Cerbelli and Giona's map is pseudo-Anosov and 9 consequences

Robert S. MACKAY

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Février 2006

IHES/M/06/06

Cerbelli and Giona's map is pseudo-Anosov and 9 consequences

R.S.MacKay
Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.

25 October 2005, revised 23 January 2006

Abstract

It is shown that a piecewise affine area-preserving homeomorphism of the 2-torus studied by Cerbelli and Giona is pseudo-Anosov. This enables one to prove various of their conjectures, quantify the multifractality of its "w-measures", calculate many other quantities for its dynamics, and construct an exact area-preserving tilt map of the cylinder with proved diffusive behaviour.

1 Introduction

In [CG], an interesting example of an area-preserving homeomorphism of a 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ was introduced, $\mathcal{H} : (x, y) \mapsto (x', y')$,

$$\begin{aligned} x' &= x + f(y) \\ y' &= y + x' \end{aligned} \tag{1}$$

with $f(y) = 2y$ if $y \in [0, \frac{1}{2}]$, $2 - 2y$ if $y \in [\frac{1}{2}, 1]$ (extended with period 1). Cerbelli and Giona calculated explicitly many of its dynamical features. In particular, they constructed explicit invariant foliations $\mathcal{F}^\pm$ whose leaves are contracted exponentially in forwards and backwards time respectively. They proved $\mathcal{H}$ to be mixing with respect to area, and to have positive Lyapunov exponent almost everywhere, $\chi_{ae} = \frac{1}{2} \log \kappa$ with $\kappa = 2 + \sqrt{3}$. They proposed it has a sign-alternation property and that it is not topologically conjugate to Anosov. The authors nominated it as an archetype of continuous area-preserving maps exhibiting "non-uniform chaos".

The purpose of this paper is to show that $\mathcal{H}$ belongs to a well established class of maps invented by Thurston, known as *pseudo-Anosov*, and to apply the associated body of theory to it. In particular, I compute the topological entropy and line stretching rate explicitly, prove that the "w-measures" of [CG] are singular, compute their dimension scaling function implicitly, prove that their product is the measure of maximal entropy, deduce a sign alternation property, compute the distribution of finite-time Lyapunov exponents and the shear rotation interval, construct a Markov factor, make a few comments about perturbations of $\mathcal{H}$ and show that $R\mathcal{H}^2$ gives an exact area-preserving map of the cylinder $\mathbb{R} \times \mathbb{T}$ with diffusive behaviour and explicitly calculated diffusion constant.

2 Pseudo-Anosov maps

A *pseudo-Anosov map* is a homeomorphism g of a compact surface for which there exists "dilation" $\lambda > 1$ and a transverse pair of invariant continuous foliations $\mathcal{F}^\pm$ carrying transverse measures $\mu^\pm$ which are expanded by precisely λ each iteration of $g^{\pm 1}$, respectively, but which possess a finite non-zero number of singularities at each of which $\mathcal{F}^\pm$ are modelled up to homeomorphism on the curves of constant real and imaginary parts (modulo sign if k is odd) of $z^{k/2}$ near $z = 0$ in $\mathbb{C}$ for some $k \in \mathbb{N} \setminus \{2\}$ (called a *k-prong* singularity).

Some authors allow the case of no singularities, but such maps are topologically conjugate to Anosov (which have their own well developed theory), and although some results apply to both, other results require at least one singularity (in any case the nomenclature "pseudo-Anosov" requires one to exclude the Anosov case). Some authors require g to be C^∞ except at the singularities, but this seems unnecessary to me. Some authors disallow 1-prongs and call a pseudo-Anosov map with non-empty set S of 1-prongs "pseudo-Anosov relative to S". Most authors allow an extension to compact surfaces with boundary, but this possibility is not required for the purposes of this note.

For introductions to pseudo-Anosov maps, see [Th, FLP, CB, Bo, Ma], of which I particularly recommend [Bo]. In this section I will just recall one class of examples and some of the most significant results.

Simple examples of pseudo-Anosov homeomorphisms are provided by choosing any hyperbolic automorphism Z of the 2-torus $\mathbb{T}^2$ and identifying points under reflection through $(0,0)$. This yields a pseudo-Anosov homeomorphism of the 2-sphere S^2. The dilation λ is the modulus of the eigenvalue of Z larger than one, $\mathcal{F}^{\pm}$ are the images of the lines parallel to the backwards and forwards contracting eigenspaces of Z under the identification, $\mu^{\pm}$ are given by length on $\mathbb{T}^2$ of a leaf of $\mathcal{F}^{\mp}$ respectively, and there are precisely 4 singularities, all 1-prongs, namely the points $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$ (which are their own reflections through $(0,0)$).

Pseudo-Anosov homeomorphisms have many interesting dynamical properties. In particular, they have a Markov partition, meaning (in this context[1]) a decomposition (up to overlap along the edges) into sets bounded by segments of leaves from $\mathcal{F}^{\pm}$ such that any $\mathcal{F}^{+}$ boundary segment is mapped by g inside another one and any $\mathcal{F}^{-}$ boundary segment is mapped by g^{-1} inside another one. This leads to a correspondence of the dynamics, up to ambiguities for boundary points, with a subshift of finite type ("topological Markov chain"). They are topologically mixing and have positive topological entropy, in fact $h_{top} = \log\lambda$. Each leaf of $\mathcal{F}^{\pm}$ is dense. If g has a singularity with odd k, then the leaves of each foliation fold back along themselves in opposite directions, arbitrarily close to any point. The product of $\mu^{\pm}$ is an invariant measure μ; it is mixing and is the measure of maximal entropy. Any continuous perturbation $\tilde{g}$ of a pseudo-Anosov map g preserving the set of 1-prongs as a finite invariant set has at least as much dynamics (more precisely, it has an invariant subset Y with a semi-conjugacy of $\tilde{g}|_Y$ to g; the construction is relatively explicit [Ha]).

3 Proof that $\mathcal{H}$ is pseudo-Anosov

In [CG], invariant foliations $\mathcal{F}^{\pm}$ were constructed for $\mathcal{H}$. The leaves of $\mathcal{F}^{\pm}$ are piecewise straight, with slopes $\frac{1\mp\sqrt{3}}{2}$ in $A = \{(x,y) \in \mathbb{T}^2 : 0 \le y \le \frac{1}{2}\}$ and $B = \{(x,y) \in \mathbb{T}^2 : 0 \le y - x \le \frac{1}{2} \le y \le 1\}$, and[2] $\frac{\sqrt{3}\pm 1}{2\sqrt{3}}$ in $C = \{(x,y) \in \mathbb{T}^2 : 0 \le x - y \le \frac{1}{2} \le y \le 1\}$. It follows that they have precisely four singularities: $(0,0)$ and $(0,\frac{1}{2})$ are 1-prongs, $(\frac{1}{2},0)$ and $(\frac{1}{2},\frac{1}{2})$ are 3-prongs (the homeomorphisms to the standard models are piecewise smooth). See Figure 1.

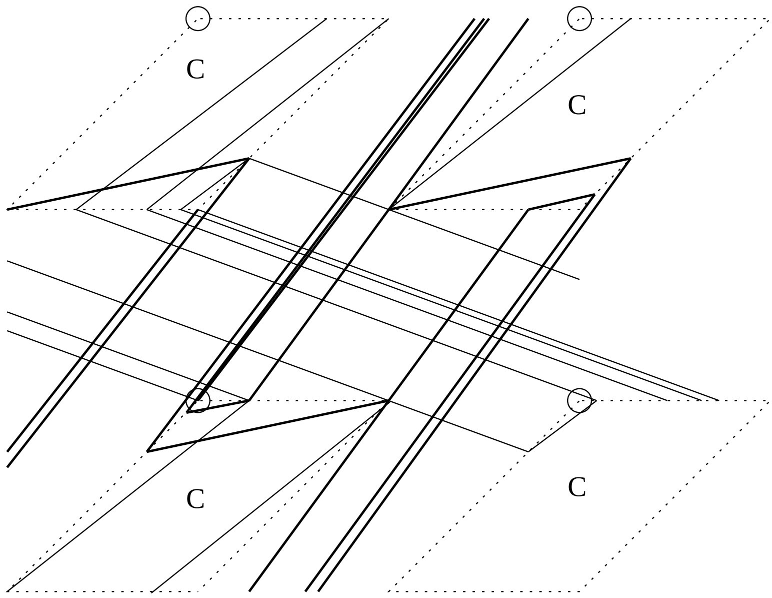


Figure 1: The four singularities of $\mathcal{F}^{\pm}$ are at the corners of the region C bounded by dotted lines. Thick lines belong to $\mathcal{F}^{-}$ and thin lines to $\mathcal{F}^{+}$. O denotes $(0,0)$.

