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# On Killing tensors and cubic vertices in higher-spin gauge theories

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The problem of determining all consistent non-Abelian local interactions is reviewed in flat space-time. The antifield-BRST formulation of the free theory is an efficient tool to address this problem. Firstly, it allows to compute all on-shell local Killing tensor fields, which are important because of their deep relationship with higher-spin algebras. Secondly, under the sole assumptions of locality and Poincaré invariance, all non-trivial consistent deformations of a sum of spin-three quadratic actions deforming the Abelian gauge algebra were determined. They are compared with lower-spin cases.

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## 1 Introduction

It has been known since the works of Wigner that first-quantized relativistic particles are in one-to-one correspondence with inequivalent unitary irreducible representations (UIRs) of the Poincaré group  $ISO(n-1, 1)$ . Therefore it is a natural programme to explore the exhaustive list of such possibilities and to determine if some exotic cases could be of physical relevance. At sufficiently high energy, any given massive field effectively behaves as a massless field. The little group<sup>1</sup>  $ISO(n-2)$  of a light-like momentum governs the classification of the massless representations which fall into two distinct categories: the particularly interesting “helicity” (i.e. finite-dimensional) representations and the generic “continuous spin” (i.e. infinite-dimensional) representations. The helicity fields are either the celebrated completely symmetric gauge fields (see e.g. [1] for introductions) or the mixed symmetry gauge fields (see for instance [2] for a review). Usually the “spin”  $s$  of a single-valued helicity representation refers to the number of columns of the Young diagram labeling the representation of the maximal compact subgroup  $SO(n-2)$ . For completely symmetric tensor gauge fields, the spin  $s$  is equal to the rank. “Lower spin” stands for spin  $s \leq 2$ , while “higher spin” refers to spin  $s > 2$ . Covariant field equations were recently obtained for each continuous spin representation from a subtle infinite spin limit of helicity field equations [3] so that “continuous spin” may somehow be thought of as the case  $s = \infty$ .

Whereas gauge theories describing free massless fields are by now well established, it still remains unclear whether non-trivial consistent self-couplings and/or cross-couplings among those fields may exist

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<sup>1</sup> In order for the little group to make sense, we will consider here space-time dimension  $n \geq 3$ .

in general, such that the deformed gauge algebra is non-Abelian. The goal of the present paper is to review a mathematically precise statement of this problem and to focus on the rather general results obtained thanks to a very useful tool: the Becchi-Rouet-Stora-Tyutin (BRST) reformulation of the Noether procedure for the determination of the cubic vertices and corresponding deformations of the gauge transformations.

## 2 The interaction problem

We review the deformation setting for the problem of constructing consistent local vertices for a free field theory possessing some gauge symmetries [4] and particularize to the case of non-Abelian deformations. The cohomological reformulation of the deformation problem in the antifield-BRST formalism [5] is briefly sketched. Note that other formulations exist (see e.g. [6] and refs therein).

### 2.1 General hypotheses

The starting point is some given action  $S_0$  that is said to be “undeformed”. We assume, as in the Noether deformation procedure, that the deformed action can be expressed as a formal power series in some coupling constants denoted collectively by  $g$ , the zeroth-order term in the expansion describing the undeformed theory  $S_0$ :

$$S = S_0 + g S_1 + g^2 S_2 + \mathcal{O}(g^3).$$

The procedure is then perturbative: one tries to construct the deformations order by order in the deformation parameters  $g$ . Some physical requirements naturally come out:

- Poincaré symmetry: We ask that the Lagrangian be *manifestly* invariant under the *Poincaré* group. Therefore, the free theory is formulated in the flat space-time  $\mathbb{R}^{n-1,1}$ , the Lagrangian should not depend explicitly on the space-time cartesian coordinates  $\{x^\mu\}$  and all space-time indices must be raised and lowered by using either the Minkowski metric or the Levi-Civita tensor.
- non-triviality: We reject *trivial* deformations arising from field-redefinitions that reduce to the identity at order zero:

$$\varphi \longrightarrow \varphi' = \varphi + g \phi(\varphi, \partial\varphi, \dots) + \mathcal{O}(g^2).$$

- Consistency: A deformation of a theory is called *consistent* if the deformed theory possesses the same number of possibly deformed (perturbatively as well) independent gauge symmetries, reducibility identities, *etc.*, as the system we started with. In other words, the number of physical degrees of freedom is unchanged.
- Locality: The deformed action  $S[\varphi]$  must be a *local* functional. The deformations of the gauge transformations, *etc.*, must be local functions, as well as the allowed field redefinitions.

