Anatomy of a gauge theory

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Abstract

We exhibit the role of Hochschild cohomology in quantum field theory with particular emphasis on gauge theory and Dyson–Schwinger equations, the quantum equations of motion. These equations emerge from Hopf- and Lie algebra theory and free quantum field theory only. In the course of our analysis we exhibit an intimate relation between the Slavnov-Taylor identities for the couplings and the existence of Hopf sub-algebras defined on the sum of all graphs at a given loop order, surpassing the need to work on single diagrams.

0 Introduction

Over the last seven years many properties of the Hopf algebra structure of renormalization have been investigated, mostly on the mathematical side, with at least one notable exception [1] which showed how to solve a non-linear Dyson–Schwinger equation exactly, as opposed to a perturbation expansion.

In this paper, we explore the elementary relations between a perturbative expansion in quantum field theory, the corresponding Hochschild cohomology and the equations of motion in the context of a generic gauge theory. A major feature of our analysis is the emphasis on free quantum field theory and locality expressed through Hochschild cohomology which combine to specify the interacting theory.

0.1 Interaction vertices from free quantum fields and locality

We start with a free quantum field theory with free propagators given by the usual requirements of free field theory for fermion and boson fields.

Before we discuss how locality determines the structure of the full Green functions we look at the tree level first, for motivation. We want to emphasize that the Feynman rules for tree-level graphs are determined by free quantum field theory (which gives Feynman rules for the edges) and locality (which implies Feynman rules for the vertices).

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So let us first remind ourselves how the quest for locality determines the interaction vertices in the context of renormalizable theories, without recurse to a classical Lagrangian at this stage.

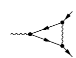
We exhibit the argument for the example of the QED vertex and aim to infer the local interaction vertex $v_\mu(q,p)$ for the interaction of a photon, with four-momentum $p-q$, with an e^+e^- pair, with four-momenta p,q. We will infere the Feynman rule for the vertex from the knowledge of the free photon, electron and positron propagator, and from the requirement that we should be able to renormalize by local counterterms.

We start with an Ansatz that at tree-level the sought-after vertex must be of the form

$$v_\mu(q,p) = \sum_{i=1}^{12} a_i f_\mu^{[i]} = a_1\gamma_\mu + a_2 \frac{\not{q} q_\mu}{q^2} + \cdots, \tag{1}$$

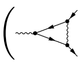
in accordance with Lorentz covariance, spin representations and dimensions of the free propagators involved. We determine $v_\mu(q,p)$ as follows.

First, we note that free quantum field theory and the requirement for renormalizability provide us with powercounting. We hence construct the 1PI graph with lowest non-vanishing loop order corresponding to our interaction:

 .

This graph is log-divergent. If we demand to have a renormalizable theory, its coefficient of log-divergence must be proportional to the sought after tree-level vertex. This is indeed the crucial clue.

By construction, the graph has no subdivergences and hence will evaluate to an expression of the form

$\phi\Big($  $\Big) \sim v_\mu(q,p)\ln\lambda + \text{finite terms},$ (2)

where we cut-off momentum integrals by λ.

Hence, we can write

$$v_\mu(q,p) \sim \lim_{\lambda\to+\infty} \frac{1}{\ln\lambda}\int_{-\lambda}^{+\lambda} d^4k \left[v_\sigma(q,k)\frac{1}{\not{k}+\not{q}} v_\mu(k+q,k+p)\frac{1}{\not{k}+\not{p}} v_\rho(k,p) D_{\sigma\rho}(k)\right]. \tag{3}$$

Inserting (1) we indeed find that

$$v_\mu(q,p) \sim \gamma_\mu, \tag{4}$$

as was to be expected.

Note that this states the obvious: in a local renormalizable quantum field theory the short-distance singularities can be absorbed by suitable modifications of parameters in the given Lagrangian. Vice-versa, choosing locality, the requirement that short-distance singularities are self-similar to the tree-level terms the starting point, the Lagrangian follows.

There is a cute message hidden here: if we settle on a given set of free fields and demand that they interact in renormalizable manner, already the lowest loop order scattering processes fix the form of the interaction vertices from the knowledge of free quantum field theory and the requirement of locality, whilst the interaction part of the Lagrangian appears as a derived quantity. Hence we will continue to explore what we can learn about QFT, in particular about a gauge theory, starting from so-determined free covariances and interaction terms, and the accompanying one-particle irreducible graphs which go with them. Our guiding principle will still be locality, in its mathematical disguise as Hochschild cohomology.

In particular, we are now interested in formal sums of graphs corresponding to a specific propagation or interaction: the evaluation under the Feynman rules of all 1PI graphs with say an external photon and an external electron pair constitutes in perturbative QFT the 1PI Green function for that amplitude. The sum over all such 1PI Green functions furnishes the effective action, by definition. From now on we will do so for a generic non-abelian gauge theory.

We hence wish to discuss the formal sums

$$\Gamma^r = \sum g^{2|\Gamma|} \frac{\Gamma}{\mathrm{sym}(\Gamma)}. \tag{5}$$

Already at this level interesting structure emerges. Our main tool will be the exploration of the Hochschild cohomology of the Hopf algebra structure which comes with such graphs. This Hochschild cohomology provides a mathematical precise formulation of locality, and will carry us far in the understanding of the structure of the theory.

0.2 Quantum equations of motion from Hochschild Cohomology

Such Hopf algebras have marvelous properties which they inherent from the universal object $\mathcal{H}_{\rm rt}$ for such algebras: the commutative Hopf algebra of non-planar rooted trees [2].

In that context, it is beneficial to study Dyson–Schwinger equations as formal constructions based on the Hochschild cohomology of such Hopf algebras. Before we justify the connection to Dyson–Schwinger equation through the study of a generic gauge theory below, let us first describe the universal set-up on rooted trees [3].

First, we settle on say a suitable Hopf algebra A which can be $\mathcal{H}_{\rm rt}$ or any suitable sub-Hopf algebra A.

Let A be any such connected graded Hopf algebra which is free or free commutative as an algebra, and $(B_+^{d_n})_{n\in\mathbb{N}}$ a collection of Hochschild 1-cocycles on it. The Dyson-Schwinger equation is

$$X = \mathbb{I} + \sum_{n=1}^{\infty} \alpha^n w_n B_+^{d_n}(X^{n+1}) \tag{6}$$

in $A[[\alpha]]$. The parameter α plays the role of a coupling constant. The w_n are scalars in k. We decompose the solution

$$X = \sum_{n=0}^{\infty} \alpha^n c_n \tag{7}$$

with $c_n \in A$.

Lemma 1 *The Dyson-Schwinger equation (6) has a unique solution described by $c_0 = \mathbb{I}$ and*

$$c_n = \sum_{m=1}^{n} w_m B_+^{d_m}\left(\sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i\geq 0} c_{k_1}\ldots c_{k_{m+1}} \right). \tag{8}$$

The c_n, coefficients in the n-th term of the perturbative expansion have a very nice property which we will rediscover in quantum field theory:

Theorem 2 *The c_n generate a Hopf subalgebra of A :*

$$\Delta(c_n) = \sum_{k=0}^{n} P_k^n \otimes c_k \tag{9}$$

where the P_k^n are homogeneous polynomials of degree $n-k$ in the c_l, $l \leq n$:

$$P_k^n = \sum_{l_1+\ldots+l_{k+1}=n-k} c_{l_1} \ldots c_{l_{k+1}}. \tag{10}$$

In particular, the P_k^n are independent of the w_n and $B_+^{d_n}$.

In this paper we want to discuss this structure when we pass from the universal object $\mathcal{H}_{\rm rt}$ to the concrete Hopf algebra of say a generic gauge theory. We aim at insights into the non-perturbative structure of QFT and should also prepare for new methods of computation. In particular we use the fact that the operadic proof of the above theorem given in [3] extends to our case once the proper insertion maps for graph insertions have been defined.

We will focus on the case of a generic non-abelian gauge theory, and exhibit how the Hochschild cohomology of the Hopf algebra of its perturbative expansion, the equations of motion and local gauge symmetry interfere. To formulate our results we first introduce the pre-Lie algebra of Feynman graphs in this context in the next section.

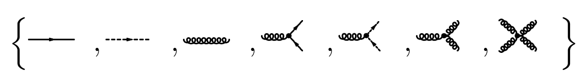
We then introduce our result and discuss it with the help of a completely worked out two-loop example.

1 Graphs

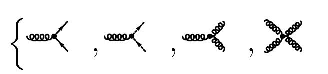
In this section we first define graphs and the accompanying pre-Lie and Hopf algebras. The material is a straightforward application of previous results to a generic gauge theory.

1.1 The set of graphs

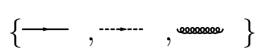
All graphs we consider are built from the following set R of edges and vertices

$R =$  . (11)

We subdivide into edges and vertices,

$R_{\rm V} =$  , (12)

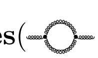
and

$R_{\rm E} =$  . (13)

Obviously, $R = R_{\rm E} \cup R_{\rm V}$. We have included edges for the free propagation of the local gauge field, corresponding ghost fields, and fermion fields as the only matter fields. We exclude the discussion of scalar matter fields coupled to gauge fields which deserve a separate discussion in future work. We thus include only the expected vertices in a generic local gauge theory: triple and quartic self-interactions of the gauge field, an interaction of the gauge field with its ghost field and minimal interaction between the gauge and matter fields - $R_{\rm V}$ is determined by $R_{\rm E}$ and locality in the spirit of the argument in the previous section.

