Noncritical osp(1|2,R) M-theory matrix model with an arbitrary time dependent cosmological constant

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Abstract: Dimensional reduction of the $D=2$ minimal super Yang-Mills to the $D=1$ matrix quantum mechanics is shown to double the number of dynamical supersymmetries, from $\mathcal{N}=1$ to $\mathcal{N}=2$. We analyze the most general supersymmetric deformation of the latter, in order to construct the noncritical $3D$ $\mathcal{M}$-theory matrix model on *generic* supersymmetric backgrounds. It amounts to adding a harmonic oscillator potential with an arbitrary time dependent coefficient or cosmological 'constant,' $\Lambda(t)$. The resulting matrix model enjoys, irrespective of $\Lambda(t)$, two dynamical supersymmetries which further reveal three hidden $\mathfrak{so}(1,2)$ symmetries. All together they form the supersymmetry algebra, $\mathfrak{osp}(1|2,\mathbf{R})$. Each $\mathfrak{so}(1,2)$ multiplet in the Hilbert space visualizes a dynamics constrained on either Euclidean or Minkowskian dS_2/AdS_2 space, depending on its Casimir. In particular, all the unitary as well as BPS multiplets have the Euclidean dS_2/AdS_2 geometry. We conjecture that the matrix model provides holographic duals to the $2D$ superstring theories on various backgrounds having the spacetime signature Minkowskian if $\Lambda(t)>0$, or Euclidean if $\Lambda(t)<0$. In particular, we argue that the choice of the negative constant Λ corresponds to the $\mathcal{N}=2$ super Liouville theory.

Keywords: $\mathcal{M}$-theory, AdS/CFT, supersymmetry.

Contents

1. Introduction

String or $\mathcal{M}$-theory dress all the known supersymmetric gauge theories with the insightful geometrical pictures by the notion of holography or AdS/CFT correspondence [1, 2]. In particular, the symmetry group of a gauge theory is identified as the isometry of the corresponding higher dimensional string/$\mathcal{M}$-theory background. Conversely, different string theories - bosonic or supersymmetric, critical or noncritical - on various backgrounds are expected to have holographic dual gauge theories.

In the present paper, we attempt to address the dual of the noncritical $2D$ superstring, which is a yet unresolved problem despite of some progress [3, 4, 5, 6, 7, 8], from the $\mathcal{M}$-theory point of view [9, 10, 11]. The spacetime dimension two is singular in the sense that the holographic dual of $2D$ superstring theory, whatever its concrete form is, should share many common features with the corresponding noncritical $\mathcal{M}$-theory matrix model.

As is well known, superstring lives in 2, 3, 4, 6 and 10 dimensions, while the supermembrane exits in dimensions one higher, i.e. 3, 4, 5, 7 and 11, since only in those spacetime dimensions the relevant Fierz identities hold. Although the pioneering work on super p-branes [12] excludes the possibility of the space-filling p-branes i.e. p-branes propagating in

$(p+1)$-dimensional target spacetime, supermembrane *does* exit in three dimensions, since the Fierz identity for the supermembrane works manifestly, from $\gamma^{012}=1$,

$$(\mathrm{d}\bar{\theta}\gamma_{\mu}\mathrm{d}\theta)(\mathrm{d}\bar{\theta}\gamma^{\mu\nu}\mathrm{d}\theta)=\epsilon^{\mu\nu\lambda}(\mathrm{d}\bar{\theta}\gamma_{\mu}\mathrm{d}\theta)(\mathrm{d}\bar{\theta}\gamma_{\lambda}\mathrm{d}\theta)=0\,, \tag{1.1}$$

where $\mathrm{d}\theta$ is a bosonic spinor. The matrix regularization [13, 14] of the supermembrane prescribes the replacement of the Poisson bracket appearing in the light cone gauged membrane action by a matrix commutator. For $3D$ supermembrane action, it leads to a *supersymmetric* and *gauged* version of a *one* matrix model, where the local gauge symmetry originates from the area preserving diffeomorphism for the Poisson bracket, and the appearance of only one matrix is due to the light cone gauge, i.e. $3-2=1$.

The resulting $N\times N$ matrix model, at least for the 'flat' $3D$ background, can be also obtained by the dimensional reduction of the $2D$ minimal super Yang-Mills[1] to $D=1$, and it is supposed to describe exactly the D0-brane dynamics of the discrete light cone momentum sector, $p_{-}=N/R$, in $\mathcal{M}$-theory compactified on a light-like circle, $x^{-}\sim x^{-}+2\pi R$, as initially proposed by Banks, Fischler, Shenker and Susskind for the critical $\mathcal{M}$-theory [16, 17]. As for the D0-branes, the local gauge symmetry is required to reflect the identical nature of the N D-particles [18].

Also for the noncritical $2D$ superstring, almost by definition, its holographic dual should be one dimensional, supersymmetric and gauged theories. In the presence of RR electric field, F, the low energy effective action of $2D$ string theory typically reads, neglecting the massless tachyon and putting $\alpha'\equiv 1$ [19, 20, 21],

$$S_{2D}=\int \mathrm{d}^2x\sqrt{-g}\left[e^{-2\Phi}\left(8+R+4\left(\nabla\Phi\right)^2\right)-\tfrac{1}{2}F^2\right]\,, \tag{1.2}$$

where $-\frac{1}{2}F^2$ plays the role of the negative cosmological constant, and the solutions are characterized by the AdS_2-like geometries.[2] Indeed, switching off the dilaton completely we have the AdS_2 solution, while turning on Φ, one has static extremal black hole-like solutions [22, 23, 6]. In the asymptotic region the latter becomes the usual linear dilaton vacuum, and in the "near-horizon" region it approaches to AdS_2 with the dilaton reaching the critical value, $\Phi_c=-\ln(\frac{1}{4}F)$.

However, the effective action, (1.2), can not be thoroughly trusted due to the α' corrections as well as the tachyon tadpoles. Necessarily one has to work on the full sigma

[1]Recently all the minimal noncritical super Yang-Mills (except $D=3$) have been identified in the noncritical superstring theories [15].

[2]In two dimensions the geometries of AdS_2 and dS_2 coincide, and we will distinguish them by the sign of the 'cosmological constant'. Also it is to be reminded that

$$k_0^2-k_1^2-k_2^2=R^2>0 \qquad :\ \text{Eucidean } AdS_2/dS_2 \quad (\text{hyperboloid of two sheets})\,,$$

$$k_0^2-k_1^2-k_2^2=-R^2<0 \qquad :\ (\text{Minkowsian})\ AdS_2/dS_2 \quad (\text{hyperboloid of one sheet})\,.$$

model (e.g. [24]) with the difficulty of dealing with background fluxes. Hence to find the exact nontrivial superstring background is not an easy task. And also for the $\mathcal{M}$-theory, the matrix regularization of the supermembrane action is not always straightforward for generic nontrivial backgrounds.

In this work, we take *supersymmetry* itself as the principal guideline to tackle the problem of constructing the noncritical $3D$ $\mathcal{M}$-theory matrix model on generic supersymmetric backgrounds. Namely after the dimensional reduction of the $D = 2$ super Yang-Mills to the $D = 1$ matrix quantum mechanics, we analyze all the possible deformations of the latter without breaking any supersymmetry. We show that the most general supersymmetric deformations simply amount to adding a harmonic oscillator potential with an arbitrary time dependent coefficient or time dependent cosmological "constant" which we denote by $\Lambda(t)$. Remarkably we find that, irrespective of $\Lambda(t)$, the resulting matrix model always enjoys two dynamical supersymmetries, not just one as in the $2D$ minimal super Yang-Mills. Namely after the dimensional reduction, the number of supersymmetry is doubled, from $\mathcal{N} = 1$ to $\mathcal{N} = 2$. Furthermore, again for arbitrary $\Lambda(t)$, these two supersymmetries reveal three hidden nontrivial bosonic symmetries. All together the five symmetries form the super Lie algebra, $\mathfrak{osp}(1|2,\mathbf{R})$, where the even part corresponds to $\mathfrak{so}(1,2)$ i.e. the isometry of the Euclidean or Minkowskian dS_2/AdS_2. We introduce a projection map from the phase space to a three dimensional '$\mathfrak{so}(1,2)$ hyperspace' associated with the bosonic symmetries. The dynamics therein is always constrained on a two dimensional rigid surface, Euclidean dS_2/AdS_2 or Minkowskian dS_2/AdS_2, depending on the sign of the $\mathfrak{so}(1,2)$ Casimir for each multiplet in the Hilbert space. The richness of the matrix model comes from the arbitrariness of the time dependent cosmological 'constant', $\Lambda(t)$, and the vast amount of supermultiplets existing in the Hilbert space each of which has its own two dimensional geometries.

The organization of the present paper is as follows. In section 2, we analyze the most general supersymmetric deformation of the matrix model having the $2D$ super Yang-Mills origin. We discuss its symmetries, Hamiltonian dynamics and the BPS configurations. We also comment on the relation to the matrix cosmology. Section 3 is devoted to the detailed analysis on the underlying supersymmetry algebra, $\mathfrak{osp}(1|2,\mathbf{R})$, both from the kinematical and dynamical point of view. In particular, we show that all the 'unitary' as well as BPS multiplets correspond to the Euclidean dS_2/AdS_2 geometry, rather than the Minkowskian one. The last section, Sec.4 includes our conclusion that the matrix model with different choices of $\Lambda(t)$ can provide holographic duals to various $2D$ superstring or superconformal theories.

