The structure of double groupoids

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Mars 2006

IHES/M/06/13

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Abstract. We give a general description of the structure of a discrete double groupoid (with an extra, quite natural, filling condition) in terms of groupoid factorizations and groupoid 2-cocycles with coefficients in abelian group bundles. Our description goes as follows: in a first step we prove that every double groupoid is obtained as an extension of its pith groupoid, which is an abelian group bundle, by its frame double groupoid. The frame satisfies that every box is determined by its edges, and thus is called a 'thin' double groupoid. In a second, independent, step we prove that every thin double groupoid with filling condition is completely determined by a factorization of a certain canonically defined 'diagonal' groupoid.

Introduction

The main result of this paper is the determination of the structure of a discrete double groupoid -satisfying a natural filling condition- in terms of groupoid data. By 'discrete' we mean here that no additional structure (differential, measurable, topological, etc.) is assumed. The problem of describing all double groupoids in terms of more familiar structures was explicitly raised by Brown and Mackenzie in [BM92, p. 271].

Double groupoids were introduced by Ehresmann [E63] in the early sixties, and later studied by several people because of their connection with different areas of mathematics, such as homotopy theory, differential geometry and Poisson-Lie groups. See for instance [B04, BJ04, BM92, L82, M92, M99, M00, P74, P77] and references therein.

A double groupoid is a groupoid object in the category of groupoids. This can be interpreted as a set of 'boxes' with two groupoid compositions –the *vertical* and *horizontal* compositions–, together with compatible groupoid compositions of the edges, obeying several conditions, in particular and most importantly the so called *interchange law*.

Date: March 21, 2006.

1991 *Mathematics Subject Classification.* 20L05; 18D05.

This work was partially supported by CONICET, Fundación Antorchas, Agencia Córdoba Ciencia, ANPCyT and Secyt (UNC).

The double groupoids we consider satisfy the following *filling condition*: for every configuration of matching edges

$$\begin{array}{c} x \\ \overline{}\!\!\rceil\, g \end{array} , \tag{0.1}$$

there is at least one box

$$\begin{array}{c} x \\ \square \; g \end{array}$$

in the double groupoid, called a 'filling' of (0.1); see (2.2). This condition is often assumed in the case of double groupoids arising in differential geometry, and is discussed by Mackenzie in [M00].

Concerning the structure of double groupoids, some very early results on 'special double groupoids with special connections' were obtained in [BS76]. For more general double groupoids, only those in a few classes were known to be determined by 'groupoid data'. One of them is the class of *vacant* double groupoids (*i.e.* those for which every configuration as in (0.1) has a unique filling): it was proved by Mackenzie in [M92, M00], that vacant double groupoids are essentially the same thing as *exact* factorizations of groupoids. Another one is the class of *transitive* or *locally trivial* double groupoids: in the paper [BM92], R. Brown and Mackenzie show that such a double groupoid is determined by its *core diagram.*

In the paper [AN05] we proved that vacant finite double groupoids gave rise, in a natural way, to a class of tensor categories. Thus the results of [AN05] generalized a well-known construction in Hopf algebra theory studied, among others, by G. I. Kac, Majid and Takeuchi.

Later, in [AN06], this result was extended to the much more general class of finite double groupoids satisfying only the filling condition (2.2). It turns out that double groupoids giving rise through this construction to a special class of tensor categories called *fusion* categories must be thin. We discuss this class of double groupoids in more detail in the last section of the paper. We plan to apply the results of this last section to the determination of the corresponding fusion categories in a subsequent publication.

This paper is organized as follows. In Section 1 we recall some basic facts and constructions on groupoids and their actions on bundles. We also recall here the definition and special features of double groupoids. In Section 2 we define the *pith* and the *frame* of a double groupoid and show that, together with some extra cohomological data, they determine completely the structure of the double groupoid, see Theorem 2.8. The pith groupoid turns out to be an abelian group bundle, while the frame is what we call a *thin* double groupoid; that is, a double groupoid in which every box is uniquely determined by its edges. In Section 3 we prove that thin double groupoids satisfying a natural filling condition are classified by groupoid factorizations. See Theorem 3.7. Finally, in Section 4 we discuss a special class of thin double groupoids motivated by the paper [AN06].

Acknowledgements

Several of the results in this paper were obtained while the authors visited the Institut des Hautes Études Scientifiques (IHES) in France. Both authors are grateful to the Institute for the excellent atmosphere. Part of the work of the first author was done while visiting Hunan University (China); he thanks Prof. Zhang Shouchuan for his warm hospitality. Part of the work of the second author was done while visiting Nanjing Agricultural University (China); she thanks Prof. Zhang Lianyun for his warm hospitality. Both authors thank Prof. Ronnie Brown for interesting discussions and many bibliographical references.

Notation. Along this paper we fix a nonempty set $\mathcal{P}$. A groupoid $\mathcal{G}$ on the base $\mathcal{P}$, with source s and target e, will be indicated by $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ or simply $\mathcal{G}$ if no confusion arises. Composition in $\mathcal{G}$ will be indicated by juxtaposition of arrows; so that if $g, h \in \mathcal{G}$, such that $e(g) = s(h)$, their composition will be denoted by $gh \in \mathcal{G}$. The cardinality of a set S will be indicated by $|S|$.

