First steps towards p-adic Langlands functoriality

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by

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Abstract. — By the theory of Colmez and Fontaine, a de Rham representation of the Galois group of a local field roughly corresponds to a representation of the Weil-Deligne group equipped with an admissible filtration on the underlying vector space. Using a modification of the classical local Langlands correspondence, we associate with any pair consisting of a Weil-Deligne group representation and a type of a filtration (admissible or not) a specific locally algebraic representation of a general linear group. We advertise the conjecture that this pair comes from a de Rham representation if and only if the corresponding locally algebraic representation carries an invariant norm. In the crystalline case, the Weil-Deligne group representation is unramified and the associated locally algebraic representation can be studied using the classical Satake isomorphism. By extending the latter to a specific norm completion of the Hecke algebra, we show that the existence of an invariant norm implies that our pair, indeed, comes from a crystalline representation. We also show, by using the formalism of Tannakian categories, that this latter fact is compatible with classical unramified Langlands functoriality and therefore generalizes to arbitrary split reductive groups.

Résumé. — Par la théorie de Colmez et Fontaine, une représentation de de Rham du groupe de Galois d'un corps local correspond essentiellement à une représentation du groupe de Weil-Deligne dont l'espace sous-jacent est muni d'une filtration admissible. En modifiant la correspondance locale de Langlands, on associe à chaque couple formé d'une représentation du groupe de Weil-Deligne et des poids d'une filtration (admissible ou pas) une représentation localement algébrique particulière d'un groupe linéaire général. On conjecture qu'un couple provient d'une représentation de de Rham si et seulement si la représentation localement algébrique correspondante possède une norme invariante. Dans le cas cristallin, la représentation du groupe de Weil-Deligne est non-ramifiée et la représentation localement algébrique associée peut s'étudier grâce à l'isomorphisme de Satake classique. En prolongeant ce dernier à une complétion de l'algèbre de Hecke, on montre que l'existence d'une norme invariante comme ci-dessus implique que le couple provient effectivement d'une représentation cristalline. On montre aussi, en utilisant le formalisme des catégories tannakiennes, que ce dernier fait est compatible avec la fonctorialité de Langlands non-ramifiée classique, et donc qu'il se généralise à tout groupe réductif déployé.

Contents

1. Introduction and notations

The authors strongly believe in the existence of a p-adic extension of the local Langlands correspondence and even the functoriality principle. Unfortunately, at the present time, there is not even a precise picture yet which two sets this extension will put into correspondence. On the Galois side we should at least have all p-adic Weil group representations. On the reductive group side the evidence points into the direction of a set related to the set of isomorphism classes of all topologically irreducible admissible Banach space representations. But it seems very difficult to construct such Banach space representations. There has been progress recently by Breuil/Berger and by Colmez only for the group $\mathrm{GL}_2(\mathbb{Q}_p)$.

If we restrict attention on the Galois side to $(d+1)$-dimensional de Rham representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ of some finite extension L of $\mathbb{Q}_p$ then the first author some time ago has put forward the following philosophy. By the theory of Colmez and Fontaine, a de Rham representation gives rise (roughly) to a vector space D which carries a filtration as well as an action of the Weil-Deligne group of L, the two being in a numerical relation called admissibility. The classical local Langlands correspondence associates with the semisimplification of the Weil-Deligne group action an irreducible smooth representation of $\mathrm{GL}_{d+1}(L)$. On the other hand, the type of the filtration can be viewed as a highest weight giving rise to an irreducible rational representation of $\mathrm{GL}_{d+1}(L)$. The tensor product of the two forms a locally algebraic representation $\Pi(D)$. Note that the construction of $\Pi(D)$ does not make use of the admissibility relation on D. Rather the admissibility of the filtration on D should be reflected by the existence of a $\mathrm{GL}_{d+1}(L)$-invariant norm on $\Pi(D)$. It is one of the purposes of this paper to turn this philosophy into a precise conjecture which is done in section 4. The problem, of course, is that the de Rham representation can be such that the smooth part of $\Pi(D)$ is trivial whereas the algebraic part is not. Then $\Pi(D)$ cannot carry an invariant norm. In order to overcome this difficulty we suggest to use a modified version of the classical local Langlands correspondence. By the Langlands classification, every irreducible smooth representation is the unique irreducible quotient of a particular parabolically induced representation. We propose to use (a twist of) the latter instead of its irreducible quotient. In this way the smooth part of $\Pi(D)$ always is infinite dimensional and, in fact, reducible in general. In section 5, we establish some special cases of our conjecture as well as some partial results in its direction.

If we restrict attention further to crystalline Galois representations then the smooth part of $\Pi(D)$ is unramified. The classical local Langlands correspondence for unramified representations is encapsulated in the Satake isomorphism which computes the spectrum of the Satake-Hecke algebra of $\mathrm{GL}_{d+1}(L)$. The Satake-Hecke algebra together with its universal module has a natural norm completion leading to a Banach-Hecke algebra acting on a universal Banach module. This was shown by the second author together with Teitelbaum in [**19**]. It was also shown in that paper that this Banach-Hecke algebra naturally is the algebra of analytic functions on an explicitly given affinoid domain. Moreover, the defining conditions for this affinoid domain turn out to be equivalent to the admissibility condition for filtrations. This means that any crystalline Galois representation satisfying a certain regularity condition on its Hodge-Tate weights gives rise to a point in such an affinoid domain and hence to a specialization in that point of the corresponding universal Banach module. Our earlier conjecture means in this picture that this Banach space representation of $\mathrm{GL}_{d+1}(L)$ obtained by specialization is non-zero. Unfortunately, in [**19**] an embedding of L into the coefficient field K of our representations was distinguished. This had the consequence that all of the above could only be shown for a subclass of the crystalline Galois representations called special ones. Even worse, the corresponding normalizations in [**19**] are, as we believe now, misleading. In sections 2 and 3, we take up this theory again in a completely general way. By working systematically with $\mathbb{Q}_p$-rational representations of $\mathrm{GL}_{d+1}(L)$, we do obtain the above results for arbitrary "regular" crystalline Galois representations.

At this point it should be stressed that, in this paper, we do not set up an actual conjectural correspondence between Galois and Banach space representations. In both pictures, our conjecture is of the form that on a certain object there exists an admissible filtration if and only if a certain Banach space representation associated to this object (either by completion or by specialization) is non-zero. But with the exception of $\mathrm{GL}_2(\mathbb{Q}_p)$, there always will be infinitely many possibilities for the admissible filtration (provided there exists at least one). Hence, in some sense, any of these conjecturally non-zero Banach space representations is responsible for a specific whole family of Galois representations.

Our picture in the crystalline case is, upon the prize of having and fixing a square root of p in the coefficient field K, very well adapted to functoriality. The theory of Banach-Hecke algebras as described above works perfectly well for arbitrary split connected reductive groups G over L (and is developed in this generality in section 2), and can in a certain sense be made functorial on the category of K-rational representations of the connected Langlands dual group $\mathbf{G}'$ of G over K. This makes it possible, by using the formalism of Tannakian categories, to associate with any specialization of a universal Banach module for G (which still conjecturally is non-zero) a family of isomorphism classes of crystalline Galois representations with values in the dual group $\mathbf{G}'(\overline{K})$ over the algebraic closure $\overline{K}$ of K. Under the already mentioned restrictions, this was done in [**19**] and is here established in general in section 6.

In fact, in [**19**] and also in section 6, we have to assume that the split group G is such that the half sum η of its positive roots is still an integral character. The appearance of this η is forced upon us by functoriality. To deal with η in general, we need a counterpart on the Galois side which is a square root of the cyclotomic character. This does not exist in general on the Galois group, which leads to the following interesting construction. The group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ has, up to isomorphism, a unique non-trivial central extension $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)_{(2)}$ with kernel of order two. Let $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$ denote the restriction of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)_{(2)}$ to L. On $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$ we always have a square root of the cyclotomic character provided the coefficient field K is large enough. As a consequence of the theory of Colmez and Fontaine, one can set up a theory of crystalline representations of the group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$. It is exactly families of such which we get in the case of split groups for which η is not integral. This is the content of section 7. We are grateful to J.-M. Fontaine for helpful discussions about the material in this section.

Throughout the paper we fix two finite extensions L (the base field) and K (the coefficient field) of $\mathbb{Q}_p$ such that $[L:\mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L,K)|$ where $\mathrm{Hom}_{\mathbb{Q}_p}(L,K)$ denotes the set of all $\mathbb{Q}_p$-linear embeddings of the field L into the field K. We assume L is contained in an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. We denote by $q = p^f$ the cardinality of the residue field of L and by $L_0 = \mathrm{Frac}(W(\mathbb{F}_q))$ its maximal unramified subfield. If $e := [L:\mathbb{Q}_p]/f$, we set $\mathrm{val}_L(x) := e\,\mathrm{val}_{\mathbb{Q}_p}(x)$ (where $\mathrm{val}_{\mathbb{Q}_p}(p) := 1$) and $|x|_L := q^{-\mathrm{val}_L(x)}$ for any x in a finite extension of $\mathbb{Q}_p$. We denote by $\mathrm{W}(\overline{\mathbb{Q}_p}/L)$ (resp. $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$) the Weil (resp. Galois) group of L and by $\mathrm{rec} : \mathrm{W}(\overline{\mathbb{Q}_p}/L)^{\mathrm{ab}} \xrightarrow{\sim} L^\times$ the reciprocity map sending the arithmetic Frobeniuses to the inverse of uniformizers. If $\lambda \in K^\times$, $\mathrm{unr}(\lambda)$ stands for the unramified character of $L^\times$ sending a uniformizer to λ.

2. Completed Satake-Hecke algebras

We extend the theory and results of [**19**, §§2-3].

We fix an L-split connected reductive group $\mathbf{G}$ over L and put $G := \mathbf{G}(L)$. Let (ρ, E) be an irreducible $\mathbb{Q}_p$-rational representation of G in a finite dimensional K-vector space E. Fixing a good maximal compact subgroup $U \subseteq G$, we let

$\rho_U := \rho|U$. The corresponding Satake-Hecke algebra $\mathcal{H}(G, \rho_U)$ is the convolution algebra over K of all compactly supported functions $\psi : G \longrightarrow \mathrm{End}_K(E)$ satisfying:

$$\psi(u_1 g u_2) = \rho(u_1) \circ \psi(g) \circ \rho(u_2)$$

for any $u_1, u_2 \in U$ and $g \in G$. The algebra $\mathcal{H}(G, \rho_U)$ can naturally be identified with the ring of G-endomorphisms of the compact induction $\mathrm{ind}_U^G(\rho_U)$. Fixing once and for all a U-invariant norm $\| \ \|$ on E, the K-vector space $\mathrm{ind}_U^G(\rho_U)$ carries the corresponding G-invariant sup-norm also denoted by $\| \ \|$. The G-action on $\mathrm{ind}_U^G(\rho_U)$ extends to an isometric G-action on the completion $B_U^G(\rho_U)$ of $\mathrm{ind}_U^G(\rho_U)$ with respect to $\| \ \|$. Using the operator norm on $\mathrm{End}_K(E)$ we also have a corresponding sup-norm $\| \ \|$ on $\mathcal{H}(G, \rho_U)$ which is submultiplicative. Its completion $\mathcal{B}(G, \rho_U)$ therefore is a K-Banach algebra. It is shown in [**19**, Lem.1.3] that one has a natural isomorphism of K-algebras:

$$\mathcal{B}(G, \rho_U) \xrightarrow{\cong} \mathrm{End}_G^{\mathrm{cont}}(B_U^G(\rho_U))$$

which is an isometry with respect to the operator norm on the right hand side (which consists of the continuous and G-equivariant endomorphisms of the Banach space $B_U^G(\rho_U)$).

Generalizing the results in [**19**, §§1-3], we want to explicitly compute the Banach algebra $\mathcal{B}(G, \rho_U)$. In order to recall what it means for ρ to be $\mathbb{Q}_p$-rational we introduce the connected reductive group:

$$\widetilde{\mathbf{G}} := (\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{G})_K$$

over K obtained by base extension from the Weil restriction from L to $\mathbb{Q}_p$ of $\mathbf{G}$. We also put:

$$\widetilde{G} := \widetilde{\mathbf{G}}(K) = \mathbf{G}(L \otimes_{\mathbb{Q}_p} K).$$

The ring homomorphism $L \longrightarrow L \otimes_{\mathbb{Q}_p} K$ which sends a to $a \otimes 1$ induces an embedding of groups $G \hookrightarrow \widetilde{G}$. We have:

$$\widetilde{\mathbf{G}} = \prod_{\sigma : L \hookrightarrow K} \mathbf{G}_\sigma$$

where $\mathbf{G}_\sigma$ denotes the base extension of $\mathbf{G}$ to K via the embedding $\sigma : L \to K$. In particular, the groups $\mathbf{G}_\sigma$ and $\widetilde{\mathbf{G}}$ are K-split.

The $\mathbb{Q}_p$-rationality of ρ means that there is an irreducible K-rational representation $\widetilde{\rho}$ of $\widetilde{\mathbf{G}}$ on E such that ρ is the pull-back of $\widetilde{\rho}$ via $G \hookrightarrow \widetilde{G}$. Since G is Zariski dense in $\widetilde{\mathbf{G}}$ ([**15**, §34.4]), the representation $\widetilde{\rho}$ is uniquely determined by ρ. We also note ([**20**, Lem.68]) that:

$$(\widetilde{\rho}, E) \cong \bigotimes_{\sigma : L \hookrightarrow K} (\rho_\sigma, E_\sigma)$$

with irreducible K-rational representations (ρ_σ, E_σ) of $\mathbf{G}_\sigma$. Conversely any such tensor product gives rise, by restriction, to an irreducible $\mathbb{Q}_p$-rational ρ.

By a variant of [**19**, Lem.1.4], one easily shows the following, where 1_U denotes the trivial representation of U.

Lemma 2.1. — *The map:*

$$\iota_\rho : \mathcal{H}(G, 1_U) \xrightarrow{\cong} \mathcal{H}(G, \rho_U)$$
$$\psi \longmapsto \psi \cdot \rho$$

is an isomorphism of K-algebras.

