{d-1,1}$ is equivalent to the classification of all unitary irreducible representations of the Poincaré algebra $\mathfrak{iso}(d-1,1)$. Moreover, Wigner proved that unitary irreducible $\mathfrak{iso}(d-1,1)$-modules are labeled by the mass-square m^2 (a continuous real parameter), the "spin" degrees of freedom s (for finite component representations, these are a finite set of non-negative (half) integers). Therefore, the **one-particle Hilbert space** $\mathcal{H}^{(1)} \subset \mathcal{H}$ of solutions decomposes into a direct sum of unitary irreducible $\mathfrak{iso}(d-1,1)$-modules

$$\mathcal{H}^{(1)} = \bigoplus_{m^2,\,s,\,i} \mathcal{H}_{(m^2,\,s,\,i)}$$

where the index i labels the particle type. A standard basis is made of plane-wave states $|\,p, s, i\,\rangle$ where p_μ is the momentum.

In second quantization, the **Hilbert space of scattering theory** is the direct sum

$$\mathsf{In} := \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n)}\,,$$

of the **n-particle Hilbert spaces** $\mathcal{H}^{(n)}$ that are subspaces of n-fold tensor products $\otimes^n \mathcal{H}^{(1)}$, determined by the generalized exclusion principle (for instance, in the case without fermions $\mathcal{H}^{(n)} = \odot^n \mathcal{H}^{(1)}$). The dual of the **in-going particle space** In is called the **out-going particle space** and will be denoted by $\mathsf{Out}(:= \mathsf{In}^*)$. The **$S$-matrix** is a unitary operator on In, *i.e.* $\mathrm{S} \in U(\mathsf{In})$. A **scattering amplitude** is a complex number given by $\langle out|\,\mathrm{S}|in\rangle$ for some elements $|in\rangle \in \mathsf{In}$ and $\langle out| \in \mathsf{Out}$.

4.2 Higher symmetries

Since the kinetic operator K is Hermitian, the linear wave equation $\mathrm{K}|\,\psi\rangle = 0$ comes from an action principle: $S[\psi] = \langle\psi|\,\mathrm{K}|\,\psi\rangle$. A **local symmetry transformation of the free action** is a unitary operator U on $\mathcal{H}$ that commutes with K and acts locally, in the sense that $\langle x, s, i|\,\mathrm{U}|\,\psi\rangle$ can be written in the form $\langle x', s', i'|\psi\rangle$ where $x' = f(x)$ is a smooth change of coordinates. The group of all such transformations is said to be the **local symmetry group of the free action**.

A **symmetry transformation of the S-matrix** is a unitary operator on In that commutes with S (so that it leaves scattering amplitudes invariant), turns one-particle states into one-particle states and acts on many-particle states as tensor products of one-particle states. The group G of such transformations is said to be the **symmetry group of the S-matrix**. The S-matrix is said to be **Poincaré invariant** if $G \supseteq ISO(d-1,1)$.

It turns out to be convenient to translate the previous definitions into Lie algebraic language: the (Lie) algebra $\mathfrak{g}$ corresponds to the symmetry group G and a

- **local symmetry generator of the free action** is a Hermitian differential operator H on $\mathcal{H}$ that

  (a) commutes with the kinetic operator, $\mathrm{HK} = \mathrm{KH}$,

  (b) is at most linear in the momentum operator P_a.

- **symmetry generator of the S-matrix** is a Hermitian operator H on In that

  (i) commutes with the S-matrix, $\mathrm{HS} = \mathrm{SH}$,

  (ii) turns one-particle states into one-particle states, $\mathrm{H}\mathcal{H}^{(1)} \subseteq \mathcal{H}^{(1)}$,

  (iii) acts additively on many-particle states (roughly, "like" the Leibnitz rule). More precisely, it acts on multiparticle spaces $\mathcal{H}^{(n)}$ as the nth tensor product module of $\mathcal{H}^{(1)}$ (see Section 1.5 for the definition).

  As shown by J.T. Lopuszanski and others, the property (iii) necessarily follows from the *locality* of the symmetry transformations (together with some natural asymptotic conditions).