We then define one-particle irreducible (1PI) graphs as usual: they remain connected after removal of any one of the internal edges. For such a 1PI graph Γ we have external legs $\Gamma^{[1]}_{\rm ext}$, internal edges $\Gamma^{[1]}_{\rm int}$ and vertices $\Gamma^{[0]}$.

For any 1PI graph Γ we let $\mathbf{res}(\Gamma)$ be the graph when all its internal edges shrink to a point

$\mathbf{res}($  $) =$. (14)

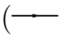
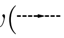
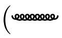
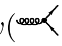
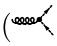
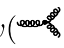
We call $\mathbf{res}(\Gamma)$ the residue of Γ to emphasize that a graph Γ provides a counterterm $S_R^\phi(\Gamma)$ which contributes $S_R^\phi(\Gamma)\phi(\mathbf{res}(\Gamma))$ to the Lagrangian

$$\mathcal{L} = \sum_{r\in R_{\mathrm{E}}} 1/\phi(r) + \sum_{r\in R_{\mathrm{V}}} \phi(r), \tag{15}$$

where we note that for $r \in R_{\mathrm{E}}$ $1/\phi(r)$ is the inverse free propagator, as it should be. We extend the notion of a residue of a graph to a product $\Pi_i(\Gamma_i)$ of graphs:

$$\mathbf{res}\,(\Pi_i\Gamma_i) = \Pi_i\mathbf{res}(\Gamma_i). \tag{16}$$

Any element r in R has a superficial degree of divergence (sdd), $w(r)$, given as follows

$w($$) = 1, w($$) = 1, w($$) = 2, w($$) = 0,$
$w($$) = 0, w($$) = -1, w($$) = 0.$ (17)

We introduce the loop number $|\Gamma|$,

$$|\Gamma| = \mathrm{rank}(H_1(\Gamma)), \tag{18}$$

where $H_1(\Gamma)$ is the first homology group of Γ, and

$$\left|\prod_i \Gamma_i\right| = \sum_i |\Gamma_i|. \tag{19}$$

For a 1PI graph Γ we let then its superficial degree of divergence $w(\Gamma)$ be

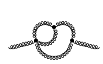
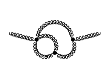
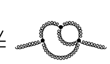
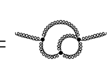
$$w(\Gamma) = -4|\Gamma| + \sum_{p\in\Gamma^{[1]}_{\mathrm{int}}\cup\Gamma^{[0]}} w(p). \tag{20}$$

Note that all 1PI graphs which have sdd ≤ 0 have residue in the above finite set R - we are dealing with a renormalizable theory. Here we are mainly interested in the structure of superficially divergent graphs, and hence do not discuss graphs and Green functions which are superficially convergent. For all $r \in R$, we let M_r be the set of graphs such that $\mathbf{res}(\Gamma) = r$.

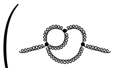
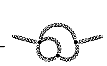
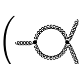
1.2 Isotopy classes of graphs

The symmetry factor of a graph Γ, $\mathbf{sym}(\Gamma)$, is defined as usual as the rank of the automorphism group of Γ.

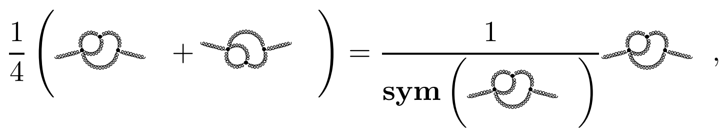
We consider graphs up to the usual isotopy, for fixed external legs:

 $=$  $\neq$  $=$  . (21)

This plays a role in the study of the pre-Lie structure on graphs below. Indeed, we note that the symmetry factor of a sum of all graphs belonging to an isotopy class is the product of the symmetry factor of the subgraphs times the symmetry factor of the cograph obtained by shrinking the subgraphs:

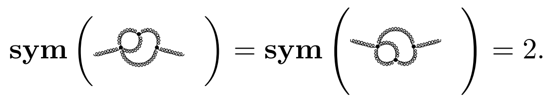
$\mathbf{sym}\Big($ $+$ $\Big) = 4 = \mathbf{sym}\Big($$\Big) \times \mathbf{sym}\Big($$\Big),$ (22)

so that



(23)

with



(24)

1.3 Combinatorial Green functions

We can now speak of the set of superficially divergent 1PI graphs and consider graphs according to their residue and loop number $|\Gamma|$.

We define the formal sums

$$\Gamma^{\underline{r}} = 1 + \sum_{\Gamma \in M_r} g^{2|\Gamma|} \frac{\Gamma}{\mathbf{sym}(\Gamma)},\ r \in R_V, \tag{25}$$

$$\Gamma^{\underline{r}} = 1 - \sum_{\Gamma \in M_r} g^{2|\Gamma|} \frac{\Gamma}{\mathbf{sym}(\Gamma)},\ r \in R_E. \tag{26}$$

We let

$$c_k^{\underline{r}} = \sum_{\substack{|\Gamma|=k \\ \mathrm{res}(\Gamma)=r}} \frac{\Gamma}{\mathbf{sym}(\Gamma)},\ r \in R, \tag{27}$$

be the sum of graphs with given residue $r \in R$ and loop number k. We have $\Gamma^{\underline{r}} = 1 + \sum_{k=1}^{\infty} g^{2k} c_k,\ r \in R_V$ and $\Gamma^{\underline{r}} = 1 - \sum_{k=1}^{\infty} g^{2k} c_k,\ r \in R_E$.

We call these formal sums $\Gamma^{\underline{r}}$ combinatorial Green functions. Each term on their rhs maps under the Feynman rules to a contribution in the perturbative expansion of the Green functions of our gauge theory. Already the algebraic structure of these combinatorial Green functions is rather interesting. Analytic consequences will be briefly discussed at the end and explored in subsequent work.

Our task in this paper is to acquaint the reader with the structure of these sums, which is amazingly rich even at this elementary combinatorial level.

1.4 Insertion places

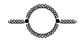
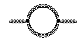
Each graph Γ has internal edges and vertices. We call those edges and vertices places of Γ. Note that each place provides adjacent edges: for a vertex these are the edges attached to it, while each interior point of an edge defines two edges adjacent edges for it: the two pieces of the edge on both sides of that point. In such places, other graphs can be inserted, using a bijection between the external edges of those graphs and the adjacent edges provided by these very places.

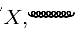
The first thing we need to do is to count the number of insertion places with respect to the graphs to be inserted. Let $X = \prod_i \Gamma_i$ be a set of graphs to be inserted. Let us introduce variables a_r for all $r \in R$. To X we assign the monomial

$$x := \prod_i a_{\mathrm{res}(\Gamma_i)}. \tag{28}$$

This monomial defines integers $n_{X,s}$ for all $s \in R$ by setting

$$x = \prod_{s \in R} a_s^{n_{X,s}}. \tag{29}$$

For example, if $X =$   , then

$n_{X,}$  $= 2,\ n_{X,s} = 0,\ s \neq$  . (30)

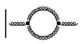
Furthermore, to a graph Γ assign the function b_Γ, and integers $m_{\Gamma,s}$ by

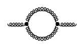
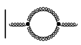
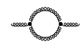
$$b_\Gamma := \prod_{v\in\Gamma^{[0]}} a_v \prod_{e\in\Gamma^{[1]}_{\text{int}}} \frac{1}{1-a_e} = \prod_{s\in R_V} a_s^{m_{\Gamma,s}} \prod_{e\in R_E} \frac{1}{[1-a_e]^{m_{\Gamma,e}}}. \tag{31}$$

Then, we define the number of insertion places for X in Γ, denoted $\Gamma|X$, by

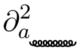
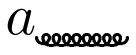
$$\Gamma|X := \prod_{s\in R_V} \binom{m_{\Gamma,s}}{n_{X,s}} \prod_{e\in R_E} \frac{\partial^{n_{X,e}} \frac{1}{[1-a_e]^{m_{\Gamma,e}}}(0)}{n_{X,e}!}. \tag{32}$$

A few examples:

 $|$  $= 3,$  $|$  $= 2,$  $|$   $) = 1,$ (33)

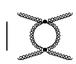
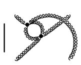
 $|$   $= 3,$ (34)

as, for the last case,

∂^2_{a}  $\frac{1}{[1-a}$  $]^2}(0) = 6.$ (35)

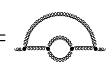
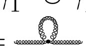
1.5 Permutation of external edges

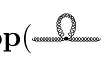
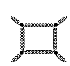
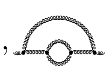
We call $|\Gamma|_\vee$ the number of distinct graphs Γ which are equal upon removal of the external edges. For example

$|$  $|_\vee = |$  $|_\vee = |$  $|_\vee = 3.$ (36)

Such graphs can be obtained from each other by a permutation of external edges.