1. Preliminaries on groupoids

1.1. **Orbits of groupoid actions.**

Let $s, e : \mathcal{G} \rightrightarrows \mathcal{P}$ be a groupoid. Let $P, Q \in \mathcal{P}$. As usual, $\mathcal{G}(P, Q)$ is the set of arrows from P to Q and $\mathcal{G}(Q) = \mathcal{G}(Q, Q)$. Recall the equivalence relation induced by $\mathcal{G}$ on $\mathcal{P}$: $P \sim Q$ if and only if $\mathcal{G}(P, Q) \neq \emptyset$. We denote by $\widetilde{Q}$ the equivalence class of Q. We set

$$\mathcal{G}_{\curvearrowright Q} = \{g \in \mathcal{G} : e(g) = Q\} = \coprod_{P \in \mathcal{P}} \mathcal{G}(P, Q) = \coprod_{P \in \widetilde{Q}} \mathcal{G}(P, Q),$$

the set of arrows with target Q. It is clear from the above that

$$|\mathcal{G}_{\curvearrowright Q}| = |\widetilde{Q}| \times |\mathcal{G}(Q)|.$$

We shall consider the fiber bundle $s : \mathcal{G}_{\curvearrowright Q} \to \mathcal{P}$.

Let K be a subgroup of $\mathcal{G}(Q)$. We define the quotient $\mathcal{G}_{\curvearrowright Q}/K := \mathcal{G}_{\curvearrowright Q}/\equiv_K$, where $\equiv_K$ is the equivalence relation in $\mathcal{G}_{\curvearrowright Q}$ given by

$$g \equiv_K h \iff g^{-1}h \in K.$$

Clearly, the source map descends to the quotient and we can consider the fiber bundle $s : \mathcal{G}_{\curvearrowright Q}/K \to \mathcal{P}$. Hence,

$$|\mathcal{G}_{\curvearrowright Q}/K| = \frac{|\widetilde{Q}| \times |\mathcal{G}(Q)|}{|K|}.$$

Let $p : \mathcal{E} \to \mathcal{P}$ be a fiber bundle. Recall that a *left action* of $\mathcal{G}$ on p is a map $\triangleright : \mathcal{G}\,{}_e\!\times_p \mathcal{E} \to \mathcal{E}$ such that

$$p(g\triangleright x) = s(g), \qquad g\triangleright(h\triangleright x) = gh\triangleright x, \qquad \mathbf{id}\, p(x) \triangleright x = x,$$

for all $g, h \in \mathcal{G}$, $x \in \mathcal{E}$ composable in the appropriate sense. Hence, if $\mathcal{E}_b := p^{-1}(b)$, then the action of $g \in \mathcal{G}$ is an isomorphism $g\triangleright\_\_ : \mathcal{E}_{t(g)} \to \mathcal{E}_{s(g)}$. For example $\mathcal{G}$ acts on $s : \mathcal{G}_{\curvearrowright Q} \to \mathcal{P}$ by left multiplication.

Assume that $\mathcal{G}$ acts on $p : \mathcal{E} \to \mathcal{P}$. Let $x \in \mathcal{E}$ and define

$$\mathcal{O}_x = \{g\triangleright x : g \in \mathcal{G}, e(g) = p(x)\}, \qquad \text{the orbit of } x,$$
$$\mathcal{G}^x = \{g \in \mathcal{G} : g\triangleright x = x\} < \mathcal{G}(p(x)), \qquad \text{the isotropy subgroup of } x.$$

The groupoid $\mathcal{G}$ still acts on the orbit $\mathcal{O}_x$. Then there is an isomorphism of $\mathcal{G}$-fiber bundles $\varphi : \mathcal{G}_{\to p(x)}/\mathcal{G}^x \to \mathcal{O}_x$ induced by $g \mapsto g\triangleright x$. In particular,

$$|\mathcal{O}_x| = \frac{|\widetilde{p(x)}| \times |\mathcal{G}(p(x))|}{|\mathcal{G}^x|}.$$

1.2. **Free product of groupoids.**

In this subsection we work in the category of groupoids over $\mathcal{P}$; morphisms are the identity on $\mathcal{P}$.

We briefly recall the basic properties of the free product construction for groupoids. We refer the reader to [H71] for a detailed exposition.

Let $\mathcal{V}$ and $\mathcal{H}$ be groupoids. Let $\mathcal{V} = \langle X|R\rangle$, $\mathcal{H} = \langle Y|S\rangle$, be presentations of $\mathcal{V}$ and $\mathcal{H}$ by generators and relations [H71, Chapter 9]. Let $\mathcal{V} * \mathcal{H} = \langle X \coprod Y | R \coprod S\rangle$ be the *free product* of the groupoids $\mathcal{V}$ and $\mathcal{H}$; $\mathcal{V} * \mathcal{H}$ is the coproduct of $\mathcal{V}$ and $\mathcal{H}$ in the category of groupoids over $\mathcal{P}$. In other words, the groupoid $\mathcal{V} * \mathcal{H}$ is characterized by the following universal property: for every groupoid $\mathcal{G}$ and groupoid maps $i : \mathcal{H} \to \mathcal{G}$, $j : \mathcal{V} \to \mathcal{G}$, there is a unique morphism of groupoids $f : \mathcal{V} * \mathcal{H} \to \mathcal{G}$ such that $f|_{\mathcal{V}} = j$, and $f|_{\mathcal{H}} = j$. In particular it does not depend on the choice of the presentations of $\mathcal{V}$ and $\mathcal{H}$.