At this point we need to introduce further notation. We fix a maximal L-split torus $\mathbf{T}$ in $\mathbf{G}$ and a Borel subgroup $\mathbf{P} = \mathbf{TN}$ of $\mathbf{G}$ with Levi component $\mathbf{T}$ and unipotent radical $\mathbf{N}$. Then $\widetilde{\mathbf{T}} := (\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{T})_K$ is a maximal K-split torus in the Borel subgroup $\widetilde{\mathbf{P}} := (\mathrm{Res}_{L/\mathbb{Q}_p}\mathbf{P})_K$ of $\widetilde{\mathbf{G}}$. We denote by P, T and N the group of L-valued points of $\mathbf{P}$, $\mathbf{T}$ and $\mathbf{N}$ respectively. The Weyl group of G is the quotient $W = N(T)/T$ of the normalizer $N(T)$ of T in G by T. We always assume that our fixed maximal compact subgroup $U \subseteq G$ is special with respect to T, and we put $T_0 := U \cap T$ and $N_0 := U \cap N$. The quotient $\Lambda := T/T_0$ is a free abelian group of rank equal to the dimension of $\mathbf{T}$ and can naturally be identified with the cocharacter group $X_*(\mathbf{T})$. Let $\lambda : T \longrightarrow \Lambda$ denote the projection map. The conjugation action of $N(T)$ on T induces W-actions on T and Λ which we denote by $t \longmapsto {}^w t$ and $\lambda \longmapsto {}^w\lambda$ respectively. We have the embedding:

$$X^*(\mathbf{T}) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{R}) =: V_\mathbb{R}$$
$$\chi \longmapsto \mathrm{val}_L \circ \chi$$

which induces an isomorphism:

$$X^*(\mathbf{T}) \otimes \mathbb{R} \xrightarrow{\cong} V_\mathbb{R}.$$

We therefore may view $V_\mathbb{R}$ as the real vector space underlying the root datum of G with respect to T. Evidently any $\lambda \in \Lambda$ defines a linear form in the dual vector space $V_\mathbb{R}^*$ also denoted by λ. Let Φ denote the set of roots of T in G and let $\Phi^+ \subseteq \Phi$ be the subset of those roots which are positive with respect to P. As usual, $\check{\alpha} \in \Lambda$ denotes the coroot corresponding to the root $\alpha \in \Phi$. The subset $\Lambda^{--} \subseteq \Lambda$ of antidominant cocharacters is defined to be the image $\Lambda^{--} := \lambda(T^{--})$ of:

$$T^{--} := \{t \in T, |\alpha(t)|_L \geq 1 \text{ for any } \alpha \in \Phi^+\}.$$

Hence, we have:

$$\Lambda^{--} = \{\lambda \in \Lambda, \mathrm{val}_L \circ \alpha(\lambda) \leq 0 \text{ for any } \alpha \in \Phi^+\}.$$

By the Cartan decomposition, G is the disjoint union of the double cosets UtU with t running over T^{--}/T_0. The norm $\| \ \|$ on $\mathcal{H}(G, \rho_U)$ corresponds therefore under the isomorphism ι_ρ from Lemma 2.1 to the norm $\| \ \|_\rho$ on $\mathcal{H}(G, 1_U)$ defined by:

$$\|\psi\|_\rho := \sup_{t \in T^{--}} (|\psi(t)|_L \cdot \|\rho(t)\|).$$

Hence ι_ρ extends to an isometric isomorphism of Banach algebras:

$$\| \; \|_\rho\text{–completion of } \mathcal{H}(G, 1_U) \xrightarrow{\cong} \mathcal{B}(G, \rho_U).$$

In order to compute this norm further, we let $\widetilde{\xi} \in X^*(\widetilde{\mathbf{T}})$ denote the highest weight (with respect to $\widetilde{\mathbf{P}}$) of the representation $\widetilde{\rho}$ and $\xi : T \longrightarrow K^\times$ its restriction to T.

Lemma 2.2. — *We have $\|\rho(t)\| = |\xi(t)|_L$ for any $t \in T^{--}$.*

Proof. — According to [**19**, Lem.3.2], we have $\|\widetilde{\rho}(\widetilde{t})\| = |\widetilde{\xi}(\widetilde{t})|_L$ for any $\widetilde{t} \in \widetilde{\mathbf{T}}(K)$ which is antidominant with respect to $\widetilde{\mathbf{P}}$. We therefore have to show that the image in $\widetilde{\mathbf{T}}(K)$ of any $t \in T^{--}$ is antidominant with respect to $\widetilde{\mathbf{P}}$. Let $\mathbf{T}_\sigma$, for any $\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(L, K)$, denote the base extension of $\mathbf{T}$ to K via the embedding σ. For simplicity we also write $\sigma : T \hookrightarrow \mathbf{T}_\sigma(K)$ for the corresponding embedding of groups. The image of t in $\widetilde{\mathbf{T}}(K) = \prod_\sigma \mathbf{T}_\sigma(K)$ then is given by $(\sigma(t))_\sigma$. On the other hand, using the natural identifications $X^*(\mathbf{T}) = X^*(\mathbf{T}_\sigma)$, any root $\widetilde{\alpha}$ of $\widetilde{\mathbf{T}}$ which is positive with respect to $\widetilde{\mathbf{P}}$ may be viewed as a tuple $(\alpha_\sigma)_\sigma$ of roots $\alpha_\sigma \in \Phi^+$, and we have $\widetilde{\alpha}(t) = \prod_\sigma \alpha_\sigma(\sigma(t)) = \prod_\sigma \sigma(\alpha_\sigma(t))$. Since $|\alpha_\sigma(t)|_L \geq 1$ by assumption, it follows that $|\widetilde{\alpha}(t))|_L \geq 1$. $\square$

The unnormalized Satake map:

$$\begin{aligned} S : \mathcal{H}(G, 1_U) &\longrightarrow K[\Lambda] \\ \psi &\longmapsto \sum_{t \in T/T_0} \sum_{n \in N/N_0} \psi(tn)\lambda(t) \end{aligned}$$

induces an isomorphism of K-algebras:

$$\mathcal{H}(G, 1_U) \xrightarrow{\cong} K[\Lambda]^{W,\gamma}$$

where the right hand side denotes the W-invariants in the group ring $K[\Lambda]$ with respect to the twisted W-action:

$$\begin{aligned} W \times K[\Lambda] &\longrightarrow K[\Lambda] \\ (w, \sum_\lambda c_\lambda \lambda) &\longmapsto w \cdot (\sum_\lambda c_\lambda \lambda) := \sum_\lambda \gamma(w, \lambda) c_\lambda {}^w\lambda \end{aligned}$$

for the K-valued cocycle:

$$\gamma(w, \lambda) := \frac{\delta^{1/2}({}^w\lambda)}{\delta^{1/2}(\lambda)}$$

with $\delta : P \longrightarrow \mathbb{Q}^\times \subseteq K^\times$ denoting the modulus character of the Borel subgroup P (compare [**19**, §§2-3]). By Lemma 2.2 together with a variant of [**19**, Prop.3.5], the norm $\| \; \|_\rho$ on $\mathcal{H}(G, 1_U)$ corresponds under this Satake isomorphism to the restriction of the norm $\| \; \|_\xi$ on $K[\Lambda]$ given by:

$$\| \sum_{\lambda \in \Lambda} c_\lambda \lambda \|_\xi := \sup_{\lambda = \lambda(t)} |\gamma(w, \lambda)\xi({}^w t) c_\lambda|_L$$

with $w \in W$ for each λ being chosen in such a way that ${}^w\lambda \in \Lambda^{--}$. One checks that this norm on $K[\Lambda]$ is submultiplicative and that the twisted W-action is isometric in this norm (compare [**19**, Lem.2.1] and Examples 1 and 2 in §2 of *loc.cit.*). Hence the completion of $K[\Lambda]$ with respect to $\| \, \|_\xi$ is a K-Banach algebra $K\langle\Lambda;\xi\rangle$ to which the twisted W-action extends. In particular, we may form the Banach algebra $K\langle\Lambda;\xi\rangle^{W,\gamma}$ of W-invariants. As a result of the discussion so far we obtain the following:

Proposition 2.3. — *The Banach algebras $\mathcal{B}(G,\rho_U)$ and $K\langle\Lambda;\xi\rangle^{W,\gamma}$ are, in a natural way, isometrically isomorphic.*

Let $\mathbf{T}'$ denote the L-torus dual to $\mathbf{T}$. Its K-valued points are given by $\mathbf{T}'(K) = \mathrm{Hom}(\Lambda, K^\times)$. The group ring $K[\Lambda]$ naturally identifies with the ring of K-valued algebraic functions on $\mathbf{T}'$. We introduce the "valuation map":

$$\mathrm{val}: \ \mathbf{T}'(K) = \mathrm{Hom}(\Lambda, K^\times) \xrightarrow{\mathrm{val}_L \circ} \mathrm{Hom}(\Lambda, \mathbb{R}) = V_{\mathbb{R}}.$$

In $V_{\mathbb{R}}$ we have the two distinguished points:

$$\xi_L := \mathrm{val}_L \circ \xi \qquad \text{and} \qquad \eta_L := [L:\mathbb{Q}_p]\cdot\eta$$

where η denotes half the sum of the positive roots in Φ^+. Let $\leq$ denote the partial order on $V_{\mathbb{R}}$ defined by Φ^+ (cf. [**3**, Chap.VI,§1.6]). Given any point $z \in V_{\mathbb{R}}$, let z^{dom} be the unique dominant point in the W-orbit of z. We put:

$$V_{\mathbb{R}}^{\xi} := \{z \in V_{\mathbb{R}}, (z+\eta_L)^{\mathrm{dom}} \leq \eta_L + \xi_L\}$$

and:

$$\mathbf{T}'_\xi := \mathrm{val}^{-1}(V_{\mathbb{R}}^{\xi}).$$

Theorem 2.4. — (i) *$V_{\mathbb{R}}^{\xi}$ is the convex hull of the points ${}^w(\eta_L+\xi_L)-\eta_L$ for $w \in W$;*
(ii) *$\mathbf{T}'_\xi$ is an open K-affinoid subdomain of the torus $\mathbf{T}'$;*
(iii) *the Banach algebra $K\langle\Lambda;\xi\rangle$ is naturally isomorphic to the ring of analytic functions on the affinoid domain $\mathbf{T}'_\xi$;*
(iv) *$K\langle\Lambda;\xi\rangle^{W,\gamma}$ is an affinoid K-algebra.*

Proof. — This is a straightforward variant of [**19**, Lem.2.3, Prop.2.4, Lem.2.7, Ex.3 of §2] (compare also the discussion before the remark in §6). □

The Weyl group W acts on the affinoid $\mathbf{T}'_\xi$ by:

$$(w,\zeta) \longrightarrow \frac{{}^w\delta^{1/2}}{\delta^{1/2}}\cdot {}^w\zeta.$$

Corollary 2.5. — *The Banach algebra $\mathcal{B}(G,\rho_U)$ is naturally isomorphic to the ring of analytic functions on the quotient affinoid $W\backslash\mathbf{T}'_\xi$.*

For later purposes (see §6), we also have to discuss briefly the subsequent renormalization but for which we are forced to assume that the coefficient field K contains a square root of q. Fix once and for all such a square root $q^{1/2}$. We then have a preferred square root $\delta^{1/2} \in \mathbf{T}'(K)$ of $\delta \in \mathbf{T}'(K)$ and as a consequence the normalized Satake isomorphism:

$$S^{\mathrm{norm}} : \mathcal{H}(G, 1_U) \xrightarrow{\cong} K[\Lambda]^W$$
$$\psi \longmapsto \sum_{t \in T/T_0} \delta^{-1/2}(t)\big(\sum_{n \in N/N_0} \psi(tn)\big)\lambda(t)$$

where now on the right hand side the W-invariants are formed with respect to the action induced by the conjugation action of $N(T)$ on T. We define:

$$V_{\mathbb{R}}^{\xi,\mathrm{norm}} := \{z \in V_{\mathbb{R}}, z^{\mathrm{dom}} \leq \eta_L + \xi_L\}$$

and:

$$\mathbf{T}'_{\xi,\mathrm{norm}} := \mathrm{val}^{-1}(V_{\mathbb{R}}^{\xi,\mathrm{norm}}).$$

This is an affinoid subdomain which is invariant under the natural W-action on $\mathbf{T}'$. As discussed in [**19**] before the remark in §6, the above corollary can be reformulated as follows.

Corollary 2.6. — *The Banach algebra $\mathcal{B}(G, \rho_U)$ is naturally isomorphic to the ring of analytic functions on the quotient affinoid $W\backslash\mathbf{T}'_{\xi,\mathrm{norm}}$.*

As a consequence of the above corollaries, we have natural identifications between the set of K-valued (continuous) characters of the Banach algebra $\mathcal{B}(G, \rho_U)$ and the sets of K-rational points $(W\backslash\mathbf{T}'_{\xi})(K)$ and $(W\backslash\mathbf{T}'_{\xi,\mathrm{norm}})(K)$ respectively.

3. Crystalline Galois representations

We focus on the special case $G = \mathrm{GL}_{d+1}(L)$, $d \geq 1$. We give a link between the constructions and results of §2 and the theory of crystalline representations, generalizing [**19**, §5] (although we do not use the same normalization as in *loc.cit.*).

We let $U := \mathrm{GL}_{d+1}(\mathcal{O}_L)$. We fix an irreducible $\mathbb{Q}_p$-rational representation (ρ, E) of G as in §2 and $(\zeta_1, \cdots, \zeta_{d+1}) \in (K^\times)^{d+1}$. Let τ be a permutation of $\{1, \cdots, d+1\}$ such that $\mathrm{val}_L(\zeta_{\tau(1)}) \leq \mathrm{val}_L(\zeta_{\tau(2)}) \leq \cdots \leq \mathrm{val}_L(\zeta_{\tau(d+1)})$. Let $\widehat{\zeta} : T \to K^\times$ be the character which sends $t := \mathrm{diag}(t_1, \cdots, t_{d+1}) \in T$ to $\prod_j (\zeta_j q^{1-j})^{\mathrm{val}_L(t_j)}$, in other words:

$$\widehat{\zeta} := \mathrm{unr}(\zeta_1) \otimes \mathrm{unr}(\zeta_2)|\ |_L \otimes \mathrm{unr}(\zeta_3)|\ |_L^2 \otimes \cdots \otimes \mathrm{unr}(\zeta_{d+1})|\ |_L^d.$$

With the notations of §2, we also denote by $\widehat{\zeta}$ the K-linear map $K[\Lambda] \to K$ induced by $\widehat{\zeta}$ and by $K_{\widehat{\zeta}}$ the one dimensional K-vector space on which $\mathcal{H}(G, \rho_U)$

acts through the character:

$$\mathcal{H}(G,\rho_U) \xrightarrow{\iota_\rho^{-1}} \mathcal{H}(G,1_U) \xrightarrow{S} K[\Lambda] \xrightarrow{\widehat{\zeta}} K.$$

For $\sigma : L \hookrightarrow K$, we let $(a_{1,\sigma},\cdots,a_{d+1,\sigma}) \in \mathbb{Z}^{d+1}$ with $a_{j,\sigma} \leq a_{j+1,\sigma}$ be the highest weight of ρ_σ with respect to the parabolic subgroup of lower triangular matrices of G (see §2).