2. Noncritical $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model

2.1 Derivation of the matrix model and supersymmetry enhancement

In two dimensional Minkowskian spacetime the fermion satisfies the Majorana-Weyl condition, resulting in only one component real spinor. After the dimensional reduction to $D = 1$, the $2D$ super Yang-Mills leads to the following supersymmetric matrix model, which can be also obtained by the matrix regularization of the $3D$ supermembrane action in the light cone gauge,

$$\mathcal{L} = \mathrm{tr}\Big[\tfrac{1}{2}D_tXD_tX + i\tfrac{1}{2}\psi D_t\psi + X\psi\psi\Big]\,, \tag{2.1}$$

where X, ψ are respectively bosonic or fermionic $N\times N$ Hermitian matrices. With a gauge potential, $A_0 = A_0{}^\dagger$, the covariant time derivative reads, in our convention,

$$D_t = \partial_t \;-i[A_0\,,\quad]\,. \tag{2.2}$$

Bosons, X, A_0, have the mass dimension 1, while the fermion, ψ, has the mass dimension $\frac{3}{2}$, so that the Lagrangian has the mass dimension, 4.

The supersymmetry transformation, $\delta_{\rm YM}$, descending from the $2D$ super Yang-Mills theory is, with a constant supersymmetry parameter, ε,

$$\delta_{\rm YM}A_0 = \delta_{\rm YM}X = i\psi\varepsilon\,, \qquad\qquad \delta_{\rm YM}\psi = D_tX\varepsilon\,. \tag{2.3}$$

Now we look for the generalizations of the above Lagrangian as well as the supersymmetry transformations. First of all, we note from

$$\mathrm{tr}\Big[i\tfrac{1}{2}\psi D_t\psi + X\psi\psi\Big] = \mathrm{tr}\Big[i\tfrac{1}{2}\psi\partial_t\psi + (X-A_0)\psi\psi\Big]\,, \tag{2.4}$$

that in order to cancel the cubic terms of ψ in any possible supersymmetry variation which will transform the bosons, $(X-A_0)$ to the fermion, it is inevitable to impose[3]

$$\delta A_0 = \delta X\,. \tag{2.5}$$

Hence, introducing a time dependent function, $f(t)$, we let the generalized supersymmetry transformation be

$$\delta A_0 = \delta X = if(t)\psi\varepsilon\,, \qquad\qquad \delta\psi = \Big(f(t)D_tX + \Delta\Big)\varepsilon\,, \tag{2.6}$$

where Δ is a bosonic quantity having the mass dimension 2, and its explicit form is to be determined shortly. After some straightforward manipulation, we obtain

$$\delta\mathcal{L} = \mathrm{tr}\Big[i\psi\varepsilon\Big(D_t\Big(\dot{f}X+\Delta\Big) - \ddot{f}X + i[X,\Delta]\Big)\Big] \;+\; \partial_t\mathcal{K}\,, \tag{2.7}$$

[3]Essentially this rigidity corresponds to the Fierz identity, $\mathrm{tr}\big(\bar{\psi}\gamma^\mu[\bar{\varepsilon}\gamma_\mu\psi,\psi]\big) = 0$, relevant to the existence of the minimal super Yang-Mills in 2, 3, 4, 6, 10 dimensions.

where the total derivative term is given by

$$\mathcal{K} = \mathrm{tr}\big(D_t X \delta X - i\tfrac{1}{2}\psi\delta\psi\big)\ . \tag{2.8}$$

Of course, the simplest case where $f(t) = 1$ and $\Delta = 0$ reduces to the supersymmetry of the original $2D$ super Yang-Mills, (2.3). For the generic cases, we are obliged to set

$$\Delta = -\dot{f} X\,, \tag{2.9}$$

and obtain the following supersymmetry invariance,

$$\delta\left[\mathcal{L} + \tfrac{1}{2}(\ddot{f}/f)\,\mathrm{tr}\big(X^2\big)\right] = \partial_t \mathcal{K}\,, \tag{2.10}$$

which essentially leads to the novel supersymmetric matrix model or the $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model, (2.11).

For a given function, $\Lambda(t)$, there exit two $f(t)$'s satisfying the second order differential equation, $\Lambda = \ddot{f}/f$. Thus, surprisingly, there are two supersymmetries in the matrix model, even for $\Lambda = 0$ case.[4] This will further reveal three nontrivial bosonic symmetries as written in the next subsection.

Rather than taking (2.9) one might attempt to close the supersymmetry invariance by adding other terms to $\mathcal{L}$. However, since there exits only one component spinor, there can not be any mass term for the fermion[5] as $\mathrm{tr}(\psi\psi) = 0$. Thus, as long as we restrict on the 'non-derivative corrections', the above generalization is the most generic one.

[4]This kind of supersymmetry enhancement after the dimensional reduction can be also noticed elsewhere. For example, in the earlier works [25, 26], we derived the effective worldvolume gauge theories for the longitudinal D5 and D2 branes on the maximally supersymmetric $11D$ *pp*-wave background. After the dimensional reductions to $D = 1$, both of them lead to a matrix quantum mechanics which is equivalent to the BMN $\mathcal{M}$-theory matrix model [27] up to field redefinitions. The formers have only four dynamical supersymmetries, while the BMN model has 32 supersymmetries, 16 dynamical and 16 kinematical. The physical reason for the enhancement is that the D-branes which the higher dimensional gauge theories describe preserve only the fraction of the full $\mathcal{M}$-theory supersymmetries, $\frac{4}{32}$. The same reasoning also holds for the present $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model having three supersymmetries, two dynamical and one kinematical. As we see shortly, all the BPS states preserve only one supersymmetry, breaking the other two. Hence, the minimal $2D$ super Yang-Mills can be interpreted as the worldvolume action of the longitudinal M2-brane which preserves only one supersymmetry. However, it remains somewhat mysterious that the total number of supersymmetries is three, a rather unusual odd number.

[5]This is a special feature only present in the matrix quantum mechanics of the $2D$ super Yang-Mills origin. In fact, in the higher dimensional cases one needs to add the fermion mass term for the supersymmetry invariance as in the BMN matrix model [27] or [28].

2.2 Noncritical $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model : Final form

With an arbitrary time dependent 'cosmological constant,' $\Lambda(t)$, having the mass dimension two, the generic form of the noncritical $3D$ $\mathcal{M}$-theory matrix model reads

$$\mathcal{L}_{\mathfrak{osp}(1|2,\mathbf{R})} = \mathrm{tr}\Big[\tfrac{1}{2}\left(D_tX\right)^2 + i\tfrac{1}{2}\psi D_t\psi + X\psi\psi + \tfrac{1}{2}\Lambda(t)X^2\Big]\,. \qquad (2.11)$$

The Lagrangian corresponds to the most general supersymmetric deformation of the '$\mathcal{N}=2$' matrix quantum mechanics of the $2D$ super Yang-Mills origin. The matrix model is to describe the noncritical $3D$ supermembrane in a controllable manner through the matrix regularization, and our claim is further that it also provides holographic duals to $2D$ superstring theories, as discussed in the last section.

The matrix model is equipped with the standard local gauge symmetry,

$$X \longrightarrow gXg^{-1}\,, \qquad \psi \longrightarrow g\psi g^{-1}\,, \qquad A_0 \longrightarrow gA_0g^{-1} - i\partial_t g g^{-1}\,, \qquad g\in \mathrm{U}(N)\,, \qquad (2.12)$$

and enjoys *two* dynamical supersymmetries,

$$\delta_\pm A_0 = \delta_\pm X = if_\pm(t)\psi\varepsilon_\pm\,, \qquad\qquad \delta_\pm\psi = \Big(f_\pm(t)D_tX - \dot{f}_\pm(t)X\Big)\varepsilon_\pm\,, \qquad (2.13)$$

where ε_+, ε_-, are two real supersymmetry parameters, and $f_\pm(t)$ are the two different solutions of the second order differential equation,

$$\ddot{f}_\pm(t) = f_\pm(t)\Lambda(t)\,. \qquad (2.14)$$

The above two dynamical supersymmetries further reveal three hidden nontrivial bosonic symmetries, which we denote by $\delta_{++}\,,\ \delta_{--}\,,\ \delta_{\{+,-\}}\,,$ in order to indicate the anti-commutator origin from the two supersymmetries,

$$\begin{array}{ll}
\delta_{++}A_0 = \delta_{++}X = f_+\Big(f_+D_tX - \dot{f}_+X\Big)\,, & \delta_{++}\psi = 0\,,\\[2ex]
\delta_{--}A_0 = \delta_{--}X = f_-\Big(f_-D_tX - \dot{f}_-X\Big)\,, & \delta_{--}\psi = 0\,,\\[2ex]
\delta_{\{+,-\}}A_0 = \delta_{\{+,-\}}X = 2f_+f_-D_tX - \Big(f_+\dot{f}_- + f_-\dot{f}_+\Big)X\,, & \delta_{\{+,-\}}\psi = 0\,.
\end{array} \qquad (2.15)$$

As we see shortly, all the five symmetries form the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra, while the three bosonic symmetries correspond to its even part, $\mathfrak{sp}(2,\mathbf{R}) \equiv \mathfrak{so}(1,2) \equiv \mathfrak{sl}(2,\mathbf{R})$.