Furthermore for graphs $\gamma_1, \gamma_2, \gamma$ we let $\mathbf{top}(\gamma_1, \gamma_2, \gamma)_p$ be the number of bijections between $\gamma^{[1]}_{2,\text{ext}}$ and a chosen place $p = \mathbf{res}(\gamma_2) \in \gamma_1^{[0]} \cup \gamma^{[1]}_{1,\text{int}}$ such that γ is obtained. This counts the number of ways to glue γ_2 into a chosen place $\in \gamma_1$ to obtain γ. This has a straightforward generalization $\mathbf{top}(\gamma_1, X, \gamma)_p$ to products of graphs X, where now p is an appropriate set of chosen residues $\in \gamma_1^{[0]} \cup \gamma^{[1]}_{1,\text{int}}$.

For example if $\gamma =$  , $\gamma_1 =$  , p the vertex  in γ_1, we have

$\mathbf{top}($  ,  ,  $)_p = 2.$ (37)

By definition, at a given place p,

$$\mathbf{top}(\gamma_1, X, \gamma)_p = \mathbf{top}(\gamma_1, \tilde{X}, \gamma)_p, \tag{38}$$

for all pairs $X, \tilde{X}$ related by a permutation of external legs.

1.6 Ramification in graphs

Above, we have counted the number of ways $\mathbf{top}(\gamma_1, X, \gamma)_p$ how to glue graph(s) X into a chosen single place $p \in \gamma_1^{[0]} \cup \gamma_{1,\text{int}}^{[1]}$ so as to obtain a given graph γ. Furthermore, there might be various different places $p_i \in \gamma_1$ which provide a bijection for X such that the same γ is obtained.

Let $\mathbf{bij}(\gamma_1, \gamma_2, \gamma)$ be the number of bijections between $\gamma_{2,\text{ext}}^{[1]}$ and adjacent edges of places $p \sim \mathbf{res}(\gamma_2)$ in γ_1 such that γ is obtained.

A graph is described by vertices, edges and relations. For any bijection as above, we understand that the relations in γ_2, together with the relations in γ_1 which remain after removal of a chosen place, and the relations provided by the bijection combine to the relations describing the graph γ.

We let $\{\mathbf{bij}\}(\gamma_1, \gamma_2, \gamma)$ be the set of such bijections and write, for each $b \in \{\mathbf{bij}\}(\gamma_1, \gamma_2, \gamma)$,

$$\gamma = \gamma_1 b \gamma_2. \tag{39}$$

$\mathbf{top}(\gamma_1, \gamma_2, \gamma)_p$ is then number of such bijections restricted to a place p in γ_1.

We have a factorization into the bijections at a given place p, and the distinct bijections which lead to the same result at that place:

$$\mathbf{bij}(\gamma_1, \gamma_2, \gamma) = \mathbf{top}(\gamma_1, \gamma_2, \gamma)\mathbf{ram}(\gamma_1, \gamma_2, \gamma). \tag{40}$$

Here, $\mathbf{ram}(\gamma_1, \gamma_2, \gamma)$ counts the numbers of different places $p \in \gamma_1^{[0]} \cup \gamma_{1,\text{int}}^{[1]}$ which allow for bijections such that

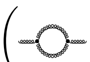
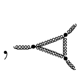
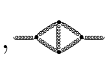
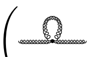
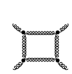
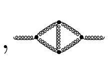
$$\gamma_1 b \gamma_2 = \gamma. \tag{41}$$

Note that for any two such places $p, \tilde{p}$ we find precisely $\mathbf{top}(\gamma_1, \gamma_2, \gamma)$ such bijections:

$$\mathbf{top}(\gamma_1, \gamma_2, \gamma) := \mathbf{top}(\gamma_1, \gamma_2, \gamma)_p = \mathbf{top}(\gamma_1, \gamma_2, \gamma)_{\tilde{p}}. \tag{42}$$

One immediately confirms that this number is indeed independent of p as we can pair off the bijections at p with the bijections at $\tilde{p}$ for any places p, p', so that the factorization (40) of $\mathbf{bij}(\gamma_1, \gamma_2, \gamma)$ is straightforward.

We call this integer $\mathbf{ram}(\gamma_1, \gamma_2; \gamma)$ the ramification index: it counts the degeneracy of inserting a graph at different places - if the ramification index is greater than one, the same graph Γ can be obtained from inserting a graph γ_2 into a graph γ_1 at different places. For example

ram ( ,  , ) $= 2$, **ram** ( ,  , ) $= 1$. (43)

The generalization replacing γ_2 by a product of graphs X is straightforward. The motivation of the name comes from a comparison with the situation in the study of number fields which will be given in future work.

1.7 pre-Lie structure of graphs

The pre-Lie product we will use sums over all bijections and places of graph insertions. Hence it gives the same result for the insertion of any two graphs related by permutation of their external legs. One could formulate the Hopf and Lie structure hence on graphs with amputated external legs, but we will stick with the usual physicists convention and work with Feynman rules which have external edges.

We define $n(\gamma_1, X, \gamma)$ as the number of ways to shrink X to its residue in X such that γ_1 remains.

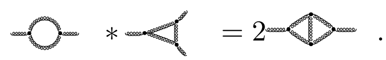
We define a bilinear map

$$\Gamma_1 * \Gamma_2 = \sum_{\Gamma} \frac{n(\Gamma_1, \Gamma_2, \Gamma)}{|\Gamma_2|_\vee} \Gamma. \tag{44}$$

This is a finite sum, as on the rhs only graphs can contribute such that

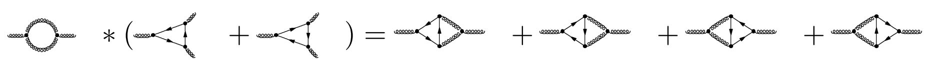
$$|\Gamma| = |\Gamma_1| + |\Gamma_2|. \tag{45}$$

For example,



(46)

while



(47)

Proposition 3 *This map is pre-Lie:*

$$(\Gamma_1 * \Gamma_2) * \Gamma_3 - \Gamma_1 * (\Gamma_2 * \Gamma_3) = (\Gamma_1 * \Gamma_3) * \Gamma_2 - \Gamma_1 * (\Gamma_3 * \Gamma_2). \tag{48}$$

Note that the graphs on the rhs have all the same residue as Γ_1. The proof is analogous to the one in [4]. For a product of graphs X we define analogously

$$\Gamma_1 * X = \sum_{\Gamma} \frac{n(\Gamma_1, X, \Gamma)}{|X|_\vee} \Gamma. \tag{49}$$

1.8 The Lie algebra of graphs

We let $\mathcal{L}$ be the corresponding Lie algebra, obtained by antisymmetrizing the pre-Lie product:

$$[\Gamma_1, \Gamma_2] = \Gamma_2 * \Gamma_1 - \Gamma_1 * \Gamma_2. \tag{50}$$

The bracket $[,]$ fulfils a Jacobi identity and we hence get a graded Lie algebra. Note that the terms generated by the Lie bracket involve graphs of different residue.

1.9 The Hopf algebra of graphs

Let $\mathcal{H}$ be the corresponding Hopf algebra. Let us quickly describe how it is found. To $\mathcal{L}$, we assign its universal enveloping algebra

$$U(\mathcal{L}) = \bigoplus_{j=0}^{\infty} T(\mathcal{L})^{(j)}, \tag{51}$$

where $T(\mathcal{L})^{(j)} = \mathcal{L}^{\otimes j}$ is the j-fold tensorproduct of $\mathcal{L}$. In $U(\mathcal{L})$ we identify

$$\Gamma_1 \otimes \Gamma_2 - \Gamma_2 \otimes \Gamma_1 = [\Gamma_1, \Gamma_2], \tag{52}$$

as usual. We let

$$\langle \Gamma_1, \Gamma_2 \rangle = \begin{cases} 0, & \Gamma_1 \neq \Gamma_2 \\ 1, & \Gamma_1 = \Gamma_2 \end{cases}. \tag{53}$$

Here we understand that entries on the lhs of $\langle \cdot, \cdot \rangle$ belong to the Lie algebra, entries on the rhs to the Hopf algebra.

We compute the coproduct from this pairing requiring

$$\langle[\Gamma_1,\Gamma_2],\Delta(\Gamma)\rangle=\langle\Gamma_1\otimes\Gamma_2-\Gamma_2\otimes\Gamma_1,\Delta(\Gamma)\rangle, \tag{54}$$

and find the usual composition into subgraphs and cographs

$$\Delta(\Gamma)=\Gamma\otimes 1+1\otimes\Gamma+\sum_\gamma \gamma\otimes\Gamma/\gamma. \tag{55}$$

The antipode $S:\mathcal{H}\to\mathcal{H}$ is

$$S(\Gamma)=-\Gamma-\sum_\gamma S(\gamma)\otimes\Gamma/\gamma. \tag{56}$$

The counit $\bar{e}$ annihilates the augmentation ideal as usual [4, 5].