Recall that an invariant norm on a locally algebraic representation of G (on a K-vector space E) is a p-adic norm $\| \ \|$ such that $\|gv\| = \|v\|$ for all $g \in G$ and $v \in E$.

Corollary 3.1. — *If the locally algebraic representation:*

$$K_{\widehat{\zeta}} \otimes_{\mathcal{H}(G,\rho_U)} \mathrm{ind}_U^G \rho_U$$

admits an invariant norm, then the following inequalities hold in $\mathbb{Q}$*:*

$$\begin{aligned}\sum_{j=i}^{d+1} \mathrm{val}_L(\zeta_{\tau(j)}) &\leq \sum_{j=i}^{d+1}\sum_{\sigma} a_{j,\sigma} + [L:\mathbb{Q}_p]\frac{d(d+1)-(i-2)(i-1)}{2}, \ 2 \leq i \leq d+1\\ \sum_{j=1}^{d+1} \mathrm{val}_L(\zeta_{\tau(j)}) &= \sum_{j=1}^{d+1}\sum_{\sigma} a_{j,\sigma} + [L:\mathbb{Q}_p]\frac{d(d+1)}{2}.\end{aligned}$$

Proof. — If the above locally algebraic representation admits an invariant norm, then necessarily the image of the unit ball of $\mathrm{ind}_U^G\rho_U$ in $K_{\widehat{\zeta}} \otimes_{\mathcal{H}(G,\rho_U)} \mathrm{ind}_U^G\rho_U$ is again a unit ball (i.e. remains a genuine $\mathcal{O}_L$-lattice). This implies that the character $\widehat{\zeta} : \mathcal{H}(G,\rho_U) \to K_{\widehat{\zeta}}$ factors through the completion $\mathcal{B}(G,\rho_U)$ of $\mathcal{H}(G,\rho_U)$, hence defines a K-point of $\mathbf{T}'_\xi$ by Corollary 2.5. By the description of those K-points, this implies that one has, with the notations of §2:

$$(\mathrm{val}_L \circ \widehat{\zeta} + \eta_L)^{\mathrm{dom}} \leq \xi_L + \eta_L,$$

that is to say:

$$\begin{aligned}\Big(\big(\mathrm{val}_L(\zeta_1), \mathrm{val}_L(\zeta_2) - [L:\mathbb{Q}_p], \cdots, \mathrm{val}_L(\zeta_{d+1}) - d[L:\mathbb{Q}_p]\big)\\ + [L:\mathbb{Q}_p]\Big(-\frac{d}{2}, -\frac{d}{2}+1, \cdots, \frac{d}{2}\Big)\Big)^{\mathrm{dom}}\\ \leq \Big(\sum_\sigma a_{1,\sigma}, \cdots, \sum_\sigma a_{d+1,\sigma}\Big) + [L:\mathbb{Q}_p]\Big(-\frac{d}{2}, -\frac{d}{2}+1, \cdots, \frac{d}{2}\Big).\end{aligned}$$

An immediate computation shows this is equivalent to:

$$\begin{aligned}\big(\mathrm{val}_L(\zeta_1), \mathrm{val}_L(\zeta_2), \cdots, \mathrm{val}_L(\zeta_{d+1})\big)^{\mathrm{dom}} + [L:\mathbb{Q}_p]\Big(-\frac{d}{2}, -\frac{d}{2}, \cdots, -\frac{d}{2}\Big)\\ \leq \Big(\sum_\sigma a_{1,\sigma}, \cdots, \sum_\sigma a_{d+1,\sigma}\Big) + [L:\mathbb{Q}_p]\Big(-\frac{d}{2}, -\frac{d}{2}+1, \cdots, \frac{d}{2}\Big),\end{aligned}$$

that is to say:

$$\big(\mathrm{val}_L(\zeta_{\tau(1)}), \mathrm{val}_L(\zeta_{\tau(2)}), \cdots, \mathrm{val}_L(\zeta_{\tau(d+1)})\big) \leq \Big(\sum_\sigma a_{1,\sigma}, \cdots, \sum_\sigma a_{d+1,\sigma}\Big) + [L:\mathbb{Q}_p](0,1,\cdots,d)$$

which is what we want. □

Denote by $\varphi_0 : L_0 \to L_0$ the absolute Frobenius. Let $D := L_0 \otimes_{\mathbb{Q}_p} K \cdot e_1 \oplus \cdots \oplus L_0 \otimes_{\mathbb{Q}_p} K \cdot e_{d+1}$ be a free $L_0 \otimes_{\mathbb{Q}_p} K$-module of rank $d+1$ and denote by F_ζ the unique $L_0 \otimes_{\mathbb{Q}_p} K$-linear automorphism of D such that:

$$F_\zeta(e_j) := \zeta_j^{-1} e_j, \quad j \in \{1, \cdots, d+1\}.$$

We call a Frobenius on D any bijective map $\varphi : D \to D$ satisfying:

$$\varphi((\ell \otimes k)\cdot d) = (\varphi_0(\ell) \otimes k) \cdot \varphi(d)$$

where $\ell \in L_0$, $k \in K$ and $d \in D$. If φ is a Frobenius on D, then φ^f is $L_0 \otimes_{\mathbb{Q}_p} K$-linear. The isomorphism $L_0 \otimes_{\mathbb{Q}_p} K \simeq \prod_{\sigma_0 : L_0 \hookrightarrow K} K$, $\ell \otimes k \mapsto (\sigma_0(\ell)k)_\sigma$ induces an isomorphism:

$$D \simeq \prod_{\sigma_0 : L_0 \hookrightarrow K} D_{\sigma_0}$$

where $D_{\sigma_0} := (0, 0, \cdots, 0, 1_{\sigma_0}, 0, \cdots, 0) \cdot D_L$. The linear map φ^f thus induces a K-linear automorphism on each D_{σ_0} and all the pairs $(D_{\sigma_0}, \varphi^f)$ are isomorphic via some power of φ. We write $(\varphi^f)^{\mathrm{ss}}$ for the semisimple part of φ^f on D. We define:

$$t_N(D) := \frac{1}{[L:\mathbb{Q}_p]} \mathrm{val}_L\big(\det_{L_0}(\varphi^f|_D)\big). \tag{1}$$

Let $D_L := L \otimes_{L_0} D$, as before the isomorphism $L \otimes_{\mathbb{Q}_p} K \simeq \prod_{\sigma : L \hookrightarrow K} K$, $\ell \otimes k \mapsto (\sigma(\ell)k)_\sigma$ induces an isomorphism:

$$D_L \simeq \prod_{\sigma : L \hookrightarrow K} D_{L,\sigma}$$

where $D_{L,\sigma} := (0, 0, \cdots, 0, 1_\sigma, 0, \cdots, 0) \cdot D_L$. To give an $L \otimes_{\mathbb{Q}_p} K$-submodule $\mathrm{Fil} D_L$ of D_L is thus the same thing as to give a collection $(\mathrm{Fil} D_{L,\sigma})_\sigma$ where $\mathrm{Fil} D_{L,\sigma}$ is a K-vector subspace of $D_{L,\sigma}$. If $(\mathrm{Fil}^i D_{L,\sigma})_{i,\sigma}$ is a decreasing exhaustive separated filtration on D_L by $L \otimes_{\mathbb{Q}_p} K$-submodules indexed by $i \in \mathbb{Z}$, we define:

$$t_H(D_L) := \sum_\sigma \sum_{i \in \mathbb{Z}} i \dim_L\big(\mathrm{Fil}^i D_{L,\sigma} / \mathrm{Fil}^{i+1} D_{L,\sigma}\big). \tag{2}$$

Recall that such a filtration is called admissible (one used to say weakly admissible) if $t_H(D_L) = t_N(D)$ and if, for any L_0-vector subspace $D' \subseteq D$ preserved by φ with the induced filtration on D'_L, one has $t_H(D'_L) \leq t_N(D')$.

For $j \in \{1, \cdots, d+1\}$, let:

(3) $$i_{j,\sigma} := -a_{d+2-j,\sigma} - (d+1-j) \in \mathbb{Z}.$$

Note that one has $i_{1,\sigma} < i_{d,\sigma} < \cdots < i_{d+1,\sigma}$.

Proposition 3.2. — *The following conditions are equivalent:*

(i) *there is a Frobenius φ on D such that $(\varphi^f)^{\mathrm{ss}} = F_\zeta$ and an admissible filtration $(\mathrm{Fil}^i D_{L,\sigma})_{i,\sigma}$ on the φ-module (φ, D) such that, $\forall\ \sigma$:*

$$\mathrm{Fil}^i D_{L,\sigma}/\mathrm{Fil}^{i+1} D_{L,\sigma} \neq 0 \Leftrightarrow i \in \{i_{1,\sigma}, \cdots, i_{d+1,\sigma}\};$$

(ii) *the following inequalities hold in $\mathbb{Q}$:*

$$\sum_{j=1}^{i}\sum_{\sigma} i_{j,\sigma} \leq -\sum_{j=d+2-i}^{d+1} \mathrm{val}_L(\zeta_{\tau(j)}),\ 1 \leq i \leq d$$

$$\sum_{j=1}^{d+1}\sum_{\sigma} i_{j,\sigma} = -\sum_{j=1}^{d+1} \mathrm{val}_L(\zeta_{\tau(j)});$$

(iii) *the (Hodge) polygon associated to:*

$$\Big(\sum_\sigma i_{1,\sigma}, \sum_\sigma i_{1,\sigma} + \sum_\sigma i_{2,\sigma}, \cdots, \sum_{j=1}^{d+1}\sum_\sigma i_{j,\sigma}\Big)$$

is under the (Newton) polygon associated to:

$$\Big(-\mathrm{val}_L(\zeta_{\tau(d+1)}), -\mathrm{val}_L(\zeta_{\tau(d+1)}) - \mathrm{val}_L(\zeta_{\tau(d)}), \cdots, -\sum_{j=1}^{d+1} \mathrm{val}_L(\zeta_{\tau(j)})\Big)$$

and both have the same endpoints.

Proof. — (iii) is just a restatement of (ii). Assume (i). By assumption, one can modify τ such that the matrix of φ^f in the basis $(e_{\tau(d+2-i)})_{1\leq i\leq d+1}$ is upper triangular with $(\zeta_{\tau(d+2-i)}^{-1})_i$ on the diagonal. Consider the subspace $D_1 := L_0 \otimes_{\mathbb{Q}_p} K \cdot e_{\tau(d+1)}$ (which is preserved by φ). Viewing D_1 (resp. $D_{1,L}$) as just an L_0-vector space (resp. L-vector space), one has by (2) $\sum_\sigma [K:L] i_{1,\sigma} = [K:L]\sum_\sigma i_{1,\sigma} \leq t_H(D_{1,L})$ and by (1) $t_N(D_1) = [L:\mathbb{Q}_p]^{-1}[K:\mathbb{Q}_p]\mathrm{val}_L(\zeta_{\tau(d+1)}^{-1}) = -[K:L]\mathrm{val}_L(\zeta_{\tau(d+1)})$. The inequality $t_H(D_{1,L}) \leq t_N(D_1)$ then implies the first inequality $\sum_\sigma i_{1,\sigma} \leq -\mathrm{val}_L(\zeta_{\tau(d+1)})$. One can proceed with the subspaces:

$$D_i := L_0 \otimes_{\mathbb{Q}_p} K \cdot e_{\tau(d+1)} \oplus \cdots \oplus L_0 \otimes_{\mathbb{Q}_p} K \cdot e_{\tau(d+2-i)}$$

for $2 \leq i \leq d+1$ (which are all preserved by φ). The inequalities $t_H(D_{i,L}) \leq t_N(D_i)$ for $i \leq d$ imply the intermediate inequalities of (ii) whereas the equality $t_H(D_L) = t_N(D)$ yields the final equality. Assume (ii). Note first that it is enough to check the admissibility conditions for $L_0 \otimes_{\mathbb{Q}_p} K$-submodules preserved by φ (instead of L_0-vector subspaces preserved by φ but not necessarily K): see [**5**, Prop.3.1.1.5]. Let $e_{i,\sigma} := (0, 0, \cdots, 0, 1_\sigma, 0, \cdots, 0) \cdot e_i$ for $\sigma : L \hookrightarrow K$. Define φ

such that, on each piece of D where $(\varphi^f)^{\mathrm{ss}}$ is scalar, say $L_0\otimes_{\mathbb{Q}_p}K\cdot e_{\tau(i)}\oplus\cdots\oplus L_0\otimes_{\mathbb{Q}_p}K\cdot e_{\tau(i-h)}$, φ^f is given by the matrix in the basis $(e_{\tau(i)},\cdots,e_{\tau(i-h)})$:

$$\begin{pmatrix} \zeta_{\tau(i)}^{-1} & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & \zeta_{\tau(i)}^{-1} \end{pmatrix}.$$

Define the filtration such that, for each σ:

$$\begin{aligned} \mathrm{Fil}^i D_{L,\sigma} &:= D_{L,\sigma} \text{ if } i\le i_{1,\sigma} \\ \mathrm{Fil}^i D_{L,\sigma} &:= Kf_{j,\sigma}\oplus\cdots\oplus Kf_{d+1,\sigma} \text{ if } i_{j-1,\sigma}+1\le i\le i_{j,\sigma} \\ \mathrm{Fil}^i D_{L,\sigma} &:= 0 \text{ if } i_{d+1,\sigma}+1\le i \end{aligned}$$

where $f_{j,\sigma}:=e_{\tau(d+2-j),\sigma}+\lambda_{j,j-1}e_{\tau(d+3-j),\sigma}+\cdots+\lambda_{j,1}e_{\tau(d+1),\sigma}$ if $1\le j\le d+1$ and the $\lambda_{j,k}\in K$ are chosen such that all the determinants for $r<j_1<j_2<\cdots<j_r$:

$$\begin{vmatrix} \lambda_{j_1,r} & \cdots & \lambda_{j_1,1} \\ \vdots & \vdots & \vdots \\ \lambda_{j_r,r} & \cdots & \lambda_{j_r,1} \end{vmatrix}$$

are non-zero (which is always generically possible). We then leave it as an exercice to the reader to check that, on each $L_0\otimes_{\mathbb{Q}_p}K$-submodule D' of D preserved by φ, the condition $t_H(D'_L)\le t_N(D')$ is satisfied (as well as $t_H(D_L)=t_N(D)$). □

Corollary 3.3. — *If the locally algebraic representation:*

$$K_{\widehat{\zeta}}\otimes_{\mathcal{H}(G,\rho_U)}\mathrm{ind}_U^G\rho_U$$

admits an invariant norm, then there is a Frobenius φ on D such that $(\varphi^f)^{\mathrm{ss}}=F_\zeta$ and an admissible filtration $(\mathrm{Fil}^iD_{L,\sigma})_{i,\sigma}$ on the φ-module (φ,D) such that, $\forall\ \sigma$, $\mathrm{Fil}^iD_{L,\sigma}/\mathrm{Fil}^{i+1}D_{L,\sigma}\neq 0\Leftrightarrow i\in\{i_{1,\sigma},\cdots,i_{d+1,\sigma}\}$.