Contemporary theoretical physics has frequently questioned the ansatz of locality and it has become common to relax locality (at tiny scales and in a controlled way!) in order to build new physical models (*e.g.* extended objects, non-commutative space-time, *etc*). With such considerations in mind, it might be interesting to speculate on the relaxation of the last conditions, which are the less natural. Therefore one may extend the previous definition to **higher(-derivative) symmetry generators of the free action** for which the condition (b) is relaxed, and **higher symmetry generators of the S-matrix** for which the last condition (iii) is removed. They are allowed to be general differential operators (i.e. smooth functions of the momentum operator P_a). More comments (and contact with enveloping algebras) will be made in Section 4.4.

4.3 No-go theorem

Most frequently, the Coleman-Mandula theorem is interpreted as the destruction of any reasonable hope for a fusion between internal and space-time symmetries. (An **internal symmetry of the S-matrix** is one that commutes with Poincaré transformations. This implies that it acts on particle-type indices only.) But another far-reaching consequence of the no-go theorem is the assertion that the maximal space-time symmetry algebra of relativistic quantum field theory in flat space-time is the (super)conformal group.

Theorem 9 (Coleman-Mandula). *Let $\mathfrak{g}$ be a Lie algebra of symmetries of the matrix S and let the following four conditions hold:*

1. *(Poincaré invariance)* $\mathfrak{g} \supseteq \mathfrak{iso}(d-1,1)$,
2. *(Particle-finiteness) All unitary $\mathfrak{iso}(d-1,1)$-modules $\mathcal{H}_{(m^2,\,s,\,i)}$ are finite-dimensional for fixed mass-square. Moreover, for any finite mass M, there are only a finite number of particle types with mass less than M,*
3. *(Weak elastic analyticity) Elastic-scattering amplitudes are analytic functions of the scattering angle at almost all energies and angles,*
4. *(Occurence of scattering) Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates of $\mathcal{H}^{(1)}$. If $|p,p'\rangle$ denotes the two-particle state of $\mathcal{H}^{(2)}$ made from these, then $\mathrm{S}|p,p'\rangle \neq |p,p'\rangle$ at almost all energies.*

Then, the S-matrix symmetry algebra $\mathfrak{g}$ is isomorphic to the direct sum

$$\mathfrak{g} \cong \mathfrak{g}_{internal} \oplus \mathfrak{g}_{space-time}$$

of an internal symmetry Lie algebra $\mathfrak{g}_{internal}$ and a space-time symmetry Lie algebra $\mathfrak{g}_{space-time}$. Moreover, the space-time symmetry algebra $\mathfrak{g}_{space-time}$ can be the conformal algebra $\mathfrak{so}(d,2)$ only if all particles are massless. In all other cases, it is the Poincaré algebra $\mathfrak{iso}(d-1,1)$.

Remarks:

☞ The S-matrix symmetry algebra $\mathfrak{g}$ is not assumed to be finite-dimensional.

☞ The extension of the theorem to the massless and supersymmetric case is actually due to Haag, Lopuszanski and Sohnius, who obtained the (super)conformal group as maximal possibility for $\mathfrak{g}_{space-time}$.

☞ For almost each hypothesis of the theorem, there exists a famous counterexample to the conclusions of Coleman-Mandula theorem which violates the corresponding assumption. For instance,

1. There are examples of Galilean-invariant models (such as the $SU(6)$-invariant model of non-relativistic quarks) which unite multiplets of different spin and thus contradict the conclusions of the theorem.

2. Tensionless strings violate the second assumption since the spectrum contains an infinite set of massless particles.

3. Symmetry groups are known which are not direct products, but which do allow scattering, although only in the forward and backward directions.

4. If the S-matrix is the identity, the underlying field theory is free and the symmetry algebra may be enhanced. Notice that the free case is still of interest if the theory is conformal[4], as was argued by Sundborg and Witten in 2000, because in the tensionless limit the AdS/CFT correspondence should relate an interacting theory in the bulk to a free theory on the boundary.

[4]Strictly speaking, for conformal field theories the S-matrix does not exist (since asymptotic states are not well defined because of the scale invariance) and has to be replaced by correlation functions.