Furthermore, we define $|\Gamma|_{\text{aug}}$ to be the augmentation degree, defined via the projection P into the augmentation ideal. We let $c^{\underline{r}}_{k,s}$ be the sum of graphs with given residue $\underline{r}$, loop number k and augmentation degree s. $\mathcal{H}_{\text{lin}}$ is the span of the linear generators of $\mathcal{H}$.

With the Hopf algebra comes its character group, which contains three distinguished objects: the Feynman rules ϕ, the $\bar{R}$ operation

$$\bar{\phi}=(S_R^\phi\otimes\phi P)\Delta, \tag{57}$$

and counterterm $-R\bar{\phi}$.

Note that the forgetfulness upon insertion wrt the external legs (38) forces us to work with with a symmetric renormalization scheme

$$S_R^\phi(\Gamma_1)=S_R^\phi(\Gamma_2), \tag{58}$$

for consistency, for all pairs Γ_1,Γ_2 which agree by a permutation of external edges.

Indeed,

$$0 = \gamma*\Gamma_1-\gamma*\Gamma_2 \tag{59}$$
$$= \bar{\phi}(\gamma*\Gamma_1-\gamma*\Gamma_2) \tag{60}$$
$$= S_R^\phi(\Gamma_1-\Gamma_2)\phi(\gamma), \tag{61}$$

as $\phi(\gamma*\Gamma_1-\gamma*\Gamma_2)=0$, and similarly for all cographs of Γ_1 and Γ_2.

1.10 External structures

In later work it will be useful to disentangle Green functions wrt to their form-factor decomposition. This can be easily achieved by the appropriate use of external structures [4].

We hence extend graphs γ to pairs (γ,σ) where σ labels the formfactor and with a forgetful rule

$$\sum_{\sigma_2}(\Gamma_1,\sigma_1)*(\Gamma_2,\sigma_2):=\sum_\Gamma\frac{n(\Gamma_1,\Gamma_2,\Gamma)}{|\Gamma_2|_\vee}(\Gamma,\sigma_1). \tag{62}$$

This allows to separate the form-factor decompositions as partitions of unity $1=\sum_{\sigma_2}$ in calculational convenient ways for which we will use in future work. If we do not sum over σ_2 we can extend our notation to marked graphs as in [4].

2 The theorems

In this section we state the main result. It concerns the role played by the maps $B_+^{k,\underline{r}}$ to be defined here: they provide the equations of motion, ensure locality, and lead us to the Slavnov–Taylor identities for the couplings.

Alas, we define a map

$$B_+^{k;\underline{r}} = \sum_{\substack{|\gamma|=k \\ |\gamma|_{\mathrm{aug}}=1 \\ \mathrm{res}(\gamma)=\underline{r}}} B_+^{\gamma}, \tag{63}$$

where B_+^{γ} is a normalized generalization of the pre-Lie insertion into γ defined by requiring $B_+^{k;\underline{r}}$ to be Hochschild closed. To achieve this, we need to count the maximal forests of a graph Γ. It is the number of ways to shrink subdivergences to points such that the resulting cograph is primitive. To define it more formally, we use Sweedler's notation to write $\Delta(X) = \sum X' \otimes X''$. If $X = c\prod \Gamma_i$ is a Hopf algebra element with Γ_i graphs and c a scalar, we let $\widehat{X} = \prod \Gamma_i$. We then can write

$$\Delta(X) = c(X', X'')\widehat{X'} \otimes \widehat{X''}, \tag{64}$$

which defines scalars $c(X', X'')$. We now set

$$\mathrm{maxf}(\Gamma) = \sum_{|\gamma|_{\mathrm{aug}}=1} \sum c(\Gamma', \Gamma'')\langle \gamma, \Gamma'' \rangle. \tag{65}$$

One shows that

$$\mathrm{maxf}(\Gamma) = \sum_{|\gamma|_{\mathrm{aug}}=1} \sum_{|X|=|\Gamma|-|\gamma|} n(\gamma, X, \Gamma)\mathbf{top}(\gamma, X, \Gamma), \tag{66}$$

as each maximal forest has a primitive cograph γ and some subdivergences X of loop number $|\Gamma| - |\gamma|$.

We have

$$\gamma * X = \sum_{\Gamma \in \mathcal{H}_{\mathrm{lin}}} \frac{n(\gamma, X, \Gamma)}{|X|_\vee}\Gamma. \tag{67}$$

Now we define

$$B_+^{\gamma}(X) = \sum_{\Gamma \in \mathcal{H}_{\mathrm{lin}}} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{|X|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{[\gamma|X]}\Gamma, \tag{68}$$

for all X in the augmentation ideal. Furthermore, $B_+^{\gamma}(\mathbb{I}) = \gamma$. Now one concludes:

Theorem 4 *(the Hochschild theorem)*

$$\Gamma^{\underline{r}} \equiv 1 + \sum_{\Gamma \in M_r} \frac{\Gamma}{\mathbf{sym}(\Gamma)} = 1 + \sum_{k=1}^{\infty} g^k B_+^{k;\underline{r}}(X_{k,\underline{r}}), \tag{69}$$

where $X_{k,\underline{r}} = \Gamma^{\underline{r}} \prod_{v \in \gamma^{[0]}} \frac{\Gamma^v}{\prod_{j \in \gamma^{[1]}} (\Gamma^j)^{1/2}}$*,*

which follows upon recognizing each bijection as a labelled composition and comparison with Theorem 2.

For the next theorem, we have to define the Slavnov–Taylor identities for the coupling.

$$X_{k,\underline{r}} = \Gamma^{\underline{r}} X^k_{\mathrm{coupl}}. \tag{70}$$

We set

$$X_{\mathrm{coupl}} = 1 + \sum_{k=1}^{\infty} g^{2k} c_k^{\mathrm{coupl}}, \tag{71}$$

which determines the c_k^{coupl} as polynomials in the c_j^r. The Slavnov–Taylor identities for the couplings read

$$\frac{\Gamma}{\Gamma} = \frac{\Gamma}{\Gamma} = \frac{\Gamma}{\Gamma} = \frac{\Gamma}{\Gamma}, \tag{72}$$

which leads to identities in every order in g^2 and leads to define

$$X_{\mathrm{coupl}} = \frac{\Gamma}{\Gamma \sqrt{\Gamma}} = \frac{\Gamma}{\Gamma \sqrt{\Gamma}} = \frac{\Gamma}{[\Gamma]^{3/2}} = \frac{\sqrt{\Gamma}}{\Gamma}. \tag{73}$$

Theorem 5 *(the gauge theory theorem)*

$$\Delta(B_+^{k;\underline{r}}(X_{k,\underline{r}})) = \mathbb{I} \otimes B_+^{k;\underline{r}}(X_{k,\underline{r}}) + (\mathrm{id} \otimes B_+^{k;\underline{r}})\Delta(X_{k,\underline{r}}). \tag{74}$$

$$\Delta(c_k^{\underline{r}}) = \sum_{j=0}^{k} \mathrm{Pol}_j^{\underline{r}}(c) \otimes c_{k-j}^{\underline{r}}, \tag{75}$$

where $\mathrm{Pol}_j^{\underline{r}}(c)$ *is a polynomial in the variables* c_m^r *of degree* j*, determined as the order* j *coefficient in the Taylor expansion of* $\Gamma^r[X_{\mathrm{coupl}}]^j$*.*

3 Two-loop Example

3.1 QCD

This section provides a crucial example. We consider our non-abelian gauge theory and first list its one-loop graphs, which provide by definition maps from $\mathcal{H} \to \mathcal{H}_{\mathrm{lin}}$.

The maps $B_+^{k,r}$, we claim, furnish the Hochschild one-cocyles and provide the Dyson–Schwinger equations, in accordance with the Hochschild and gauge theorems. We study this for the self-energy of the gauge boson to two-loops. We want in particular exhibit the fact that each such two-loop graph is a sum of terms each lying in the image of such a map and want to understand the role of Hochschild cohomology.