We remind the reader that a local function of some set of fields  $\varphi$  is a smooth function of the fields  $\varphi$  and their derivatives  $\partial\varphi, \partial^2\varphi, \dots$  up to some *finite* order, say  $k$ , in the number of derivatives. Such a set of variables  $\varphi, \partial\varphi, \dots, \partial^k\varphi$  will be collectively denoted by  $[\varphi]$ . Therefore, a local function of  $\varphi$  is denoted by  $f([\varphi])$ . A local  $p$ -form ( $0 \leq p \leq n$ ) is a differential  $p$ -form the components of which are local functions:

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p}(x, [\varphi]) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}.$$

A local functional  $\Omega[\varphi]$  of the fields  $\varphi$  is the integral  $\Omega = \int \omega$  of a local  $n$ -form  $\omega = \omega(x, [\varphi])$ .

The general problem of consistently deforming a given action  $S_0$  is of course highly non-trivial. It is very fruitful to investigate it order by order in  $g$  and to exploit the undeformed gauge symmetries that strongly restrict the consistent possibilities. Indeed, one can easily show that non-trivial first order consistent local deformations  $S_1$  are, on-shell, non-vanishing gauge invariant local functionals (with respect to

the undeformed equations of motion and gauge symmetries). To reformulate the problem in the antifield-BRST setting it is enough to observe [5] that the latter functionals are in one-to-one correspondence with elements of the local BRST cohomology group  $H^{n,0}(s_0|d)$  in top form degree and in vanishing ghost number, where  $s_0$  is the BRST differential corresponding to the undeformed action  $S_0$ . Moreover, the obstructions to the existence of a second-order deformation  $S_2$  corresponding to  $S_1$  are encoded in the local BRST cohomology group  $H^{n,1}(s_0|d)$  in ghost number one [5].

## 2.2 Non-Abelian deformations

The conventional local free theories corresponding to UIRs of the helicity group  $SO(n-2)$  that are completely symmetric tensors have been constructed by Fronsdal a while ago [7] for arbitrary rank  $s$ . To have Lorentz invariance manifest, the theory is expressed in terms of completely symmetric tensor gauge fields  $\varphi_{\mu_1 \dots \mu_s} = \varphi_{(\mu_1 \dots \mu_s)}$  of rank  $s > 0$ , the gauge transformation of which reads

$$\delta_\varepsilon \varphi_{\mu_1 \dots \mu_s} = s \partial_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)},$$

where the curved (square) bracket denotes complete (anti)symmetrization with strength one<sup>2</sup> and the Greek indices run over  $n$  values ( $n \geq 3$ ). The gauge parameter  $\varepsilon$  is a completely symmetric tensor field of rank  $s-1$ . For spin  $s=1$  the gauge field  $\varphi_\mu$  represents the photon with  $U(1)$  gauge symmetry while for spin  $s=2$  the gauge field  $\varphi_{\mu\nu}$  represents the graviton with linearized diffeomorphism invariance. The gauge field theories corresponding to tensorial helicity representations labeled by one-column Young diagrams are the usual  $p$ -form (*i.e.* completely antisymmetric tensor) gauge theories. Analogous gauge field theories corresponding to arbitrary spin-two (*i.e.* two-column Young diagrams) helicity representations [8] have been studied recently [9, 10] by using multiform and hyperform calculus [11].