[1] For a general definition, see [S1].

[2] Incidentally, the vector $\mathbf{u}_2$ in figure 3 of [CG] should be drawn in the opposite direction.

To identify the dilation factor λ and determine the transverse measures $\mu^{\pm}$, note that it is possible to construct a Markov partition by extending leaves of $\mathcal{F}^{\pm}$ from the singularities judiciously. Figure 2 shows the one to be used here, though it might be possible to simplify it (e.g. reduce the number of partition elements).

Let Γ be the directed graph indicating which partition elements are traversed by the images of each under $\mathcal{H}$ (shown in Figure 3). If the image of one region were to cross another one more than once, the crossings should be counted individually and without sign, but for this example all traversals are simple.

Let M be the matrix indicating the numbers M_{ij} of traversals of the image of region i across region j:

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{2}$$

Since M is non-negative (has all elements non-negative), irreducible (between any two vertices of Γ there is a path in each direction) and aperiodic (the highest common factor of the lengths of cycles in Γ is 1), the Perron-Frobenius theorem shows it has a unique eigenvalue λ of maximum modulus and λ is positive (in fact greater than 1 because $\det M \in \mathbb{Z} \setminus \{0\}$). This is the dilation factor for the map. It evaluates to $\lambda \approx 2.296630262$. To specify λ exactly, it is the largest root of the factor $t^4 - 2t^3 - 2t + 1$ of the characteristic polynomial of M (the characteristic polynomial also has two factors of $t-1$, suggesting that the Markov partition could be reduced by two elements). Since this is palindromic, it can be factorised to $(t^2 - (1+\sqrt{3})t + 1)(t^2 - (1-\sqrt{3})t + 1)$ and so λ is the largest root of the first factor: $\lambda = (1 + \sqrt{3} + \sqrt{2\sqrt{3}})/2$.

Furthermore the eigenvalue λ is simple and has an eigenvector with all components positive. The components $(\lambda^3 - \lambda^2 - 1, \lambda, \lambda, \lambda^2 - \lambda, 1, \lambda^2)/(\lambda^3 + \lambda^2 + \lambda)$ of the eigenvector[3] give the weights μ_j^+ for the bundle of leaves of $\mathcal{F}^+$ crossing the region j (normalised to sum to 1). To three significant figures, the weights are $(0.297, 0.117, 0.117, 0.151, 0.051, 0.268)$. To determine the weight of an arbitrary subinterval I of leaves of $\mathcal{F}^+$ crossing a given region j, apply $\mathcal{H}^n$ for $n > 0$ to the subset of region j representing I and let $\mu^+(I, n)$ be the product of λ^{-n} and the sum of the weights of the regions fully traversed by $\mathcal{H}^n(I)$; then $\mu^+(I) := \lim_{n\to\infty} \mu^+(I, n)$ exists and is independent of the choice of initial region. This is an adaptation of the construction of "Parry measure" for a subshift of finite type (the measure of maximal entropy) [Pa, KH].

The measure μ^- can be constructed similarly, using the transition matrix for $\mathcal{H}^{-1}$ (which is the transpose of M, so the same λ is obtained). Alternatively, note that $\mathcal{H}$ is reversible with respect to $S(x, y) = (-x, y - x)$, i.e. $\mathcal{H}^{-1} = S^{-1}\mathcal{H}S$, so μ^- can be obtained from μ^+ by applying S.

Thus, the defining properties of a pseudo-Anosov homeomorphism are verified for Cerbelli and Giona's map $\mathcal{H}$.

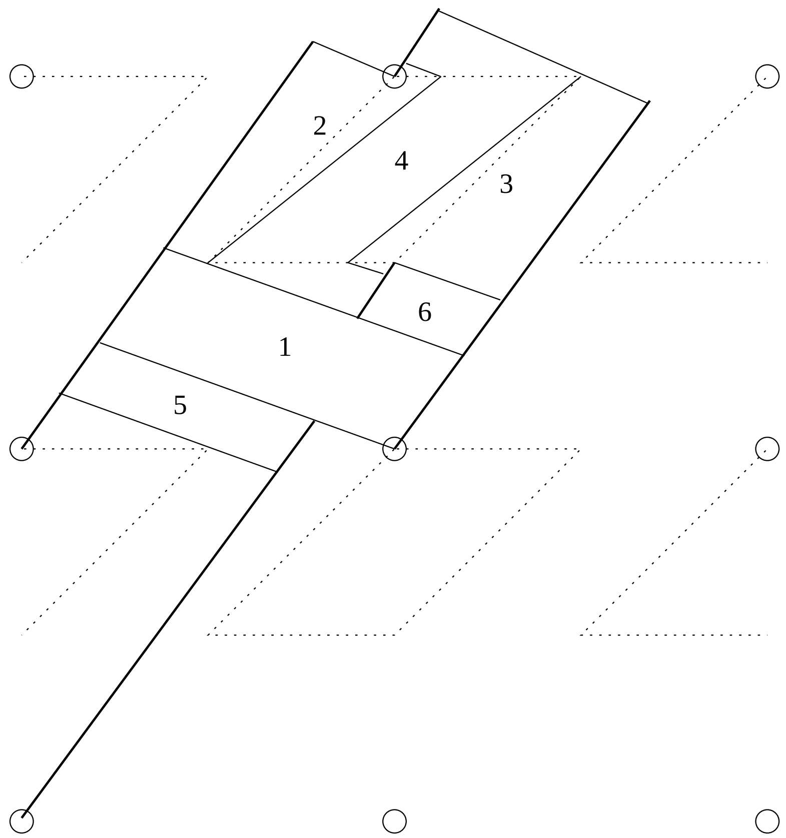
4 Some Consequences

4.1 Topological entropy and line-stretching rate

One consequence of the above proof is that it gives the exact value for the topological entropy of $\mathcal{H}$, namely $h_{top} = \log \lambda \approx 0.831443$, for which [CG] obtained a numerical approximation (agreeing well with this) and a lower bound. Roughly speaking, the topological entropy of a map g is the growth rate of the number of distinguishable orbit segments with number of iterations.

Closely related is the line-stretching rate $h_{line} = \sup_\gamma h(\gamma)$ where the supremum is taken over closed C^1 arcs γ, $h(\gamma) = \limsup_{n\to\infty} \frac{1}{n} \log \ell(g^n\gamma)$, and ℓ denotes arc length. For sufficiently smooth 2D maps, h_{line} has been proved equal to h_{top} [Ne, Yo], and the proof extends to piecewise smooth maps. For pseudo-Anosov maps, equality of h_{line} to $\log \lambda$ (and hence to h_{top}) is immediate from the action on a Markov partition. Indeed, $\frac{1}{n} \log \ell(g^n\gamma) \to \log \lambda$ as $n \to \infty$ for all C^1 arcs except those contained in a leaf of $\mathcal{F}^+$.

[3] Using $\lambda^2 - (1+\sqrt{3})\lambda + 1 = 0$ one could simplify these expressions if desired.



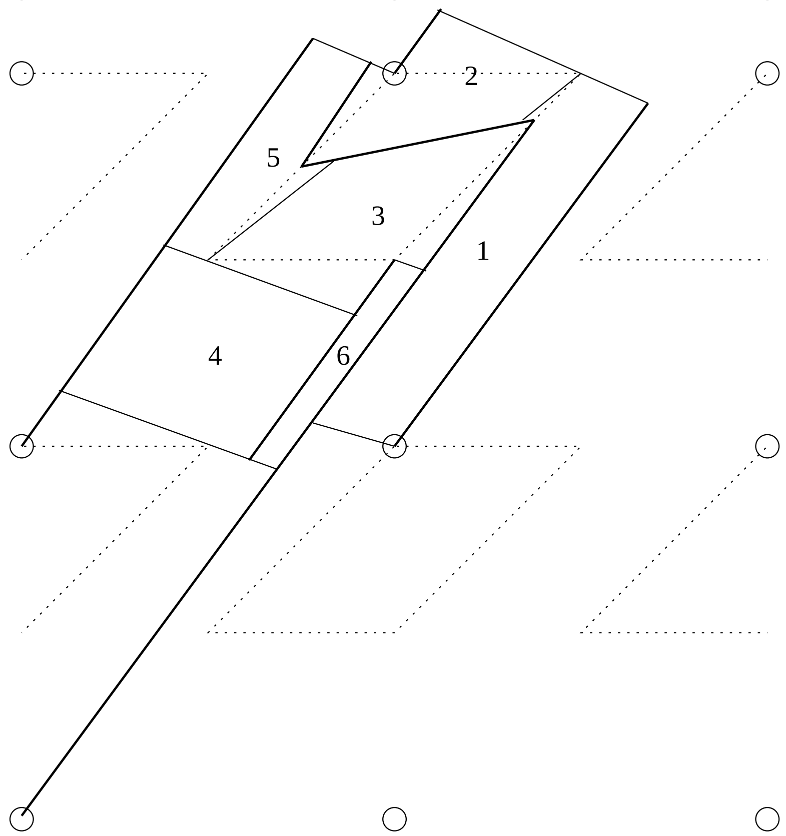


Figure 2: A Markov partition for $\mathcal{H}$ (top), and its image (bottom). Thick lines belong to $\mathcal{F}^-$ and thin lines to $\mathcal{F}^+$.