Note that our free product of $\mathcal{V}$ and $\mathcal{H}$ is close to, but not the same as, the free product in [H71, Chapter 9]; precisely, it is the free product with amalgamation of identities from *loc. cit.*

An alternative way of describing the free product is the following. Consider the set $\operatorname{Path}(\mathcal{Q})$ of all paths of the quiver $\mathcal{Q} = \mathcal{H} \coprod \mathcal{V}$. An element in $\operatorname{Path}(\mathcal{Q})$ is either a an element $P \in \mathcal{P}$ that will be indicated by $[P]$, or a sequence $(u_1, \dots, u_n)$, $n \geq 1$, with $u_i \in \mathcal{Q}$, $e(u_i) = s(u_{i+1})$.

A path $U \in \operatorname{Path}(\mathcal{Q})$ is called *reduced* if either $U = [P]$, $P \in \mathcal{P}$, or $U = (u_1, \dots, u_n)$, $n \geq 1$, $u_i \in \mathcal{Q}$, and the following conditions hold:

- no u_i is an identity arrow,
- u_i and u_{i+1} do not belong to the same groupoid $\mathcal{H}$ or $\mathcal{V}$.

For instance, the horizontal identity $\operatorname{id}_{\mathcal{H}} P \in \mathcal{H}$, $P \in \mathcal{P}$, is a path which is not reduced.

Every path $(u_1, u_2, \dots, u_n)$, with $n > 0$, can be transformed into a reduced path by means of a finite number of reductions, that is, operations of one of the following types:

- removing u_i if u_i is an identity arrow and $n > 1$,
- replacing (u_1) by $[P]$ if $u_1 = \mathrm{id}_{\mathcal{H}} P$ or $\mathrm{id}_{\mathcal{V}} P$, $P \in \mathcal{P}$,
- replacing $(u_1, \dots, u_i, u_{i+1}, \dots, u_n)$ by $(u_1, \dots, u_i u_{i+1}, \dots, u_n)$, if u_i and u_{i+1} belong to the same groupoid $\mathcal{H}$ or $\mathcal{V}$.

These two operations generate an equivalence relation in $\mathrm{Path}(\mathcal{Q})$. Following the lines of the proof of [H71, Theorem 5, Chapter 11], it is possible to see that in any equivalence class there is a unique reduced path.

The set of all reduced paths on $\mathcal{Q}$ forms a groupoid under the operation of concatenation followed by reduction. Compare also with the analogous construction for groups [S82, p. 186].

Also, the set of all reduced paths on $\mathcal{Q}$ with this product is isomorphic to $\mathcal{V} * \mathcal{H}$. Clearly $\mathcal{V} * \mathcal{H}$ contains both $\mathcal{V}$ and $\mathcal{H}$ as wide subgroupoids. In conclusion, any element u of $\mathcal{V} * \mathcal{H}$ has a unique standard form, namely:

- $u \in \mathcal{P}$ (elements of length 0), or
- $u = u_1 u_2 \dots u_n$, where the u_i's belong alternatively to different groupoids $\mathcal{V}$ or $\mathcal{H}$, no u_i is an identity (elements of length $n > 0$).

In such case we shall say that u_1, respectively u_n, is the *first*, respectively the *last*, letter of u.

Lemma 1.1. *Let $p = p_1 \dots p_N$, $q = q_1 \dots q_M$ be reduced paths in $\mathcal{V} * \mathcal{H}$ of lengths N and M respectively.*

(i). If p_N and q_1 belong to different groupoids $\mathcal{V}$ or $\mathcal{H}$, then length $(pq) = N + M$.

(ii). If p_N and q_1 belong to the same groupoid $\mathcal{V}$ or $\mathcal{H}$, but $p_N \neq (q_1)^{-1}$ then length $(pq) = N + M - 1$.

(iii). If p_N and q_1 belong to the same groupoid $\mathcal{V}$ or $\mathcal{H}$, $p_N = (q_1)^{-1}$ but $p_{N-1} \neq (q_2)^{-1}$ then length $(pq) = N + M - 2$. □

2. Double groupoids

Let $\mathcal{B}$ be a *double groupoid* [E63, BS76]; we follow the conventions and notations from [AN05, Section 2] and [AN06, Section 1]. As usual, we represent $\mathcal{B}$ in the form of four related groupoids

$$\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$$

subject to a set of axioms. The source and target maps of these groupoids are indicated by $t, b : \mathcal{B} \to \mathcal{H}$; $r, l : \mathcal{B} \to \mathcal{V}$; $r, l : \mathcal{H} \to \mathcal{P}$; $t, b : \mathcal{V} \to \mathcal{P}$

('top', 'bottom', 'right' and 'left'). An element $A \in \mathcal{B}$ is depicted as a box

$$A = l \underset{b}{\overset{t}{\square}} r$$

where $t(A) = t$, $b(A) = b$, $r(A) = r$, $l(A) = l$, and the four vertices of the square representing A are $tl(A) = lt(A)$, $tr(A) = rt(A)$, $bl(A) = lb(A)$, $br(A) = rb(A)$. The notation $A|B$ means that $r(A) = l(B)$ (A and B are horizontally composable); the corresponding horizontal product is denoted AB. Similarly, $\frac{A}{B}$ means that $b(A) = t(B)$ (A and B are vertically composable) and the vertical product is denoted $\genfrac{}{}{0pt}{}{A}{B}$.