Let us denote by  $S_0[\varphi_Y]$  the Poincaré-invariant, local, second-order, quadratic, gauge-invariant ghost-free actions mentioned above where the Young diagram  $Y$  labels the corresponding representations. Now, the interaction problem reviewed here can be formulated in a mathematically precise way as follows:

**Non-Abelian interaction problem:** *List all Poincaré-invariant, non-trivial consistent local deformations*

$$S[\varphi] = S_0[\varphi] + g S_1[\varphi] + g^2 S_2[\varphi] + \mathcal{O}(g^3)$$

*of a finite positive sum*

$$S_0[\varphi] = \sum_{Y, a} S_0[\varphi_Y^a]$$

*of a collection  $\varphi \equiv \varphi_Y^a$  (labeled by some index  $a$  for each given Young diagram  $Y$ ) of free gauge field theories such that the deformed local gauge transformations*

$$\delta\varphi^a = \delta_0\varphi^a + g F^a([\varphi^b], [\varepsilon^c]) + \mathcal{O}(g^2)$$

*are non-Abelian at first order in the coupling constants  $g$ .*

Of course, this problem is too complicated to be addressed in full generality with the techniques known at the moment. The restriction of the interaction problem to symmetric tensor gauge fields of rank two, is sometimes referred to as the ‘‘Gupta programme’’. The generalization of the latter to a collection of symmetric tensor gauge fields with arbitrary values of the rank  $s$  was proposed in [7] and is thereby frequently called the ‘‘Fronsdal programme’’ or ‘‘higher-spin interaction problem’’.

## 2.3 Fronsdal’s programme

This old programme is still far away from completion even though encouraging progresses have been obtained over the years. On the one hand, the problem of consistent interactions among only higher-spin

<sup>2</sup> For example,  $\Phi_{(\mu\nu)} = \frac{1}{2}(\Phi_{\mu\nu} + \Phi_{\nu\mu})$  and  $\Phi_{[\mu\nu]} = \frac{1}{2}(\Phi_{\mu\nu} - \Phi_{\nu\mu})$ .

gauge fields (hence without gravity) in Minkowski space-time  $\mathbb{R}^{n-1,1}$  was addressed in [12–16] (and refs therein) where some positive results have been obtained at first order in the perturbation. In the light-cone gauge, three-point couplings between completely symmetric gauge fields with arbitrary spins  $s > 2$  were constructed in [12]. For the pure spin-3 case, a cubic vertex was obtained in a covariant form by Berends, Burgers and van Dam (BBvD) [13] while new explicit vertices were obtained very recently in [15, 16]. The BBvD interaction, however, leads to inconsistencies when pushed at the next orders in powers of  $g$ , as was demonstrated in [13–15]. On the other hand, the first explicit attempts to introduce minimal coupling between higher-spin gauge fields and gravity encountered severe problems [17]. Very early, the idea was proposed that a consistent higher-spin gauge theory could exist, provided all spins are taken into account [7]. In order to overcome the gravitational coupling problem, it was also suggested to perturb around a maximally-symmetric curved background, like for example  $AdS_n$ , in which directions interesting results have indeed been obtained, such as cubic vertices consistent at first order [18] and equations of motion formally consistent at all orders [19] (see also [20] and refs therein).

If there is a lesson to learn from decades of efforts on the higher-spin interaction problem, it certainly is the unusual character of the possible interactions. For instance, the cubic vertices contain more than two derivatives.<sup>3</sup> In order to remove any prejudice on the form of the interactions, it is natural to attack the Fronsdal programme on exhaustive and purely algebraic grounds such as the antifield-BRST deformation procedure.

### 3 Killing tensor fields

A problem of physical interest for a better understanding of the higher-spin symmetries is the determination of all Killing tensor fields on Minkowski space-time, that is, the symmetric tensor fields satisfying the following Killing-like equation  $\partial_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)}(x) = 0$ , so that the corresponding Abelian gauge transformations vanish:  $\delta_\varepsilon \varphi = 0$ . The most general smooth solution of this equation is

$$\varepsilon_{\mu_1 \dots \mu_{s-1}}(x) = \sum_{t=0}^{s-1} \lambda_{\mu_1 \dots \mu_{s-1}, \nu_1 \dots \nu_t} x^{\nu_1} \dots x^{\nu_t}, \quad \lambda_{(\mu_1 \dots \mu_{s-1}, \nu_1) \nu_2 \dots \nu_t} = 0 \quad (1)$$

where each coefficient of the term of given homogeneity degree in the coordinates  $\{x^\mu\}$  is a constant tensor  $\lambda_{\mu_1 \dots \mu_{s-1}, \nu_1 \dots \nu_t}$ , the symmetries of which are labeled by a two-row Young diagram.