It is worth to note that the three bosonic symmetries are still valid in the bosonic matrix model obtained after putting $\psi\equiv 0$,

$$\mathcal{L}_{\mathfrak{so}(1,2)} = \mathrm{tr}\Big[\tfrac{1}{2}\left(D_tX\right)^2 + \tfrac{1}{2}\Lambda(t)X^2\Big]\,. \qquad (2.16)$$

Apart from them there is the usual kinematical supersymmetry,

$$\delta A_0 = \delta X = 0\,, \qquad\qquad \delta\psi = \varepsilon\,1\,. \qquad (2.17)$$

2.3 Hamiltonian

The Euler-Lagrangian equations read

$$\begin{array}{ll} D_tD_tX-\psi\psi-\Lambda(t)X=0\,, & D_t\psi+i\,[X,\psi]=0\,, \\ [D_tX,X]+i\psi\psi=0 & :\ \text{Gauss constraint}\,. \end{array} \tag{2.18}$$

Up to the Gauss constraint or the first-class constraint, the cubic vertex term vanishes, $\mathrm{tr}\,(X\psi\psi)\simeq 0$, so that the Hamiltonian becomes simply a harmonic oscillator type, being free of the fermion,

$$H=\mathrm{tr}\Big[\tfrac{1}{2}P^2-\tfrac{1}{2}\Lambda(t)X^2\Big]\,,\qquad\qquad P:=D_tX\,. \tag{2.19}$$

In fact, for any gauge invariant object,

$$\mathcal{F}=\mathrm{tr}\left[F(X,P,\psi,t)\right]\,, \tag{2.20}$$

the Euler-Lagrangian equations, (2.18), imply

$$\begin{aligned} \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t}&=\mathrm{tr}\Big(P\frac{\partial}{\partial X}+\Lambda(t)X\frac{\partial}{\partial P}\Big)\,\mathcal{F}-i\mathrm{tr}\Big([X,F]\Big)+\frac{\partial\mathcal{F}}{\partial t} \\ &=[\mathcal{F}\,,\,H\,\}_{D.B.}+\frac{\partial\mathcal{F}}{\partial t}\,. \end{aligned} \tag{2.21}$$

The Dirac bracket for our matrix model is given by, after taking care of the primary second-class constraint for the fermion [29, 30],

$$[\mathcal{F}\,,\,\mathcal{G}\,\}_{D.B.}=\frac{\partial\mathcal{F}}{\partial X^a{}_b}\frac{\partial\mathcal{G}}{\partial P^b{}_a}-\frac{\partial\mathcal{F}}{\partial P^a{}_b}\frac{\partial\mathcal{G}}{\partial X^b{}_a}+i(-1)^{\#\mathcal{F}}\frac{\partial\mathcal{F}}{\partial\psi^a{}_b}\frac{\partial\mathcal{G}}{\partial\psi^b{}_a}\,, \tag{2.22}$$

where a,b are the $N\times N$ matrix indices, while $\#\mathcal{F}=0$ or 1, depending on the spin statistics of $\mathcal{F}$, i.e. 0 for the boson and 1 for the fermion.

Due to the five symmetries of the action, there are five conserved quantities given by the Noether charges. For their explicit expressions we refer (3.29) and (3.30).

After the quantization, $[\mathcal{F}\,,\,\mathcal{G}\,\}_{D.B.}\rightarrow -i\,[\mathcal{F}\,,\,\mathcal{G}\,\}$, the matrix model is exactly solvable, as it essentially reduces to the harmonic oscillator problems. One only needs to restrict on the gauge singlets, which is nothing but taking the 'trace' of the $\mathrm{U}(N)$ indices [31]. The quantum states in the Hilbert space form supermultiplets of the underlying superalgebra, $\mathfrak{osp}(1|2,\mathbf{R})$, which we analyze in the next section. Further discussions on the dynamics including the projection to the '$\mathfrak{so}(1,2)$ hyperspace' will be carried out in the last section.

2.4 BPS states and the cosmological principle

From the supersymmetry transformations of the fermion, the BPS equations are

$$f_{\pm}(t)D_tX = \dot{f}_{\pm}(t)X + \kappa 1\,, \tag{2.23}$$

where the constant κ accounts for the kinematical supersymmetry (2.17), corresponding to the supersymmetry transformation of the "center of mass" i.e. U(1) sector. The general BPS solutions are then decompose into the traceless and U(1) parts,

$$X(t) = f_+(t)Y + h_+(t)1 \qquad \text{or} \qquad X(t) = f_-(t)Y + h_-(t)1\,, \tag{2.24}$$

where Y is an arbitrary traceless constant matrix, and $h_{\pm}(t)$ are the solutions of the first order differential equation, $f_{\pm}\dot{h}_{\pm} = \dot{f}_{\pm}h_{\pm} + \kappa$, corresponding to the center of mass position, $N^{-1}\mathrm{tr}X(t) = h_{\pm}(t)$.

Since $f_+(t) \neq f_-(t)$, the BPS state preserves only one supersymmetry out of three supersymmetries (two dynamical and one kinematical). It is interesting to note that for an arbitrary time dependent function, say $f_+(t)$, there exits a supersymmetric matrix model where $X(t) = f_+(t)Y + h_+(t)1$ corresponds to a BPS state, and furthermore there exits always its "twin" BPS state given by '$+ \rightarrow -$'.

Utilizing the gauge symmetry (2.12), one can diagonalize Y in order to show the positions of the N D-particles in the BPS sector,

$$X(t) = \mathrm{diag}\Big(x_1(t),\, x_2(t),\, \cdots,\, x_N(t)\Big) = f_{\pm}(t)\,\mathrm{diag}\Big(y_1,\, y_2,\, \cdots,\, y_N\Big) + h_{\pm}(t)1\,. \tag{2.25}$$

A remarkable fact is that all D-particles have precisely the same relative movement, same position, same velocity, same acceleration, *etc.* up to the constant scaling factors which entirely depend on their initial positions or so called the co-moving coordinates. This matches precisely with the "homothetic ansatz" adopted in the cosmology literature in order to incorporate the *cosmological principle* [32, 33]. In fact, the second order differential equation, $\ddot{f}_{\pm} = f_{\pm}\Lambda$, (2.14) can be identified as the Raychaudhuri's equation in cosmology, where Λ is indeed the time dependent cosmological "constant". Also, in the matrix approach to the cosmology [33, 34, 35], it is natural to associate Λ as the non-relativistic cosmological constant, and associate $\Lambda > 0$ and $\Lambda < 0$ with the *de-Sitter* and *Anti-de-Sitter* space respectively accounting the repulsive and attractive potential. Thus, although the geometries of dS_2 and AdS_2 coincide, we distinguish them by the sign of Λ, throughout the paper.

3. $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra

From the standard prescription, $[\mathcal{F}\,,\,\mathcal{G}\,\}_{D.B.}\rightarrow -i\,[\mathcal{F}\,,\,\mathcal{G}\,\}$, the quantization of the present $\mathfrak{osp}(1|2,\mathbf{R})$ matrix model leads to the following 'Heisenberg $\oplus$ Clifford' algebra,

$$\Big[X^{a}{}_{b}\,,\,P^{c}{}_{d}\Big]=i\,\delta^{a}{}_{d}\,\delta^{c}{}_{b}\,,\qquad\qquad \{\psi^{a}{}_{b}\,,\,\psi^{c}{}_{d}\,\}=\delta^{a}{}_{d}\,\delta^{c}{}_{b}\,. \tag{3.1}$$

In Sec. 3.1, utilizing the above algebra alone, we construct explicitly the generators of the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra.[6] The number of odd generators is two, and this is consistent with the fact that there are two dynamical supersymmetries in the matrix model, rather than one. Sec. 3.2 is devoted to the analysis on the unitary irreducible representations of the superalgebra, $\mathfrak{osp}(1|2,\mathbf{R})$. Further analysis on the superalgebra from the dynamical point of view is given in Sec. 3.3, for each case $\Lambda=0$, $\Lambda>0$, $\Lambda<0$ separately.

3.1 $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra - kinematical point of view

There are five *real* generators in $\mathfrak{osp}(1|2,\mathbf{R})$ which we take as

$$Q_{P}:=\mathrm{tr}(\psi P)\;,\qquad\qquad Q_{X}:=\mathrm{tr}(\psi X)\;, \tag{3.2}$$

and

$$K_{0}:=\tfrac{1}{2}\mathrm{tr}\big(P^{2}+X^{2}\big)\;,\qquad K_{1}:=\tfrac{1}{2}\mathrm{tr}\big(P^{2}-X^{2}\big)\;,\qquad K_{2}:=\tfrac{1}{2}\mathrm{tr}(XP+PX)\;. \tag{3.3}$$

All the super-commutator relations[7] of the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra then follow simply from the 'Heisenberg $\oplus$ Clifford' algebra, (3.1),

$$\begin{array}{lll}
{Q_{P}}^{2}=\frac{1}{2}\,(K_{0}+K_{1})\;, & {Q_{X}}^{2}=\frac{1}{2}\,(K_{0}-K_{1})\;, & \{Q_{P},Q_{X}\}=K_{2}\,,\\[2ex]
[K_{0},Q_{P}]=+iQ_{X}\,, & [K_{0},Q_{X}]=-iQ_{P}\,, & [K_{1},Q_{P}]=-iQ_{X}\,,\\[2ex]
[K_{1},Q_{X}]=-iQ_{P}\,, & [K_{2},Q_{P}]=+iQ_{P}\,, & [K_{2},Q_{X}]=-iQ_{X}\,,\\[2ex]
[K_{1},K_{2}]=-2iK_{0}\,, & [K_{0},K_{1}]=+2iK_{2}\,, & [K_{2},K_{0}]=+2iK_{1}\,.
\end{array} \tag{3.4}$$

The Casimir of the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra reads

$$\mathcal{C}_{\mathfrak{osp}(1|2,R)}=\mathcal{C}_{\mathfrak{so}(1,2)}+i\,[Q_{P},Q_{X}]\;,\qquad\qquad \Big[\mathcal{C}_{\mathfrak{osp}(1|2,R)}\,,\,\mathrm{anything}\Big]=0\,, \tag{3.5}$$

where the $\mathfrak{so}(1,2)$ Casimir is given by

$$\begin{aligned}
\mathcal{C}_{\mathfrak{so}(1,2)}\;&=K_{0}^{2}-K_{1}^{2}-K_{2}^{2}\\[1ex]
&=\tfrac{1}{2}\left\{\mathrm{tr}\big(P^{2}\big)\,,\mathrm{tr}\big(X^{2}\big)\right\}-\tfrac{1}{4}\,[\mathrm{tr}(XP+PX)]^{2}\\[1ex]
&=2\left\{{Q_{X}}^{2},{Q_{P}}^{2}\right\}-\{Q_{X},Q_{P}\}^{2}\;.
\end{aligned} \tag{3.6}$$

[6] For the construction of other various algebras, see [36].