The notation $A = \overline{\square}$ means that $t(A)$ is an identity; analogously, $B = \|\square$ means that $l(B)$ is an identity, etc.

These four groupoids should satisfy certain axioms, see e. g. [AN05]. In particular, $\mathbf{id}\,\mathrm{id}_{\mathcal{H}}\, P = \mathbf{id}\,\mathrm{id}_{\mathcal{V}}\, P$, for any $P \in \mathcal{P}$; this box is denoted Θ_P and clearly it is of the form $\boxed{\square}$.

2.1. Core groupoids.

We recall the core groupoid $\mathbf{E}$ of $\mathcal{B}$ introduced by Brown and Mackenzie [M00, BM92]. See also [AN06]. Let

$$\mathbf{E} := \{E \in \mathcal{B} : \; r(E), t(E) \in \mathcal{P}\}.$$

Thus, elements of $\mathbf{E}$ are of the form $\overline{\square}\|$. There is a groupoid structure $s, e : \mathbf{E} \rightrightarrows \mathcal{P}$, $s(E) = bl(E)$, $e(E) = br(E)$, $E \in \mathbf{E}$, identity map $\mathrm{id} : \mathcal{P} \to \mathbf{E}$, $P \mapsto \Theta_P$, and composition $\mathbf{E}\,{}_e\!\times_s \mathbf{E} \to \mathbf{E}$, given by

$$E \circ M := \left\{ \genfrac{}{}{0pt}{}{\mathbf{id} l(M)}{E} \quad \genfrac{}{}{0pt}{}{M}{\mathbf{id} b(M)} \right\}. \tag{2.1}$$

$M, E \in \mathbf{E}$. The inverse of $E \in \mathbf{E}$ is $E^{(-1)} := (E\mathbf{id}b(E)^{-1})^v = \left\{ \genfrac{}{}{0pt}{}{\mathbf{id}l(E)^{-1}}{E^h} \right\}$.

If $Q \in \mathcal{P}$, the group $\mathbf{E}(Q)$ consists of boxes in $\mathcal{B}$ of the form $h\underset{y}{\overline{\square}\|}$, with $y \in \mathcal{H}(Q)$, $h \in \mathcal{V}(Q)$.

2.2. Corner functions.

Let $\mathcal{B}$ be a finite double groupoid. We discuss the 'corner' functions defined in [AN06]. There are four of them but it is enough to consider one. If $(x, g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}$ and $B \in \mathcal{B}$ then we set

$$\mathfrak{UR}(x, g) = \left\{ U \in \mathcal{B} : \; U = \overset{x}{\square}\, g \right\}, \quad \mathfrak{UR}(B) = \mathfrak{UR}(t(B), r(B)).$$

Let $\urcorner : \mathcal{H}\,{}_r\!\times_t \mathcal{V} \to \mathbb{N} \cup \{0\}$ and $\urcorner : \mathcal{B} \to \mathbb{N}$ be given by $\urcorner(x,g) = |\mathfrak{UR}(x,g)|$ and $\urcorner(B) = |\mathfrak{UR}(B)|$. The other corner functions are defined similarly.

Recall that the *filling condition* on $\mathcal{B}$ is

$$\urcorner(x,g) > 0 \quad \text{for any } (x,g) \in \mathcal{H}\,{}_r\!\times_t \mathcal{V}. \tag{2.2}$$

We now interpret the corner functions in terms of orbits of an action of the core groupoid. Let $\gamma : \mathcal{B} \to \mathcal{P}$ be the 'left-bottom' vertex, $\gamma(B) = lb(B)$.

For the next result the finiteness of $\mathcal{B}$ is not required.

Proposition 2.1. *(a). There is an action of* $\mathbf{E}$ *on* $\gamma : \mathcal{B} \to \mathcal{P}$ *given by*

$$E \rightharpoondown A := \begin{Bmatrix} \mathbf{id}l(A) & A \\ E & \mathbf{id}b(A) \end{Bmatrix}, \quad A \in \mathcal{B}, E \in \mathbf{E}. \tag{2.3}$$

(b). Let $B \in \mathcal{B}$*. Then* $\mathcal{O}_B = \mathfrak{UR}(B)$ *and* $\mathbf{E}^B$ *is trivial.*

Proof. (a) is straightforward. (b): it follows from Definition (2.3) that $\mathcal{O}_B \subseteq \mathfrak{UR}(B)$. Then observe that for any $C \in \mathfrak{UR}(B)$, there exists a unique $E \in \mathbf{E}$ such that $E \rightharpoondown B = C$, namely $E = \begin{Bmatrix} B^v \\ C \end{Bmatrix} \operatorname{id} b(B)^{-1}$. $\square$