We have for example

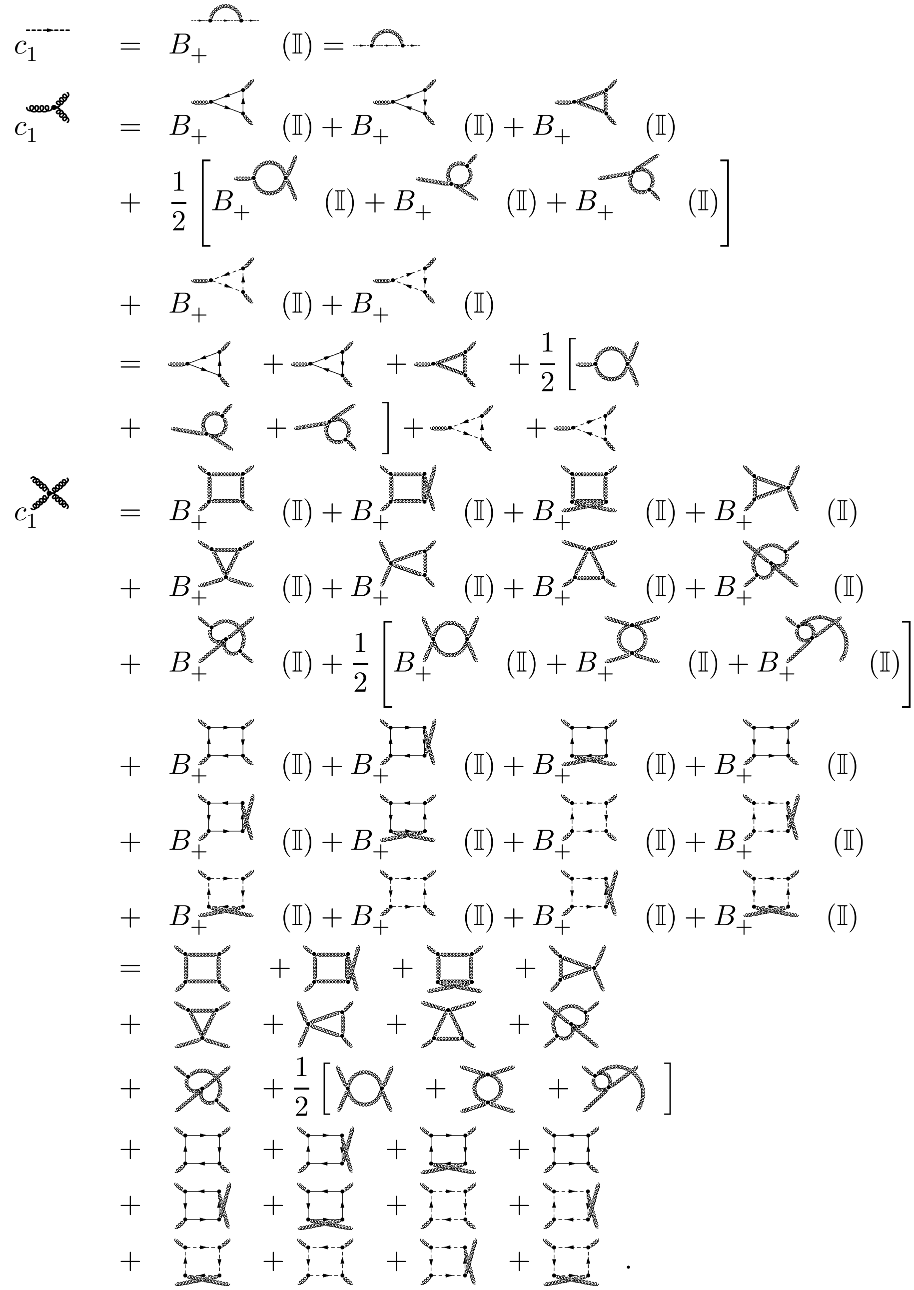
$$B_+^{\ ,1} = \frac{1}{2}B_+ + \frac{1}{2}B_+ + B_+ + B_+ . \tag{76}$$

To find the one-loop graphs we simply have to apply these maps to the unit of the Hopf algebra of graphs, which is trivial:

$$\begin{aligned} c_1 \quad &= \quad B_+ (\mathbb{I}) + \frac{1}{2}B_+ (\mathbb{I}) + \frac{1}{2}B_+ (\mathbb{I}) + B_+ (\mathbb{I}) \\ &= \quad + \frac{1}{2} + \frac{1}{2} + \end{aligned} \tag{77}$$

and similarly

$$c_1 \quad = \quad B_+ (\mathbb{I}) = \tag{78}$$



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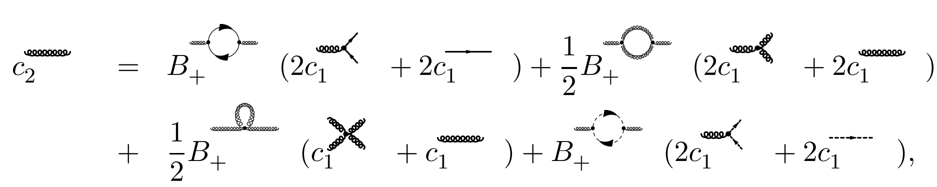
(81)

We now want to calculate



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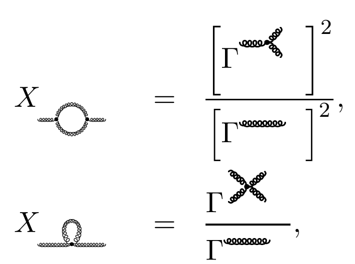
upon expanding



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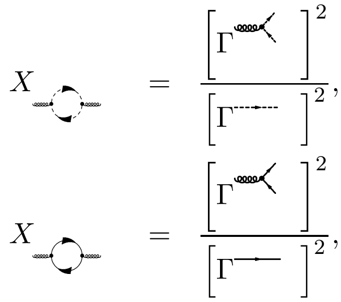

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(86)

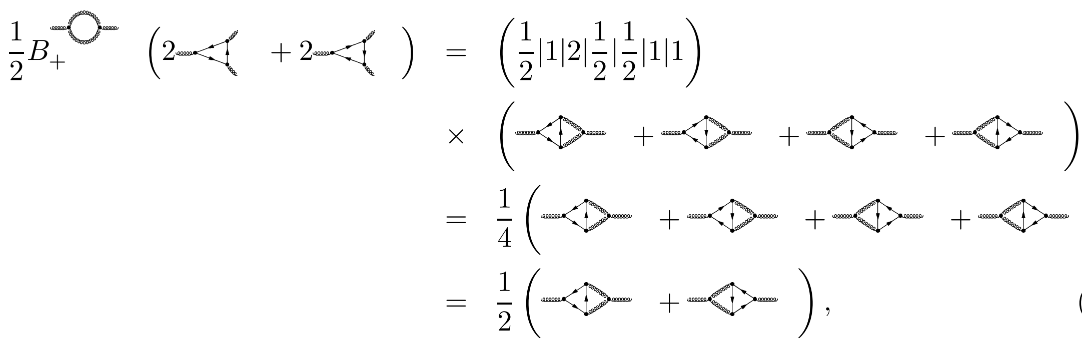
to order g^2.

Let us do this step by step. Adding up the contributions, we should find precisely the two-loop contributions to the gauge-boson self-energy, and the coproduct

$$\Delta(c_2) = c_2 \otimes \mathbb{I} + \mathbb{I} \otimes c_2 + 2c_1^{\text{coupl}} \otimes c_1 \quad . \tag{87}$$

Insertions in $\frac{1}{2}B_+$

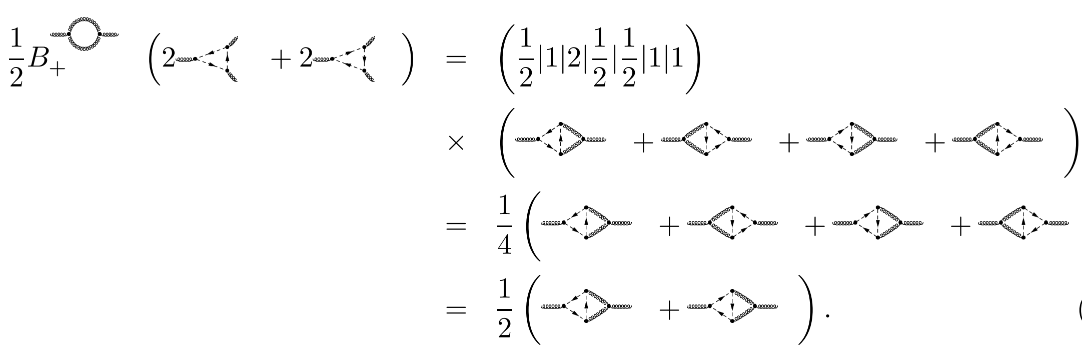
Below, we will give coefficients like $\left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right)$ in the next equation, where the first entry is the symmetry factor of the superscript γ of B_+^γ, the second entry the symmetry factor of the graphs in the argument X, the third entry the integer weight of that argument, the fourth entry the number of insertion places, the fifth entry the number of maximal forests of the graphs Γ on the rhs, the sixth entry is $\mathbf{top}(\gamma, X, \Gamma)$ and the seventh entry is $\mathbf{ram}(\gamma, X, \Gamma)$. We start



$$\frac{1}{2}B_+ \left(2 + 2\right) = \left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right) \times \left(+ + + \right) = \frac{1}{4}\left(+ + + \right) = \frac{1}{2}\left(+ \right), \tag{88}$$

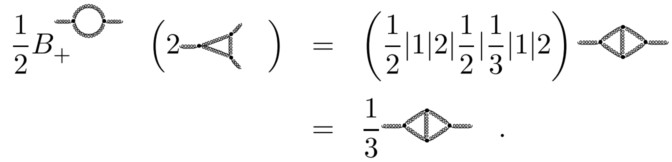
where indeed the symmetry factor for is 1/2, the symmetry factor for the graphs appearing as argument is 1, and they appear with weight two. We have two three-gluon vertices in , and hence two insertion places. Each graph on the right has two maximal forests, and for each graph the inserted subgraph can be reduced in a unique way to obtain , so the ramification index is one, and the topological weight is unity as well.