[7] Although the $\mathfrak{osp}(1,|2,\mathbf{R})$ super-commutator relations above are direct consequences of the 'Heisenberg $\oplus$ Clifford' algebra, the way to express the generators in terms of X, P and ψ is not unique. In fact, $\mathfrak{so}(1,2)$ algebra was identified thirty years ago [37] using a 'non-polynomial' basis in the conformal matrix model having the inverse square potential, and based on the observation, Strominger proposed that the conformal matrix model is dual to $2D$ type 0A string theory on AdS_{2} [21] (see also [38, 39]).

The *root structure* of the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra can be identified by complexifying the generators as[8]

$$\begin{aligned} Q_+ &:= Q_P + iQ_X\,, & Q_- &:= Q_P - iQ_X = Q_+^\dagger\,,\\ K_+ &:= K_1 + iK_2\,, & K_- &:= K_1 - iK_2 = K_+^\dagger\,. \end{aligned} \tag{3.7}$$

The Cartan subalgebra has only one element, K_0, and all others are either raising, Q_+, K_+, or lowering, Q_-, K_-, operators to satisfy

$$\begin{array}{lll} Q_+^2 = K_+\,, & Q_-^2 = K_-\,, & \{Q_-, Q_+\} = 2K_0\,,\\ [K_0, Q_+] = +Q_+\,, & [K_+, Q_+] = 0\,, & [K_-, Q_+] = +2Q_-\,,\\ [K_0, Q_-] = -Q_-\,, & [K_+, Q_-] = -2Q_+\,, & [K_-, Q_-] = 0\,,\\ [K_0, K_+] = +2K_+\,, & [K_0, K_-] = -2K_-\,, & [K_-, K_+] = 4K_0\,. \end{array} \tag{3.8}$$

In the Cartan basis, the Casimir operators, (3.5), (3.6), read

$$\mathcal{C}_{\mathfrak{osp}(1|2,R)} = \mathcal{C}_{\mathfrak{so}(1,2)} + \tfrac{1}{2}\left[Q_-, Q_+\right]\,, \qquad \mathcal{C}_{\mathfrak{so}(1,2)} = K_0^2 - \tfrac{1}{2}\left\{K_-, K_+\right\}\,. \tag{3.9}$$

The $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra can be represented by $(2{+}1)\times(2{+}1)$ real supermatrices, M, satisfying

$$M^T\mathcal{J} + \mathcal{J}M = 0\,, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix}\,, \tag{3.10}$$

so that its generic form reads, with the even and odd real Grassmann entries, x, θ,

$$M = \begin{pmatrix} x_2 & -x_+ & \theta_1\\ x_- & -x_2 & \theta_2\\ -\theta_2 & \theta_1 & 0 \end{pmatrix}\,. \tag{3.11}$$

Note that the 2×2 bosonic part corresponds to $\mathfrak{sp}(2,\mathbf{R}) \equiv \mathfrak{so}(1,2) \equiv \mathfrak{sl}(2,\mathbf{R})$, as it corresponds to $x_\mu\gamma^\mu$, where $x_\pm = x_0 \pm x_1$, and γ^μ is the $\mathfrak{so}(1,2)$ gamma matrix,

$$\begin{gathered} \gamma^0 := \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 := \begin{pmatrix} 0 & -1\\ -1 & 0 \end{pmatrix}, \qquad \gamma^2 := \begin{pmatrix} 1 & 0\\ 0 & -1 \end{pmatrix},\\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\,, \qquad \eta = \mathrm{diag}(-++)\,. \end{gathered} \tag{3.12}$$

[8]For further analysis by us on the root structures of super Lie algebras, see [40, 41].

In fact, with the notion of a $(1+2)$-dimensional two component Majorana spinor and its 'charge conjugate',

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} Q_P \\ Q_X \end{pmatrix}, \qquad \bar{Q} := Q^T \gamma^0 = (\, Q_X \quad -Q_P \,) \,, \tag{3.13}$$

the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra, (3.4), can be rewritten in a compact form,

$$\big\{ Q \,, \bar{Q} \,\big\} = \gamma^\mu K_\mu \,, \qquad [\, K_\mu \,, \, Q \,] = i\gamma_\mu Q \,, \qquad [\, K_\mu \,, \, K_\nu \,] = 2i \, \epsilon_{\mu\nu\lambda} K^\lambda \,, \tag{3.14}$$

where $\epsilon_{\mu\nu\lambda}$ is the usual three form with $\epsilon_{012} \equiv 1$. Note also, from $\bar{Q}Q = [Q_X, Q_P]$, that the $\mathfrak{osp}(1|2,\mathbf{R})$ Casimir operator, $\mathcal{C}_{\mathfrak{osp}(1|2,R)}$, expressed in (3.5) is indeed manifestly $\mathrm{SL}(2,\mathbf{R})$ invariant.

Finally, in a similar fashion to above, one can also equip the bosonic operators with the $\mathrm{SL}(2,\mathbf{R})$ covariant structure,

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{pmatrix} := \begin{pmatrix} P \\ X \end{pmatrix}, \qquad K_\mu = \tfrac{1}{2} \bar{\mathcal{V}} \gamma_\mu \mathcal{V} \,. \tag{3.15}$$

3.2 Unitary irreducible representations of $\mathfrak{osp}(1|2,\mathbf{R})$

In order to analyze the *unitary irreducible representations* or *unitary supermultiplets* of the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra spanned by the five *real* generators, (3.2), (3.3), one needs to take K_0 as the 'good' quantum number operator to diagonalize it. Different choice of the good quantum number operator, e.g. K_2, is not compatible with the unitarity, as it would lead to the raising and lowering operators with the pure imaginary unit, such as $[K_2, (K_1 \pm K_0)] = \pm 2i(K_1 \pm K_0)$.

Any $\mathfrak{osp}(1|2,\mathbf{R})$ supermultiplet decomposes into $\mathfrak{so}(1,2)$ multiplets. We first review briefly the general properties of the unitary irreducible representations of $\mathfrak{so}(1,2)$.[9] From (3.9) and the commutator relations, we get

$$\mathcal{C}_{\mathfrak{so}(1,2)} + 1 + K_\pm K_\mp = (K_0 \mp 1)^2 \,. \tag{3.16}$$

From the Hermitian conjugacy property, $K_+ = K_-^\dagger$, the third term on the left hand side, $K_\pm K_\mp$, is positive semi-definite, while the possible minimum value of the right hand side for the states in a unitary irreducible representation may lie

$$0 \leq \min\big[(K_0 \mp 1)^2\big] \leq 1 \,, \tag{3.17}$$

if the raising or lowering operators act nontrivially ever. But, this is impossible when $\mathcal{C}_{\mathfrak{so}(1,2)} > 0$. In this case, the unitary representation is infinite dimensional and characterized by the existence of either the lowest weight state obeying

$$K_-|l,l\rangle = 0 \,, \qquad K_0|l,l\rangle = l|l,l\rangle \,, \qquad \mathcal{C}_{\mathfrak{so}(1,2)} = l(l-2) > 0 \,, \qquad l > 2 \,, \tag{3.18}$$

[9] For further analysis see e.g. [42].

or the highest weight state obeying

$$K_+|h,h\rangle = 0\,, \qquad K_0|h,h\rangle = h|h,h\rangle\,, \qquad \mathcal{C}_{\mathfrak{so}(1,2)} = h(h+2) > 0\,, \qquad h < -2\,. \tag{3.19}$$

When $\mathcal{C}_{\mathfrak{so}(1,2)} = 0$, there exists only one trivial state, $|0,0\rangle$, satisfying

$$K_\pm|0,0\rangle = 0\,, \qquad K_0|0,0\rangle = 0\,. \tag{3.20}$$

When $-1 \leq \mathcal{C}_{\mathfrak{so}(1,2)} < 0$, the representation is called the 'continuous principal series'. It is infinite dimensional, and the lowest or highest weight state may or may not exit. If there is a lowest or highest weight state, then its good quantum number is $+1 \pm \sqrt{\mathcal{C}_{\mathfrak{so}(1,2)}+1}$ or $-1 \pm \sqrt{\mathcal{C}_{\mathfrak{so}(1,2)}+1}$, respectively but not simultaneously. When $\mathcal{C}_{\mathfrak{so}(1,2)} < -1$, there must be neither lowest nor highest weight state, and the representation is called the 'continuous supplementary series'.