Proof. — Use (3) to replace the $a_{j,\sigma}$ by the $i_{j,\sigma}$ in the inequalities of Corollary 3.1 and then use Proposition 3.2. □

Using [**7**], we thus get that the existence of an invariant norm on $K_{\widehat{\zeta}}\otimes_{\mathcal{H}(G,\rho_U)}\mathrm{ind}_U^G\rho_U$ implies the existence of at least one crystalline representation V of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ of dimension $d+1$ over K such that the eigenvalues of φ^f on $D_{\mathrm{cris}}(V):=(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}V)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)}$ are the ζ_j^{-1} and such that the Hodge-Tate weights of V are the $-i_{j,\sigma}$.

4. A general conjecture for de Rham representations

We keep $G = \mathrm{GL}_{d+1}(L)$, $d \geq 1$. The aim of this section is to state a conjecture which significantly generalizes and strengthens Corollary 3.3 (in its statement). We keep the notations of §3.

Let L' be a finite Galois extension of L and L'_0 its maximal unramified subfield. We assume $[L'_0 : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L'_0, K)|$ and we let $p^{f'}$ be the cardinality of the residue field of L'_0 and φ'_0 be the Frobenius on L'_0 (raising to the p each component of the Witt vectors). Consider the following two categories:

(i) the category $\mathrm{WD}_{L'/L}$ of representations (r, N, V) of the Weil-Deligne group of L ([**9**, §8]) on a K-vector space V of finite dimension such that r is unramified when restricted to $\mathrm{W}(\overline{\mathbb{Q}_p}/L')$;

(ii) the category $\mathrm{MOD}_{L'/L}$ of quadruples $(\varphi, N, \mathrm{Gal}(L'/L), D)$ where D is a free $L'_0 \otimes_{\mathbb{Q}_p} K$-modules of finite rank endowed with a Frobenius $\varphi : D \to D$ as in §3, an $L'_0 \otimes_{\mathbb{Q}_p} K$-linear endomorphism $N : D \to D$ such that $N\varphi = p\varphi N$ and an action of $\mathrm{Gal}(L'/L)$ commuting with φ and N such that $g((\ell \otimes k) \cdot d) = (g(\ell) \otimes k) \cdot g(d)$ ($g \in \mathrm{Gal}(L'/L)$, $\ell \in L'_0$, $k \in K$, $d \in D$).

Note that in (ii) N is necessarily nilpotent ([**13**, §1.1.3]).

There is a functor (due to Fontaine):

$$\mathrm{WD} : \mathrm{MOD}_{L'/L} \to \mathrm{WD}_{L'/L}$$

defined as follows (see [**13**]). Choose an embedding $\sigma'_0 : L'_0 \hookrightarrow K$ and let $V := D_{\sigma'_0}$ (with $D_{\sigma'_0}$ as in §3). As N is $L'_0 \otimes_{\mathbb{Q}_p} K$-linear, it induces a nilpotent K-linear endomorphism again denoted $N : V \to V$. For $w \in \mathrm{W}(\overline{\mathbb{Q}_p}/L)$, define $r(w) := \overline{w} \circ \varphi^{-\alpha(w)}$ where $\overline{w}$ is the image of w in $\mathrm{Gal}(L'/L)$ and $\alpha(w) \in f\mathbb{Z}$ is the unique integer such that the image of w in $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ is the $\alpha(w)$-th power of the absolute arithmetic Frobenius. We immediately see that $r(w)$ is $L'_0 \otimes_{\mathbb{Q}_p} K$-linear and thus induces a map again denoted $r(w) : V \to V$. It is not difficult to check that (r, N, V) is an object of $\mathrm{WD}_{L'/L}$. Moreover, up to non-natural isomorphism, the representation (r, N, V) doesn't depend on the choice of σ'_0 (see [**5**, Lem.2.2.1.2]).

Proposition 4.1. — *The functor* $\mathrm{WD} : \mathrm{MOD}_{L'/L} \to \mathrm{WD}_{L'/L}$ *is an equivalence of categories.*

Proof. — We build a quasi-inverse. Let (r, N, V) be an object of $\mathrm{WD}_{L'/L}$. As an $L'_0 \otimes_{\mathbb{Q}_p} K$-module, we take:

$$D := \bigoplus_{n=0}^{f'-1} V_{\sigma'_0 \circ \varphi'_0{}^{-n}}$$

where $V_{\sigma'_0\circ\varphi'_0{}^{-n}} = V$ but with L'_0 acting via $\sigma'_0 \circ \varphi'_0{}^{-n}$. We define $\varphi : D \to D$ by mapping identically $V_{\sigma'_0\circ\varphi'_0{}^{-n}}$ to $V_{\sigma'_0\circ\varphi'_0{}^{-n-1}}$ if $0 \leq n < f' - 2$ and by mapping $V_{\sigma'_0\circ\varphi'_0{}^{1-f}}$ to $V_{\sigma'_0}$ by any geometric Frobenius of $\mathrm{W}(\overline{\mathbb{Q}_p}/L')$. We define $N : V_{\sigma'_0} \to V_{\sigma'_0}$ as the endomorphism N on V and $N : V_{\sigma'_0\circ\varphi'_0{}^{-n}} \to V_{\sigma'_0\circ\varphi'_0{}^{-n}}$ for $1 \leq n \leq f'-1$ as $p^n\varphi^n \circ N \circ \varphi^{-n}$. One checks that $N\varphi = p\varphi N$ on D. For any $g \in \mathrm{Gal}(L'/L)$, let $w \in \mathrm{W}(\overline{\mathbb{Q}_p}/L)$ be a lifting of g and define $g : V_{\sigma'_0\circ\varphi'_0{}^{-n}} \to V_{\sigma'_0\circ\varphi'_0{}^{-n-\alpha(w)}}$ for $0 \leq n \leq f'-1$ as $r(w) \circ \varphi^{\alpha(w)}$ (where $r(w) : V_{\sigma'_0\circ\varphi'_0{}^{-n-\alpha(w)}} \to V_{\sigma'_0\circ\varphi'_0{}^{-n-\alpha(w)}}$ is the action of $w \in \mathrm{W}(\overline{\mathbb{Q}_p}/L)$ on V). As $r\,|_{\mathrm{W}(\overline{\mathbb{Q}_p}/L')}$ is unramified, one checks that this doesn't depend on the choice of the lifting w and that the action of g commutes with φ and N. The final details are left to the reader. □

If D is an object of $\mathrm{MOD}_{L'/L}$, we define similarly to (1):

$$t_N(D) := \frac{1}{[L : L_0]f'}\mathrm{val}_L\big(\det_{L'_0}(\varphi^{f'}|_D)\big). \tag{4}$$

For $\sigma : L \hookrightarrow K$, let:

$$D_{L',\sigma} := D_{L'} \otimes_{L'\otimes_{\mathbb{Q}_p}K} (L' \otimes_{L,\sigma} K).$$

One has again $D_{L'} \simeq \prod_{\sigma:L\hookrightarrow K} D_{L',\sigma}$. To give an $L' \otimes_{\mathbb{Q}_p} K$-submodule $\mathrm{Fil}D_{L'}$ of $D_{L'}$ preserved by $\mathrm{Gal}(L'/L)$ is thus the same thing as to give a collection $(\mathrm{Fil}D_{L',\sigma})_\sigma$ where $\mathrm{Fil}D_{L',\sigma}$ is a free $L' \otimes_{L,\sigma} K$-submodule of $D_{L',\sigma}$ (hence a direct factor as $L'\otimes_{L,\sigma}K$-modules) preserved by the action of $\mathrm{Gal}(L'/L)$. If $(\mathrm{Fil}^iD_{L',\sigma})_{i,\sigma}$ is a decreasing exhaustive separated filtration on $D_{L'}$ by $L' \otimes_{\mathbb{Q}_p} K$-submodules indexed by $i \in \mathbb{Z}$ and preserved by $\mathrm{Gal}(L'/L)$, we define similarly to (2):

$$t_H(D_{L'}) := \sum_\sigma \sum_{i\in\mathbb{Z}} i\dim_{L'}\big(\mathrm{Fil}^iD_{L',\sigma}/\mathrm{Fil}^{i+1}D_{L',\sigma}\big). \tag{5}$$

Recall that such a filtration is called admissible if $t_H(D_L) = t_N(D)$ and if, for any L'_0-vector subspace $D' \subseteq D$ preserved by φ and N with the induced filtration on $D'_{L'}$, one has $t_H(D'_{L'}) \leq t_N(D')$.

Fix a choice of $q^{1/2}$ in $\overline{\mathbb{Q}_p}$. If (r, N, V) is an object of $\mathrm{WD}_{L'/L}$ such that r is semisimple, we denote by π^{unit} the smooth irreducible representation of G over $\overline{\mathbb{Q}_p}$ corresponding to (r, N, V) by the unitary local Langlands correspondence normalized so that the central character of π^{unit} is $\det(r, N, V) \circ \mathrm{rec}^{-1}$. Note that π^{unit} depends on the choice of $q^{1/2}$.

We now modify the unitary local Langlands correspondence as follows.

Assume first that π^{unit} is generic ([**18**, §2.3]). The representation:

$$\pi^{\mathrm{unit}} \otimes_{\overline{\mathbb{Q}_p}} |\det|_L^{-d/2}$$

is the extension of scalars from K to $\overline{\mathbb{Q}_p}$ of a unique irreducible admissible representation over K which doesn't depend any-more on the choice of $q^{1/2}$ (use

[**6**, Conj.4.4] proved in [**14**, §7] and [**6**, Prop.3.2]). Denote by π this irreducible generic representation over K.

Assume now that π^{unit} is not generic. The Langlands classification tells us that π^{unit} is the unique quotient of a normalized parabolic induction:

$$\mathrm{Ind}_Q^G L(b_1,\tau_1)\otimes\cdots\otimes L(b_s,\tau_s) \tag{6}$$

where the τ_i are irreducible supercuspidal representations of $\mathrm{GL}_{n_i}(L)$, the b_i are positive integers, the $L(b_i,\tau_i)$ are the associated generalized Steinberg (same notation as in [**6**, §3.1]) and Q is the upper parabolic subgroup of G of Levi subgroup isomorphic to $\mathrm{GL}_{b_1n_1}(L)\times\cdots\times\mathrm{GL}_{b_sn_s}(L)$. In fact, writing $(r,N,V)=\oplus_i(r_i,N_i,V_i)$ over $\overline{\mathbb{Q}_p}$ where (r_i,N_i,V_i) is indecomposable, $L(b_i,\tau_i)$ corresponds to (r_i,N_i,V_i) by the above unitary local Langlands correspondence. In (6), the $L(b_i,\tau_i)$ are ordered so that the "does not precede" condition of [**18**, Def.1.2.4] holds and the representation (6) doesn't depend on such an order (the proof of this fact is the same as that of [**22**, Prop.6.4] using [**22**, Th.9.7(a)] instead of [**22**, Th.4.2]).

Lemma 4.2. — *The representation:*

$$\big(\mathrm{Ind}_Q^G L(b_1,\tau_1)\otimes\cdots\otimes L(b_s,\tau_s)\big)\otimes_{\overline{\mathbb{Q}_p}}|\det|_L^{-d/2}$$

admits a unique model over K which doesn't depend on the choice of $q^{1/2}$.

Proof. — Let $\mathcal{L}(b_i,\tau_i):=L(b_i,\tau_i)\otimes_{\overline{\mathbb{Q}_p}}|\det|_L^{(1-b_in_i)/2}$, then $\mathcal{L}(b_i,\tau_i)$ doesn't depend on the choice of $q^{1/2}$ and one has:

$$\mathrm{Ind}_Q^G L(b_1,\tau_1)\otimes\cdots\otimes L(b_s,\tau_s)\otimes_{\overline{\mathbb{Q}_p}}|\det|_L^{-d/2} = \mathrm{ind}_Q^G\mathcal{L}(b_1,\tau_1)\otimes\mathcal{L}(b_2,\tau_2)|\det|_L^{-b_1n_1}\otimes\cdots\otimes\mathcal{L}(b_s,\tau_s)|\det|_L^{-\sum_{j=1}^{s-1}b_jn_j} \tag{7}$$

where the induction on the right hand side is an unnormalized parabolic induction (no twist by any modulus). The group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ acts on the set of isomorphism classes of smooth representations of G over $\overline{\mathbb{Q}_p}$ by sending a representation to the class of its twist by an element of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ (see [**6**, §3.1]). As r is a K-representation and as the correspondence $(r,N,V)\mapsto\pi^{\mathrm{unit}}\otimes_{\overline{\mathbb{Q}_p}}|\det|_L^{-d/2}$ commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ ([**14**, §7]), one has that $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ permutes the representations $\mathcal{L}(b_i,\tau_i)$ without changing the values of b_i and n_i. One can then reorder the $\mathcal{L}(b_i,\tau_i)$ in the right hand side of (7) according to the orbits of this action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$ and rewrite the parabolic induction as a step by step parabolic induction, each inducing representation corresponding to an orbit. Using [**22**, Th.9.7], one checks that each inducing representation is irreducible and fixed by $\mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$. By [**6**, Prop.3.2], each such inducing representation is then the extension of scalars to $\overline{\mathbb{Q}_p}$ of a unique model defined over K. Inducing these models, one gets like this a model over K of the representation (7). Finally, using the Schur lemma for the representation (7) (which holds because it is of finite length and has a unique irreducible quotient which occurs with multiplicity 1, see [**22**, §§7-9]), the same proof as in [**6**, Prop.3.2] shows that this model over K is unique. It doesn't depend on the choice of $q^{1/2}$ as no representation in this proof does. □

We call π the unique model over K given by Lemma 4.2.