Another motivation is that non-trivial on-shell local Killing tensor fields are in one-to-one correspondence with cocycles of the local Koszul-Tate cohomology group  $H_2^n(\delta_0|d)$  in top form degree and antifield number two, the knowledge of which is an important ingredient in the computation of the local BRST cohomology group  $H^{n,0}(s_0|d)$ .

**Constant-curvature space-time Killing tensors** [21, 22]: *All on-shell Killing tensor fields  $\varepsilon_{\mu_1 \dots \mu_{s-1}}(x, [\varphi])$  of the completely symmetric tensor gauge field theory on constant-curvature space-times can be represented by off-shell Killing tensor fields that are independent of the gauge field  $\varphi$  and that are solutions of the Killing-like equation  $\nabla_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)}(x) = 0$ .*

Generally speaking, the global symmetries of a solution of some field equation correspond to the space of gauge parameters leaving the gauge fields invariant under gauge transformations evaluated at the solution. Furthermore, for the flat vacuum solution they are expected to correspond to the full rigid symmetry algebra of the theory. More specifically, the Minkowski Killing tensors of the infinite tower of higher-spin fields should be related to a higher-spin algebra in flat space-time.

The higher-spin gauge symmetry algebras might eventually find their origin in the general procedure of “gauging” some global higher-symmetry algebras of free theories, as was argued in [21, 23] and as we briefly sketch here. All linear relativistic wave equations  $K|\phi\rangle = 0$  (corresponding to some finite-dimensional UIR of the little group) can be derived from an action taking the form of an inner product  $\int \langle \phi | K | \phi \rangle$ . Let  $\{T_i\}$  be Hermitian operators spanning some symmetry Lie algebra. This means that they

<sup>3</sup> The full non-linear higher-spin theory exposed in [19] is even expected to be non-local.

commute with the kinetic operator so that  $\{iT_i\}$  generate, via exponentiation, unitary operators preserving the quadratic action. But exactly the same is true for any Weyl-ordered polynomial  $P(T_i)$  of such symmetry generators so that the symmetry algebra may become infinite-dimensional. If the symmetry algebra is a finite-dimensional internal algebra then the latter procedure does not produce anything interesting in general. The case of interest is when one deals with a space-time symmetry algebra generated by vector fields. In such a case, the polynomials in the basis elements are differential operators and their exponentiation leads to non-local unitary operators [23].

**Minkowski higher-spin algebra** [21]: *The algebra of Weyl-ordered polynomials in the Killing vector fields  $\partial_\mu$  and  $x_{[\mu}\partial_{\nu]}$  of Minkowski space-time is isomorphic to the algebra of differential operators given by  $\varepsilon_{\mu_1\dots\mu_{s-1}}(x)\partial^{\mu_1}\dots\partial^{\mu_{s-1}}$  defined by the infinite tower of Minkowski Killing tensor fields ( $0 < s < \infty$ ).*

From its definition, it follows directly that this Minkowski higher-spin algebra can also be obtained via an Inönü-Wigner contraction of the  $(A)dS_n$  higher-spin algebras of Vasiliev [19] in the flat limit  $\Lambda \rightarrow 0$ . To end this section, we underline that we have not discussed at all here the subtle issue of trace conditions and their relation with the factorization of the higher-spin algebras which has been debated recently [20] (in the specific context of Minkowski Killing tensors, it was also discussed in [21]).

## 4 Non-Abelian gauge transformations

The results on one one-column [24] and on two-column [10, 25] Young-diagram gauge fields together with the spin-three case [15, 16] may be summarized in the following theorem in a form which suggests itself a conjecture for an arbitrary Young diagram.

**Deformations of the algebra:** *For a collection of gauge fields  $\varphi_Y^a$  ( $a = 1, \dots, N$ ) labeled by a fixed Young diagram  $Y$  with three columns or less, the non-Abelian interaction problem does not possess any non-trivial solution if the Young diagram  $Y$  is made of more than one row. In the completely symmetric tensor case, at first order in some smooth deformation parameter, Poincaré-invariant deformations of the (Abelian) gauge algebra exist. The deformed gauge algebra may always be assumed to be closed off-shell. Two cases arise depending on the parity-symmetry property of the first-order deformation.*