Let $B \in \mathcal{B}$ and $Q = \gamma(B) = bl(B)$. Consider the relation $\sim$ on $\mathcal{P}$ induced by $\mathbf{E}$. Recall that $\theta(Q)$ is the common value

$$\lrcorner(\operatorname{id}_{\mathcal{V}} Q, \operatorname{id}_{\mathcal{H}} Q) = \urcorner(\operatorname{id}_{\mathcal{V}} Q, \operatorname{id}_{\mathcal{H}} Q) = \llcorner(\operatorname{id}_{\mathcal{V}} Q, \operatorname{id}_{\mathcal{H}} Q) = \ulcorner(\operatorname{id}_{\mathcal{V}} Q, \operatorname{id}_{\mathcal{H}} Q).$$

The preceding proposition, together with the discussion in Subsection 1.1, implies the following formula for the corner function:

$$\urcorner(B) = |\widetilde{Q}| \times |\mathbf{E}(Q)|. \tag{2.4}$$

Applied to $B = \Theta_Q$, the formula implies that

$$\theta(Q) = \urcorner(\Theta_Q) = |\widetilde{Q}| \times |\mathbf{E}(Q)| = \urcorner(B).$$

Hence $\theta(Q)$ is also given by (2.4). That is, the corner functions on a box depend only on the vertex 'opposite' to the corner of that box. Formula (2.4) provides easy alternative proofs of the following facts– see [AN06]:

(a) Let $P, Q \in \mathcal{P}$. If $P \sim Q$, then $\theta(P) = \theta(Q)$.

(b) Let $L, M, N \in \mathcal{B}$. Suppose that $\begin{array}{c|c} L & M \\ \hline N & \end{array}$. Then

$$\ulcorner(L) = \urcorner(M), \quad \llcorner(L) = \lrcorner(M), \quad \ulcorner(L) = \llcorner(N), \quad \urcorner(L) = \lrcorner(N).$$

(c) Let $X, Y, Z \in \mathcal{B}$ such that $\begin{array}{c|c} X & Y \\ \hline Z & \end{array}$. Then

$$\urcorner(XY) = \urcorner(X), \quad \urcorner\begin{pmatrix} X \\ Z \end{pmatrix} = \urcorner(Z), \quad \llcorner(XY) = \llcorner(Y), \quad \llcorner\begin{pmatrix} X \\ Z \end{pmatrix} = \llcorner(X).$$

(d) The double groupoid is vacant if and only if the core groupoid is trivial.

2.3. **Pith groupoids.**

The intersection $\mathbf{K}$ of all four core groupoids is the *pith* groupoid of $\mathcal{B}$:

$$\mathbf{K} := \{K \in \mathcal{B} :\ t(K), b(K), l(K), r(K) \in \mathcal{P}\}.$$

Thus a box is in $\mathbf{K}$ if and only if it is of the form $\boxed{\boxed{}}$. Let $p : \mathbf{K} \to \mathcal{P}$ be the 'common vertex' function, say $p(K) = lb(K)$. For any $P \in \mathbf{K}$, let $\mathbf{K}(P)$ be the fiber at P; $\mathbf{K}(P)$ is an abelian group under vertical composition, that coincides with horizontal composition. This is just the well-known fact: "a double group is the same as an abelian group". Indeed, apply the interchange law

$$\begin{matrix}(KL)\\(MN)\end{matrix} = \begin{pmatrix}K\\M\end{pmatrix}\begin{pmatrix}L\\N\end{pmatrix}$$

to four boxes $K, L, M, N \in \mathbf{K}(P)$: if $L = M = \Theta_P$, this says that $\begin{matrix}K\\N\end{matrix} = KN$ and the two operations coincide. If, instead, $K = N = \Theta_P$, this says that $\begin{matrix}L\\M\end{matrix} = ML$, hence the composition is abelian. Note that this operation in $\mathbf{K}$ coincides also with the core multiplication (2.1). In short, $\mathbf{K}$ is an abelian group bundle over $\mathcal{P}$.

The vertical and the horizontal groupoids $\mathcal{V}$ and $\mathcal{H}$ act on $\mathbf{K}$ by vertical, respectively horizontal, conjugation:

(2.5) If $g \in \mathcal{V}(Q, P)$ and $K \in \mathbf{K}(P)$ then $g \cdot K := \begin{matrix}\operatorname{id} g\\ K\\ \operatorname{id} g^{-1}\end{matrix} \in \mathbf{K}(Q)$.

(2.6) If $x \in \mathcal{H}(Q, P)$ and $K \in \mathbf{K}(P)$ then $x \cdot K = \operatorname{id} x K \operatorname{id} x^{-1} \in \mathbf{K}(Q)$.

Both actions are by group bundle automorphisms.

2.4. **Frame of a double groupoid.**

Let $\mathcal{P}$ be a set and $\mathcal{V}$, $\mathcal{H}$ be groupoids over $\mathcal{P}$ denoted vertically and horizontally, respectively. Let $\square(\mathcal{V}, \mathcal{H})$ be the set of quadruples $\begin{pmatrix} & x & \\ f & & g \\ & y & \end{pmatrix}$ with $x, y \in \mathcal{H}$, $f, g \in \mathcal{V}$ such that

$$l(x) = t(f), \quad r(x) = t(g), \quad l(y) = b(f), \quad r(y) = b(g).$$

If no confusion arises, we shall denote a quadruple as above by a box $h \overset{x}{\underset{y}{\square}} g$.