If (r, N, V) is an object of $\mathrm{WD}_{L'/L}$, we denote by $(r, N, V)^{\mathrm{ss}} \in \mathrm{WD}_{L'/L}$ its F-semisimplification (i.e. the underlying Weil representation is the semisimplification of r, see [**9**, §8.5]).

We now fix:

(i) an object (r, N, V) of $\mathrm{WD}_{L'/L}$ such that r is semisimple;

(ii) for each $\sigma : L \hookrightarrow K$, a list of $d+1$ integers $i_{1,\sigma} < \cdots < i_{d+1,\sigma}$.

From (i), we define as above the smooth admissible representation π. From (ii), we define for $\sigma : L \hookrightarrow K$ and $j \in \{1, \cdots, d+1\}$:

$$a_{j,\sigma} := -i_{d+2-j,\sigma} - (j-1) \tag{8}$$

(note that $a_{1,\sigma} \leq a_{2,\sigma} \leq \cdots \leq a_{d+1,\sigma}$) and we denote by ρ the unique $\mathbb{Q}_p$-rational representation of G over K such that $\widetilde{\rho} = \otimes_\sigma \rho_\sigma$ with ρ_σ of highest weight $(a_{1,\sigma}, \cdots, a_{d+1,\sigma})$ (see §2).

We now state our main conjecture:

Conjecture 4.3. — *Fix (r, N, V) as in (i), $(i_{j,\sigma})_{j,\sigma}$ as in (ii) and define π and ρ as above. The following conditions are equivalent:*

(i) *there is an invariant norm on $\rho \otimes_K \pi$;*

(ii) *there is an object $(\varphi, N, \mathrm{Gal}(L'/L), D)$ of $\mathrm{MOD}_{L'/L}$ such that:*

$$\mathrm{WD}\big(\varphi, N, \mathrm{Gal}(L'/L), D\big)^{\mathrm{ss}} = (r, N, V)$$

and an admissible filtration $(\mathrm{Fil}^i D_{L',\sigma})_{i,\sigma}$ preserved by $\mathrm{Gal}(L'/L)$ on $D_{L'}$ such that, $\forall\ \sigma$:

$$\mathrm{Fil}^i D_{L',\sigma}/\mathrm{Fil}^{i+1} D_{L',\sigma} \neq 0 \Leftrightarrow i \in \{i_{1,\sigma}, \cdots, i_{d+1,\sigma}\}.$$

Using [**7**], this conjecture predicts that the existence of an invariant norm on $\rho \otimes_K \pi$ is *equivalent* to the existence of a potentially semi-stable representation V of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ that has dimension $d+1$ over K, such that its Hodge-Tate weights are the $-i_{j,\sigma}$ and such that the F-semisimplification of its associated Weil-Deligne representation ([**13**, §2.3.7]) has π as Langlands parameter (modified as above).

Remark 4.4. — Note that we *do not* require the stronger statement that $\mathrm{WD}(\varphi, N, \mathrm{Gal}(L'/L), D) = (r, N, V)$ in Conjecture 4.3. Indeed, the conjecture would be false in that case: take $d = 1$, $L = L' = \mathbb{Q}_p$, $N = 0$ and r scalar, then there is no module (φ, D) with an admissible filtration such that $\mathrm{WD}(\varphi, D) = (r, V)$ (as φ is scalar), but, at least for small weights, there is an invariant norm on $\rho \otimes_K \pi$ ([**4**, Th.1.3]).

Remark 4.5. — Replacing L'_0 by the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}}$ of $\mathbb{Q}_p$ and assuming that K is a finite extension of the p-adic completion of $\mathbb{Q}_p^{\mathrm{nr}}$, one can state an equivalence of categories analogous to that of Proposition 4.1 without having to specify L'. However, it doesn't seem to be written in the literature that any admissible filtration preserved by $\mathrm{Gal}(L'/L)$ on $D_{L'}$ still corresponds to a p-adic semi-stable representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ with the right dimension over K. Although it should be probably easy to derive such a proof from [**7**], we have preferred to limit ourselves to the statement as in 4.3, depending on some L'.

Assume for simplicity that the Jordan-Hölder constituents (over $\overline{\mathbb{Q}_p}$) of the Weil representation r are pairwise non-isomorphic, so that there is only one object $(\varphi, N, \mathrm{Gal}(L'/L), D)$ of $\mathrm{MOD}_{L'/L}$ such that $\mathrm{WD}\big(\varphi, N, \mathrm{Gal}(L'/L), D\big)^{\mathrm{ss}} = (r, N, V)$, namely the one given by Proposition 4.1. Then a natural question suggested by Conjecture 4.3 would be to ask if a *given* invariant norm on $\rho \otimes_K \pi$ corresponds to a *specific* admissible filtration on $D_{L'}$, and conversely. This seems (roughly) to hold at least in the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$, but we lack examples so far for more general cases.

5. Partial cases of the conjecture

We keep the notations of §4. We check here several special cases of (weak forms of) Conjecture 4.3. We start with an observation on the central character of $\rho \otimes_K \pi$.

Proposition 5.1. — *We keep the same notations as in Conjecture 4.3. The following conditions are equivalent:*

(i) *there is an invariant norm on the central character of* $\rho \otimes_K \pi$*;*

(ii) *for any (or equivalently one) object* $(\varphi, N, \mathrm{Gal}(L'/L), D)$ *of* $\mathrm{MOD}_{L'/L}$ *such that* $\mathrm{WD}(\varphi, N, \mathrm{Gal}(L'/L), D)^{\mathrm{ss}} = (r, N, V)$ *and any (or equivalently one) filtration* $(\mathrm{Fil}^i D_{L',\sigma})_{i,\sigma}$ *preserved by* $\mathrm{Gal}(L'/L)$ *on* $D_{L'}$ *such that,* $\forall\ \sigma$*:*

$$\mathrm{Fil}^i D_{L',\sigma}/\mathrm{Fil}^{i+1} D_{L',\sigma} \neq 0 \Leftrightarrow i \in \{i_{1,\sigma}, \cdots, i_{d+1,\sigma}\},$$

one has $t_H(D_{L'}) = t_N(D)$*.*

Proof. — Let χ_ρ (resp. χ_π) be the central character of ρ (resp. π). There is an invariant norm on the central character of $\rho \otimes_K \pi$ if and only if:

$$\mathrm{val}_L(\chi_\rho)(\pi_L) + \mathrm{val}_L(\chi_\pi)(\pi_L) = 0. \tag{9}$$

Choose an embedding $\sigma_0' : L_0' \hookrightarrow K$, one has:

$$\begin{aligned}
\mathrm{val}_L(\chi_\rho(\pi_L)) &= \sum_\sigma \sum_{j=1}^{d+1} a_{j,\sigma} \\
&= \sum_{j=1}^{d+1} \sum_\sigma a_{j,\sigma} \\
\mathrm{val}_L(\chi_\pi(\pi_L)) &= -\mathrm{val}_L\big((\det_K(r))(\text{Frob. arith.})\big) + [L:\mathbb{Q}_p]\frac{d(d+1)}{2} \\
&= \frac{f}{f'}\mathrm{val}_L\big(\det_K(\varphi^{f'}|_{D_{\sigma_0'}})\big) + [L:\mathbb{Q}_p]\frac{d(d+1)}{2}.
\end{aligned}$$

On the other hand, for any object $(\varphi, N, \mathrm{Gal}(L'/L), D)$ and any filtration as in the statement, one has by (5) and (4):

$$\begin{aligned}
t_H(D_{L'}) &= \sum_\sigma \sum_{j=1}^{d+1} [K:L] i_{j,\sigma} \\
&\overset{(8)}{=} -[K:L]\sum_{j=1}^{d+1}\sum_\sigma a_{j,\sigma} - [K:\mathbb{Q}_p]\frac{d(d+1)}{2} \\
t_N(D) &= \frac{1}{[L:L_0]f'}\mathrm{val}_L\big(\det_{L_0'}(\varphi^{f'}|_D)\big) \\
&= \frac{[K:\mathbb{Q}_p]}{[L:L_0]f'}\mathrm{val}_L\big(\det_K(\varphi^{f'}|_{D_{\sigma_0'}})\big) \\
&= [K:L]\frac{f}{f'}\mathrm{val}_L\big(\det_K(\varphi^{f'}|_{D_{\sigma_0'}})\big).
\end{aligned}$$

One sees that:

$$t_N(D) - t_H(D_{L'}) = [K:L]\big(\mathrm{val}_L(\chi_\rho)(\pi_L) + \mathrm{val}_L(\chi_\pi)(\pi_L)\big)$$

which implies the proposition by (9). □

We first look at the supercuspidal case of Conjecture 4.3 which turns out to be easy.

Theorem 5.2. — *When r is absolutely irreducible, Conjecture 4.3 is true.*

Proof. — In that case, π is a supercuspidal representation. Hence π, and thus also $\rho \otimes \pi$, can be written as compact inductions from a compact open subgroup modulo the center. It is easily checked that such a compact induction admits an invariant norm if and only if its central character admits an invariant norm.

On the Galois side, it is enough to check the admissibility conditions for $L'_0 \otimes_{\mathbb{Q}_p} K$-submodules of D preserved by φ and $\mathrm{Gal}(L'/L)$ (see [**5**, Prop.3.1.1.5] and [**12**, Prop.4.4.9]). But because of Proposition 4.1 and the assumption, the only non-zero such module is D itself, hence the admissibility conditions fall down to just $t_H(D_{L'}) = t_N(D)$. The result then follows from Proposition 5.1, using $D_{L',\sigma} = L' \otimes_L (D_{L',\sigma})^{\mathrm{Gal}(L'/L)}$ (Hilbert 90) to build free $L' \otimes_{L,\sigma} K$-submodules $\mathrm{Fil}^i D_{L',\sigma} \subseteq D_{L',\sigma}$ preserved by $\mathrm{Gal}(L'/L)$. □

We then turn to the more general case of generalized Steinberg representations.

Proposition 5.3. — *With (r, N, V), $(i_j)_{j,\sigma}$, $\rho \otimes_K \pi$ as in Conjecture 4.3 assume that (r, N, V) is indecomposable over $\overline{\mathbb{Q}_p}$ but not irreducible. Let $(r, N, V) = \mathrm{WD}\big(\varphi, N, \mathrm{Gal}(L'/L), D\big)$ with $(\varphi, N, \mathrm{Gal}(L'/L), D)$ in $\mathrm{MOD}_{L'/L}$. Then the following conditions are equivalent:*

(i) *there is an invariant norm on the central character of $\rho \otimes_K \pi$;*

(ii) *there is an admissible filtration $(\mathrm{Fil}^i D_{L',\sigma})_{i,\sigma}$ preserved by $\mathrm{Gal}(L'/L)$ on $D_{L'}$ such that, $\forall\ \sigma$:*

$$\mathrm{Fil}^i D_{L',\sigma}/\mathrm{Fil}^{i+1} D_{L',\sigma} \neq 0 \Leftrightarrow i \in \{i_{1,\sigma}, \cdots, i_{d+1,\sigma}\}.$$

Proof. — Note first that the condition $\mathrm{WD}\big(\varphi, N, \mathrm{Gal}(L'/L), D\big) = (r, N, V)$ is here equivalent to the condition $\mathrm{WD}\big(\varphi, N, \mathrm{Gal}(L'/L), D\big)^{\mathrm{ss}} = (r, N, V)$. We have already seen in Proposition 5.1 that (ii) implies (i). Assume (i) and let $D_0 := \mathrm{Ker}(N : D \to D)$ which is, by assumption and using Proposition 4.1, a simple object of $\mathrm{MOD}_{L'/L}$ (with the induced actions of φ and $\mathrm{Gal}(L'/L)$). The assumptions on r imply that we necessarily have $D_0 \subsetneq D$. Let $d_0 + 1$ be the rank of D_0 over $L'_0 \otimes_{\mathbb{Q}_p} K$ and $s \in \mathbb{N}$ such that $(s+1)(d_0+1) = d+1$. In $\mathrm{MOD}_{L'/L}$, $(\varphi, N, \mathrm{Gal}(L'/L), D)$ can be described as $D_0 \oplus D_0(1) \oplus \cdots \oplus D_0(s)$ where $D_0(n) := D_0$ but with φ multiplied by p^n (and same action of $\mathrm{Gal}(L'/L)$) and where $N : D \to D$ is 0 on D_0 and sends $D_0(n)$ to $D_0(n-1)$ by the identity map if $n > 0$. The only subobjects of D in $\mathrm{MOD}_{L'/L}$ are thus $D_0 \oplus D_0(1) \oplus \cdots \oplus D_0(n)$ for $0 \leq n \leq s$. Let $(\mathrm{Fil}^i D_{L',\sigma})_{i,\sigma}$ be any decreasing separated exhaustive filtration on $D_{L'}$ preserved by $\mathrm{Gal}(L'/L)$ such that, for $0 \leq j \leq s$ and all $\sigma : L \hookrightarrow K$:

$$\mathrm{Fil}^{i_{j(d_0+1)+1,\sigma}} D_{L',\sigma} := D_0(j)_{L',\sigma} \oplus \cdots \oplus D_0(s)_{L',\sigma}.$$

Then it is not difficult to check that, for $0 \leq i \leq s$:

$$t_N(D_0 \oplus D_0(1) \oplus \cdots \oplus D_0(n)) = [K : L]\Big((n+1)\frac{t_N(D_0)}{[K : L]} + [L : \mathbb{Q}_p](d_0+1)(1 + 2 + \cdots + n)\Big)$$

and that by (5):

$$t_H((D_0 \oplus D_0(1) \oplus \cdots \oplus D_0(n))_{L'}) = [K : L]\Bigg(\sum_{j=1}^{d_0+1} \sum_{\sigma} i_{j,\sigma} + \sum_{j=d_0+2}^{2d_0+2} \sum_{\sigma} i_{j,\sigma} + \cdots \\ \cdots + \sum_{j=n(d_0+1)+1}^{(n+1)(d_0+1)} \sum_{\sigma} i_{j,\sigma} \Bigg).$$

Applying Lemma 5.4 below with:

$$\begin{aligned} i_n &:= \sum_{j=n(d_0+1)+1}^{(n+1)(d_0+1)} \sum_{\sigma} i_{j,\sigma} \\ c &:= [K : L]^{-1} t_N(D_0) \\ h &:= [L : \mathbb{Q}_p](d_0 + 1) \end{aligned}$$

yields the inequalities for $0 \leq n \leq s$:

$$t_H((D_0 \oplus D_0(1) \oplus \cdots \oplus D_0(n))_{L'}) \leq t_N(D_0 \oplus D_0(1) \oplus \cdots \oplus D_0(n))$$