- (i) *The first-order parity-invariant deformations of the gauge algebra are in one-to-one correspondence with the structure constant tensors  $C^a_{bc} = (-)^s C^a_{cb}$  of an (anti)commutative internal algebra, that may be taken as deformation parameters;*
- (ii) *The first-order parity-breaking deformations of the gauge algebra are characterized by structure constant tensors  $C^a_{bc} = (-)^s (\delta_3^n - \delta_5^n) C^a_{cb}$ , where  $n$  is the space-time dimension and  $s > 1$ .*

In other words, one may conjecture that there exists no solution to the non-Abelian interaction problem for any finite collection of mixed symmetry gauge fields (at least for fixed symmetry properties). Therefore, from now on we focus on the case of a collection  $\varphi_{\mu_1\dots\mu_s}^a$  ( $a = 1, \dots, N$ ) of completely symmetric tensor gauge fields with fixed spin  $s$ . We also review the lower spin cases  $s = 1, 2$  in order to compare them with the higher-spin case  $s = 3$  and look for similarities or novelties.

**Deformations of the transformations:** *For a collection  $\varphi_{\mu_1\dots\mu_s}^a$  ( $a = 1, \dots, N$ ) of completely symmetric tensor gauge fields with fixed spin  $s = 1, 2, 3$ , the most general Poincaré and parity-invariant gauge transformations deforming the gauge algebra at first order in the structure constants are equal to, up to gauge transformations that either are trivial or do not deform the gauge algebra at first order,*

$$s = 1: \text{ the Yang-Mills gauge transformation } \delta\varphi_\mu^a = \partial_\mu\varepsilon^a - C^a_{bc}\varphi_\mu^b\varepsilon^c;$$

$$s = 2: \text{ the "multi-diffeomorphisms" } \delta\varphi_{\mu\nu}^a = 2\partial_{(\mu}\varepsilon_{\nu)}^a - C^a_{bc}\eta^{\rho\sigma}(2\partial_{(\mu}\varphi_{\nu)\rho}^b - \partial_\rho\varphi_{\mu\nu}^b)\varepsilon_\sigma^c + \mathcal{O}(C^2);$$

$s = 3$ : *the spin-3 gauge transformations decomposing into two categories. More precisely, the structure constant tensor  $C$  splits into  $f$  and  $g$  and the gauge transformations are of the schematic form*

$$\delta\varphi_{\mu\nu\rho}^a = 3\partial_{(\mu}\varepsilon_{\nu\rho)}^a + f^a_{bc}(\partial\varphi^b\partial\varepsilon^c)_{\mu\nu\rho} + g^a_{bc}(\partial^2\varphi^b\partial^2\varepsilon^c)_{\mu\nu\rho} + \mathcal{O}(C^2),$$

where the structure constant tensor  $g^a{}_{bc}$  vanishes in space-time dimension  $n < 5$ .

In length units, the coupling constants  $C^a{}_{bc}$  have dimension  $n/2 + s - 3$ , except for  $g^a{}_{bc}$  which has dimension  $n/2 + 2$ . The spin-three deformation associated with the tensor  $f^a{}_{bc}$  was obtained in [13] while the spin-three deformation corresponding to  $g^a{}_{bc}$  was obtained in [15].

*In the spin-2 and spin-3 parity-breaking cases, the first-order deformations of the gauge transformations schematically read (for the detailed expressions, see the second ref. of [25] and ref. [16])*

$$s = 2: \quad \delta\varphi^a{}_{\mu\nu} = 2\partial_{(\mu}\varepsilon^a{}_{\nu)} + \delta_3^n \tilde{f}^a{}_{bc} \varepsilon^{\mu\nu\rho} (\partial\varphi^b\partial\varepsilon^c)_{\mu\nu\rho} + \delta_5^n \tilde{g}^a{}_{bc} \varepsilon^{\mu\nu\rho\sigma\lambda} (\partial\varphi^b\partial\varepsilon^c)_{\mu\nu\rho\sigma\lambda},$$

$$s = 3: \quad \delta\varphi^a{}_{\mu\nu\rho} = 3\partial_{(\mu}\varepsilon^a{}_{\nu\rho)} + \delta_3^n \hat{f}^a{}_{bc} \varepsilon^{\mu\nu\rho} (\partial\varphi^b\varepsilon^c + \varphi^b\partial\varepsilon^c)_{\mu\nu\rho} + \delta_5^n \hat{g}^a{}_{bc} \varepsilon^{\mu\nu\rho\sigma\lambda} (\partial^2\varphi^b\partial\varepsilon^c)_{\mu\nu\rho\sigma\lambda}.$$

## 5 Non-Abelian cubic vertices

In order to provide an intrinsic characterization of the conditions on the constant tensors characterizing the deformations, let us start by briefly reviewing some basics of abstract algebra.