0. The most celebrated counter-example to Coleman-Mandula's theorem is supersymmetry which violates the initial "hidden" assumption that the algebra is a Lie algebra instead of the more general possibility of a Lie *superalgebra*.

☞ The no-go theorem has direct application only for symmetries of the S-matrix, *i.e.* "physical" or "visible" symmetries. In other words, gauge symmetries are beyond the scope of the no-go theorem. Nevertheless, local symmetry groups contain rigid symmetries as a subgroup. As a conclusion, the theorem restricts the possibilities of reasonable candidates for rigid symmetries to be gauged. Indeed the standard model corresponds to the gauging of an internal symmetry group while (super)gravity roughly corresponds to the gauging of the (super)Poincaré group.

☞ As a corollary, the S-matrix no-go theorems impose some constraints on the spin of the gauge fields associated with the gauged symmetries. Indeed, gauging a symmetry group G requires the introduction of a connection taking values in the symmetry algebra $\mathfrak{g}$. On the one hand, when the algebra is internal the generators T_i do not carry any space-time index, hence the gauge field is a vector gauge field A^i_μ. On the other hand, for a space-time symmetry algebra the generators do carry space-time indices and the Poincaré generator P_a is associated to the vielbein e^a_μ which in turn leads to a spin-two field: the metric $g_{\mu\nu}$ (correspondingly, the supersymmetry generator leads to a Rarita-Schwinger field of spin three-half). The restriction on the space-time symmetry algebra therefore rules out gauge fields of higher (*i.e.* greater than two) spin.

4.4 Yes-go conjecture

To evade the conclusions of the S-matrix no-go theorems one should remove (at least) one of its assumptions. Symmetries mixing particles with different spin were already obtained by introducing graded symmetry generators: the supersymmetry generators are somehow "squareroots" of the translation generators. Thus, it is tempting to try the reverse procedure by introducing new generators that are "powers" $\mathrm{P}_{a_1}\dots\mathrm{P}_{a_m}$ of the translation generators. The corresponding connections $e^{a_1\dots a_m}_\mu$ may lead to a symmetric tensor gauge fields $g_{\mu\,\mu_1\dots\mu_m}$ of rank ("spin") equal to $m+1$. Including Lorentz generators in the game naturally leads to the enveloping algebra of the space-time symmetry algebra! The main point is that such powers of the generators are no more standard symmetries but rather higher symmetries (defined at the end of Section 4.1) generating space-time symmetry transformations which are *non-local*.

Exercise 7: Show that Weyl-ordered polynomials of Hermitian operators satisfying the properties or (i)-(iii) are Hermitian and do obey the properties (i)-(ii) but violate[5] the axiom (iii). Idem with the properties (a)-(b). ✍

Therefore, one may propose the following rather general way to circumvent usual no-go theorems on maximal symmetry groups:

◇ Given a symmetry group $G \subset U(\mathcal{H})$ of unitary operators acting on some Hilbert space $\mathcal{H}$, one first builds the enveloping algebra $Env(\mathfrak{g}\,;\mathcal{H}) \subset \mathfrak{u}(\mathcal{H})$ of the realization of $\mathfrak{g}$ as Hermitian operators acting on $\mathcal{H}$. By Proposition 3, this associative algebra $Env(\mathfrak{g}\,;\mathcal{H})$ is isomorphic to the quotient $\mathcal{U}(\mathfrak{g})/Ann(\mathcal{H})$ of the universal enveloping algebra of the symmetry algebra $\mathfrak{g}$ by the annihilator of the $\mathcal{U}(\mathfrak{g})$-module $\mathcal{H}$.

◇ Secondly, one makes use of Proposition 6 to define its commutator algebra[6] $[Env(\mathfrak{g}\,;\mathcal{H})]$ that I propose to call **higher-symmetry algebra** of the $\mathfrak{g}$-module $\mathcal{H}$ and to denote by $\mathtt{hs}(\mathfrak{g}\,;\mathcal{H})$.