(the last one being an equality) which exactly mean that the filtration $(\mathrm{Fil}^i D_{L',\sigma})_{i,\sigma}$ is admissible. □

Lemma 5.4. — *Let $s \in \mathbb{N}$, $i_0, \cdots, i_s \in \mathbb{Z}$, $h \in \mathbb{Z}$ and $c \in \mathbb{Q}$. Assume that:*

$$i_{n-1} + h \leq i_n \text{ for } 1 \leq n \leq s \tag{10}$$

$$i_0 + \cdots + i_s \leq (s + 1)c + h(1 + 2 + \cdots + s). \tag{11}$$

Then, for $0 \leq n \leq s$, one has the inequalities:

$$i_0 + \cdots + i_n \leq (n + 1)c + h(1 + 2 + \cdots + n).$$

Proof. — From (10), we get:

$$i_0 + \cdots + i_s \geq i_0 + (i_0 + h) + \cdots + (i_0 + sh) = (s + 1)i_0 + h(1 + 2 + \cdots + s).$$

Using (11), we immediately deduce $c \geq i_0$ (case $n = 0$ of the above inequalities). Assume that, for some $n \in \{1, \cdots, s\}$, we have:

$$i_0 + \cdots + i_n > (n + 1)c + h(1 + 2 + \cdots + n) \tag{12}$$

and choose the smallest such n. As $nc + h(1 + 2 + \cdots + n - 1) \geq i_0 + \cdots + i_{n-1}$, we have:

$$i_0 + \cdots + i_n > i_0 + \cdots + i_{n-1} + c + hn,$$

hence $i_n > c + hn$. Using (10), we thus get $i_{n+1} > c + h(n + 1), \cdots, i_s > c + hs$, hence:

$$\begin{aligned} i_{n+1} + \cdots + i_s &> c + h(n + 1) + \cdots + c + hs \\ &= (s - n)c + h(n + 1 + \cdots + s). \end{aligned} \tag{13}$$

Adding (12) and (13), we get:

$$i_0 + \cdots + i_s > (s+1)c + h(1 + 2 + \cdots s)$$

which is in contradiction with (11). □

Combining Proposition 5.3 with Conjecture 4.3, we get in particular the following conjecture:

Conjecture 5.5. — *Let ρ be an irreducible algebraic $\mathbb{Q}_p$-rational representation of G over K and let π be a generalized Steinberg representation of G over K. Then $\rho \otimes_K \pi$ admits an invariant norm if and only if its central character is integral.*

To deduce Conjecture 5.5 from Conjecture 4.3, note that (r, N, V) is indecomposable over $\overline{\mathbb{Q}_p}$ if and only if π is a generalized Steinberg (and thus always generic), that $\rho \otimes_K \pi$ admits an invariant norm if and only if $\rho \otimes_K \pi \otimes_K K'$ does (where K' is a finite extension of $\mathbb{Q}_p$) and then use Proposition 5.3 (replacing K by some K' containing L'_0 if necessary).

Finally, we look at the case of unramified principal series.

Theorem 5.6. — *Assume (r, N, V) is an unramified K-split Weil representation, then (i) implies (ii) in Conjecture 4.3.*

Proof. — By assumption $N = 0$, $L' = L$ and (r, V) sends any arithmetic Frobenius of $\mathrm{W}(\overline{\mathbb{Q}_p}/L)$ to $\mathrm{diag}(\zeta_1, \cdots, \zeta_{d+1})$ for some $(\zeta_1, \cdots, \zeta_{d+1}) \in (K^\times)^{d+1}$. Let $\widehat{\zeta}$ be as in §3. Then, an examination of the proof of [**8**, Lem.3.1] shows that one has $K_{\widehat{\zeta}} \otimes_{\mathcal{H}(G,\rho_U)} \mathrm{ind}_U^G \rho_U$ isomorphic to $\rho \otimes_K \pi$. Hence (i) is equivalent to the existence of an invariant norm on $K_{\widehat{\zeta}} \otimes_{\mathcal{H}(G,\rho_U)} \mathrm{ind}_U^G \rho_U$. Going back through the definition of the functor WD, one sees that the result then exactly follows from Corollary 3.3. □

When (r, N, V) is a Weil representation that is a direct sum of characters, the (ii) $\Rightarrow$ (i) sense in Conjecture 4.3 seems much deeper (even if the characters are unramified). The only known case so far is $d = 1$, $L = \mathbb{Q}_p$ and r non-scalar up to twist ([**1**]).

Remark 5.7. — As in Proposition 3.2, starting from $(r, N, V) \in \mathrm{WD}_{L'/L}$, one can give explicit conditions which are equivalent to the existence of an object $(\varphi, N, \mathrm{Gal}(L'/L), D)$ equipped with an admissible filtration as in Conjecture 4.3, at least for K large enough. Assume one can write:

$$(r, N, V) = \oplus_{i \in \{1, \cdots, s\}} (r_i, N_i, V_i)$$

over K with (r_i, N_i, V_i) absolutely indecomposable and denote simply by D_i the object of $\mathrm{MOD}_{L'/L}$ such that $\mathrm{WD}(D_i) = (r_i, N_i, V_i)$. Let $t_{N,i} := t_N(D_i)$ as in (4) and $d_i := \dim_K(r_i)$. Order the set of representations (r_i, N_i, V_i) so that $t_{N,i}$

increases when i grows, and inside each subset where $t_{N,i}$ is constant so that d_i decreases with i grows. Then, as in Proposition 3.2, one can prove using Lemma 5.4 that there exists an object $(\varphi, N, \mathrm{Gal}(L'/L), D)$ of $\mathrm{MOD}_{L'/L}$ equipped with an admissible filtration as in Conjecture 4.3 if and only if the polygon associated to:

$$\Big(\sum_{j=1}^{d_1}\sum_{\sigma} i_{j,\sigma}, \sum_{j=1}^{d_1+d_2}\sum_{\sigma} i_{j,\sigma}, \cdots, \sum_{j=1}^{d+1}\sum_{\sigma} i_{j,\sigma}\Big)$$

is under the polygon associated to $\big(t_{N,1}, t_{N,1}+t_{N,2}, \cdots, \sum_{j=1}^{s} t_{N,j}\big)$ and both have the same endpoints. Using Conjecture 4.3, this gives a conjectural necessary and sufficient explicit condition for $\rho \otimes_K \pi$ to admit an invariant norm.

6. Towards a p-adic unramified functoriality I

In this and the following section we show that the results of §3 are functorial in rational representations of the Langlands dual group and therefore generalize to arbitrary split groups.

We use the notations and assumptions of §2. In particular $G = \mathbf{G}(L)$ with $\mathbf{G}$ an L-split connected reductive group over L, $\widetilde{\xi} \in X^*(\widetilde{\mathbf{T}})$ is the highest weight of a $\mathbb{Q}_p$-rational representation ρ of G in a K-vector space and $\mathbf{T}'$ is the K-torus dual to $\mathbf{T}$. We assume throughout this section that $q^{1/2} \in K$. By the normalized Satake isomorphism, any point $\zeta \in \mathbf{T}(K)$ gives rise to a K-valued character of the Satake-Hecke algebra $\mathcal{H}(G, \rho_U)$ which we view as a one dimensional module K_ζ for $\mathcal{H}(G, \rho_U)$ as in §3. By specialization we may form the locally algebraic G-representation:

$$H_{\xi,\zeta} := K_\zeta \otimes_{\mathcal{H}(G,\rho_U)} \mathrm{ind}_U^G(\rho_U)$$

which is of finite length and has a unique irreducible quotient $V_{\xi,\zeta}$.

Let us first look at the case $\xi = 1$. Then the G-representation $V_{1,\zeta}$ is smooth. Let $\mathbf{G}'$ be the connected Langlands dual group over K of $\mathbf{G}$. It contains $\mathbf{T}'$ as a maximal K-split torus. Hence our point $\zeta \in \mathbf{T}'(K)$ defines a K-split semisimple conjugacy class in $\mathbf{G}'(K)$ which we may also view as an isomorphism class of unramified homomorphisms $\mathrm{W}(\overline{\mathbb{Q}_p}/L) \longrightarrow \mathbf{G}'(K)$. In the limit over K the correspondence:

$$\zeta \longmapsto V_{1,\zeta}$$

therefore is a manifestation of the unramified local Langlands functoriality principle (compare [**2**, Chap.II & III]).

Going back to the case of a general ξ, let us assume that the point ζ lies in the affinoid subdomain $\mathbf{T}'_{\xi,\mathrm{norm}}$ of $\mathbf{T}'$. Then the corresponding character of $\mathcal{H}(G, \rho_U)$ extends to a (continuous) character of the Banach algebra $\mathcal{B}(G, \rho_U)$ (§2) and we may form, using the completed tensor product, the specialization:

$$B_{\xi,\zeta} := K_\zeta \,\widehat{\otimes}_{\mathcal{B}(G,\rho_U)}\, B_U^G(\rho_U).$$

It is a unitary Banach space representation of G.

Conjecture 6.1. — *The Banach space $B_{\xi,\zeta}$ is non-zero.*

If $G = \mathrm{GL}_{d+1}(L)$ then this conjecture is a special case of Conjecture 4.3 (see the proof of Theorem 5.6).

Following the case $G = \mathrm{GL}_{d+1}(L)$ of §3, we will construct, given a pair (ξ, ζ) with $\zeta \in \mathbf{T}'_{\xi,\mathrm{norm}}(K)$, a family of p-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ with values in $\mathbf{G}'(\overline{K})$. One naive hope would be that this family parametrizes the topologically irreducible quotients of $B_{\xi,\zeta}$ (which, if true, could be seen as a p-adic extension of unramified Langlands functoriality). Note that, for general G, we are obliged to use the *normalized* Satake isomorphism (there is no Tate normalization available).

It is useful to begin in a more general setting. We view $\zeta \in \mathbf{T}'(K) \subseteq \mathbf{G}'(K)$ as a point of the dual group. Via the natural identification $X^*(\widetilde{\mathbf{T}}) = X_*(\widetilde{\mathbf{T}}')$, we view our highest weight $\widetilde{\xi}$ as a rational cocharacter of the K-torus $\widetilde{\mathbf{T}}'$ dual to $\widetilde{\mathbf{T}}$. Obviously, one has $X_*(\widetilde{\mathbf{T}}') \subseteq X_*(\widetilde{\mathbf{G}}')(K)$ where the right hand side denotes the group of K-rational cocharacters of the connected Langlands dual group $\widetilde{\mathbf{G}}'$ of $\widetilde{\mathbf{G}}$. The latter satisfies:

$$\widetilde{\mathbf{G}}' = \prod_{\sigma: L \hookrightarrow K} \mathbf{G}'$$

so that we have:

$$X_*(\widetilde{\mathbf{G}}') = \prod_{\sigma: L \hookrightarrow K} X_*(\mathbf{G}').$$

Hence $\widetilde{\xi}$ gives rise to a family of cocharacters $(\widetilde{\xi}_\sigma)_\sigma$ where $\widetilde{\xi}_\sigma \in X_*(\mathbf{G}')(K)$.

We fix now more generally any pair (ν, b) where $\nu = (\nu_\sigma)_\sigma$ with $\nu_\sigma \in X_*(\mathbf{G}')(K)$ and $b \in \mathbf{G}'(K)$. Let $\mathrm{REP}_K(\mathbf{G}')$ denote the neutral Tannakian category of all K-rational representations of $\mathbf{G}'$ and $\mathrm{FIC}_{L,K}$ the additive tensor category of all filtered isocrystals over L with coefficients in K. An object of the latter is a triple $(D, \varphi, \mathrm{Fil}^\cdot D_L)$ consisting of a free $L_0 \otimes_{\mathbb{Q}_p} K$-module D of finite rank, a φ_0-linear automorphism φ of D and an exhaustive, separated and decreasing filtration $\mathrm{Fil}^\cdot D_L$ on $D_L = L \otimes_{L_0} D$ by $L \otimes_{\mathbb{Q}_p} K$-submodules (see §3). The pair (ν, b) gives rise to an additive tensor functor:

$$I_{(\nu,b)} : \mathrm{REP}_K(\mathbf{G}') \longrightarrow \mathrm{FIC}_{L,K}$$

as follows (depending on the choice of an embedding $\sigma_0 \in \mathrm{Hom}_{\mathbb{Q}_p}(L_0, K)$). Let $\rho' : \mathbf{G}' \longrightarrow \mathrm{GL}(E')$ be a K-rational representation. We put $D := L_0 \otimes_{\mathbb{Q}_p} E'$ and let:

$$D = \prod_{\sigma:L_0\hookrightarrow K} D_\sigma$$

be the L_0-isotypic decomposition where L_0 acts on D_σ through the embedding σ (as in §3). Correspondingly we have the decomposition:

$$1 \otimes \rho'(b) = \prod_{\sigma:L_0\hookrightarrow K} \rho'(b)_\sigma.$$

We now define the Frobenius $\varphi_{\rho'(b)}$ on D by $\varphi_{\rho'(b)} := (\varphi_0 \otimes 1) \circ \varphi'$ where:

$$\varphi'|D_\sigma := \begin{cases} \rho'(b)_{\sigma_0} & \text{if } \sigma = \sigma_0, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

For each cocharacter ν_σ we may decompose E' into its weight spaces:

$$E' = \oplus_{i\in\mathbb{Z}} E'(\rho' \circ \nu_\sigma, i)$$

with respect to the cocharacter $\rho' \circ \nu_\sigma$ and define a filtration on E':

$$\mathrm{Fil}^i_{\rho'\circ\nu_\sigma} E' := \oplus_{j\geq i} E'(\rho' \circ \nu_\sigma, j).$$

Using the L-isotypic decomposition (as in §3):

$$D_L = \prod_{\sigma:L\hookrightarrow K} D_{L,\sigma}$$

and the composed K-linear isomorphisms $E' \longrightarrow D_L \xrightarrow{\mathrm{pr}} D_{L,\sigma}$, we first transport $\mathrm{Fil}^{\cdot}_{\rho'\circ\nu_\sigma} E'$ to a filtration $\mathrm{Fil}^{\cdot}_{\rho'\circ\nu_\sigma} D_{L,\sigma}$ on $D_{L,\sigma}$ and define $\mathrm{Fil}^{\cdot}_{\rho'\circ\nu} D_L := (\mathrm{Fil}^{\cdot}_{\rho'\circ\nu_\sigma} D_{L,\sigma})_\sigma$.