The collection $\begin{matrix} \square(\mathcal{V},\mathcal{H}) & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ forms a double groupoid, called the *coarse double groupoid* with sides in $\mathcal{V}$ and $\mathcal{H}$, with horizontal and vertical compositions given by

$$h \underset{y}{\overset{x}{\square}} g \; g \underset{y'}{\overset{x'}{\square}} g' = h \underset{yy'}{\overset{xx'}{\square}} g', \qquad \begin{matrix} h \underset{y}{\overset{x}{\square}} g \\ h' \underset{y'}{\overset{y}{\square}} g' \end{matrix} = hh' \underset{y'}{\overset{x}{\square}} gg',$$

for all $x, y, x', y' \in \mathcal{H}$, $g, h, g', h' \in \mathcal{V}$ appropriately composable.

Let $\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ be a double groupoid. There is a map $\Pi : \mathcal{B} \to \square(\mathcal{V},\mathcal{H})$ given by

$$\Pi\left(f \underset{y}{\overset{x}{\square}} g \right) = \left(f \underset{y}{\overset{x}{}} g \right), \qquad f \underset{y}{\overset{x}{\square}} g \in \mathcal{B}.$$

Clearly, Π induces a morphism of double groupoids $\mathcal{B} \to \square(\mathcal{V},\mathcal{H})$.

Definition 2.2. We shall say that $\mathcal{B}$ is *thin* if Π is injective (any box is determined by its sides).

Let $\mathcal{F}$ be the image of Π. The *frame* of $\mathcal{B}$ is the thin double groupoid

$$\begin{matrix} \mathcal{F} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows . \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$$

Let $\mathcal{B}$ be a double groupoid. Several properties on $\mathcal{B}$ are controlled by its frame $\mathcal{F}$. Recall the following definitions [BM92, Definition 2.3].

(a) $\mathcal{B}$ is *horizontally transitive* if every configuration of matching sides $\sqcup$ can be completed to at least one box in $\mathcal{B}$.
(b) $\mathcal{B}$ is *vertically transitive* if every configuration of matching sides $\urcorner$ can be completed to at least one box in $\mathcal{B}$.
(c) $\mathcal{B}$ is *transitive* or *locally trivial* if it is both vertically and horizontally transitive.

Remark 2.3. Let $\mathcal{B}$ be a double groupoid. Then

(i) $\mathcal{B}$ satisfies the filling condition (2.2) if and only if so does $\mathcal{F}$.

(ii) $\mathcal{B}$ is horizontally (vertically) transitive if and only if so is $\mathcal{F}$.

(iii) If $\mathcal{B}$ is vacant then it is thin.

Our aim is to show that $\mathcal{B}$ is determined as an extension of its frame double groupoid by its pith groupoid. Let us fix a section $\mu : \mathcal{F} \to \mathcal{B}$ of Π. Recall the action (2.3) of the core groupoid.

Lemma 2.4. *For all $B \in \mathfrak{LR}(\mu(F))$, there is a unique $K \in \mathbf{K}$ such that $B = K \rightharpoondown \mu(F)$. In other words, the map $\Psi : \mathbf{K}\, {}_p\!\times_\gamma \mathcal{F} \to \mathcal{B}$ given by $\Psi(K, F) = K \rightharpoondown \mu(F)$ is a bijection.*

Proof. Note that $\Pi(K \rightharpoondown \mu(F)) = F$. Hence, $K \rightharpoondown \mu(F) = K' \rightharpoondown \mu(F')$ implies $F = F'$; thus $K \rightharpoondown \mu(F) = K' \rightharpoondown \mu(F)$, and $K = K'$ by Proposition 2.1 (b). That is, Ψ is injective. We show that it is surjective. Let $B \in \mathcal{B}$ and let $F = \Pi(B)$. Since B and $\mu(F)$ have the same sides, there exists $K \in \mathbf{E}$ such that $B = K \rightharpoondown \mu(F)$, again by Proposition 2.1. But clearly $K \in \mathbf{K}$. $\square$

We next introduce vertical and horizontal cocycles to control the lack of multiplicativity of the section μ. We define $\tau : \mathcal{F}\, {}_r\!\times_l \mathcal{F} \to \mathbf{K}$ and $\sigma : \mathcal{F}\, {}_b\!\times_t \mathcal{F} \to \mathbf{K}$ by

$$\mu(F)\mu(G) = \tau(F,G) \rightharpoondown \mu(FG), \qquad r(F) = l(G), \tag{2.7}$$

$$\begin{matrix}\mu(F)\\ \mu(G)\end{matrix} = \sigma(F,G) \rightharpoondown \mu\begin{pmatrix}F\\ G\end{pmatrix}, \qquad b(F) = t(G). \tag{2.8}$$

That is,

$$\mu(F)\mu(G) = \begin{matrix}\operatorname{id} l(F) & \mu(FG)\\ \tau(F,G) & \operatorname{id} b(FG)\end{matrix},$$

$$\begin{matrix}\mu(F)\\ \mu(G)\end{matrix} = \begin{matrix}\operatorname{id} l(F)l(G) & \mu\begin{pmatrix}F\\ G\end{pmatrix}\\ \sigma(F,G) & \operatorname{id} b(G)\end{matrix},$$

for appropriate F, $G \in \mathcal{F}$. The cocycles σ and τ are well-defined in virtue of Lemma 2.4. If we assume that $\mu(\operatorname{id} x) = \operatorname{id} x$ and $\mu(\operatorname{id} g) = \operatorname{id} g$ for any $x \in \mathcal{H}$ and $g \in \mathcal{V}$ then σ and τ are normalized:

$$\tau(F, \operatorname{id} r(F)) = \Theta_{bl(F)} = \tau(\operatorname{id} l(F), F), \tag{2.9}$$

$$\sigma(F, \operatorname{id} b(F)) = \Theta_{bl(F)} = \sigma(\operatorname{id} t(F), F). \tag{2.10}$$