As for the present $\mathfrak{osp}(1|2,\mathbf{R})$ matrix model, K_0 is positive definite as

$$\begin{gathered} K_0 = \tfrac{1}{2}\mathrm{tr}\big(P^2+X^2\big) = \mathrm{tr}\left(A^\dagger A\right) + \tfrac{1}{2}N^2 \,\geq\, \tfrac{1}{2}N^2\,, \\ A := \tfrac{1}{\sqrt{2}}\left(P - iX\right)\,, \qquad \left[A^a{}_b\,,\, A^{\dagger c}{}_d\right] = \delta^a{}_d\,\delta^c{}_b\,. \end{gathered} \tag{3.21}$$

Thus there exits always a lowest weight state in any $\mathfrak{so}(1,2)$ multiplet, and from (3.16), the $\mathfrak{so}(1,2)$ Casimir is bounded below[10]

$$\begin{aligned} &\mathcal{C}_{\mathfrak{so}(1,2)} \geq \tfrac{1}{4}N^2(N^2-4) && \text{for} \quad N \geq 2\,, \\ &\mathcal{C}_{\mathfrak{so}(1,2)} > 0 \quad \text{or} \quad \mathcal{C}_{\mathfrak{so}(1,2)} = -\tfrac{3}{4} && \text{for} \quad N = 1\,. \end{aligned} \tag{3.22}$$

Now as for the $\mathfrak{osp}(1|2,\mathbf{R})$ unitary supermultiplet, we first note that the odd roots, $Q_\pm$, shift the 'good' quantum number by one unit, half of what $K_\pm$ do. Hence the odd roots move one $\mathfrak{so}(1,2)$ multiplet to another inside a $\mathfrak{osp}(1|2,\mathbf{R})$ supermultiplet, but at most once due to $Q_\pm^2 = K_\pm$. Similar to (3.16), we also have

$$\mathcal{C}_{\mathfrak{osp}(1|2,R)} + K_\pm K_\mp \pm Q_\pm Q_\mp = K_0\left(K_0 \mp 1\right)\,. \tag{3.23}$$

After all, utilizing all the facts above, we conclude that any unitary irreducible representation of the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra satisfying the positiveness, (3.21), is infinite dimensional and characterized by the existence of the super-lowest weight state obeying

$$Q_-|l_s,l_s\rangle = 0\,, \qquad K_0|l_s,l_s\rangle = l_s|l_s,l_s\rangle\,, \qquad \mathcal{C}_{\mathfrak{osp}(1|2,R)} = l_s(l_s-1)\,, \qquad l_s \geq \tfrac{1}{2}N^2\,. \tag{3.24}$$

[10] In fact, from (3.6), expressing $\mathcal{C}_{\mathfrak{so}(1,2)}$ in terms of the odd generator, the trace of $\mathcal{C}_{\mathfrak{so}(1,2)}$ also 'formally' shows the positiveness,

$$\mathrm{Tr}\,\mathcal{C}_{\mathfrak{so}(1,2)} \approx \mathrm{Tr}\left(-\left[Q_X, Q_P\right]^2\right) \,\geq\, 0\,.$$

The subtlety is due to the infinite sum over the infinite dimensional $\mathfrak{so}(1,2)$ multiplet.

Furthermore, the $\mathfrak{osp}(1|2,\mathbf{R})$ unitary supermultiplet always decomposes into two $\mathfrak{so}(1,2)$ multiplets whose lowest weight states are given by

$$|l_s, l_s\rangle \qquad \text{and} \qquad |l_s+1, l_s+1\rangle = \frac{1}{\sqrt{2l_s}}\, Q_+|l_s, l_s\rangle\,. \tag{3.25}$$

3.3 $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra - dynamical point of view

The Noether charges corresponding to the two dynamical supersymmetries, (2.13), are given by

$$\mathrm{tr}(i\psi\delta_\pm\psi) = if_\pm(t)\hat{\mathcal{Q}}_\pm\varepsilon_\pm\,, \qquad \hat{\mathcal{Q}}_\pm = \mathrm{tr}\Big[\psi\Big(P - g_\pm(t)X\Big)\Big]\,, \tag{3.26}$$

where we put

$$g_\pm(t) := \frac{\dot{f}_\pm(t)}{f_\pm(t)}\,, \qquad \dot{g}_\pm + g_\pm^2 = \Lambda(t)\,. \tag{3.27}$$

Because the Hamiltonian as well as the above two supercharges can be expressed in terms of the previous "kinematical" basis, Q_X, Q_P, K_0, K_1, K_2, (3.2), (3.3), the underlying supersymmetry algebra must correspond to $\mathfrak{osp}(1|2,\mathbf{R})$, no matter what the dynamics is. However, the use of the above supercharges, $\hat{\mathcal{Q}}$, will not lead to simple expressions for the superalgebra. For example, from the conservation of the Noether charge and Eq.(2.21), the commutator relation between the Hamiltonian and the supercharge reads in a less economic manner,

$$\Big[H\,,\,\hat{\mathcal{Q}}_\pm\Big] = ig_\pm\hat{\mathcal{Q}}_\pm + i\,\frac{\dot{g}_\pm}{g_+ - g_-}\Big(\hat{\mathcal{Q}}_+ - \hat{\mathcal{Q}}_-\Big)\,. \tag{3.28}$$

Henceforth, in order to analyze the underlying $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra in a simple fashion but still to keep track of the dynamical properties, we slightly modify the basis of the odd generators and keep the Hamiltonian explicitly as a $\mathfrak{so}(1,2)$ generator. Note that the change of basis requires the time dependent coefficients due to $\Lambda(t)$, as $\hat{\mathcal{Q}}_\pm = Q_P - g_\pm(t)Q_X$. Hence, only with specific time dependent coefficients we can write down the time independent conserved quantities, as one can expect from (2.21). All together there are five conserved "true" Noether charges corresponding to the five symmetries, (2.13), (2.15). Namely we have the two fermionic conserved Noether charges for the two dynamical supersymmetries,

$$f_\pm\hat{\mathcal{Q}}_\pm = f_\pm Q_P - \dot{f}_\pm Q_X\,, \tag{3.29}$$

and three bosonic conserved Noether charges for the $\mathfrak{so}(1,2)$ symmetries (2.15),

$$\begin{aligned}\Big(f_\pm\hat{\mathcal{Q}}_\pm\Big)^2 &= \tfrac{1}{2}\Big(f_\pm^2 + \dot{f}_\pm^2\Big)K_0 + \tfrac{1}{2}\Big(f_\pm^2 - \dot{f}_\pm^2\Big)K_1 - f_\pm\dot{f}_\pm K_2\,,\\ \Big\{f_+\hat{\mathcal{Q}}_+\,,\,f_-\hat{\mathcal{Q}}_-\Big\} &= \Big(f_+f_- + \dot{f}_+\dot{f}_-\Big)K_0 + \Big(f_+f_- - \dot{f}_+\dot{f}_-\Big)K_1 - \Big(f_+\dot{f}_- + f_-\dot{f}_+\Big)K_2\,.\end{aligned} \tag{3.30}$$

Apart from the above five Noether charges, both of the $\mathfrak{osp}(1|2,\mathbf{R})$ and $\mathfrak{so}(1,2)$ Casimir operators, $\mathcal{C}_{\mathfrak{osp}(1|2,R)}$ (3.5) and $\mathcal{C}_{\mathfrak{so}(1,2)}$ (3.6), are also conserved time independent quantities, since they do not include any explicit time dependency and they commute with the Hamiltonian, for sure.

3.3.1 $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra when $\Lambda(t)=0$

When $\Lambda=0$, the two supercharges corresponding to $f_+=1$ and $f_-=t$ are

$$\mathcal{Q}^{+}_{\Lambda=0}=\mathrm{tr}[\psi P]\,,\qquad\qquad \mathcal{Q}^{-}_{\Lambda=0}=\mathrm{tr}\big[\psi(P-t^{-1}X)\big]\,,\tag{3.31}$$

while the Hamiltonian is given by

$$H=\tfrac{1}{2}\left(P^2\right)=\tfrac{1}{2}\left(K_0+K_1\right)\,.\tag{3.32}$$

Rather than $\mathcal{Q}^{\pm}_{\Lambda=0}$, we adopt the kinematical odd generators, (3.2),

$$Q_P=\mathrm{tr}[\psi P]=\mathcal{Q}^{+}_{\Lambda=0}\,,\qquad\qquad Q_X=\mathrm{tr}[\psi X]=t\left(\mathcal{Q}^{+}_{\Lambda=0}-\mathcal{Q}^{-}_{\Lambda=0}\right)\,,\tag{3.33}$$

and write the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra in terms of the real basis,

$$\begin{array}{lll}
{Q_P}^2=H\,, & {Q_X}^2=V:=\frac{1}{2}\mathrm{tr}\big(X^2\big)\,, & \{Q_P,Q_X\}=K_2\,,\\[2mm]
[H,Q_X]=-iQ_P\,, & [H,Q_P]=0\,, & [V,Q_X]=0\,,\\[2mm]
[V,Q_P]=+iQ_X\,, & [K_2,Q_X]=-iQ_X\,, & [K_2,Q_P]=+iQ_P\,,\\[2mm]
[H,V]=-iK_2\,, & [K_2,H]=+2iH\,, & [K_2,V]=-2iV\,.
\end{array}\tag{3.34}$$

In particular, the $\mathfrak{so}(1,2)$ Casimir operator, (3.6), reads

$$\mathcal{C}_{\mathfrak{so}(1,2)}=K_0^2-K_1^2-K_2^2=2(HV+VH)-K_2^2=(H+V)^2-(H-V)^2-K_2^2\,.\tag{3.35}$$

3.3.2 $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra when $\Lambda(t)>0$

In the case of $\Lambda(t)>0$, we first set two real operators as[11]

$$C_\pm(t):=\frac{P\mp\sqrt{\Lambda(t)}X}{\sqrt{2}\,\Lambda(t)^{\frac{1}{4}}}=C_\pm(t)^\dagger\,,\tag{3.36}$$

further to define two real even generators in $\mathfrak{osp}(1|2,\mathbf{R})$,

$$L_\pm(t):=\mathrm{tr}\big(C_\pm(t)^2\big)=L_\pm(t)^\dagger\,,\tag{3.37}$$

as well as two real odd generators,

$$\mathcal{Q}_\pm(t):=\mathrm{tr}(\psi C_\pm(t))=\mathcal{Q}_\pm(t)^\dagger\,.\tag{3.38}$$

Note that, up to the overall factor, $\sqrt{2}\Lambda(t)^{\frac{1}{4}}$, $\mathcal{Q}_\pm$ coincide with the actual supercharges, $\hat{\mathcal{Q}}_\pm$, (3.26), provided that $\Lambda(t)$ is positive constant.