Similarly for ghosts



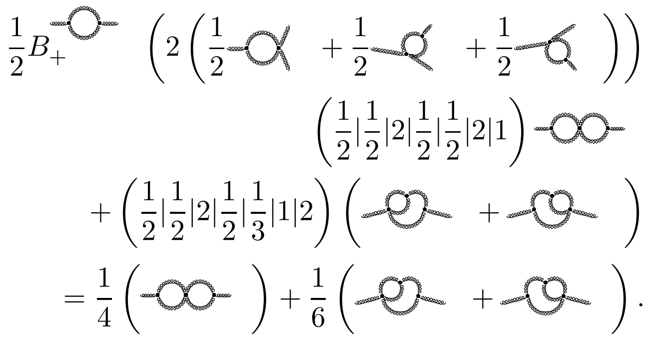
$$\frac{1}{2}B_+ \left(2 + 2\right) = \left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right) \times \left(+ + + \right) = \frac{1}{4}\left(+ + + \right) = \frac{1}{2}\left(+ \right). \tag{89}$$

Next,



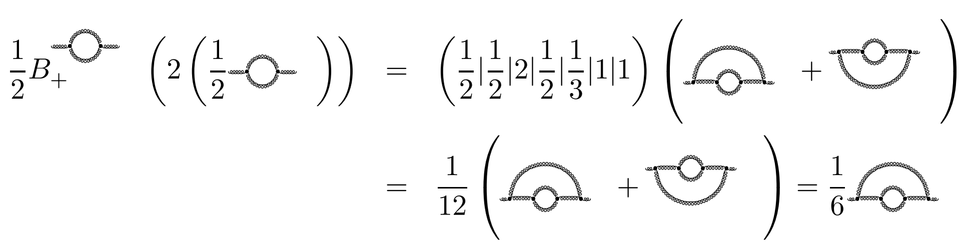
This is more interesting. There are three maximal forests in , a one-loop three-point vertex-graph to the left and to the right, and also the four gluon propagators which form the square give a one-loop log-divergent four-point graph. Also, we can form by inserting the argument into either vertex of , and hence the ramification index is two. The topological index is 1. Note that the total weight 1/3 of the graph is not its contribution to c_2 . We expect the same graph to be generated from inserting into , as we will confirm soon. This is generally true: only in the Hochschild closed sum over insertions in all components of c_1 will we see the correct weights emerging.

We continue.



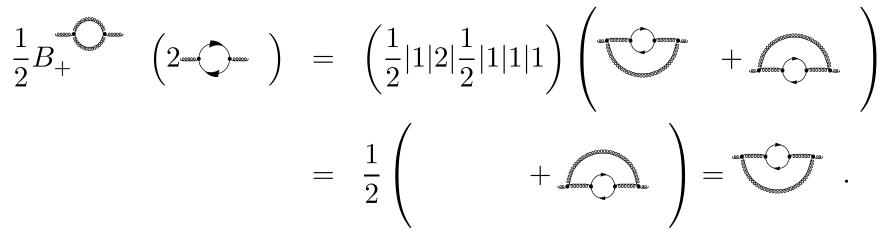
Here, the first graph on the rhs has two maximal forests, a ramification index of two as the graph can be obtained by insertion in either vertex of , and is generated by one bijection, while the other two graphs have three maximal forests, 1 as a ramification index and 2 as a topological index: there are two different bijections leading to each of them.

So far we inserted 3-point one-loop vertex corrections. Now we insert propagator corrections.



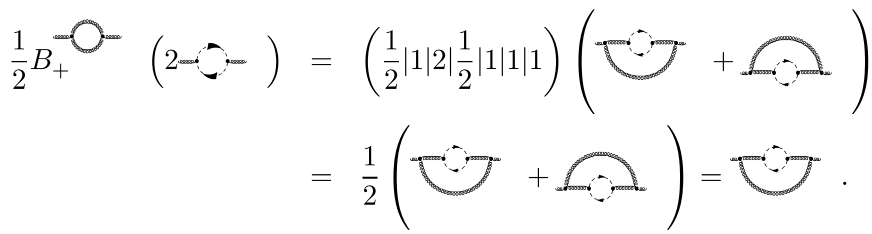
Note that the graph allows for three maximal forests: apart from the inserted one-loop self-energy graph it has two more maximal forests, corresponding to the two one-loop four-point vertex-subgraphs in , obtained by opening an internal edge in the subgraph.

Next we insert a fermion loop:



(93)

Indeed, no ramification, just one maximal forest and a single bijection. Similarly, the ghost loop



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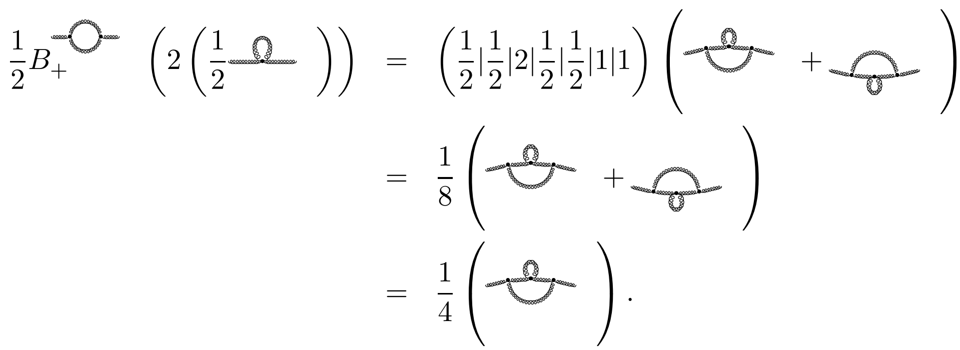
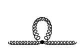
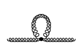
Finally,

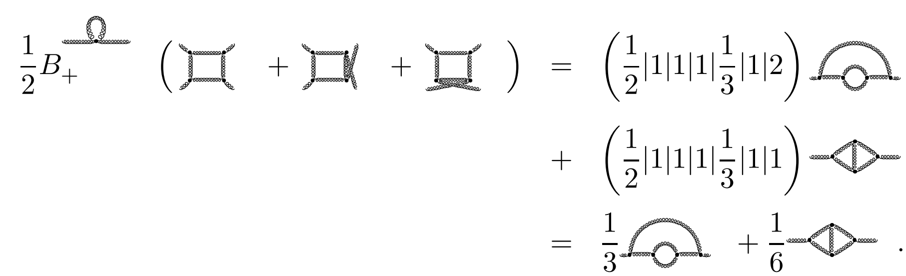


(95)

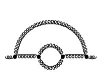
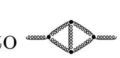
A single bijection, no ramification and two maximal forests in  . This concludes insertions into  .

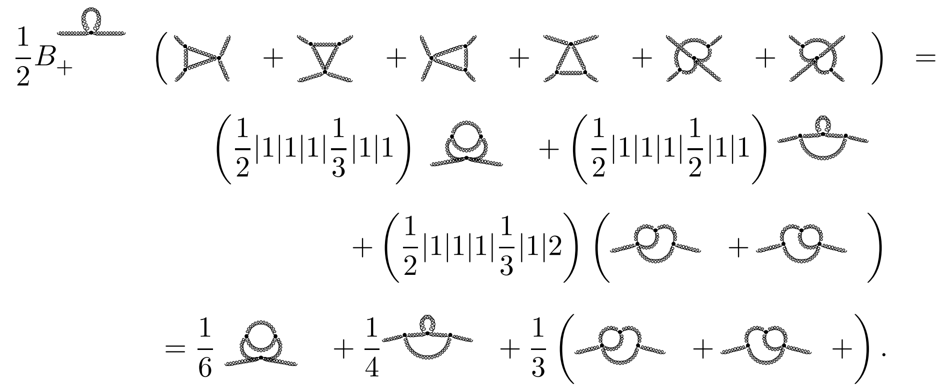
Insertions into $\frac{1}{2}B_+$ 

We come to insertions into  .

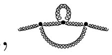
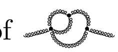


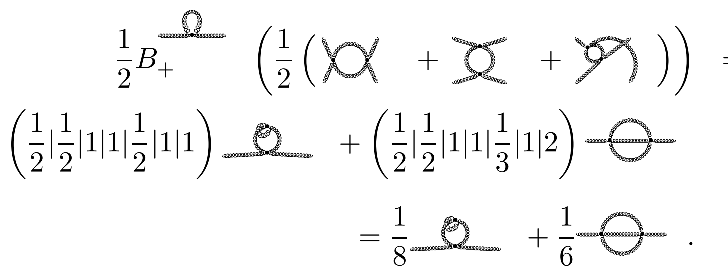
(96)

Indeed,  has three maximal forests, no ramification as there is only a single insertion place and two of the three bijections lead to it, while one bijection leads to  , which also has three maximal forests.