Our functor now is:

$$I_{(\nu,b)}(\rho', E') := (D, \varphi_{\rho'(b)}, \mathrm{Fil}^{\cdot}_{\rho'\circ\nu} D_L).$$

Definition 6.2. — *The pair (ν, b) is called L-admissible if, for any (ρ', E') in $\mathrm{REP}_K(\mathbf{G}')$, the filtered isocrystal $I_{(\nu,b)}(\rho', E')$ is admissible.*

One checks that this definition is independent of the choice of the embedding σ_0 which was used in the construction of the functor $I_{(\nu,b)}$. Suppose that (ν, b) is L-admissible. Then $I_{(\nu,b)}$ can be viewed as a functor:

$$I_{(\nu,b)} : \mathrm{REP}_K(\mathbf{G}') \longrightarrow \mathrm{FIC}^{\mathrm{adm}}_{L,K}$$

into the full subcategory $\mathrm{FIC}^{\mathrm{adm}}_{L,K}$ of admissible filtered isocrystals which again is a tensor category. Moreover, letting $\mathrm{Rep}^{\mathrm{con}}_K(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L))$ denote the category of finite dimensional K-linear continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$, the inverse of the functor D_{cris} induces a faithful tensor functor ([**7**]):

$$\mathrm{FIC}^{\mathrm{adm}}_{L,K} \longrightarrow \mathrm{Rep}^{\mathrm{con}}_K(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)).$$

By composing these two functors, we obtain a faithful tensor functor between K-linear neutral Tannakian categories:

$$\Gamma_{(\nu,b)} : \mathrm{REP}_K(\mathbf{G}') \longrightarrow \mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L))$$

(but which is not compatible with the obvious fiber functors). Nevertheless, as explained in [**19**, §6], by the Tannakian formalism this latter functor gives rise to a continuous homomorphism of groups:

$$\gamma_{\nu,b} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/L) \longrightarrow \mathbf{G}'(\overline{K})$$

which is determined by the pair (ν, b) up to conjugation in $\mathbf{G}'(\overline{K})$. So we see that any L-admissible pair (ν, b) determines an isomorphism class of "Galois parameters" $\gamma_{\nu,b}$.

We assume in this section that η is integral, i.e. lies in $X^*(\mathbf{T})$. Via the obvious diagonal embedding:

$$X^*(\mathbf{T}) = X_*(\mathbf{T}') \hookrightarrow X_*(\widetilde{\mathbf{T}}')$$

we may form the product cocharacter $\widetilde{\xi}\eta = (\widetilde{\xi}_\sigma \eta)_\sigma \in X_*(\widetilde{\mathbf{G}}')(K)$. We have $\mathrm{val}_L \circ \big((\widetilde{\xi}\eta)|_T\big) = \xi_L + \eta_L$.

Theorem 6.3. — *Suppose that η is integral, let $\widetilde{\xi} \in X^*(\widetilde{\mathbf{T}})$ be dominant, and let $\zeta \in \mathbf{T}'(K)$. Then there exists an L-admissible pair (ν, b) (and hence a Galois parameter $\gamma_{\nu,b}$) such that ν lies in the $\widetilde{\mathbf{G}}'(K)$-orbit of $\widetilde{\xi}\eta$ and b has semisimple part ζ if and only if $\zeta \in \mathbf{T}'_{\xi,\mathrm{norm}}(K)$.*

Proof. — This is a straightforward generalization of the proof of [**19**, Prop.6.1]. It might be more in the spirit of that proof and hence helpful to note that the category $\mathrm{FIC}_{L,K}$ can equivalently be described as follows. For any natural number $\ell > 0$, let $\mathrm{F}^\ell\mathrm{IC}_K$ denote the category of ℓ-fold filtered K-isocrystals whose objects are finite dimensional K-vector spaces equipped with a K-linear automorphism and a family of ℓ exhaustive, separated and decreasing filtrations by K-subspaces. Fixing again an embedding $\sigma_0 \in \mathrm{Hom}_{\mathbb{Q}_p}(L_0, K)$, a construction as in the definition of $I_{(\nu,b)}$ establishes an equivalence of categories $\mathrm{FIC}_{L,K} \simeq \mathrm{F}^{[L:\mathbb{Q}_p]}\mathrm{IC}_K$. We leave it to the reader as an exercise to work out the weak admissibility conditions for objects in the category $\mathrm{F}^\ell\mathrm{IC}_K$ (compare Proposition 3.2 and its proof). □

We remark that the element b in the statement of Theorem 6.3 can be chosen to be a regular element in $\mathbf{G}'(K)$ whose semisimple part is ζ. Note that to make this theorem compatible with §3, one has to consider L-admissible pairs (ν, b) such that ν lies in the $\widetilde{\mathbf{G}}'(K)$-orbit of $(\widetilde{\xi}\eta)^{-1}$ and b has semisimple part ζ^{-1}.

7. Towards a p-adic unramified functoriality II

We keep the notations of §6. We would like here to drop the integrality assumption on η in Theorem 6.3. For this, we have to introduce the additive tensor category $\mathrm{FIC}_{L,K,2}$ of 2-filtered isocrystals over L with coefficients in K. These are triples $(D, \varphi, \mathrm{Fil}^{\cdot} D_L)$ defined exactly as before except that the filtration $\mathrm{Fil}^{\cdot} D_L$ is allowed to be indexed by $\frac{1}{2}\mathbb{Z}$. The notion of weak admissibility also extends, with literally the same definition, to the objects in this larger category leading to the full subcategory $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$.

Proposition 7.1. — *The category* $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$ *is K-linear neutral Tannakian.*

Proof. — This is essentially contained in [**21**]. □

Let $\mathbb{D}$ denote the protorus with character group $\mathbb{Q}$. The elements $\nu \in (X_*(\mathbf{G}')\otimes \frac{1}{2}\mathbb{Z})(K)$ can be viewed as K-rational homomorphisms $\nu : \mathbb{D} \longrightarrow \mathbf{G}'$ whose weights in any given K-rational representation of $\mathbf{G}'$ lie in $\frac{1}{2}\mathbb{Z}$. Hence our earlier construction of an additive tensor functor $I_{(\nu,b)}$ makes sense for any pair $(\nu, b) \in (X_*(\widetilde{\mathbf{G}}') \otimes \frac{1}{2}\mathbb{Z})(K) \times \mathbf{G}'(K)$ producing a functor:

$$I_{(\nu,b)} : \mathrm{REP}_K(\mathbf{G}') \longrightarrow \mathrm{FIC}_{L,K,2}.$$

We continue to call the pair (ν, b) L-admissible if this functor has values in $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$. Since $\eta \in X^*(\mathbf{T}) \otimes \frac{1}{2}\mathbb{Z}$, the following variant of Theorem 6.3 holds true with literally the same proof.

Theorem 7.2. — *Let $\widetilde{\xi} \in X^*(\widetilde{\mathbf{T}})$ be dominant, and let $\zeta \in \mathbf{T}'(K)$. Then there exists an L-admissible pair (ν, b) such that ν lies in the $\widetilde{\mathbf{G}}'(K)$-orbit of $\widetilde{\xi}\eta$ and b has semisimple part ζ if and only if $\zeta \in \mathbf{T}'_{\xi,\mathrm{norm}}(K)$.*

For the rest of this section, we fix a pair $(\widetilde{\xi}, \zeta)$ such that $\zeta \in \mathbf{T}'_{\xi,\mathrm{norm}}(K)$ and a pair (ν, b) as in Theorem 7.2. We have the faithful tensor functor:

$$I_{(\nu,b)} : \mathrm{REP}_K(\mathbf{G}') \longrightarrow \mathrm{FIC}^{\mathrm{adm}}_{L,K,2}.$$

Our goal is to associate with (ν, b) similarly as before an isomorphism class of Galois parameters. It is not clear how to relate the category $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$ to the category $\mathrm{Rep}^{\mathrm{con}}_K(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L))$, but the functor $I_{(\nu,b)}$ has values in a particular subcategory of $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$ which will turn out to be related to the category of continuous representations of a group close to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$.

For any embedding $\sigma_0 \in \mathrm{Hom}_{\mathbb{Q}_p}(L_0, K)$, recall that we have the σ_0-tautological fiber functor on $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$ which sends the isocrystal D to its L_0-isotypic component D_{σ_0}. Let $\mathbb{G}$ (resp. $\mathbb{G}_2$) denote the affine K-group scheme of $\otimes$-automorphisms of the σ_0-tautological fiber functor on $\mathrm{FIC}^{\mathrm{adm}}_{L,K}$ (resp. on $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$). The inclusion of Tannakian categories $\mathrm{FIC}_{L,K}^{\mathrm{adm}} \subseteq \mathrm{FIC}_{L,K,2}^{\mathrm{adm}}$ corresponds to a faithfully flat K-rational homomorphism $\mathbb{G}_2 \longrightarrow \mathbb{G}$ ([**10**, Prop.2.21(a)]). Under the functor $I_{(\nu,b)}$ (whose construction involved a choice of σ_0), the tautological fiber functor on $\mathrm{REP}_K(\mathbf{G}')$ corresponds to the σ_0-tautological fiber functor on $\mathrm{FIC}_{L,K,2}^{\mathrm{adm}}$. Hence $I_{(\nu,b)}$ is induced by a homomorphism of K-group schemes:

$$i_{(\nu,b)} : \mathbb{G} \longrightarrow \mathbf{G}'.$$

The homomorphism:

$$\begin{aligned} L^\times &\longrightarrow K^\times \\ a &\longmapsto |a|_L \cdot N_{L/\mathbb{Q}_p}(a) \end{aligned}$$

extends to the profinite completion of $L^\times$ and hence, composed with the reciprocity map of local class field theory, defines a continuous character:

$$\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}_p}/L) \longrightarrow K^\times$$

which is nothing else than the p-adic cyclotomic character restricted to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$. Define the following object $\underline{K}$ in $\mathrm{FIC}_{L,K}^{\mathrm{adm}}$: its underlying module is $L_0 \otimes_{\mathbb{Q}_p} K$, its filtration is:

$$L \otimes_{\mathbb{Q}_p} K = \mathrm{Fil}^{-1}(L \otimes_{\mathbb{Q}_p} K) \supseteq 0 = \mathrm{Fil}^0(L \otimes_{\mathbb{Q}_p} K)$$

and its Frobenius on $L_0 \otimes_{\mathbb{Q}_p} K$ is $\varphi_0 \otimes p^{-1}$. Equivalently, writing $L_0 \otimes_{\mathbb{Q}_p} K = \prod_{\sigma_0} K$ and rescaling, the Frobenius is the circular permutation between the components with multiplication by q^{-1} after one round.

The following lemma is well known:

Lemma 7.3. — *The representation ε and the filtered module $\underline{K}$ correspond to each other under the functor D_{cris}.*

The object $\underline{K}$ in $\mathrm{FIC}_{L,K}^{\mathrm{adm}}$ also corresponds to a K-rational character:

$$\underline{\varepsilon} : \mathbb{G} \longrightarrow \mathbb{G}_m.$$

We now introduce the fiber product $\mathbb{G}_{(2)}$ of affine K-group schemes:

$$\begin{array}{ccc} \mathbb{G}_{(2)} & \longrightarrow & \mathbb{G} \\ {\scriptstyle \underline{\varepsilon}_2}\downarrow & & \downarrow{\scriptstyle \underline{\varepsilon}} \\ \mathbb{G}_m & \xrightarrow{(.)^2} & \mathbb{G}_m \end{array}$$

and note that the horizontal arrows are faithfully flat. In particular $\mathrm{FIC}_{L,K}^{\mathrm{adm}} \simeq \mathrm{REP}_K(\mathbb{G})$ is a full subcategory of $\mathrm{REP}_K(\mathbb{G}_{(2)})$. The kernel of the upper horizontal arrow is central and isomorphic to the group of order two μ_2. Using the decomposition into eigenspaces with respect to μ_2, one concludes that any object M in $\mathrm{REP}_K(\mathbb{G}_{(2)})$ decomposes uniquely as:

$$M = M_0 \oplus (\underline{\varepsilon}_2 \otimes M_1)$$

with objects M_0, M_1 in $\mathrm{REP}_K(\mathbb{G})$.

In $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$, the object $\underline{K}$ has a tensor square root $\underline{K}_2$ defined similarly but with $q^{-1/2}$ instead of q^{-1} and with the filtration $L \otimes_{\mathbb{Q}_p} K = \mathrm{Fil}^{-1/2}(L \otimes_{\mathbb{Q}_p} K) \supseteq 0 = \mathrm{Fil}^0(L \otimes_{\mathbb{Q}_p} K)$. We therefore have a unique K-rational homomorphism:

$$\mathbb{G}_2 \longrightarrow \mathbb{G}_{(2)}$$

such that the composite $\mathbb{G}_2 \longrightarrow \mathbb{G}_{(2)} \xrightarrow{\underline{\varepsilon}_2} \mathbb{G}_m$ classifies $\underline{K}_2$ and the composite $\mathbb{G}_2 \longrightarrow \mathbb{G}_{(2)} \longrightarrow \mathbb{G}$ is the natural map. Let:

$$\mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$$

denote the Tannakian subcategory of $\mathrm{FIC}^{\mathrm{adm}}_{L,K,2}$ generated by $\mathrm{FIC}^{\mathrm{adm}}_{L,K}$ and the object $\underline{K}_2$. By direct inspection, one checks that any object D in $\mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$ decomposes uniquely as:

$$D = D_0 \oplus (\underline{K}_2 \otimes D_1)$$

with objects D_0, D_1 in $\mathrm{FIC}^{\mathrm{adm}}_{L,K}$. It follows that the homomorphism $\mathbb{G}_2 \longrightarrow \mathbb{G}_{(2)}$ is faithfully flat and induces an equivalence of categories $\mathrm{REP}_K(\mathbb{G}_{(2)}) \simeq \mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$ so that $\underline{\varepsilon}_2$ corresponds to $\underline{K}_2$.

Proposition 7.4. — *The homomorphism $i_{(\nu,b)} : \mathbb{G} \longrightarrow \mathbf{G}'$ factorizes through $\mathbb{G}_{(2)}$.*

Proof. — We have to show that $I_{(\nu,b)}((\rho', E'))$ lies in $\mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$ for any (ρ', E') in $\mathrm{REP}_K(\mathbf{G}')$.