Now we can reconstruct the horizontal and vertical products of $\mathcal{B}$ in terms of the pith groupoid $\mathbf{K}$, the frame thin double groupoid $\mathcal{F}$, the actions (2.6), (2.5) and the cocycles σ and τ. If $K, L \in \mathbf{K}$, $F, G \in \mathcal{F}$ then

$$\big(K \rightharpoondown \mu(F)\big)\big(L \rightharpoondown \mu(G)\big) = \big(K(b(F)\cdot L)\tau(F,G)\big) \rightharpoondown \mu(FG), \tag{2.11}$$

if $r(F) = l(G)$ and

$$\begin{Bmatrix}K \rightharpoondown \mu(F)\\ L \rightharpoondown \mu(G)\end{Bmatrix} = \big((l(g)^{-1}\cdot K)\, L\, \sigma(F,G)\big) \rightharpoondown \mu\begin{pmatrix}F\\ G\end{pmatrix} \tag{2.12}$$

if $b(F) = t(G)$.

2.5. **Extensions of double groupoids by abelian group bundles.** The description in the preceding subsection suggests the following construction. Let $\begin{matrix} \mathcal{F} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ be any double groupoid (not necessarily thin) and let $\gamma : \mathbf{K} \to \mathcal{P}$ be any abelian group bundle. Assume that $\mathcal{V}$ and $\mathcal{H}$ act on $\mathbf{K}$ by group bundle isomorphisms. Let $\tau : \mathcal{F}\, {}_r\!\times_l \mathcal{F} \to \mathbf{K}$ and $\sigma : \mathcal{F}\, {}_b\!\times_t \mathcal{F} \to \mathbf{K}$ be maps such that

$$\gamma(\sigma(F,G)) = bl(G), \qquad \text{if } b(F) = t(G), \tag{2.13}$$

$$\gamma(\tau(F,G)) = bl(F), \qquad \text{if } r(F) = l(G), \tag{2.14}$$

normalized by (2.9) and (2.10). Consider the collection $\begin{matrix} \mathbf{K}\, {}_p\!\times_\gamma \mathcal{F} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$, where:

- The maps t, b, l, r on $\mathbf{K}\, {}_p\!\times_\gamma \mathcal{F}$ are defined by those in $\mathcal{F}$: $t(K,F) = t(F)$ and so on.
- The horizontal and vertical products in $\mathbf{K}\, {}_p\!\times_\gamma \mathcal{F}$ are given by

$$(K,F)(L,G) = \big(K\,(b(F)\cdot L)\,\tau(F,G), FG\big), \qquad \text{if } F|G, \tag{2.15}$$

$$\frac{(K,F)}{(L,G)} = \left((l(G)^{-1}\cdot K)\,L\,\sigma(F,G), \frac{F}{G}\right), \qquad \text{if } \frac{F}{G}. \tag{2.16}$$

- The identity maps $\mathrm{id} : \mathcal{V} \to \mathbf{K}\, {}_p\!\times_\gamma \mathcal{F}$, $\mathrm{id} : \mathcal{H} \to \mathbf{K}\, {}_p\!\times_\gamma \mathcal{F}$ are given by $\mathrm{id}\, g = (\Theta_{b(g)}, \mathrm{id}\, g)$, $\mathrm{id}\, x = (\Theta_{l(x)}, \mathrm{id}\, x)$, $g \in \mathcal{V}$, $x \in \mathcal{H}$.
- The inverse of (K,F) with respect to the horizontal and vertical products are respectively given by

$$(K,F)^h = \left(b(F)^{-1}\cdot\left(K^{-1}\tau(F,F^h)^{-1}\right), F^h\right), \tag{2.17}$$

$$(K,F)^v = \left((l(F)\cdot K)^{-1}\,\sigma(F,F^v)^{-1}, F^v\right). \tag{2.18}$$

Proposition 2.5. *$\mathbf{K}\, {}_p\!\times_\gamma \mathcal{F}$ is a double groupoid if and only if, for all $F, G, H \in \mathcal{F}$,*

$$\tau(F,G)\,\tau(FG,H) = \tau(F,GH)\,\big(b(F)\cdot\tau(G,H)\big), \quad F|G|H; \tag{2.19}$$

$$\sigma(G,H)\,\sigma\left(F, \frac{G}{H}\right) = \big(l(H)^{-1}\cdot\sigma(F,G)\big)\,\sigma\left(\frac{F}{G}, H\right), \quad \begin{array}{c} F \\ \hline G \\ \hline H \end{array}; \tag{2.20}$$

$$l(H)^{-1}\cdot(t(H)\cdot L) = b(H)\cdot(r(H)^{-1}\cdot L), \quad L \in \mathbf{K}(tr(H)); \tag{2.21}$$

$$\begin{aligned}(2.22)\quad &\big(l(H)^{-1}\cdot\tau(F,G)\big)\tau(H,J)\sigma(FG,HJ)\\ &= \big(b(H)\cdot\sigma(G,J)\big)\sigma(F,H)\tau\begin{pmatrix}F & G\\ H, & J\end{pmatrix}, \quad \begin{array}{c|c} F & G \\ \hline H & J\end{array}.\end{aligned}$$

If these conditions hold, we say that the double groupoid $\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ is an *abelian extension* of the abelian group bundle $\mathbf{K}$ by $\mathcal{F}$.