◇ Thirdly, since the latter algebra is a Lie algebra one may exponentiate it (at least formally) to get an **higher-symmetry group** $\mathtt{HS}(G\,;\mathcal{H})$ of the group G.

$$
\begin{tikzcd}
G \arrow[rr, hook] \arrow[d, "\log"'] & & \mathtt{HS}(G\,;\mathcal{H}) \\
\mathfrak{g} \arrow[rr, hook] \arrow[dr, "i"'] & & \mathtt{hs}(\mathfrak{g}\,;\mathcal{H}) \arrow[u, "\exp"'] \\
& Env(\mathfrak{g}\,;\mathcal{H}) \arrow[ur, "{[\;]}"'] &
\end{tikzcd}
$$

To conclude this more speculative section, one may summarize the previous considerations in the following loose conjecture:

[5] This last property is intimately related to the fact that the universal enveloping algebra has a canonical Hopf algebra structure (used in the construction of the non-commutative theory of gravity presented by F. Meyer in his talk). In this context, the property (iii) is christened "primitivity" and it is known that the space of primitive elements of $\mathcal{U}(\mathfrak{g})$ is precisely $\mathfrak{g}$.

[6] Commutator of enveloping algebras were first considered in physics by E. S. Fradkin and V. Y. Linetsky in 1990.

If one allows higher symmetries, then there exist counter-examples to the conclusions of symmetry no-go theorems, and the "gauging" of higher-space-time symmetry algebras $\mathtt{hs}(\mathfrak{g}_{space-time};\mathcal{H})$ *should correspond to some consistent higher-spin gauge theories.*

Remarks:

☞ Actually, these speculations are a mere retrospective viewpoint on the higher-spin gauge theories introduced by M. A. Vasiliev where the gauge fields take values in the higher-symmetry algebra of the defining representation of the anti de Sitter isometry algebra $\mathfrak{o}(d-1,2)$ (*i.e.* the enveloping algebra of AdS_d Killing tensors). Nevertheless, the present perspective underlines how general such constructions might be and suggests that the rigid counterpart of the gauge symmetries are actually realized as genuine (higher-derivative) space-time symmetries of the corresponding theories.

☞ If these rough speculations are correct, then an interesting issue is whether higher-spin symmetries are either realized on quadratic actions and satisfy the Leibnitz rule, or violate the Leibnitz rule and be realized on non-linear matter field theories (i.e. be free action or interacting S-matrix higher-symmetries).

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Appendices

A Proof of Berezin's formula

We decompose the proof into some lemmas that we glue together at the end.

Lemma 2. *Let*

$$P(t_i)=\sum_{k=0}\Pi^{i_1\dots i_k}\,t_{i_1}\dots t_{i_k}$$

be a symbol of $\mathbb{K}[t_i]$ *written with coefficients* $\Pi^{i_1\dots i_n}\in\mathbb{K}$ *which are symmetric* *over all indices. The* ***stared symbol*** *defined as*

$$P^\star(t_i) := \sum_{k=0} \Pi^{i_1\dots i_k}\, t_{i_1} \star \dots \star t_{i_k}$$

is equal to the original symbol: $P^\star(t_i) = P(t_i)$ *in* $\star(\mathfrak{g})$*.*

Proof: Indeed, their Weyl polynomials are equal, $W[P^*] = W[P]$. More explicitly,

$$\begin{aligned} W[P^*(t)] &= \sum_{k=0} \Pi^{i_1\dots i_k}\, W[t_{i_1} \star \dots \star t_{i_k}] \\ &= \sum_{k=0} \Pi^{i_1\dots i_k}\, W[t_{i_1}] \dots W[t_{i_k}] = P_W(T)\,, \end{aligned}$$

by linearity, the algebra homomorphism property and the definition of the Weyl map, since $W(t_i) = T_i$. $\square$

Lemma 3. *The product of two polynomial symbols of* $\star(\mathfrak{g})$ *is equal to*

$$P(t) \star Q(t) = P_W(t\star)\, Q(t)\,,$$

where the right-hand-side should be interpreted as the action of the linear operator $P_W(t\star)$ *on the polynomial* $Q(t)$*. The linear operator* $P_W(t\star)$ *is defined as the Weyl-ordered polynomial function of the elementary operators of left multiplication* $t_i\star$ *that act as* $t_i \star Q(t)$*.*