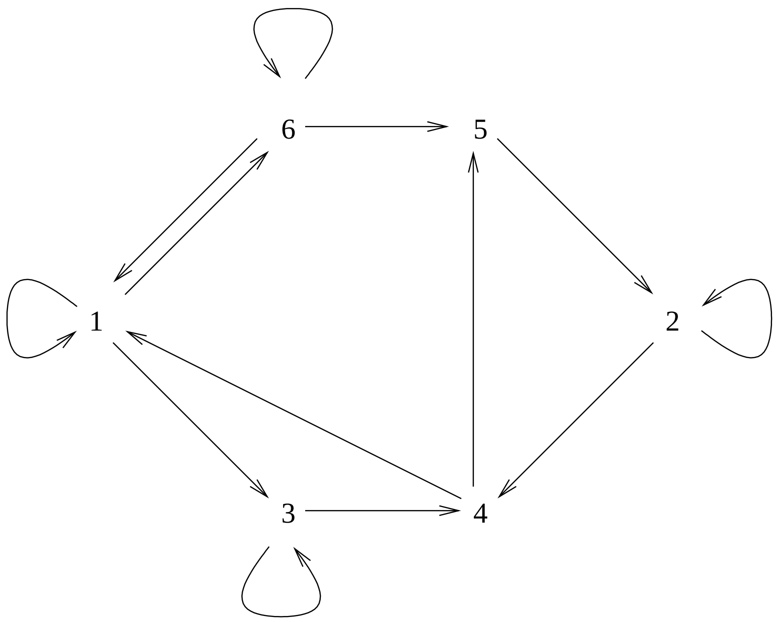


Figure 3: The graph Γ of allowed transitions for the Markov partition of Figure 2.

4.2 Singularity of the w-measures

A second consequence of the proof is that it allows analysis of the "w-measures" of [CG]. The w^+-measure describes the asymptotic distribution of any long segment of any leaf of $\mathcal{F}^+$ with respect to arc length, and similarly w^- for $\mathcal{F}^-$. As argued in [GA], they are relevant for mixing of fluids, because they determine where diffusion can be expected to play the strongest role. As can be seen in Figure 4 of [CG], they are highly non-uniform; [CG] conjectured they are singular and "multifractal".

This is indeed so. The $w^\pm$-measures for $\mathcal{H}$ are the extensions to the torus of the transverse measures $\mu^\pm$ by multiplying by arc length along the leaves, and $\mu^\pm$ are singular and multifractal. I will prove these statements for $+$ case (the same analysis applied to $\mathcal{H}^{-1}$ proves them for $-$ case) .

To prove the first statement, let us call a connected component of the intersection of a leaf of $\mathcal{F}^+$ with a region of our Markov partition a *leaf segment*. Then the preimage by $\mathcal{H}$ of any leaf segment in a region j is a union of leaf segments, one per region antecedent to j in the graph Γ. By definition, w^+ gives them weight proportional to their length, and μ^+ gives them equal weight because they all go to the same leaf segment under $\mathcal{H}$. Thus w^+ is the extension to $\mathbb{T}^2$ of μ^+ by arc length along $\mathcal{F}^+$.

To prove that μ^+ is singular, meaning that in any smooth curve transverse to $\mathcal{F}^+$ there is a subset of full μ^+-measure and zero Lebesgue measure, note that $\mathcal{H}$ has periodic points with different positive Lyapunov exponent and adapt Corollary 1 in section 4 of [S2] (although for Anosov diffeomorphisms, Sinai's result depends only on the existence of a Markov partition and piecewise smoothness). To see the existence of periodic orbits with different exponents, the analysis of [CG] shows that the positive Lyapunov exponent of any orbit is $\phi \log \kappa$ where $\kappa = 2+\sqrt{3}$ and ϕ is the fraction of time spent in $A = \{(x,y) \in \mathbb{T}^2 : y \in [0, \frac{1}{2}]\}$, and that $\phi \in [\frac{1}{3}, 1]$. The extremes (and every rational in between: see next subsection) are realised by periodic orbits, albeit ones on discontinuities of the derivative of the map, but they still have well defined Lyapunov exponents, which suffices for Sinai's argument. Specifically, the $\mathcal{F}^-$ prong from $(0,0)$ is expanded by κ each iteration, so has $\phi = 1$, and the set of $\mathcal{F}^-$ prongs from the period-2 orbit $(\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$ perform a 6-cycle with expansion κ^2 per cycle, so have $\phi = \frac{1}{3}$. A more direct argument that μ^+ is singular will be given in the next subsection.

To prove that μ^+ is multifractal, meaning that there are points x of a transverse smooth arc L to $\mathcal{F}^+$ for which the pointwise dimension

$$\delta(x) = \lim_{n\to\infty} \frac{\log \mu^+(I_n)}{\log |I_n|} \tag{3}$$

(for sequences of intervals I_n in L around x with length $|I_n| \to 0$) exists and take different values, we can just use the intersection with L of the $\mathcal{F}^+$ leaves of the periodic points of the previous paragraph. If $x \in L$ is in the $\mathcal{F}^+$ leaf of a period q point with positive Lyapunov exponent χ_x then taking an interval I_0 of L containing x and lying in the same region of the Markov partition and I_n to be the intersection with L of the leaf segments of the component of $\mathcal{H}^{-nq}(I_0)$ containing x, one obtains

$$\delta(x) = \frac{\log \lambda}{\chi_x}$$

(where λ is the dilation of $\mathcal{H}$). Thus different Lyapunov exponents produce different pointwise dimension for μ^+.

4.3 Quantification of multifractality

The multifractal structure of μ^+ can be analysed further, in particular to determine the Hausdorff dimension $F(\alpha)$ of the set of points x of a transverse smooth section L with pointwise dimension $\delta(x) = \alpha$. As above, this is the set of points $x \in L$ with positive Lyapunov exponent $\chi_x = \frac{\log \lambda}{\alpha}$, so it is convenient to consider $D(\chi) = F(\frac{\log \lambda}{\alpha})$, the dimension of the set of points of L with positive Lyapunov exponent χ. From [CG], the possible values of χ range from $\frac{1}{3} \log \kappa$ to $\log \kappa$ where $\kappa = 2 + \sqrt{3} \approx 3.73$, and the almost everywhere value is $\chi_{ae} = \frac{1}{2} \log \kappa \approx 0.6584789$.

I calculate here the functions D and F implicitly (i.e. in parametric form), using results of [BR] which is based on the "thermodynamic formalism" for dynamical systems [Ru]. First I convert the problem to an expanding map of a graph I call a "skeleton", next I make a side-step to show how this gives a direct argument for singularity of μ^+, and then I give the calculation of D and F.

I define the *skeleton* for a Markov partition of a pseudo-Anosov map to be the topological space L of leaf segments (connected components of the intersections of leaves of $\mathcal{F}^+$ with regions of the Markov partition). Thus from Figure 2 for $\mathcal{H}$, we obtain the skeleton of Figure 4. The arcs in a

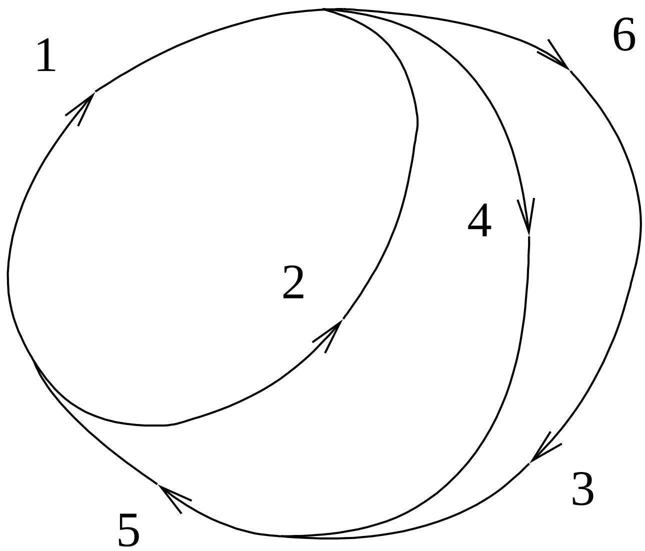


Figure 4: The skeleton L for $\mathcal{H}$.

skeleton are shown joining tangentially to indicate how the leaf segments join or split on leaving each region. They are also assigned orientations; here I have chosen to orient them according to their closest intersection with the $\mathcal{F}^-$ ray from O. There is a natural measure on the skeleton for $\mathcal{H}$, namely the area of the corresponding set of leaf segments, but instead I chose to use an equivalent measure, namely length on the first intersection with the $\mathcal{F}^-$ ray from O, which I denote L_i for region i. The concept of skeleton is closely related to that of "train track" for pseudo-Anosov maps, but there are at least two definitions of train track [CB, BH], and in contrast to them I do not need the skeleton to be embedded in the surface.