[11] In fact, $C_\pm$ correspond to the generators of W_∞ algebra in the Hamiltonian system, $H_\infty=\frac{1}{2}P^2-\frac{1}{2}X^2$ [43].

The Hamiltonian is then

$$H=\sqrt{\Lambda(t)}\,L_{H}(t)\,,\qquad\qquad L_{H}(t):=\tfrac{1}{2}\mathrm{tr}\Big[C_{+}(t)C_{-}(t)+C_{-}(t)C_{+}(t)\Big]\,, \tag{3.39}$$

and from the quantization relation,

$$\Big[C_{-}(t)^{a}{}_{b}\,,\,C_{+}(t)^{c}{}_{d}\Big]=+i\,\delta^{a}{}_{d}\,\delta^{c}{}_{b}\,, \tag{3.40}$$

we obtain such as

$$[L_{H},C_{\pm}]=\pm iC_{\pm}\,,\qquad [L_{-},C_{+}]=+2iC_{-}\,,\qquad [L_{+},C_{-}]=-2iC_{+}\,. \tag{3.41}$$

Now, the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra for $\Lambda>0$ case reads in terms of $\mathcal{Q}_{\pm},L_{\pm},L_{H}$,

$$\begin{array}{lll}
\mathcal{Q}_{+}^{2}=\tfrac{1}{2}L_{+}\,, & \mathcal{Q}_{-}^{2}=\tfrac{1}{2}L_{-}\,, & \{\mathcal{Q}_{+},\mathcal{Q}_{-}\}=L_{H}\,,\\
{[L_{-},\mathcal{Q}_{+}]}=+2i\mathcal{Q}_{-}\,, & [L_{+},\mathcal{Q}_{-}]=-2i\mathcal{Q}_{+}\,, & [L_{H},\mathcal{Q}_{\pm}]=\pm i\mathcal{Q}_{\pm}\,,\\
{[L_{-},L_{+}]}=+4iL_{H}\,, & [L_{H},L_{\pm}]=\pm 2iL_{\pm}\,, & [L_{\pm},\mathcal{Q}_{\pm}]=0\,.
\end{array} \tag{3.42}$$

The $\mathfrak{so}(1,2)$ Casimir operator, (3.6), can be reexpressed as

$$\mathcal{C}_{\mathfrak{so}(1,2)}=K_{0}^{2}-K_{1}^{2}-K_{2}^{2}=\tfrac{1}{2}\{L_{+},L_{-}\}-{L_{H}}^{2}=\left[\tfrac{1}{2}\,(L_{+}+L_{-})\right]^{2}-\left[\tfrac{1}{2}\,(L_{+}-L_{-})\right]^{2}-{L_{H}}^{2}\,, \tag{3.43}$$

which shows that $\frac{1}{2}\,(L_{+}\pm L_{-})$, L_{H} and K_0,K_1,K_2 are related to each other by a SO(1,2) transformation.

It is crucial to note that the Hamiltonian can not be diagonalized, , as it would mean the existence of the raising and lowering operators by the imaginary unit, $i\sqrt{\Lambda}$. The unitary representation is only possible when one takes $\frac{1}{2}\,(L_{+}+L_{-})$ as the good quantum number operator. Physically, this amounts to the fact that the matrix model describes the Fermi sea and there is no normalizable state (see e.g. [44, 35]).

3.3.3 $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra when $\Lambda(t)<0$

In the case of $\Lambda<0$, following the standard harmonic oscillator analysis, we first set a pair of complex operators,

$$A_{\pm}(t):=\frac{P\pm i\sqrt{|\Lambda(t)|}X}{\sqrt{2}\,|\Lambda(t)|^{\frac{1}{4}}}=A_{\mp}(t)^{\dagger}\,, \tag{3.44}$$

and define a pair of complexified even generators in $\mathfrak{osp}(1|2,\mathbf{R})$,

$$J_{\pm}(t):=\mathrm{tr}\big(A_{\pm}(t)^{2}\big)=J_{\mp}(t)^{\dagger}\,, \tag{3.45}$$

as well as a pair of complexified odd generators,

$$\mathbf{Q}_{\pm}(t):=\mathrm{tr}(\psi A_{\pm}(t))=\mathbf{Q}_{\mp}(t)^{\dagger}\,. \tag{3.46}$$

Note that, up to the overall factor, $\sqrt{2}|\Lambda(t)|^{\frac{1}{4}}$, $\mathbf{Q}_{\pm}$ coincide with the actual supercharges, $\hat{\mathcal{Q}}_{\pm}$, (3.26), provided that $\Lambda(t)$ is negative constant.

The Hamiltonian is then

$$H = \sqrt{|\Lambda(t)|}\, J_{H}(t)\,, \qquad J_{H}(t) := \tfrac{1}{2}\mathrm{tr}\Big[A_{+}(t)A_{-}(t) + A_{-}(t)A_{+}(t)\Big]\,, \tag{3.47}$$

and from the quantization relation,

$$\Big[A_{-}(t)^{a}{}_{b}\,,\, A_{+}(t)^{c}{}_{d}\Big] = \delta^{a}{}_{d}\,\delta^{c}{}_{b}\,, \tag{3.48}$$

we obtain such as

$$[J_{H}, A_{\pm}] = \pm A_{\pm}\,, \qquad [J_{-}, A_{+}] = +2A_{-}\,, \qquad [J_{+}, A_{-}] = -2A_{+}\,. \tag{3.49}$$

Now, the $\mathfrak{osp}(1|2,\mathbf{R})$ superalgebra for $\Lambda < 0$ case reads in terms of $\mathbf{Q}_{\pm}, J_{\pm}, J_{H}$,

$$\begin{array}{lll}
\mathbf{Q}_{+}^{2} = \tfrac{1}{2}J_{+}\,, & \mathbf{Q}_{-}^{2} = \tfrac{1}{2}J_{-}\,, & \{\mathbf{Q}_{+}, \mathbf{Q}_{-}\} = J_{H}\,, \\
{[J_{-}, \mathbf{Q}_{+}]} = +2\mathbf{Q}_{-}\,, & [J_{+}, \mathbf{Q}_{-}] = -2\mathbf{Q}_{+}\,, & [J_{H}, \mathbf{Q}_{\pm}] = \pm\mathbf{Q}_{\pm}\,, \\
{[J_{-}, J_{+}]} = +4J_{H}\,, & [J_{H}, J_{\pm}] = \pm 2J_{\pm}\,, & [J_{\pm}, \mathbf{Q}_{\pm}] = 0\,.
\end{array} \tag{3.50}$$

Especially, the $\mathfrak{so}(1,2)$ Casimir operator, (3.6), can be reexpressed as

$$\mathcal{C}_{\mathfrak{so}(1,2)} = K_{0}^{2} - K_{1}^{2} - K_{2}^{2} = {J_{H}}^{2} - \tfrac{1}{2}\{J_{+}, J_{-}\} = {J_{H}}^{2} - (\mathrm{Re}J_{+})^{2} - (\mathrm{Im}J_{+})^{2}\,, \tag{3.51}$$

which shows that $\mathrm{Re}J_{+}$, $\mathrm{Im}J_{+}$, J_{H} and K_{0}, K_{1}, K_{2} are related to each other by a $\mathrm{SO}(1,2)$ transformation.

4. Discussion and conclusion

We have derived a $\mathcal{N} = 2$ supersymmetric matrix model, (2.11), with a harmonic oscillator potential whose coefficient, $\Lambda(t)$, can be an arbitrary time dependent function. The resulting matrix model corresponds to the most general supersymmetric deformation of the matrix quantum mechanics having the $2D$ super Yang-Mills origin. We have shown that, for arbitrary $\Lambda(t)$ the matrix model enjoys two dynamical supersymmetries, Q_{1}, Q_{2}, and three bosonic symmetries, K_{0}, K_{1}, K_{2}, which amount to the superalgebra, $\mathfrak{osp}(1|2,\mathbf{R})$, (3.14),

$$\big\{Q\,,\,\bar{Q}\,\big\} = \gamma^{\mu}K_{\mu}\,, \qquad [\,K_{\mu}\,,\,Q\,] = i\gamma_{\mu}Q\,, \qquad [\,K_{\mu}\,,\,K_{\nu}\,] = 2i\,\epsilon_{\mu\nu\lambda}K^{\lambda}\,. \tag{4.1}$$

If the matrix model had only one supersymmetry as in the $2D$ minimal super Yang-Mills, the $\mathfrak{osp}(1|2,\mathbf{R})$ structure would be absent.

At the quantum level, the matrix model is exactly solvable, as it essentially reduces to the usual harmonic oscillator problem. In order to have a unifying description for arbitrary $\Lambda(t)$, the full Hilbert space of the $\mathcal{M}$-theory matrix model should include not only the *normalizable states* but also the *non-normalizable states*. The former is relevant to the case $\Lambda(t) < 0$, and the latter is so for the case $\Lambda(t) \geq 0$. Each sector then decomposes into irreducible representations or supermultiplets of $\mathfrak{osp}(1|2,\mathbf{R})$. In particular, the supermultiplets in the normalizable sector correspond to the unitary irreducible representations of $\mathfrak{osp}(1|2,\mathbf{R})$ which we discussed in Sec. 3.2.