### 5.1 Algebraic preliminaries

Let  $\mathcal{A}$  be a real algebra of dimension  $N$  with a basis  $\{T_a\}$ . Its multiplication law  $*$  :  $\mathcal{A}^2 \rightarrow \mathcal{A}$  obeys  $a * b = (-) b * a$  if it is (anti)commutative, which is equivalent to the fact that the structure constant tensor defined by  $T_b * T_c = C^a{}_{bc} T_a$  is (anti)symmetric in the covariant indices:  $C^a{}_{bc} = (-) C^a{}_{cb}$ . Moreover, let us assume that the algebra  $\mathcal{A}$  is an Euclidean space, i.e. it is endowed with a scalar product  $\langle , \rangle$  :  $\mathcal{A}^2 \rightarrow \mathbb{R}$  with respect to which the basis  $\{T_a\}$  is orthonormal,  $\langle T_a, T_b \rangle = \delta_{ab}$ . For an (anti)commutative algebra, the scalar product is said to be *invariant* (under the left or right multiplication) if and only if  $\langle a * b, c \rangle = \langle a, b * c \rangle$  for any  $a, b, c \in \mathcal{A}$ , and the latter property is equivalent to the complete (anti)symmetry of the trilinear form

$$C : \mathcal{A}^3 \rightarrow \mathbb{R} : (a, b, c) \mapsto C(a, b, c) = \langle a, b * c \rangle$$

or, in components, to the complete (anti)symmetry property of the covariant tensor  $C_{abc} := \delta_{ad} C^d{}_{bc}$ . For that reason, the former algebras are said to be (anti)symmetric. An anticommutative algebra satisfying the Jacobi identity  $a * (b * c) + b * (c * a) + c * (a * b) = 0$  is called a Lie algebra and the product  $*$  is called a Lie bracket. In components it reads  $C^e{}_{d[a} C^d{}_{bc]} = 0$ . Furthermore, the Killing form of a (compact) semisimple Lie algebra endows it with an (Euclidean) antisymmetric algebra structure. Eventually, an algebra is said to be associative if  $a * (b * c) = (a * b) * c$  which, for (anti)commutative algebras, reads in components  $C^d{}_{b[c} C^e{}_{a]d} = 0$ . For anticommutative algebras, the associativity is much stronger than the Jacobi identity.

### 5.2 Cubic vertices

An important physical question is whether or not these first-order gauge symmetry deformations possess some Lagrangian counterpart. The following theorem provides a sufficient condition [15, 16, 24, 25].

**Cubic vertices:** *Let the constant tensor  $C_{abc} := \delta_{ad} C^d{}_{bc}$  be completely (anti)symmetric, then the non-Abelian interaction problem for a quadratic local action  $S_0[\varphi^{\alpha}_{\mu_1 \dots \mu_s}]$  (rank  $s$  fixed) admits first-order solutions which are local functionals  $C_{abc} S_1^{abc}[\varphi^{\alpha}_{\mu_1 \dots \mu_s}]$  such that the deformation  $S = S_0 + C_{abc} S_1^{abc} + \mathcal{O}(C^2)$  is invariant under the aforementioned gauge transformations at first order in  $C$ . They*

$$\mathbf{s=1:} \quad \text{are equal to the Yang-Mills cubic vertex } S_1[\varphi^d_{\mu}] = C_{[abc]} \int d^n x \partial^{[\mu} \varphi^{\nu]a} \varphi^b_{\mu} \varphi^c_{\nu}.$$

$$\mathbf{s=2:} \quad \text{decompose as a sum } S_1[\varphi^d_{\mu\nu}] = C_{(abc)} R^{abc} + \delta_3^n \tilde{f}_{(abc)} \tilde{U}^{abc} + \delta_5^n \tilde{g}_{[abc]} \tilde{V}^{abc} \text{ of cubic functionals respectively containing two, three and three derivatives.}$$

$$\mathbf{s=3:} \quad \text{decompose as a sum } S_1[\varphi^d_{\mu\nu\rho}] = f_{[abc]} S^{abc} + g_{[abc]} T^{abc} + \delta_3^n \hat{f}_{[abc]} \hat{U}^{abc} + \delta_5^n \hat{g}_{(abc)} \hat{V}^{abc} \text{ of cubic functionals respectively containing three, five, two and four derivatives.}$$