The map $\mathcal{H}$ on $\mathbb{T}^2$ induces a map u on the skeleton. Its graph (with respect to Lebesgue measure and the chosen orientations on the L_i) is drawn in Figure 5. It is piecewise affine with slopes κ and $-1/\kappa$. Although u appears to be discontinuous, it is in fact a continuous map of the skeleton. The allowed transitions for u between the arcs L_i are the same as for the regions in Figure 3. The map u is expanding, meaning some iterate expands all lengths by at least some common factor. Specifically, u^5 expands all lengths by at least κ, because u expands by κ everywhere except on transitions $3 \to 4$ and $5 \to 2$ where it multiplies by $-1/\kappa$, but each of these is followed by a transition with expansion κ and the only case without another subsequent expansion κ is $3 \to 4 \to 5 \to 2$

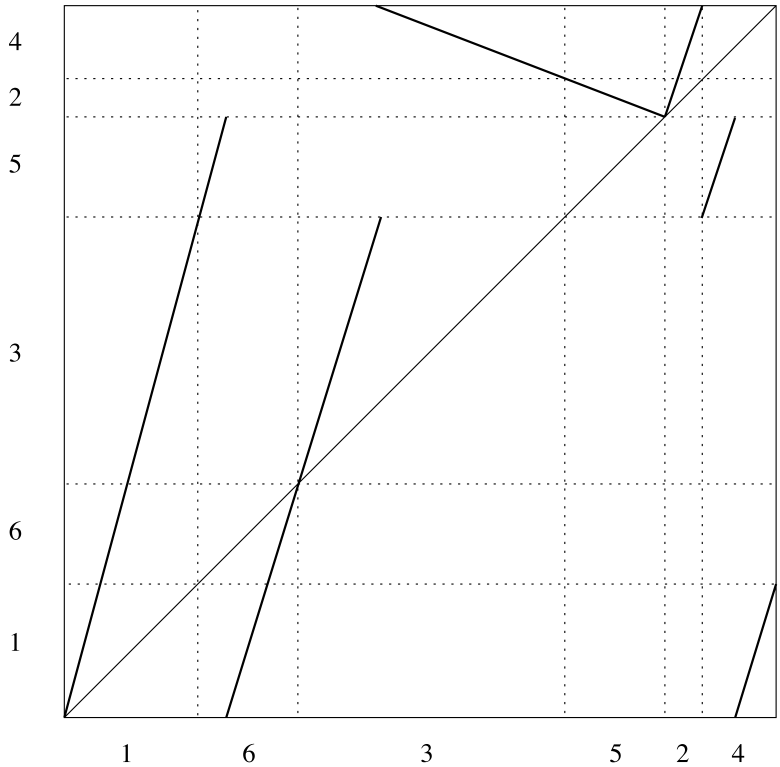


Figure 5: The graph of the map u of $L = \bigcup_i L_i$ to itself.

and this is always preceded and followed by two expansions by κ. Note that existence of periodic orbits with Lyapunov exponent $\phi \log \kappa$ for any rational $\phi \in [\frac{1}{3}, 1]$ follows, e.g. the sequence $3^p(413)^q$ repeated periodically gives a periodic orbit with $\chi = \frac{p+q}{p+3q} \log \kappa$.

From Section 3, the measure μ^+ is obtained by assigning $\mu^+(L_j) = \mu_j^+$ and $\mu^+(\Sigma) = \lambda^{-n}\mu_j^+$ to any subinterval Σ for which $u^n(\Sigma) = L_j$.

As promised, I now give a direct argument for singularity of μ^+. The n^{th} preimage by u of the partition of L into the L_i consists of order λ^n subintervals Σ. If $u^n(\Sigma) = L_i$ then Σ has $\mu^+(\Sigma) = \lambda^{-n}\mu_i^+$, but length

$$|\Sigma| = |L_i| \prod_{m=0}^{n-1} |u'(u^m(\Sigma))|^{-1}.$$

Thus

$$\frac{\mu^+(\Sigma)}{|\Sigma|} = \frac{\mu_i^+}{|L_i|}\lambda^{-n} \prod_{m=0}^{n-1} |u'(u^m(\Sigma))|.$$

This behaves like $\exp n(\chi - \log \lambda)$ as $n \to \infty$, where χ is the Lyapunov exponent of the chosen point. Thus at points with $\chi > \log \lambda$ (for example whose orbits make the transitions $3 \to 4$ or $5 \to 2$ rarely, so χ is near $\log \kappa$) the measure μ^+ has infinite density, and at points with $\chi < \log \lambda$ (for example whose orbits use those transitions nearly once every three iterations, so χ is near $\frac{1}{3} \log \kappa$) it has zero density. This corresponds well with the densities in figure 4(f) of [CG], which for example shows w^+ to be weak in region 3 and to have strong bands across regions 1 and 6 (presumably corresponding to the full shift on this pair, available in Γ). Although length on L is not invariant under u, rescaling it by a bounded factor to the area of the corresponding bundle of leaf segments converts it to an invariant measure for u. By Pesin's formula $h_\mu = \int \chi d\mu$ for the entropy of an invariant measure μ of an almost everywhere differentiable map [Pe, QZ], this measure has entropy equal to its almost everywhere Lyapunov exponent $\chi_{ae} = \frac{1}{2} \log \kappa$. Then by Shannon's theorem [Bi], for all $\varepsilon > 0$ there is a subset of order $e^{n(\chi_{ae}+\varepsilon)}$ of the subintervals of the n^{th} preimage partition which cover all but an exponentially small fraction of the length of L. Since they each have μ^+ measure equal to $\lambda^{-n}\mu_i^+$ and $\chi_{ae} < \log \lambda$, we can choose ε so that their union has exponentially small μ^+ measure, about $\lambda^{-n}e^{n(\chi_{ae}+\varepsilon)}$. Thus we make subsets of L of arbitrarily small μ^+ measure and arbitrarily close to full length. So μ^+ is singular with respect to length on L.

Finally, to compute D and F, for $\beta \in \mathbb{R}$ let $M(\beta)$ be the 6×6 matrix with entries $|u'|^{-\beta}$ for the allowed transitions. Thus the matrix elements are

$$\nu = \kappa^{-\beta}$$

for all allowed transitions except $1/\nu$ for $3 \to 4$ and $5 \to 2$, and $M(0)$ is our previous M. Let $P(\beta)$ be the logarithm of the largest eigenvalue of $M(\beta)$ (which is unique and positive). $P(\beta)$ is known as the "topological pressure" of the function $-\beta \log |u'|$ for the dynamical system u. The function P is convex and analytic. Let S be the Legendre transform of $-P$, i.e. $S(\chi) = \sup_{\beta\in\mathbb{R}}(P(\beta)+\chi\beta)$. Note that χ and β are related by $P'(\beta) = -\chi$, $S'(\chi) = \beta$. S is defined for χ in the interval $R = (-P'(+\infty), -P(-\infty))$, is analytic and concave on R and has $S' = \pm\infty$ at the ends of R. It can be interpreted as the topological entropy of the set of points with Lyapunov exponent χ [BR]. Finally, let $D(\chi) = S(\chi)/\chi$ and $F(\alpha) = D(\frac{\log \lambda}{\alpha})$, adapting [BR].

This calculation can be done explicitly, in parametric form. First write

$$M(\beta) = \nu N(\beta) \text{ and } a = \nu^{-2} = \kappa^{2\beta},$$

so

$$N(\beta) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4}$$

The characteristic polynomial of $N(\beta)$ has a factor $(t-1)^2$ and a remaining factor $t^4-2t^3-2at+a$. We can solve this for a as a function of t:

$$a = \frac{t^3(t-2)}{2t-1}. \tag{5}$$

It has derivative

$$\frac{da}{dt} = \frac{6t^2(t-1)^2}{(2t-1)^2},$$

so the largest eigenvalue t of N goes monotonically and smoothly from 2 to $+\infty$ as a goes from 0 to $+\infty$. We shall compute everything as functions of this t.