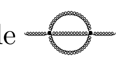


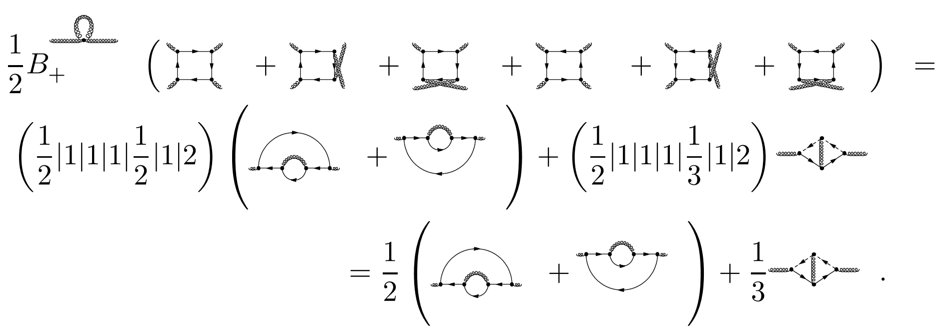
(97)

Note that  has three maximal forests and comes from one bijection,  has two maximal forests and comes as well from one bijection, while each of  and  come from two bijections and have three maximal forests.

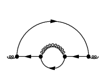
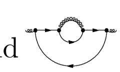
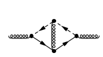


(98)

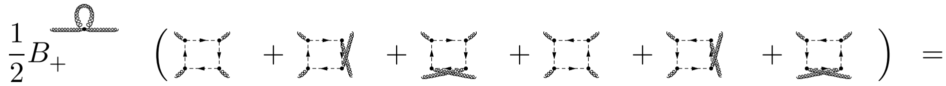
This time,  has two maximal forests and comes from one bijection while  has three maximal forts and the two remaining bijections leading to it.

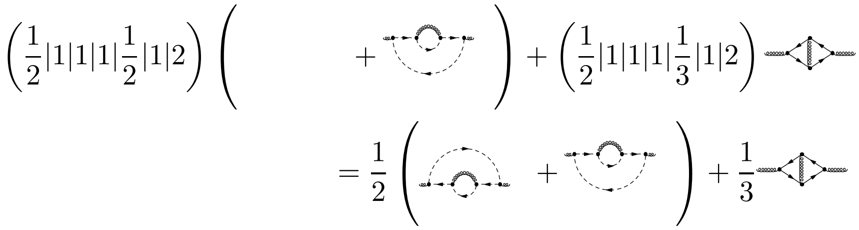


(99)

Indeed,  and  have both two maximal forests and two bijections leading to them each, while  has three maximal forests and is generated from two bijections.

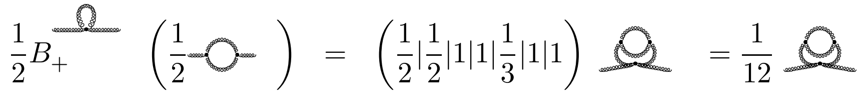
Similarly for ghosts





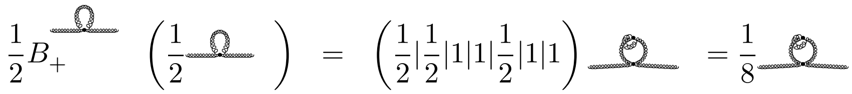
$$\left(\frac{1}{2}|1|1|1|\frac{1}{2}|1|2\right)\left(\quad + \quad \right) + \left(\frac{1}{2}|1|1|1|\frac{1}{3}|1|2\right) \quad = \frac{1}{2}\left(\quad + \quad \right) + \frac{1}{3} \quad . \qquad (100)$$

Now insertion of self-energies.



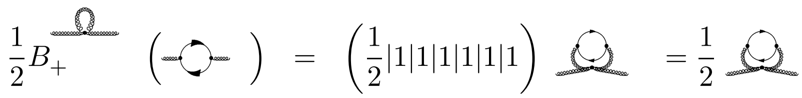
$$\frac{1}{2}B_+ \left(\frac{1}{2} \quad \right) = \left(\frac{1}{2}|\frac{1}{2}|1|1|\frac{1}{3}|1|1\right) \quad = \frac{1}{12} \quad , \qquad (101)$$

straightforward.



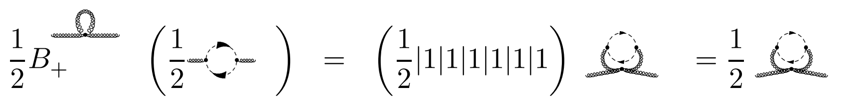
$$\frac{1}{2}B_+ \left(\frac{1}{2} \quad \right) = \left(\frac{1}{2}|\frac{1}{2}|1|1|\frac{1}{2}|1|1\right) \quad = \frac{1}{8} \quad , \qquad (102)$$

dito. Next,



$$\frac{1}{2}B_+ \left(\quad \right) = \left(\frac{1}{2}|1|1|1|1|1|1\right) \quad = \frac{1}{2} \qquad (103)$$

and similar for the ghost-loop

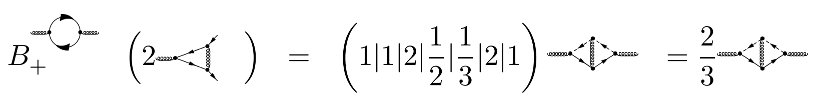


$$\frac{1}{2}B_+ \left(\frac{1}{2} \quad \right) = \left(\frac{1}{2}|1|1|1|1|1|1\right) \quad = \frac{1}{2} \quad . \qquad (104)$$

This concludes insertions into .

Insertions into B_+ **and** B_+

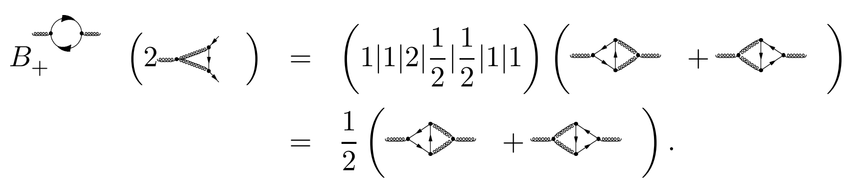
It remain the insertions into and .



$$B_+ \left(2 \quad \right) = \left(1|1|2|\frac{1}{2}|\frac{1}{3}|2|1\right) \quad = \frac{2}{3} \quad . \qquad (105)$$

Indeed, there are three maximal forests, a ramification index of two and just a single bijection for each place.

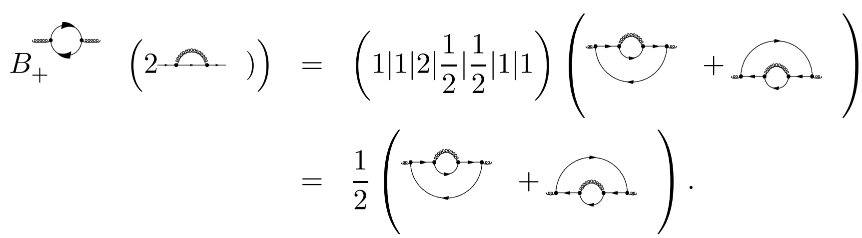
Next



$$B_+ \left(2 \quad \right) = \left(1|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right)\left(\quad + \quad \right) = \frac{1}{2}\left(\quad + \quad \right). \qquad (106)$$

This time we have no ramification and two maximal forests.

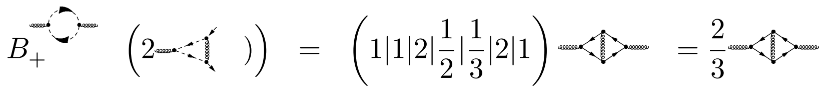
Next the self-energy,



$$B_+ \left(2 \quad \right) = \left(1|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right)\left(\quad + \quad \right) = \frac{1}{2}\left(\quad + \quad \right). \qquad (107)$$

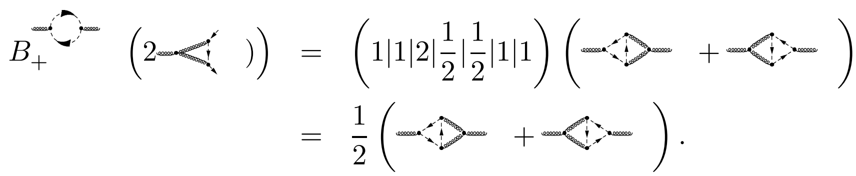
Again, two maximal forests, single bijection and no ramification.