The vertices in the first-order deformations are determined uniquely by the structure constants, modulo vertices that do not deform the gauge algebra. Moreover, the (anti)symmetry of the internal algebra is not only a sufficient but also a necessary requirement for all known cases (i.e. this issue is open only for the spin-three case with the deformation associated with  $g^a{}_{bc}$ ).

The first-order covariant cubic deformation  $S^{abc}[\varphi^d_{\mu\nu\rho}]$  is the BBvD vertex [13]. We do not know yet whether the antisymmetry condition on the structure constant  $g^a{}_{bc}$  is actually necessary or not for the existence of a consistent vertex at first order but, looking at all other cases, it seems very plausible. One may thus conjecture that, for any fixed helicity  $s$ , the existence of a local Lagrangian counterpart to the non-Abelian gauge symmetries requires that the structure constants define an (anti)symmetric internal algebra. The fact that the internal algebra is Euclidean is hidden in the fact that in the free limit the action is a *positive* sum of quadratic *ghost-free* actions.

## 6 Consistency at second order

Of course, the next issue is whether the first order deformations can be pushed further or if they are obstructed. Consistency of the gauge algebras (only by itself and already at second order) constrains rather strongly the parity-invariant possibilities [4, 14, 15, 24, 25]. In the spin-2 and spin-3 parity-breaking cases, the issue is more subtle and a detailed comparative discussion is given in the conclusion of [16].

**Consistency of the algebra:** *At second order in  $C^a{}_{bc}$ , the parity-invariant deformation of the gauge algebra can be assumed to close off-shell without loss of generality, and for  $s = 1, 2, 3$  it is not obstructed if and only if the structure constants  $C^a{}_{bc}$  define an internal algebra which is*

**s=1:** *a Lie algebra.*

**s=2:** *an associative algebra.*

**s=3:** *an associative algebra for  $f^a{}_{bc}$ , or if the space-time dimension is equal to  $n = 3$ .*

We emphasize that the existence of a cubic vertex corresponding to the non-Abelian gauge transformations was not necessary to derive this theorem. In order to combine the latter result with the former one for the existence of a Lagrangian counterpart, one may use the following well-known lemma.<sup>4</sup>

**Lemma:** *If a finite-dimensional (anti)commutative (anti)symmetric Euclidian algebra is associative, then it is the direct sum of one-dimensional ideals.*

This lemma leads to stringent restrictions on the deformations which are consistent till second order.

**Corollary:** *For a collection  $\varphi^a_{\mu_1\dots\mu_s}$  ( $a = 1, \dots, N$ ) of completely symmetric tensor gauge fields with fixed rank  $s = 1, 2, 3$ , the non-Abelian parity-invariant interaction problem is such that the deformed gauge algebra is*

**s=1:** *a finite-dimensional internal Lie algebra endowed with an antisymmetric algebra structure. For semi-simple compact Lie algebras, the scalar product is naturally identified with the Killing form.*

**s=2:** *given by the direct sum of diffeomorphism (i.e. vector field) Lie algebras.*

**s=3:** *inconsistent if  $f^a{}_{bc} \neq 0$  and  $n > 3$ .*

To conclude, the recent (modest but exciting) observations on the spin-three non-Abelian interaction problem are that, at second order at the level of the gauge algebra, the new deformations corresponding to the structure constants  $g_{[abc]}$  [15] and  $\hat{g}_{(abc)}$  [16] both pass the consistency requirement where the BBvD vertex fails, and that the BBvD gauge symmetries are not obstructed in three-dimensional flat space-time (this new result is proper to the present paper). Unfortunately, we do not know yet whether there exist second-order gauge transformations that are consistent at this order.

<sup>4</sup> The proofs are elementary and were given in the corresponding references.

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