In particular,

$$\beta = \frac{\log a}{2\log\kappa} = \frac{1}{2\log\kappa}\log\frac{t^3(t-2)}{2t-1},$$

and

$$P(\beta) = \log\nu + \log t = \log t - \beta\log\kappa = -\frac{1}{2}\log\frac{t(t-2)}{2t-1}.$$

Then $P'(\beta) = \frac{1}{t}\frac{dt}{da}\frac{da}{d\beta} - \log\kappa$, so

$$\chi = -P'(\beta) = \left(1 - \frac{(t-2)(2t-1)}{3(t-1)^2}\right)\log\kappa.$$

Next

$$S(\chi) = P(\beta) + \chi\beta = \log t - \frac{(t-2)(2t-1)}{6(t-1)^2}\log\frac{t^3(t-2)}{2t-1}.$$

Finally, $D(\chi) = \frac{S(\chi)}{\chi}$ and $F(\alpha) = D(\chi)$ with $\alpha = \frac{\log\lambda}{\chi}$.

Figures 6–9 show parametric plots of P against β, S against χ, D against χ, and F against α (to ensure the ends were treated well by Maple, I set $t = \frac{2+s^2}{1-s}$ and plotted for $s = 0..1$). Note various features, which are consistent with all that we already know. P has asymptotic slopes $-\log\kappa$ and $-\frac{1}{3}\log\kappa$, so the domain R of possible Lyapunov exponents χ is $[\frac{1}{3}\log\kappa, \log\kappa]$. $P(0) = \log\lambda$ and $P(1) = 0$. The asymptotes of P have intercepts $\log 2$ and $\frac{1}{3}\log 2$ so S takes these values at the ends of R, and as a consequence D takes the value $\frac{\log 2}{\log\kappa}$ at both ends. These values can be understood: when $a = 0$ the transitions $3 \to 4$ and $5 \to 2$ in Γ have no weight, so the topological entropy is reduced to that for the full shift on $\{1, 6\}$, $\log 2$; for $a = \infty$, all the weight comes from the subshift consisting of the two 3-cycles $4 \to 1 \to 3$ and $4 \to 5 \to 2$ and switches between them, which has topological entropy $\frac{1}{3}\log 2$. The slopes of S, D and F are infinite at the ends of their ranges, corresponding to $\beta = \pm\infty$. The maximum value of S is $\log\lambda$, the topological entropy for $\mathcal{H}$ (although it might

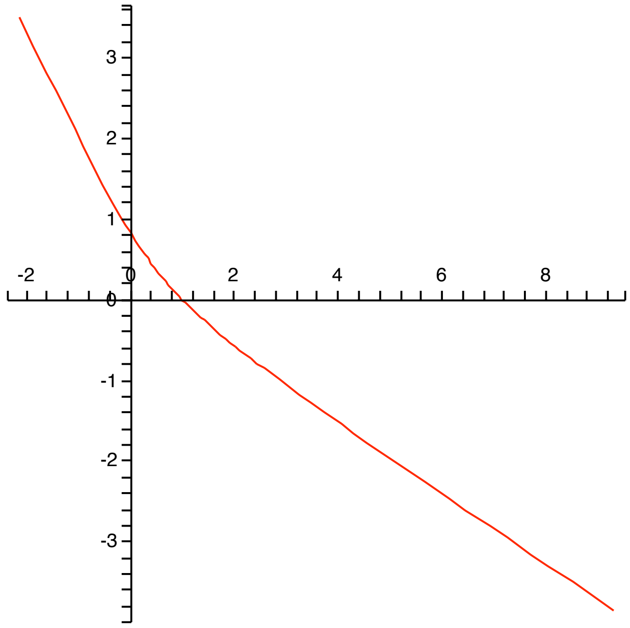


Figure 6: The topological pressure $P(\beta)$ of the function $-\beta \log |u'|$.

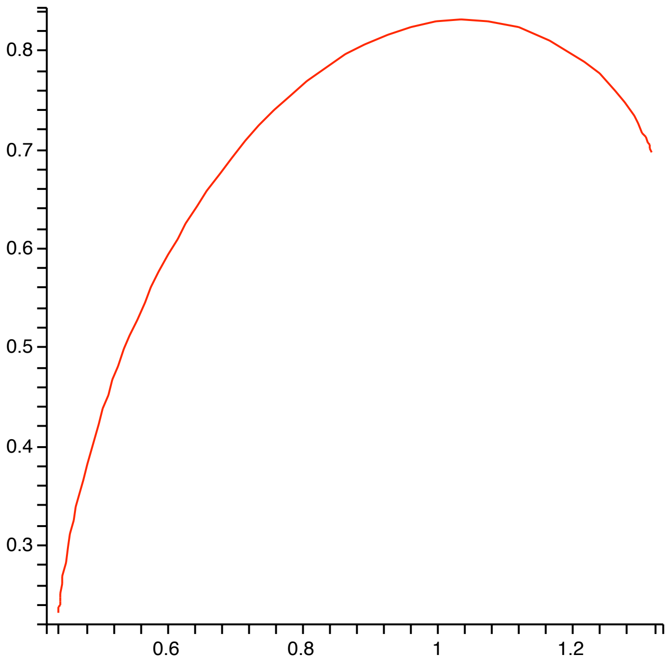


Figure 7: The topological entropy $S(\chi)$ of the set with Lyapunov exponent χ.

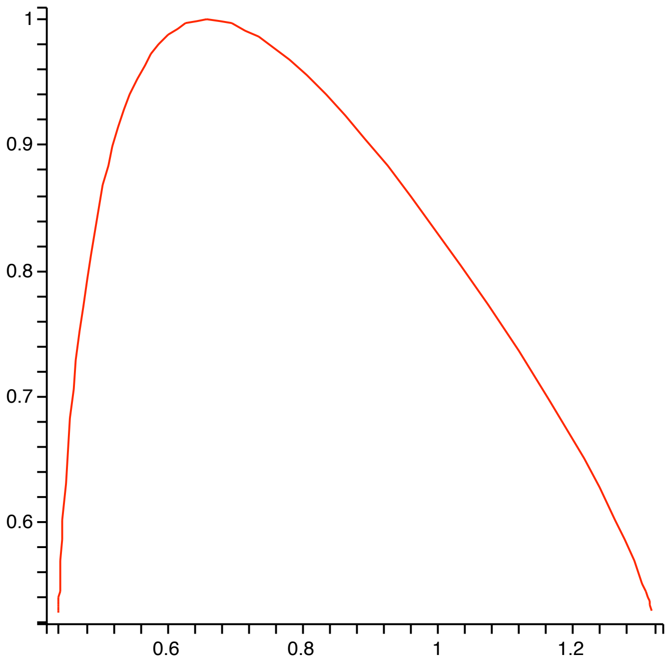


Figure 8: The Hausdorff dimension $D(\chi)$ of the set with Lyapunov exponent χ.

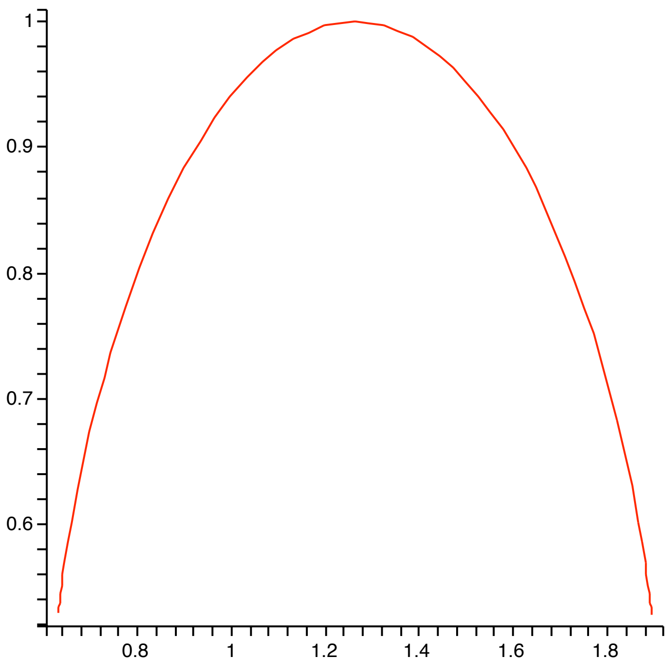


Figure 9: The Hausdorff dimension $F(\alpha)$ of the set for which μ^+ has pointwise dimension α.

appear this occurs at $\chi = 1$, in fact it occurs at $\frac{\lambda^2-\lambda+1}{3(\lambda-1)^2}\log\kappa = \frac{\log\kappa}{3-\sqrt{3}} \approx 1.03865$). The graph of S is tangent to the diagonal at $\chi = \chi_{ae}$. The maximum of D occurs at $\chi = \chi_{ae} = \frac{1}{2}\log\kappa$ and has value 1. The domain of F is $(\frac{\log\lambda}{\log\kappa}, 3\frac{\log\lambda}{\log\kappa})$. The maximum of F occurs at $\alpha = 2\frac{\log\lambda}{\log\kappa} \approx 1.26267$ and has value 1: this corresponds to the feature already observed, that for Lebesgue almost all points of S the w^+-measure has pointwise dimension $2\frac{\log\lambda}{\log\kappa}$ and since this exceeds 1 the w^+-measure has density 0 there and so is singular.