Step 1: We first consider the special case where the derived group of $\mathbf{G}$ is simply connected. Then there exists an element $\chi \in X^*(\mathbf{G}) \otimes \frac{1}{2}\mathbb{Z}$ such that $\chi\eta \in X^*(\mathbf{T})$ is integral ([**11**, §8]). We claim that, with (ν, b), also the pair $(\chi\nu, \chi^2(q^{1/2})b)$ is L-admissible. Note that $\chi|\mathbf{T}$ viewed as a $\mathbf{T}'$-valued cocharacter in fact has values in the connected center $\mathbf{Z}' \subseteq \mathbf{T}' \subseteq \mathbf{G}'$ of $\mathbf{G}'$. Hence $\chi\eta \in X_*(\widetilde{\mathbf{G}}')(K)$ via the diagonal embedding and $\chi\nu$ lying in the $\widetilde{\mathbf{G}}'(K)$-orbit of $\widetilde{\xi}\chi\eta \in X_*(\widetilde{\mathbf{G}}')(K)$ is integral as well. Let now $\rho' : \mathbf{G}' \longrightarrow \mathrm{GL}(E')$ be any K-rational representation. By additivity we may assume that ρ' is irreducible. It then follows from Schur's lemma ([**16**, Prop. II.2.8]) that the image of $\rho' \circ \chi$ lies in the center of $\mathrm{GL}(E')$. Hence there is an $n_{E'} \in \frac{1}{2}\mathbb{Z} \subseteq X^*(\mathbb{D})$ such that:

$$\begin{aligned} \rho' \circ \chi : \mathbb{D} &\longrightarrow \mathbb{G}_m = \text{center of } \mathrm{GL}(E') \\ a &\longmapsto a^{n_{E'}} \end{aligned}$$

and:

$$\begin{aligned} \rho' \circ \chi^2 : \mathbb{D} &\longrightarrow \mathbb{G}_m \\ a &\longmapsto a^{2n_{E'}}. \end{aligned}$$

We conclude that:

$$\mathrm{Fil}^{\bullet}_{\rho'\circ(\chi\nu)}(L\otimes_{\mathbb{Q}_p}E')=\mathrm{Fil}^{\bullet-n_{E'}}_{\rho'\circ\nu}(L\otimes_{\mathbb{Q}_p}E')$$

and that $\rho'(\chi^2(q^{1/2})b)=q^{n_{E'}}\rho'(b)$. This obviously means that:

$$I_{(\chi\nu,\chi^2(q^{1/2})b)}(\rho',E')=\underline{K}_2^{\otimes -2n_{E'}}\otimes I_{(\nu,b)}(\rho',E').$$

We in particular obtain that $I_{(\nu,b)}(\rho',E')$ lies in $\mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$.

Step 2: For a general $\mathbf{G}$, we choose some L-split z-extension $f:\mathbf{H}\longrightarrow\mathbf{G}$ of $\mathbf{G}$ ([**17**, Lem.1.1]). This is a surjective L-rational homomorphism of L-split connected reductive groups whose kernel is an L-split torus $\mathbf{S}$ which is central in $\mathbf{H}$ and such that the derived group of $\mathbf{H}$ is simply connected. By functoriality ([**2**, §I.2.5]) we obtain a short exact sequence of Langlands dual groups:

$$1\longrightarrow\mathbf{G}'\xrightarrow{f'}\mathbf{H}'\longrightarrow\mathbf{S}'\longrightarrow 1.$$

Clearly we have the commutative diagram of functors:

$$\begin{array}{ccc}\mathrm{REP}_K(\mathbf{H}') & & \\ & \searrow^{I_{(f'(\nu),f'(b))}} & \\ \downarrow{\scriptstyle \mathrm{res}} & & \mathrm{FIC}_{L,K,2}. \\ & \nearrow_{I_{(\nu,b)}} & \\ \mathrm{REP}_K(\mathbf{G}') & & \end{array}$$

This implies that, with (ν,b), also $(f'(\nu),f'(b))$ is L-admissible and that, in fact, we have the commutative diagram:

$$\begin{array}{ccc}\mathrm{REP}_K(\mathbf{H}') & & \\ & \searrow^{I_{(f'(\nu),f'(b))}} & \\ \downarrow{\scriptstyle \mathrm{res}} & & \mathrm{FIC}^{\mathrm{adm}}_{L,K,2}. \\ & \nearrow_{I_{(\nu,b)}} & \\ \mathrm{REP}_K(\mathbf{G}') & & \end{array}$$

By Step 1, the upper oblique arrow has values in $\mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$. But by the theory of dominant weights the perpendicular restriction functor is surjective on objects. Hence the lower oblique arrow has values in $\mathrm{FIC}^{\mathrm{adm}}_{L,K,(2)}$ as well. □

On the Galois side, we may imitate the construction of $\mathbb{G}_{(2)}$ as follows.

First, we need the *assumption* that any element of $\mathbb{Q}_p^\times$ is a square in $K^\times$. We then introduce the fiber product:

$$\begin{array}{ccc}\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)} & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}_p}/L) \\ \downarrow{\scriptstyle \varepsilon_2} & & \downarrow{\scriptstyle \varepsilon} \\ K^\times & \xrightarrow{(.)^2} & K^\times. \end{array}$$

This produces a central extension (Ext_L) of the form:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}_p}/L) \longrightarrow 1$$

and we have $\varepsilon_2^2 = \varepsilon$.

Lemma 7.5. — *The above extension is split if and only if* $[L : \mathbb{Q}_p]$ *is even.*

Proof. — Extensions of the above form are classified by the Galois cohomology group $H^2(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L), \mathbb{Z}/2\mathbb{Z}) = H^2(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L), \mu_2)$ which, by local class field theory, is isomorphic to the 2-torsion subgroup $\mathrm{Br}(L)_2$ in the Brauer group of L and hence has order two. One easily checks that the specific extension ($\mathrm{Ext}_{\mathbb{Q}_p}$) is non-split and that (Ext_L) is the restriction to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ of ($\mathrm{Ext}_{\mathbb{Q}_p}$). But, again by local class field theory, the restriction map $\mathrm{Br}(\mathbb{Q}_p)_2 \to \mathrm{Br}(L)_2$ is the multiplication by $[L : \mathbb{Q}_p]$ and hence is the zero map if and only if $[L : \mathbb{Q}_p]$ is even. □

Any representation V in $\mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)})$ decomposes into its eigenspaces $V = V_+ \oplus V_-$ with respect to the action of the subgroup $\{\pm 1\}$. Moreover, V_+ and $\varepsilon_2 \otimes V_-$ lie in $\mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L))$. The inverse of D_{cris} therefore extends to a faithful tensor functor:

$$D_{\mathrm{cris}}^{-1} : \mathrm{FIC}_{L,K,(2)}^{\mathrm{adm}} \longrightarrow \mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)})$$

uniquely characterized by the property that it sends $\underline{K}_2$ to ε_2. To give a more intrinsic description, we pick a generator t of $\mathbb{Z}_p(1)$ inside $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ and put:

$$B_{\mathrm{cris},2} := B_{\mathrm{cris}}[X]/(X^2 - t) \subseteq B_{\mathrm{dR},2} := B_{\mathrm{dR}}[X]/(X^2 - t).$$

Let $t^{1/2}$ denote the image of X in $B_{\mathrm{cris},2}$. The Galois action on $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ extends to an action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$ on $B_{\mathrm{cris},2} \otimes_{\mathbb{Q}_p} K \subseteq B_{\mathrm{dR},2} \otimes_{\mathbb{Q}_p} K$ by the requirement that $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$ acts on $t^{1/2}$ through the character ε_2, i.e. $g(t^{1/2} \otimes 1) := t^{1/2} \otimes \varepsilon_2(g)$ for any $g \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$. The Frobenius φ_0 on B_{cris} is extended to $B_{\mathrm{cris},2} \otimes_{\mathbb{Q}_p} K$ by:

$$\varphi_0(t^{1/2} \otimes 1) := t^{1/2} \otimes p^{1/2}.$$

The filtration $\mathrm{Fil}^{\cdot} B_{\mathrm{dR}}$ is extended to a filtration on $B_{\mathrm{dR},2}$ indexed by $\frac{1}{2}\mathbb{Z}$ by the requirement that $t^{1/2} \in \mathrm{Fil}^{1/2} B_{\mathrm{dR},2}$. For any V in $\mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)})$ we define:

$$D_{\mathrm{cris}}(V) := (B_{\mathrm{cris},2} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}} = \big((B_{\mathrm{cris},2} \otimes_{\mathbb{Q}_p} K) \otimes_K V\big)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}}$$

and:

$$D_{\mathrm{dR}}(V) := (B_{\mathrm{dR},2} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}}$$

as the invariants of the respective diagonal action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$. The former is a free $L_0 \otimes_{\mathbb{Q}_p} K$-module of finite rank equipped with a φ_0-linear Frobenius automorphism. The latter is an $L \otimes_{\mathbb{Q}_p} K$-module equipped with a filtration. Via the inclusion $L \otimes_{L_0} D_{\mathrm{cris}}(V) \hookrightarrow D_{\mathrm{dR}}(V)$ this filtration induces a filtration on $D_{\mathrm{cris}}(V)_L$ indexed by $\frac{1}{2}\mathbb{Z}$. Hence we have a functor:

$$D_{\mathrm{cris}} : \mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}) \longrightarrow \mathrm{FIC}_{L,K}.$$

Definition 7.6. — *A representation V in $\mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)})$ is called crystalline if the $L \otimes_{L_0} K$-rank of $D_{\mathrm{cris}}(V)$ is equal to $\dim_K V$.*

Let $\mathrm{Rep}_K^{\mathrm{cris}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)})$ denote the full subcategory of all crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}$. For general V we compute:

$$\begin{aligned} D_{\mathrm{cris}}(V) = D_{\mathrm{cris}}(V_+ \oplus V_-) &= [(B_{\mathrm{cris}} \oplus t^{1/2}B_{\mathrm{cris}}) \otimes_{\mathbb{Q}_p} (V_+ \oplus V_-)]^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}} \\ &= [(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_+) \oplus (t^{1/2}B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_-)]^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)} \\ &= D_{\mathrm{cris}}(V_+) \oplus (\underline{K}_2^* \otimes D_{\mathrm{cris}}(\varepsilon_2 \otimes V_-)). \end{aligned}$$

It follows that V is crystalline if and only if V_+ and $\varepsilon_2 \otimes V_-$ are crystalline in the usual sense. As an immediate consequence of the main result of [**7**] we therefore obtain the following result.

Proposition 7.7. — *The functor D_{cris} restricts to an equivalence of tensor categories:*

$$\mathrm{Rep}_K^{\mathrm{cris}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)}) \xrightarrow{\sim} \mathrm{FIC}_{L,K,(2)}^{\mathrm{adm}}.$$

By Proposition 7.4, the composite:

$$\Gamma_{(\nu,b)} := D_{\mathrm{cris}}^{-1} \circ I_{(\nu,b)} : \mathrm{REP}_K(\mathbf{G}') \longrightarrow \mathrm{Rep}_K^{\mathrm{con}}(\mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)})$$

is well defined as a faithful tensor functor between K-linear neutral Tannakian categories. It gives rise, as before, to an isomorphism class of "Galois parameters":

$$\gamma_{\nu,b} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)_{(2)} \longrightarrow \mathbf{G}'(\overline{K})$$

which is uniquely determined by the L-admissible pair (ν, b).

References

[1] Berger L., Breuil C., *Sur quelques représentations potentiellement cristallines de* $\mathrm{GL}_2(\mathbb{Q}_p)$, preprint (2004, revised 2006), available at http://www.ihes.fr/~breuil/publications.html

[2] Borel A., *Automorphic L-Functions*, Proc. Symp. Pure Math. 33 (1979), American Math. Soc., 27–61.

[3] Bourbaki N., *Groupes et algèbres de Lie (chapitres 4, 5 et 6)*, Masson (1981).

[4] Breuil C., *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbb{Q})$ *II*, J. Institut Math. Jussieu 2 (2003), 23–58.

[5] Breuil C., Mézard A., *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbb{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *en* $\ell = p$, Duke Math. J. 115 (2002), 205–310.

[6] Clozel L., *Motifs et formes automorphes : applications du principe de fonctorialité*, Perspectives in Math. 10 (1990), Academic Press, 77–159.

[7] Colmez P., Fontaine J.-M., *Construction des représentations p-adiques semi-stables*, Inv. Math. 140 (2000), 1–43.

[8] Dat J.-F, *Caractères à valeurs dans le centre de Bernstein*, J. reine angew. Math. 508 (1999), 61–83.

[9] Deligne P., *Les constantes des équations fonctionnelles des fonctions* L, Lecture Notes in Math. 349 (1973), Springer-Verlag, 501–597.

[10] Deligne P., Milne J. S., *Tannakian categories*, Lecture Notes in Math. 900 (1982), Springer, 101–228.

[11] Gross B.H., *On the Satake isomorphism*, London Math. Soc. Lect. Notes 254 (1998), Cambridge Univ. Press, 223–237.

[12] Fontaine J.-M., *Représentations p-adiques semi-stables*, Astérisque 223 (1994), Soc. Math. de France, 113–184.

[13] Fontaine J.-M., *Représentations* ℓ*-adiques potentiellement semi-stables*, Astérisque 223 (1994), Soc. Math. de France, 321–347.

[14] Henniart G., *Sur la conjecture de Langlands locale pour* GL_n, J. Th. des Nombres Bordeaux 13 (2001), 167–187.

[15] Humphreys J.E., *Linear Algebraic Groups*, Springer-Verlag (1987).

[16] Jantzen J.C., *Representations of Algebraic Groups*, Academic Press (1987).

[17] Kottwitz R., *Rational conjugacy classes in reductive groups*, Duke Math. J. 49 (1982), 785–806.

[18] Kudla S., *The local Langlands correspondence: the non-Archimedean case*, Proceedings of Symposia in Pure Math. 55 (1994), American Math. Soc., 365–410.

[19] Schneider P., Teitelbaum J., *Banach-Hecke algebras and p-adic Galois representations*, preprint (2005), available at http://www.uni-muenster.de/math/u/schneider

[20] Steinberg R., *Lectures on Chevalley Groups*, Yale (1967).

[21] Totaro B., *Tensor products in p-adic Hodge theory*, Duke Math. J. 83 (1996), 79–104.

[22] Zelevinsky A., *Induced representations of reductive p-adic groups II*, Ann. Scien. Éc. Norm. Sup. 13 (1980), 165–210.

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