Among the three generators of $\mathfrak{so}(1,2)$, only K_0 can give the 'good' quantum number, along with the Casimir, $\mathcal{C}_{\mathfrak{osp}(1|2,R)}$. When $\Lambda < 0$, K_0 coincides with $H/\sqrt{|\Lambda|}$ up to $\mathrm{SO}(1,2)$ rotations, otherwise not. Within the normalizable sector, relevant to the case $\Lambda < 0$, the energy spectrum is discretized by the unit $\sqrt{|\Lambda(t)|}$, with the zero point vacuum energy $\frac{1}{2}N^2\,|\Lambda(t)|$. The vacuum has the degeneracy, $2^{[N^2/2]}$, due to the fermions. The non-vanishing zero point energy refers to the existing two other bosonic charges in the superalgebra apart from the Hamiltonian. The non-normalizable sector for $\Lambda > 0$ corresponds to the quantum states of the Fermi sea [44, 35], and the Hamiltonian is not diagonalizable there.

The matrix model is to describe the noncritical $3D$ $\mathcal{M}$-theory on generic supersymmetric backgrounds in a controllable manner through the matrix regularization, and our claim is further that, with the arbitrariness of $\Lambda(t)$, it also provides holographic duals to various two dimensional superstring theories, as we argue below.

To start, we project the full phase space into the '$\mathfrak{so}(1,2)$ hyperspace' given by the "coordinates", K_0, K_1, K_2, (3.3). The induced dynamics therein is subject to

$$\begin{aligned} &\dot{K}_\mu = i\,[H,K_\mu] = 2\epsilon_{\mu\nu\lambda}\mathcal{T}^\nu K^\lambda\,, \qquad && \mathcal{T}^\nu := \left(\,\tfrac{1}{2}(1-\Lambda)\,,\,\tfrac{1}{2}(1+\Lambda)\,,\,0\,\right)\,, \\ &\mathcal{T}^\mu K_\mu = H\,, && \mathcal{T}^\mu\mathcal{T}_\mu = \Lambda\,. \end{aligned} \tag{4.2}$$

Naturally the '$\mathfrak{so}(1,2)$ hyperspace' is equipped with the $\mathfrak{so}(1,2)$ metric, $\eta = \mathrm{diag}(-++)$ (3.12). First of all, the $\mathfrak{so}(1,2)$ Casimir, $\mathcal{C}_{\mathfrak{so}(1,2)}$, (3.6) highlights the geometrical picture,

$$\mathcal{C}_{\mathfrak{so}(1,2)} = K_0^2 - K_1^2 - K_2^2\,. \tag{4.3}$$

Since, from (4.2), $K^\mu\dot{K}_\mu = 0$, $\mathcal{C}_{\mathfrak{so}(1,2)}$ is a conserved time independent quantity. Therefore, we observe that *for each* $\mathfrak{so}(1,2)$ *multiplet in the Hilbert space, the corresponding* $\mathfrak{so}(1,2)$ *hyperspace dynamics is constrained on a two dimensional rigid surface* such that

$$\begin{array}{lll} \text{Euclidean } dS_2/AdS_2 & \text{if} & \mathcal{C}_{\mathfrak{so}(1,2)} > 0\,, \\[2ex] \text{Minkowskian } dS_2/AdS_2 & \text{if} & \mathcal{C}_{\mathfrak{so}(1,2)} < 0\,, \\[2ex] \textit{Null cone} & \text{if} & \mathcal{C}_{\mathfrak{so}(1,2)} = 0\,. \end{array} \tag{4.4}$$

Surely the specific value of the Casimir for each multiplet is to be superselected just like any boundary condition in quantum field theories. This also fits into the $3D$ $\mathcal{M}$-theory picture, to include or provide holographic dual descriptions to all the superstring theories. The richness of the $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model originates from the arbitrariness of the cosmological constant, $\Lambda(t)$, and also the vast amount of existing $\mathfrak{so}(1,2)$ multiplets in the Hilbert space each of which has its own two dimensional geometry.

However, if we restrict on the unitary irreducible representations, i.e. the normalizable sector relevant to the case $\Lambda(t)<0$, we have the bound for the Casimir, (3.22),

$$\mathcal{C}_{\mathfrak{so}(1,2)} \geq \tfrac{1}{4}N^2(N^2-4) \qquad \text{if} \quad N \geq 2\,, \tag{4.5}$$

and the corresponding geometry is always Euclidean dS_2/AdS_2. As for the non-normalizable sector, relevant to $\Lambda(t)\geq 0$, the above bound does not hold.

On the other hand, for any classical BPS configurations, (2.24), it is straightforward to show that the $\mathfrak{so}(1,2)$ Casimir is positive semi-definite, irrespective of $\Lambda(t)$,

$$\mathcal{C}_{\mathfrak{so}(1,2)} = \left(\frac{\kappa}{f_\pm}\right)^2 \left[N\,\mathrm{tr}\big(X^2\big) - (\mathrm{tr}(X))^2\right] = \kappa^2 N\,\mathrm{tr}\big(Y^2\big) \geq 0\,. \tag{4.6}$$

When $\kappa \neq 0$, the equality only holds when all the D-particles are on the same position, the center of mass. Namely, further relative separations of the BPS D-particles mean the larger radius of dS_2/AdS_2. Thus, the generic BPS states in the $\mathcal{M}$-theory matrix model always correspond to the dynamics on the Euclidean dS_2/AdS_2 space. Nevertheless, when $\Lambda(t)\geq 0$, the corresponding quantum BPS states in the Hilbert space are not normalizable.

From (4.2), a 'dispersion relation' follows

$$\dot{K}_\mu \dot{K}^\mu = 4\left(H^2 - \Lambda K_\mu K^\mu\right) = 4\left(H(t)^2 + \Lambda(t)\mathcal{C}_{\mathfrak{so}(1,2)}\right)\,, \tag{4.7}$$

which shows that the 'mass' is conserved if $\Lambda(t)$ is constant. Furthermore, from (3.43), (3.51), we have the positive semi-definite bound,

$$\begin{array}{lll} \text{For}\ \ \Lambda(t)=0, & \dot{K}_\mu \dot{K}^\mu = 4H(t)^2 & \geq 0\,, \\[2ex] \text{For}\ \ \Lambda(t)>0, & \dot{K}_\mu \dot{K}^\mu = 8\Lambda(t)\left\{\mathcal{Q}_+^2, \mathcal{Q}_-^2\right\} & \geq 0\,, \\[2ex] \text{For}\ \ \Lambda(t)<0, & \dot{K}_\mu \dot{K}^\mu = -2\Lambda(t)\left\{J_+, J_-\right\} & \geq 0\,. \end{array} \tag{4.8}$$

Thus, the velocity vector, $\dot{K}_\mu$ is space-like, which is natural for the Euclidean geometry. But for the Minkowskian space it implies the superluminar behavior, i.e. "tachyon". As shown above, all the BPS configurations correspond to the Euclidean geometry, and hence not tachyonic.

To summarize, the normalizable sector in the Hilbert space of the $\mathfrak{osp}(1|2,\mathbf{R})$ matrix model is characterized by the Euclidean dS_2/AdS_2 geometry, while the $\mathfrak{so}(1,2)$ multiplet in the non-normalizable sector can be any, either Euclidean or Minkowskian. All the BPS states always correspond to the Euclidean geometry, and if $\Lambda(t) > 0$ then the BPS state is not normalizable. Especially when $\Lambda(t) > 0$ and $\mathcal{C}_{\mathfrak{so}(1,2)} < 0$, i.e. Minkowskian *de-Sitter* geometry, from (4.8), the particles in the $\mathfrak{so}(1,2)$ hyperspace are tachyonic and can not be supersymmetric.

Various matrix models with potentials having a single maximum have been proposed as dual candidates of $2D$ string theories on AdS_2-type backgrounds with the rolling tachyon or the linear dilaton [45, 46, 47, 19, 21, 8]. The continuum or so called the double scaling limit in the matrix models zoom in on the maximum of the potential, effectively leaving a single upside down harmonic potential [48, 49, 50], precisely the same feature as our $\mathfrak{osp}(1|2,\mathbf{R})$ matrix model shares when $\Lambda > 0$. Furthermore, the Hermitian matrix itself is supposed to represent the non-Abelian open string tachyon [45], and this is manifest in our dispersion relation, (4.8), for the case of $\Lambda > 0$ and $\mathcal{C}_{\mathfrak{so}(1,2)} < 0$. Thus, we conclude that when $\Lambda(t)$ is positive, the $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model provides holographic duals to the two dimensional Minkowskian superstring theories. The relevant sector in the matrix model Hilbert space is then the non-normalizable one with the constraint, $\mathcal{C}_{\mathfrak{so}(1,2)} < 0$. From (4.8), the choice of the decreasing $\Lambda(t)$, like $\Lambda(t) = e^{-t/t_o}$, seems appropriate for the description of the tachyon condensation [57] or the D-brane decay [45]. Further investigation is to be required.

On the other hand, for the constant positive Λ, the generic BPS configurations (2.24) are given by the hyperbolic functions,

$$X(t) = \cosh\left(\sqrt{\Lambda}\,t\right) X(0) + \frac{\kappa}{\sqrt{\Lambda}} \sinh\left(\sqrt{\Lambda}\,t\right) 1\,, \tag{4.9}$$

while for the constant negative Λ, they are the usual harmonic oscillators,

$$X(t) = \cos\left(\sqrt{|\Lambda|}\,t\right) X(0) + \frac{\kappa}{\sqrt{|\Lambda|}} \sin\left(\sqrt{|\Lambda|}\,t\right) 1\,. \tag{4.10}$$

From (4.6), when $\kappa \neq 0$, the former and the latter correspond to the D0-branes in Euclidean dS_2 and Euclidean AdS_2 space respectively. If $\kappa = 0$, the center of mass is fixed and the D-particles are on the *Null cone* of the $\mathfrak{so}(1,2)$ hyperspace. However, when $\Lambda > 0$, there is no corresponding normalizable quantum BPS state for (4.9). This is also consistent with the Euclidean $2D$ superstring theory or the $\mathcal{N} = 2$ super Liouville theory results [51, 52, 53, 54, 55, 56]. The classical shape of the so called FZZT brane (falling Euclidean D0-brane), which is given as time dependent boundary state, precisely matches with (4.10). Thus, we expect that when $\Lambda(t)$ is negative, the $\mathfrak{osp}(1|2,\mathbf{R})$ $\mathcal{M}$-theory matrix model provides holographic dual description of $2D$ Euclidean superstring theories or superconformal theories. In particular, if Λ is negative constant, it corresponds to the $\mathcal{N} = 2$ super Liouville theory, with the relation to the 'Liouville background charge', $Q_{\text{Liouville}} = 2\,|\Lambda|$.