To aid interpretation of the figures, some correspondences are collected in Table 1.

t	β	χ
2	$-\infty$	$\log\kappa$
λ	0	$\frac{\log\kappa}{3-\sqrt{3}}$
κ	1	$\chi_{ae} = \frac{1}{2}\log\kappa$
$+\infty$	$+\infty$	$\frac{1}{3}\log\kappa$

Table 1: Correspondence between t, β and χ at some special points.

4.4 Sign alternation

A third consequence is that one can prove a sign-alternation property for $\mathcal{H}$. Two definitions of this are introduced in [CG], but as pointed out to me by Khanin they fail to capture what I believe the authors intended[4]. I think it is better to define a 2D map with invariant foliations to have the sign-alternation property if the foliations are not orientable. Then for $\mathcal{H}$ this is a standard consequence of the foliation having a singularity of odd order [FLP].

It would be interesting to calculate the associated "cancellation exponent" and dimension spectrum [DO].

4.5 Finite-time Lyapunov exponents

A fourth consequence of the proof is that one can compute the large n asymptotics of the probability distribution (with respect to Lebesgue measure) of the finite-time Lyapunov exponent $\chi_n = \frac{1}{n}\log||D\mathcal{H}^n||$, as done numerically in [CG]. It is the same as for the finite-time Lyapunov exponent of the skeleton map u with respect to the area measure, which is equivalent to length. The elements of the n^{th} preimage partition have lengths ℓ_n within a bounded factor of $e^{-n\chi_n}$, using the value of χ_n starting from any point of the partition element. So those partition elements with χ_n near a given value χ have lengths near $\ell_n(\chi) = e^{-n\chi}$. From the construction of the function D, the number of partition elements with χ_n near a given value χ is of order $\ell_n(\chi)^{-D(\chi)} = e^{nD(\chi)\chi}$. So their total length is of order $e^{n(D(\chi)-1)\chi} = e^{n(S(\chi)-\chi)}$. Thus the area of the torus for which χ_n is near χ is asymptotically $e^{nI(\chi)}$ with "large deviations rate function" $I(\chi) = S(\chi) - \chi$. Its graph is plotted in Figure 10.

In particular, it follows that the distribution of χ_n is asymptotically normal[5] for large n, as observed numerically in [CG], with mean χ_{ae} and variance $\frac{v}{n}$ with $v = 1/|I''(\chi_{ae})|$. Since $S'(\chi) = \beta$ and $\beta = 1$ at $\chi = \chi_{ae}$, we obtain

$$I''(\chi_{ae}) = S''(\chi_{ae}) = \frac{d\beta}{d\chi} = \frac{d\beta/dx}{d\chi/dx}$$

at $x = \kappa$. Inserting the previous formulae for $\beta(x)$ and $\chi(x)$, v evaluates to about 0.250336, which compares well with the numerical value 0.25 obtained by [CG].

4.6 Measure of maximal entropy

A fifth consequence of the proof is that the product of $\mu^\pm$ is the measure of maximal entropy. This result for a general pseudo-Anosov map is quoted in [Bo]; it is a direct consequence of the construction of $\mu^\pm$. Alternatively, a proof can be made along the lines of [S1], which (as in subsection 4.2)

[4]Contrary to their claim, Definition 1 of [CG] is satisfied for nearly all Anosov toral diffeomorphisms, so is too weak; for example it holds for all asymmetric Anosov toral automorphisms, because then the invariant foliations are not perpendicular and a vector $\mathbf{w}$ with the specified properties can be chosen. On the other hand, their Definition 2 is not satisfied for any map with piecewise smooth invariant foliations (including $\mathcal{H}$), so is too strong; take δ such that $\partial B_\delta(y)$ is tangent to Γ and choose $\mathbf{x}$ to be the point of tangency, then the specified property does not hold.

[5]This also follows from standard central limit results for piecewise smooth systems with a Markov partition, e.g. [Ch].

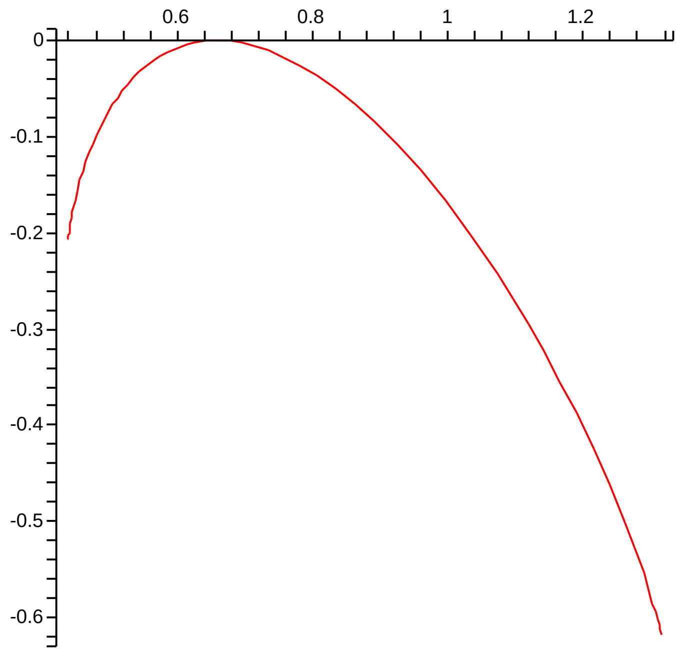


Figure 10: The rate function $I(\chi)$ for the distribution of finite-time Lyapunov exponents χ.

although for Anosov diffeomorphisms, depends only on the existence of a Markov partition and piecewise smoothness.

4.7 Shear rotation interval

A sixth consequence is that one can compute the rotational behaviour of orbits. The action of $\mathcal{H}$ on first homology is a vertical shear

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(to see this, homotope the function f in (1) to 0). Incidentally, this provides the easiest way to see that $\mathcal{H}$ is not topologically conjugate to Anosov: else the action on first homology would be hyperbolic.

Thus although the amount of vertical translation made by an orbit segment on the torus is not well defined, the amount of horizontal translation is. The *shear rotation rate* of an orbit is the average horizontal translation per iteration if defined (which is the case for almost every point with respect to any invariant measure). The *shear rotation interval* of a map with vertical shear action on first homology can be defined to be the set of shear rotation rates of its orbits (proved to be an interval or single point [Do]).

The shear rotation interval is easily computed for $\mathcal{H}$. The set of points making the transition from region i of the Markov partition to region j on $\mathbb{T}^2$ actually move from region i into a translate of region j on the cylindrical cover $\mathbb{R} \times \mathbb{T}$ by an integer ϕ_{ij}. Label each edge $i \to j$ of the graph Γ (Figure 3) by ϕ_{ij}. Then the edges from 3, 4 and 6 are labelled by $+1$ and those from 1, 2 and 5 by 0. Thus all shear rotation rates are contained in $[0, +1]$ and the extremes are realised, e.g. 0 by the sequence $\ldots 111 \ldots$ and $+1$ by the sequence $\ldots 333 \ldots$. Hence the shear rotation interval is $[0, +1]$.

The mean shear rotation rate with respect to Lebesgue measure is easily computed to be $\frac{1}{2}$, by applying the ergodic theorem to the horizontal translation function f of (1). One can also work out the mean shear rotation rate with respect to the product of μ^+ and μ^-; it is $\mu_3^+ + \mu_4^+ + \mu_6^+ = \frac{2\lambda}{\lambda^2+\lambda+1} = \frac{2}{2+\sqrt{3}} \approx 0.536$.

If desired, one could also compute the dimension of the set of points with specified shear rotation rate. More interesting for present purposes is the large deviations rate function for the distribution of finite-time shear rotation rates, which I shall now discuss.