Finally, the ghosts bring nothing new:



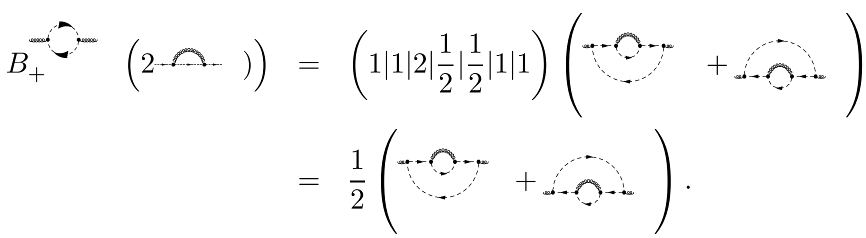
(108)

And



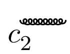
(109)

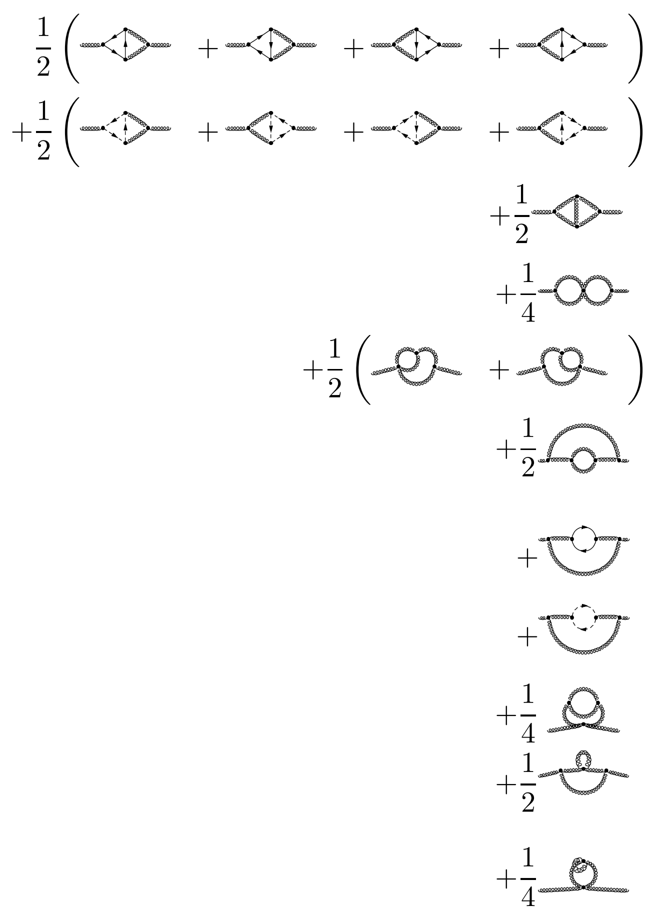
Also,

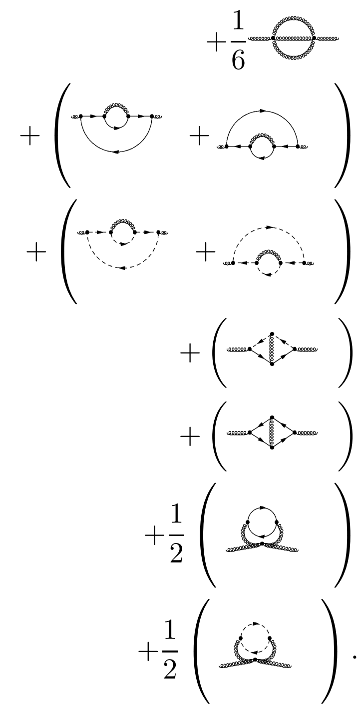


(110)

3.2 Adding up

Now we indeed confirm that the results adds up to c_2 . Adding up, we find





We indeed confirm that the result is

$$c_2 = \sum_{\substack{|\Gamma|=2 \\ \mathbf{res}(\Gamma)=}} \frac{\Gamma}{\mathbf{sym}(\Gamma)}. \tag{111}$$

This confirms the Hochschild theorem.

Furthermore, we find that

$$\Delta'(c_2) = \left(2c_1 + 2c_1\right) \otimes \frac{1}{2}B_+ \quad (\mathbb{I}) \tag{112}$$

$$+ \left(c_1 + c_1\right) \otimes \frac{1}{2}B_+ \quad (\mathbb{I}) \tag{113}$$

$$+ \left(2c_1 + 2c_1\right) \otimes B_+ \quad (\mathbb{I}) \tag{114}$$

$$+ \left(2c_1 + 2c_1\right) \otimes B_+ \quad (\mathbb{I}). \tag{115}$$

We now impose the Slavnov–Taylor identity, which allows us to write the above as

$$2c_1^{\text{coupl}} \otimes B_+^{1,}, \tag{116}$$

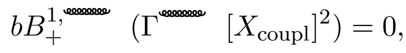
by expanding (72) to order g^2. Vice versa, if we require that the coproduct defines a sub Hopf algebra on the c_j^r, we reobtain the Slavnov–Taylor identities

$$2c_1 + 2c_1 = c_1 + c_1 = 2c_1 + 2c_1 = 2c_1 + 2c_1 . \tag{117}$$

Hence we recover the Slavnov Taylor identities for the couplings from the above requirement. Summarizing, we indeed find

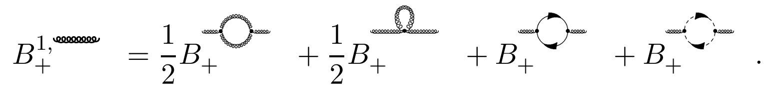
$$\Delta'(c_2) = \left[2c_1 + 2c_1\right] \otimes c_1 = 2c_1^{\text{coupl}} \otimes c_1 . \tag{118}$$

Note that the above indeed implies

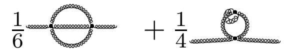


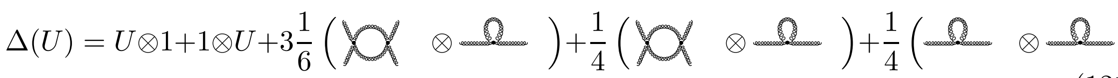
(119)

where

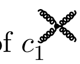


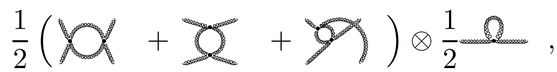
(120)

It is instructive to see how the Hochschild closedness comes about. Working out the coproduct on the combination  $=: U$ say we find



(121)

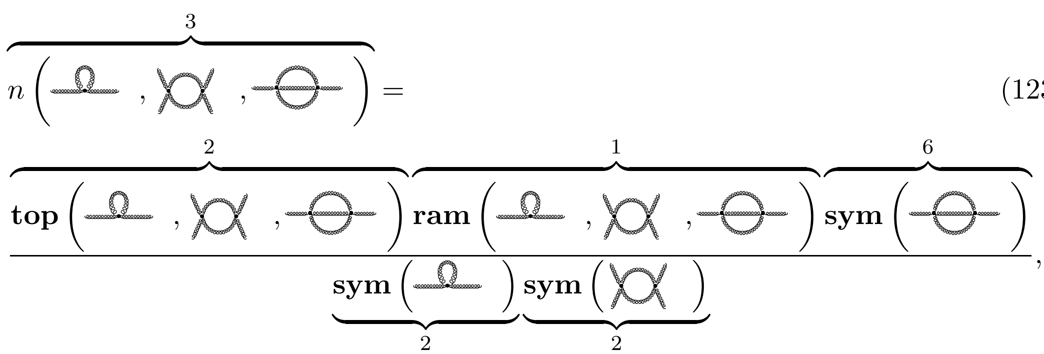
On the other hand, looking at the definition of , we find a mixed term



(122)

and we now see why we insist on a symmetric renormalization point.

Furthermore, we confirm



(123)

as it must by our definitions.

4 Discussion

We shortly discuss the major consequences of our set-up. This is largely meant as an outlook to upcoming results obtained by the author and collaborators.

4.1 Locality and Finiteness

The first result concerns the proof of locality of counterterms and finiteness of renormalizad Hopf algebra. The structure

$$\Gamma^{\underline{r}} = 1 + \sum_k g^{2k} B_+^{k;r}(\Gamma^{\underline{r}}[X_{\text{coupl}}]^k) \tag{124}$$

allows to prove locality of counterterms and finiteness of renormalized Green function via induction over the augmentation degree, involving nothing more than an elementary application of Weinberg's theorem to primitive graphs [5]. It unravels in that manner the source of equisingularity in the corresponding Riemann–Hilbert correspondences [6]. For the DSE equations, this implies that we can define renormalized Feynman rules via the choice of a suitable boundary condition. This leads to an analytic study of the properties of the integral kernels of $\phi(B_+^{k;r}(\mathbb{I}))$ to be given in future work.

4.2 Expansions in the conformal anomaly

The form of the arguments $X_{r,k} = \Gamma^{\underline{r}} X^k_{\text{coupl}}$ allows for a systematic expansion in the coefficients of the β-function which relates the renormalization group to the lower central series of the Lie algebra $\mathcal{L}$. Indeed, if the β function vanishes X_{coupl} is mapped under the Feynman rules to a constant, and hence the resulting DSE become linear, by inspection. One immediately confirms that the resulting Hopf algebra structure is cocommutative, and the Lie algebra hence abelian.

4.3 Central Extensions

The sub-Hopf algebras underlying the gauge theory theorem remain invariant upon addition of new primitive elements - beyond the one-loop level they obtain the form of a hierarchy of central extensions, which clearly deserves further study.

4.4 Radius of convergence

The above structure ensures that the Green function come as a solution to a recursive equation which naturally provides one primitive generator in each degree. This has remarkable consequences for the radius of convergence when we express perturbation theory as a series in the coefficients c^r_k, upon utilizing properties of generating functions for recursive structures [7].

4.5 Motivic picture

The primitives themselves relate naturally to motivic theory [8]. Each primitive generator is transcendentally distinguished, with the one-loop iterated integral providing the rational seed of the game.

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