Proof: From Lemma 2, one gets the first line of the chain of equalities

$$\begin{aligned} P(t) \star Q(t) &= \Big(\sum \Pi^{i_1\dots i_k}\, t_{i_1} \star \dots \star t_{i_k}\Big) \star Q(t) \\ &= \sum \Pi^{i_1\dots i_k}\, (t_{i_1} \star \dots \star t_{i_k}) \star Q(t) \\ &= \sum \Pi^{i_1\dots i_k}\, \Big(t_{i_1} \star \Big(\dots \star \Big(t_{i_k} \star Q(t)\Big) \dots\Big)\Big) \\ &= \sum \Pi^{i_1\dots i_k}\, (t_{i_1}\star) \dots (t_{i_k}\star)\, Q(t) \\ &= P_W(t\star)\, Q(t)\,, \end{aligned}$$

where we used the linearity and associativity properties of the star product $\star$ of symbols and of the composition $\circ$ of operators. $\square$

As a consequence of Lemma 3, in order to compute the product of any symbols, it is sufficient to know explicitly the action of the elementary operators of left multiplication $t_i\star$. Actually, the latter result was the formula obtained originally by Berezin:

Lemma 4. *The elementary operators of left multiplication in $\star(\mathfrak{g})$ are explicitly given by*

$$t_i\star \;=\; t_j\,\mathcal{L}^j{}_i\Big(\frac{\partial}{\partial t}\Big)$$

where $\mathcal{L}^j{}_i(Y)$ are the component of the matrix of the left shift operator $\mathcal{L}(Y) = \mathcal{B}(ad_Y)$.

Proof: The computation is performed in terms of Weyl polynomials, that is one evaluate the product of the two Weyl polynomials

$$T_i = i\int d^n u\,\frac{\partial}{\partial u^i}\Big(\delta(u)\Big)\;\exp(i\,u^j T_j)$$

and

$$Q_W(T) = \int d^n v\;\widetilde{Q}(v)\;\exp(i\,v^j T_j)\,,$$

where $\widetilde{Q}$ is the Fourier transform of the polynomial Q. By definition of the left-shift operator, one knows that

$$\begin{aligned}\exp(i\,u^j T_j)\;\exp(i\,v^j T_j) &= \exp\Big(i\,\big(v^j + u^i\mathcal{L}^j{}_i\,(\,i\,v\,) + \mathcal{O}(u^2)\,\big)T_j\Big)\\ &= \Big(1 + u^i\mathcal{L}^j{}_i\,(\,i\,v\,)\frac{\partial}{\partial v^j} + \mathcal{O}(u^2)\Big)\exp(i\,v^j T_j)\end{aligned}$$

Therefore, the product of the Weyl polynomials is equal to

$$\begin{aligned}&T_i\,Q_W(T) =\\ =\;& i\int d^n u\,d^n v\,\frac{\partial}{\partial u^i}\Big(\delta(u)\Big)\;\widetilde{Q}(v)\;\Big(1 + u^k\mathcal{L}^j{}_k\,(\,i\,v\,)\frac{\partial}{\partial v^j} + \mathcal{O}(u^2)\Big)\exp(i\,v^j T_j)\\ =\;& -i\int d^n v\;\widetilde{Q}(v)\;\mathcal{L}^j{}_i\,(\,i\,v\,)\frac{\partial}{\partial v^j}\exp(i\,v^j T_j)\\ =\;& i\int d^n v\;\frac{\partial}{\partial v^j}\Big(\widetilde{Q}(v)\;\mathcal{L}^j{}_i\,(\,i\,v\,)\Big)\exp(i\,v^j T_j)\,.\end{aligned}$$