For $\gamma \in \mathbb{R}$, define a matrix Y by $Y_{ij} = |u'|^{-1} \exp \gamma\phi_{ij}$ if the transition $i \to j$ is allowed, 0 otherwise. Then the ij element of the matrix Y^n is the sum over paths $i = i_0, \ldots, i_n = j$ of length n from i to j of $\ell \exp \gamma \sum_{k=0}^{n-1} \phi_{i_k i_{k+1}}$ where ℓ (up to a bounded factor) is the area of the set of points on $\mathbb{T}^2$ following this path in the Markov partition. The asymptotic growth rate of $(Y^n)_{ij}$ with n is given by $Q(\gamma)$, the logarithm of the largest eigenvalue of Y. Take (minus) the Legendre transform $J(\rho) = \inf_{\gamma\in\mathbb{R}}(Q(\gamma) - \gamma\rho)$ and note the interchange of variables is given by $J'(\rho) = -\gamma$, $Q'(\gamma) = \rho$. Then for any $\rho_1 < \rho_2$,

$$\frac{1}{n} \log P\{\frac{1}{n}\sum_{k=0}^{n-1} \phi_{i_k i_{k+1}} \in [\rho_1, \rho_2]\} \to \max_{\rho\in[\rho_1,\rho_2]} J(\rho) \tag{6}$$

as $n \to \infty$, where P denotes Lebesgue measure on the torus.

The characteristic polynomial of Q has two easy roots $t = \frac{g}{k}, \frac{1}{k}$, where $g = \exp \gamma$, but the largest is the largest root of

$$\kappa t^4 - (1 + g)t^3 - (1 + g)gt + g^2/\kappa = 0. \tag{7}$$

In contrast to the case of Lyapunov exponents, this is not so easy to treat. Nevertheless, in principle it determines $Q(\gamma) = \log t$ and hence the function J, which is automatically concave.

We know already the most important point, namely the maximum of J: $\gamma = 0$, $\rho = \frac{1}{2}$, $Q(0) = 0$, $Q'(0) = \frac{1}{2}$, $J(\frac{1}{2}) = 0$, $J'(\frac{1}{2}) = 0$. We can also see that Q has asymptotic slopes 0 and 1 as $\gamma \to \mp\infty$, so the interval of definition of J is $(0, 1)$, with slopes $\pm\infty$ at the ends. With a little work we can determine the next most useful feature, namely the curvature of J at the maximum. This is useful because a simple consequence of (6) is that the distribution of the finite-time shear rotation rate on a scale $\frac{1}{\sqrt{n}}$ is asymptotically normal as $n \to \infty$ with mean $\frac{1}{2}$ and variance $\frac{-1}{nJ''(\frac{1}{2})} = \frac{Q''(0)}{n}$.

Now

$$Q''(\gamma) = g\frac{d}{dg}(\frac{g}{t}\frac{dt}{dg})$$

(treating t as a function of g via (7)) and

$$\frac{dt}{dg} = \frac{t^3 + (1 + 2g)t - 2g/\kappa}{4\kappa t^3 - 3(1 + g)t^2 - g(1 + g)}.$$

Evaluating at $g = t = 1$ gives $Q''(0) \approx 0.14434$.

4.8 Markovian dynamics

A seventh consequence is that the dynamics with respect to Lebesgue measure has a projection to a Markov process. Restricted to subsets which are measurable unions of leaf segments, $\mathcal{H}$ is measure-theoretically equivalent to the Markov process on the graph Γ of Figure 3 with transition matrix $p_{ij} = \frac{1}{l_i} M'_{ij} l_j$ from state i to j, where $M' = M(1)$ (i.e. evaluated at $\beta = 1$, $a = \kappa^2$, $\nu = \kappa^{-1}$) and l is the Perron-Frobenius eigenvector of M' with all components positive (and eigenvalue +1). The components of l represent the conversion factors from length on the train track to area of the corresponding union of leaf segments. Thus p_{ij} represents the fraction of area of region i making the transition to j. The Perron-Frobenius eigenform π for p (satisfying $\pi p = \pi$, normalised to sum to 1) is its stationary distribution, hence π_i is the area of region i.

In particular, the relaxation to equilibrium, restricted to subsets of this form, is exponential with rates given by the logarithms of the remaining eigenvalues of p. Now $p\xi = \eta\xi$ if and only if $M'\xi' = \eta\xi'$ with $\xi'_i = l_i\xi_i$, so the eigenvalues r of p are the same as those of M'. To evaluate the eigenvalues of M' it is simplest as in subsection 4.3 to evaluate the characteristic polynomial for $N' = \kappa M'$ and then divide its roots by κ. N' has a double eigenvalue $t = 1$, so p has a double eigenvalue $r = \kappa^{-1} \approx 0.2679$. Using $a = \kappa^2$, (5) gives $t^4 - 2t^3 - 2\kappa^2 t + \kappa^2 = 0$ for the other eigenvalues of N'. Writing in terms of $r = \frac{t}{\kappa}$, extracting the expected root $r = 1$ and using $\kappa^2 - 4\kappa + 1 = 0$, this reduces to $\kappa^2 r^3 + (2\kappa - 1)(r^2 + r) - 1 = 0$, which numerically has roots $r \approx -0.298 \pm 0.674i, 0.1322$. Adding in the eigenvalues 1 and 0.2679(twice) already found, the slowest decay is given by the complex conjugate pair, which have modulus $|r| \approx 0.737$. We see that the mixing rate ($-\log 0.737 \approx 0.305$) is slower than infinitesimal stretching ($\chi_{ae} = \frac{1}{2}\log\kappa \approx 0.658$, which is itself slower than the line stretching rate $h_{top} = \log\lambda \approx 0.831$, as already discussed).

Froyland pointed out to me that the corresponding eigenforms on $\mathbb{T}^2$ (eigenvectors of the transfer operator) must be singular distributions in the contracting direction and that it is likely that the full spectrum in appropriate function spaces consists of the above eigenvalues union essential spectrum contained in a narrow annulus (cf. [Fr]). It would be interesting to study these two points.

4.9 Perturbations

An eighth consequence of the proof is that it allows an understanding of continuous perturbations $\tilde{\mathcal{H}}$ of $\mathcal{H}$ fixing the 1-prongs (meaning preserving them as an invariant set), e.g. Wilbrink's smoothing of the triangle wave [Wi] affinely transformed to have minimum value 0 and maximum +1. Firstly, they possess at least all the dynamics of $\mathcal{H}$ (in the sense of Section 2). Secondly, smooth (even real analytic [Ge]) perturbations can be made which remain pseudo-Anosov [GK] (though the dynamics on the skeleton ceases to be piecewise affine and the associated stochastic process is in general Gibbsian rather than Markov), or which fatten chosen leaves ending on singularities into strips containing additional dynamics ("derived from pseudo-Anosov" maps; see [Cou] for a nice illustration of an example of Plykin derived from a pseudo-Anosov map of S^2).

4.10 An exact area-preserving map with diffusion

Finally, $\mathcal{H}$ can be used to make an exact area-preserving tilt map of the cylinder with diffusion.

To achieve this, note that in addition to the torus, $\mathcal{H}$ also defines an area-preserving map of the cylinder $\mathbb{R} \times \mathbb{T}$ because it is homotopic to a vertical shear. Furthermore, it is a twist map since $\partial y'/\partial x > 0$ (the coordinates are the opposite way round from usual). Although $\mathcal{H}$ has net flux $\frac{1}{2}$, the composition $R\mathcal{H}^2$ has zero net flux, where $R(x, y) = (x - 1, y)$. R commutes with $\mathcal{H}$ on $\mathbb{R} \times \mathbb{T}$, so $R\mathcal{H}^2$ has equivalent dynamics to $\mathcal{H}^2$ modulo integer translations. $R\mathcal{H}^2$ is not a twist map because its derivative rotates the tangent vector $(1, 0)$ to $(-1, 0)$ on parts of the state space, yet it is a "positive tilt map" (more specifically, a composition of two positive twist maps $\mathcal{H}$ and $R\mathcal{H}$), so inherits many of the properties of twist maps (in particular, the whole of Aubry-Mather theory applies, e.g. [Go]). For all $n > 0$, the time-nT map of any T-periodic optical Hamiltonian system on $T^*\mathbb{T}$ is an area-preserving positive tilt map of the cylinder with zero net flux. Thus $R\mathcal{H}^2$ provides an example of such which is ergodic and has positive metric entropy ($\log \kappa$) (since the cylinder has infinite area, this is best interpreted by taking the quotient by R to $\mathbb{T}^2$). In this context, the coordinate x along the cylinder should be interpreted as a momentum, and the shear rotation rate of an orbit as its average acceleration (or in general, force). $R\mathcal{H}^2$ has well defined diffusion in x on a large scale, because the auto-correlation of displacements in x decays exponentially (e.g. apply [DP]). In particular, variance of the finite-time shear rotation rate of $\mathcal{H}$ asymptotically $\frac{Q''(0)}{n}$ (from subsection 4.7) implies diffusion coefficient $D = Q''(0) \approx 0.14434$ for $R\mathcal{H}^2$ (two factors of 2 cancel).