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References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[3] E. Marinari and G. Parisi, Phys. Lett. B **240** (1990) 375.

[4] J. McGreevy, S. Murthy and H. L. Verlinde, JHEP **0404** (2004) 015 [arXiv:hep-th/0308105].

[5] A. Dabholkar, Nucl. Phys. B **368** (1992) 293.

[6] S. Gukov, T. Takayanagi and N. Toumbas, JHEP **0403** (2004) 017 [arXiv:hep-th/0312208].

[7] T. Takayanagi, JHEP **0411** (2004) 030 [arXiv:hep-th/0408086].

[8] H. L. Verlinde, "Superstrings on AdS(2) and superconformal matrix quantum mechanics," arXiv:hep-th/0403024.

[9] P. K. Townsend, "The M(atrix) model/AdS(2) correspondence," arXiv:hep-th/9903043.

[10] P. Horava and C. A. Keeler, "Noncritical M-theory in 2+1 dimensions as a nonrelativistic Fermi liquid," arXiv:hep-th/0508024.

[11] M. McGuigan, "Noncritical M-theory: Three dimensions," arXiv:hep-th/0408041.

[12] A. Achucarro, J. M. Evans, P. K. Townsend and D. L. Wiltshire, Phys. Lett. B **198** (1987) 441.

[13] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B **305**, 545 (1988).

[14] J. Hoppe, "Membranes and matrix models," arXiv:hep-th/0206192.

[15] S. K. Ashok, S. Murthy and J. Troost, "D-branes in non-critical superstrings and minimal super Yang-Mills in various dimensions," arXiv:hep-th/0504079.

[16] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D **55** (1997) 5112 [hep-th/9610043].

[17] L. Susskind, "Another conjecture about M(atrix) theory," arXiv:hep-th/9704080.

[18] J.-H. Park, Phys. Lett. A **307** (2003) 183 [arXiv:hep-th/0203017].

[19] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg, [arXiv:hep-th/0307195].

[20] D. M. Thompson, Phys. Rev. D **70** (2004) 106001 [arXiv:hep-th/0312156].

[21] A. Strominger, JHEP **0403** (2004) 066 [arXiv:hep-th/0312194].

[22] T. Banks and M. O'Loughlin, Phys. Rev. D **48**, 698 (1993) [arXiv:hep-th/9212136].

[23] N. Berkovits, S. Gukov and B. C. Vallilo, Nucl. Phys. B **614**, 195 (2001) [arXiv:hep-th/0107140].

[24] Y. Kazama and H. Suzuki, Nucl. Phys. B **321** (1989) 232.

[25] S. Hyun and J.-H. Park, JHEP **0211** (2002) 001 [arXiv:hep-th/0209219].

[26] S. Hyun, J.-H. Park and S. H. Yi, JHEP **0303** (2003) 004 [arXiv:hep-th/0301090].

[27] D. Berenstein, J. M. Maldacena and H. Nastase, JHEP **0204** (2002) 013 [hep-th/0202021].

[28] N. Kim and J.-H. Park, "Noncritical $\mathcal{M}$-theory matrix models in $3,4,5,7$ dimensions" in preparation.

[29] Paul A. M. Dirac, *Lectures on quantum mechanics,* Yeshiva University, 1964.

[30] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems,* Princeton University Press, 1992.

[31] J.-H. Park, Class. Quant. Grav. **19** (2002) L11 [arXiv:hep-th/0108145].

[32] E. Alvarez and P. Meessen, Phys. Lett. B **426**, 282 (1998) [arXiv:hep-th/9712136].

[33] D. Z. Freedman, G. W. Gibbons and M. Schnabl, AIP Conf. Proc. **743**, 286 (2005) [arXiv:hep-th/0411119].

[34] G. W. Gibbons and C. E. Patricot, Class. Quant. Grav. **20** (2003) 5225 [arXiv:hep-th/0308200].

[35] S. R. Das, J. L. Davis, F. Larsen and P. Mukhopadhyay, Phys. Rev. D **70**, 044017 (2004) [arXiv:hep-th/0403275].

[36] M. Gunaydin and C. Saclioglu, Commun. Math. Phys. **87**, 159 (1982).
I. Bars and M. Gunaydin, Commun. Math. Phys. **91**, 31 (1983).
M. Gunaydin and S. Hyun, J. Math. Phys. **29**, 2367 (1988).
M. Gunaydin and R. J. Scalise, J. Math. Phys. **32**, 599 (1991).
M. Gunaydin, J. Math. Phys. **29**, 1275 (1988).
S. Fernando, M. Gunaydin and S. Hyun, [arXiv:hep-th/0411281].

[37] V. de Alfaro, S. Fubini and G. Furlan, Nuovo Cim. A **34**, 569 (1976).

[38] P. M. Ho, JHEP **0405**, 008 (2004) [arXiv:hep-th/0401167].

[39] O. Aharony and A. Patir, "The Conformal Limit of the 0A Matrix Model and String Theory on AdS(2)," [arXiv:hep-th/0509221].

[40] N. Kim and J.-H. Park, Phys. Rev. D **66** (2002) 106007 [arXiv:hep-th/0207061].

[41] S. Lee and J.-H. Park, JHEP **0406** (2004) 038 [arXiv:hep-th/0404051].

[42] L. J. Dixon, M. E. Peskin and J. Lykken, Nucl. Phys. B **325** (1989) 329;
M. S. Plyushchay, J. Math. Phys. **34** (1993) 3954;
H. Nicolai and D. I. Olive, Lett. Math. Phys. **58** (2001) 141 [arXiv:hep-th/0107146];
J. M. Maldacena and H. Ooguri, J. Math. Phys. **42** (2001) 2929;
B. Morariu and A. P. Polychronakos, "Quantum mechanics on noncommutative Riemann surfaces," [hep-th/0201070].

[43] J. Avan and A. Jevicki, Phys. Lett. B **266**, 35 (1991);
J. Avan and A. Jevicki, Phys. Lett. B **272**, 17 (1991);
S. R. Das, A. Dhar, G. Mandal and S. R. Wadia, Int. J. Mod. Phys. A **7**, 5165 (1992) [arXiv:hep-th/9110021];
S. R. Das, A. Dhar, G. Mandal and S. R. Wadia, Mod. Phys. Lett. A **7**, 71 (1992) [arXiv:hep-th/9111021];
E. Witten, Nucl. Phys. B **373**, 187 (1992) [arXiv:hep-th/9108004].

[44] J. L. Karczmarek and A. Strominger, JHEP **0404** (2004) 055 [arXiv:hep-th/0309138].

[45] J. McGreevy and H. L. Verlinde, JHEP **0312**, 054 (2003) [arXiv:hep-th/0304224].

[46] I. R. Klebanov, J. Maldacena and N. Seiberg, JHEP **0307**, 045 (2003) [arXiv:hep-th/0305159].

[47] T. Takayanagi and N. Toumbas, JHEP **0307**, 064 (2003) [arXiv:hep-th/0307083].

[48] D. J. Gross and N. Miljkovic, Phys. Lett. B **238**, 217 (1990).

[49] E. Brezin, V. A. Kazakov and A. B. Zamolodchikov, Dimension," Nucl. Phys. B **338**, 673 (1990).

[50] P. H. Ginsparg and J. Zinn-Justin, Phys. Lett. B **240**, 333 (1990).

[51] T. Eguchi and Y. Sugawara, "Modular bootstrap for boundary N = 2 Liouville theory," JHEP **0401**, 025 (2004) [arXiv:hep-th/0311141].

[52] C. Ahn, M. Stanishkov and M. Yamamoto, "One-point functions of N = 2 super-Liouville theory with boundary," Nucl. Phys. B **683**, 177 (2004) [arXiv:hep-th/0311169].

[53] D. Kutasov, "D-brane dynamics near NS5-branes," [arXiv:hep-th/0405058].

[54] C. Ahn, M. Stanishkov and M. Yamamoto, "ZZ-branes of N = 2 super-Liouville theory," JHEP **0407**, 057 (2004) [arXiv:hep-th/0405274].

[55] Y. Nakayama, Y. Sugawara and H. Takayanagi, JHEP **0407** (2004) 020 [arXiv:hep-th/0406173].

[56] J. M. Lapan and W. Li, "Falling D0-branes in 2D superstring theory," [arXiv:hep-th/0501054].

[57] A. Sen, JHEP **0405**, 076 (2004) [arXiv:hep-th/0402157];
A. Sen, Int. J. Mod. Phys. A **18**, 4869 (2003) [arXiv:hep-th/0209122];
A. Sen, Mod. Phys. Lett. A **17**, 1797 (2002) [arXiv:hep-th/0204143];
A. Sen, JHEP **0207**, 065 (2002) [arXiv:hep-th/0203265];
A. Sen, JHEP **0204**, 048 (2002) [arXiv:hep-th/0203211].