Hence, the corresponding symbol is by definition

$$\begin{aligned} t_i \star Q(t) &= i \int d^n v \, \frac{\partial}{\partial v^j} \Big(\widetilde{Q}(v) \; \mathcal{L}^j{}_i \, (\, i\, v\,) \Big) \exp(i\, v^j t_j) \\ &= t_j \int d^n v \; \widetilde{Q}(v) \; \mathcal{L}^j{}_i \, (\, i\, v\,) \; \exp(i\, v^j t_j) \\ &= t_j \, \mathcal{L}^j{}_i \left(\frac{\partial}{\partial t} \right) \int d^n v \; \widetilde{Q}(v) \; \exp(i\, v^j t_j) \\ &= t_j \, \mathcal{L}^j{}_i \left(\frac{\partial}{\partial t} \right) Q(t)\,. \end{aligned}$$

$\square$

B Proof of BCH's product

The computation is performed in terms of Weyl polynomials, that is one evaluate the product of the two Weyl polynomials

$$P_W(T) = \int d^n r \; \widetilde{P}(r) \; \exp(i\, r^j T_j)\,,$$

and

$$Q_W(T) = \int d^n s \; \widetilde{Q}(s) \; \exp(i\, s^j T_j)\,,$$

where $\widetilde{P}$ and $\widetilde{Q}$ are the Fourier transform of the respective polynomials P and Q. From Theorem 1, one knows that

$$\exp(i\, r^j T_j) \; \exp(i\, s^j T_j) = \exp \Big[\big(\, i\, (r^j + s^j) + m^j(\, ir \,,\, is \,) \big) \, T_j \Big]\,,$$

where the functions m^j are defined in Theorem 5. Therefore, the product of the Weyl polynomials is equal to

$$P_W(T)\, Q_W(T) = \int d^n r \, d^n s \, \widetilde{P}(r) \; \widetilde{Q}(s) \; \exp \Big[\big(\, i\, (r^j + s^j) \; + \; m^j(\, ir \,,\, is \,) \big) \, T_j \Big]\,,$$

and the corresponding symbol is by definition the star product

$$
\begin{aligned}
& P(t) \star Q(t) = \\
= \; & \int d^n r \, d^n s \, \widetilde{P}(r) \; \widetilde{Q}(s) \; \exp\big(i \, (r^j + s^j) \, t_j \big) \; \exp\big(t_j \, m^j(\, ir \, , \, is \,) \big) \\
= \; & \frac{1}{(2\pi)^{2n}} \int d^n r \, d^n s \, d^n u' \, d^n v' \, P(u') \; Q(v') \; \times \\
& \qquad \exp\big(- i \, r^j (u'_j - t_j) \big) \; \exp\big(- i \, s^j (v'_j - t_j) \big) \exp\big(t_j \, m^j(\, ir \, , \, is \,) \big) \\
= \; & \frac{1}{(2\pi)^{2n}} \int d^n r \, d^n s \, d^n u \, d^n v \, P(t+u) \; Q(t+v) \; \times \\
& \qquad \qquad \exp\big(t_j \, m^j(\, ir \, , \, is \,) \big) \; \exp\big(- i \, (r^j u_j + s^j v_j) \big) \\
= \; & \frac{1}{(2\pi)^{2n}} \int d^n r \, d^n s \, d^n u \, d^n v \, P(t+u) \; Q(t+v) \; \times \\
& \qquad \qquad \exp\Big(t_j \, m^j \big(-\frac{\partial}{\partial u} \, , -\frac{\partial}{\partial v} \, \big) \Big) \; \exp\big(- i \, (r^j u_j + s^j v_j) \big)
\end{aligned}
$$

where we performed the changes of integration variables $u_j := u'_j - t_j$ and $v_j := v'_j - t_j$ to obtain the fourth line. Integrating by part, one gets

$$
\begin{aligned}
& P(t) \star Q(t) = \\
= \; & \frac{1}{(2\pi)^{2n}} \int d^n r \, d^n s \, d^n u \, d^n v \, \exp\big(- i \, (r^j u_j + s^j v_j) \big) \; \times \\
& \qquad \exp\Big(t_j \, m^j \big(\frac{\partial}{\partial u} \, , \frac{\partial}{\partial v} \, \big) \Big) P(t+u) \; Q(t+v)
\end{aligned}
$$

which achieves the proof of Theorem 5 since the integration over r and s provides Dirac's deltas over u and v. $\square$

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