Wojtkowski's example [W1] already satisfies all these conditions except that I don't know if it is proved it has sufficiently fast decay of correlations for the large scale motion to look like diffusion. A big advantage of $R\mathcal{H}^2$ is that it can be smoothed. Following section 4.9, $\mathcal{H}$ can be perturbed to a topologically conjugate C^∞ diffeomorphism $\tilde{\mathcal{H}}$ which preserves area, commutes with R, has net flux $\frac{1}{2}$ and metric entropy close to $\frac{1}{2} \log \kappa$. Thus $R\tilde{\mathcal{H}}^2$ gives a C^∞ area-preserving tilt map of the cylinder of zero net flux which is ergodic, has metric entropy close to $\log \kappa$ and well defined diffusion coefficient close to 0.14434.

5 Final Remarks

Boyland suggested to me that one might alternatively seek to prove $\mathcal{H}$ is pseudo-Anosov by using a theorem of Lewowicz [Le] and Hiraide [Hi] that every expansive homeomorphism of a surface is pseudo-Anosov (a continuous map g is *expansive* if for given metric d there exists $C > 0$ such that $d(g^n x, g^n x') < C$ for all $n \in \mathbb{Z}$ implies $x = x'$). These authors, however, require a stronger definition of pseudo-Anosov that allows 1-prongs on the boundary only (for good reason, as the expansive property fails near internal 1-prongs of pseudo-Anosov maps, as pointed out by a referee). Nevertheless, by blowing up the two 1-prongs of $\mathcal{H}$ to circles I think it probable that the analysis of [CG] would suffice to prove expansivity. On the other hand, the explicit verification of the pseudo-Anosov property given here allows one to calculate much more.

It would be interesting to investigate whether some other examples, like those of Wojtkowski [W1, W2], might be pseudo-Anosov for selected parameter values. Note that Burton and Easton's [BE] are not (they are essentially Anosov with a fixed point blown up into a square of fixed points).

It should be pointed out that the phenomena of singular w-measures, non-trivial distribution of Lyapunov exponents, asymptotically normal distribution of finite-time Lyapunov exponents (and indeed analogues for most time-averages) occur for all typical Anosov diffeomorphisms too (though not sign-alternation).

Four aspects that merit further investigation are cancellation exponents (a result of sign-alternation) and related dimension spectra [DO], the effect of the distribution of w-measure scaling exponents on progress of chemical reactions when small diffusion is allowed (cf. [Cox]), the combined effect of the above two on magnetic fields when small resistivity is allowed (cf. [DO]), and the full Perron-Frobenius spectrum (cf. [Fr]).

Acknowledgements

I am grateful to Max Giona for permitting me a copy of [CG] before its publication, and to Mark Pollicott, Konstantin Khanin, Philip Boyland and Gary Froyland for comments. In particular, Khanin suggested calculating the distribution of finite-time Lyapunov exponents and Boyland the shear rotation interval. Much of the paper was completed at the Fields Institute, for whose hospitality I am grateful. Some additions were made while a guest of MASCOS at the University of New South Wales and final touches at IHES in France, and I am grateful for their hospitality too. Finally, I thank one of the referees for detailed comments and corrections and providing additional references, which have led to an improved final version.

References

[BH] Bestvina M, Handel M, Train-tracks for surface homeomorphisms, Topology 34 (1995) 109–140.

[Bi] Billingsley P, Ergodic theory and information (Wiley, 1965).

[BR] Bohr T, Rand DA, The entropy function for characteristic exponents, Physica D 25 (1987) 387–98.

[Bo] Boyland P, Topological methods in surface dynamics, Topology and Applns 58 (1994) 223–98.

[BE] Burton R, Easton RW, Ergodicity of linked twist maps, Springer Lect Notes in Math 819 (1980) 35–49.

[CB] Casson A, Bleiler S, Automorphisms of surfaces after Nielsen and Thurston, LMS Student Texts 9 (Cambridge University Press, 1988).

[CG] Cerbelli S, Giona M, A continuous archetype of nonuniform chaos in area-preserving dynamical systems, J Nonlinear Sci 15 (2005) 387–421.

[Ch] Chernov NI, Limit theorems and Markov approximations for chaotic dynamical systems, Probab Thy Rel Fields 101 (1995) 321–362.

[Cou] Coudène Y, Pictures of hyperbolic dynamical systems, Not Am Math Soc 53 (2006) 8–13.

[Cox] Cox SM, Chaotic mixing of a competitive-consecutive reaction, Physica D 199 (2004) 369–386.

[DP] Denker M, Phillips W, Approximation by Brownian motion for Gibbs measures and flows under a function, Ergod Th Dyn Sys 4 (1984) 541–552.

[Do] Doeff E, Rotation measures for homeomorphisms of the torus homotopic to a Dehn twist, Ergod Th Dyn Sys 17 (1997) 1–17.

[DO] Du Y, Ott E, Fractal dimension of fast dynamo magnetic fields, Physica D 67 (1993) 387–417.

[FLP] Fathi A, Laudenbach F, Poénaru V, Travaux de Thurston sur les surfaces, Astérisque 66–67 (1979) 1–284.

[Fr] Froyland G, A hyperbolic map whose Perron-Frobenius operator possesses an isolate eigenvalue (UNSW preprint, 2005).

[Ge] Gerber M, Conditional stability and real analytic pseudo-Anosov maps, Mem Am Math Soc 54 (1985) no.321, iv+116pp.

[GK] Gerber M, Katok A, Smooth models of Thurston's pseudo-Anosov maps, Ann Sci Ec Norm Sup 15 (1982) 173–204.

[GA] Giona M, Adrover A, Non-uniform stationary measure of the invariant unstable foliation in Hamiltonian and fluid mixing systems, Phys Rev Lett 81 (1998) 3864–7.

[Go] Golé C, Symplectic twist maps: global variational techniques (World Sci, 2001).

[Ha] Handel M, Global shadowing of pseudo-Anosov homeomorphisms, Ergod Th Dyn Sys 5 (1985) 373–7.

[Hi] Hiraide K, Expansive homeomorphisms of compact surfaces are pseudo-Anosov, Osaka J Math 27 (1990) 117–162.

[KH] Katok A, Hasselblatt B, Introduction to the modern theory of dynamical systems (Cambridge U Press, 1995).

[Le] Lewowicz J, Expansive homeomorphisms of surfaces, Bol Soc Brasil Mat 20 (1989) 113–133.

[Ma] MacKay RS, Complicated dynamics from simple topological hypotheses, Phil Trans Roy Soc London A 359 (2001) 1479–96.

[Ne] Newhouse S, Entropy and volume, Ergod Th Dyn Sys 8 (1988) 283–299.

[Pa] Parry W, Intrinsic Markov chains, Trans Am Math Soc 112 (1964) 55–66, and Symbolic dynamics and transformations of the unit interval, Trans Am Math Soc 122 (1966) 368–378.

[Pe] Pesin Ya B, Characteristic Lyapunov exponents and smooth ergodic theory, Russ Math Surv 32:4 (1977) 55–114.

[QZ] Qian M, Zhu S, SRB measures and Pesin's entropy formula for endomorphisms, Trans Am Math Soc 354 (2002) 1453–71.

[Ru] Ruelle D, Thermodynamic formalism (Addison-Wesley, 1978); and The thermodynamic formalism for expanding maps, Commun Math Phys 125 (1989) 239–262.

[S1] Sinai Ya G, Markov partitions and C-diffeomorphisms, Funct Anal Appl 2:1 (1968) 64–89.

[S2] Sinai Ya G, Gibbs measures in ergodic theory, Russ Math Surv 27:4 (1972) 21–69.

[Th] Thurston WP, On the geometry and dynamics of diffeomorphisms of surfaces, Bull Am Math Soc 19 (1988) 417–431.

[Wi] Wilbrink J, Erratic behavior of invariant circles in standard-like mappings, Physica D 26 (1987) 358–68.

[W1] Wojtkowski M, Linked twist mappings have the K-property, in: Nonlinear Dynamics, ed Helleman RHG, Annals NY Acad Sci 357 (1980) 65–76.

[W2] Wojtkowski M, A model problem with the coexistence of stochastic and integrable behaviour, Commun Math Phys 80 (1981) 453–64.

[Yo] Yomdin Y, Volume growth and entropy, Israel J Math 57 (1987) 285–317.