Proof. The associativity of the horizontal and vertical compositions are respectively equivalent to (2.19) and (2.20).

We have to check the axioms of double groupoid as in [BS76]; we follow [AN05, Lemma 1.2]. All the axioms are consequences of the definitions (since the axioms hold in $\mathcal{F}$) except the interchange law, which is equivalent to (2.21) and (2.22). Indeed, let $H \in \mathcal{F}$ and $L \in \mathbf{K}(tr(H))$. Computing $\begin{Bmatrix}\operatorname{id} t(H) & L\\ H & \operatorname{id} r(H)\end{Bmatrix}$ in two different ways, we see that (2.21) is equivalent to the interchange law in this case. Next, consider

$$(K, F),\ (L, G),\ (M, H),\ (N, J) \in \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F} \text{ such that } \begin{array}{c|c} F & G \\ \hline H & J\end{array}.$$

Compute $\begin{matrix}\{(K,F)(L,G)\}\\ \{(M,H)(N,J)\}\end{matrix}$ and $\begin{Bmatrix}(K,F)\\(M,H)\end{Bmatrix}\begin{Bmatrix}(L,G)\\(N,J)\end{Bmatrix}$. The resulting expressions are equal if and only if

$$\begin{aligned}&l(H)^{-1}\cdot(b(F)\cdot L)l(H)^{-1}\cdot\tau(F,G)\tau(H,J)\sigma(FG,HJ)\\ &= b(H)\cdot(l(J)^{-1}\cdot L)b(H)\cdot\sigma(G,J)\sigma(F,H)\tau\begin{pmatrix}F & G\\ H, & J\end{pmatrix}\end{aligned}$$

It is not difficult to see that this is equivalent to (2.21) and (2.22). □

Remark 2.6. As we shall see later, see Section 3, the thin double groupoid $\mathcal{F}$ determines a 'diagonal' groupoid $\mathcal{D}$ endowed with groupoid maps $j : \mathcal{V} \to \mathcal{D}$, $i : \mathcal{H} \to \mathcal{D}$. In terms of this groupoid, Condition (2.21) means that the actions of $\mathcal{H}$ and $\mathcal{V}$ come from an action of $\mathcal{D}$ on $\mathbf{K}$.

Remark 2.7. Conditions (2.19) and (2.20) are cocycle conditions on the horizontal and vertical composition groupoids. Together with (2.22) they give a cocycle condition in the double complex associated to the double groupoid $\mathcal{F}$ as considered in [AN05, AM06].

Assume that the hypotheses in Proposition 2.5 are fulfilled. Let us identify $\mathbf{K}$ with a subset of $\mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$ via $K \mapsto (K, \Theta_P)$, if $K \in \mathbf{K}(P)$. Also, let

$\mu : \mathcal{F} \to \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}$, $\mu(F) = (\Theta_{bl(F)}, F)$. Then

$$(K, F) = \begin{matrix} \operatorname{id} l(F) & \mu(F) \\ K & \operatorname{id} b(F) \end{matrix}, \qquad \text{for any } (K, F) \in \mathbf{K}\,{}_p\!\times_\gamma \mathcal{F}.$$

Hence the formulas (2.15) and (2.16) are equivalent to (2.11) and (2.12), respectively. In particular we have

Theorem 2.8. *Any double groupoid is an abelian extension of its pith group bundle by its frame.* □

3. Thin double groupoids and factorizations of groupoids

Let $\mathcal{V}$ and $\mathcal{H}$ be groupoids over $\mathcal{P}$. In this section we describe all thin double groupoids satisfying the filling condition (2.2) whose groupoids of vertical and horizontal edges coincide with $\mathcal{V}$ and $\mathcal{H}$, respectively.

3.1. Double groupoid associated to a diagram of groupoids.

Let us say that a *diagram* $(\mathcal{D}, j, i)$ over $\mathcal{V}$ and $\mathcal{H}$ is a groupoid $\mathcal{D}$ over $\mathcal{P}$ endowed with groupoid maps over $\mathcal{P}$

$$i : \mathcal{H} \to \mathcal{D}, \quad j : \mathcal{V} \to \mathcal{D}.$$

The class of all diagrams $(\mathcal{D}, j, i)$ over $\mathcal{V}$ and $\mathcal{H}$ is a category with morphisms $(\mathcal{D}, j, i) \to (\mathcal{D}', j', i')$ being morphisms $f : \mathcal{D} \to \mathcal{D}'$ of groupoids over $\mathcal{P}$ such that $fi = i'$ and $fj = j'$.

Consider the full subcategory of diagrams $(\mathcal{D}, j, i)$ with $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$; that is, such that every arrow in $\mathcal{D}$ can be written as a product $j(g)i(x)$, for some $g \in \mathcal{V}$, $x \in \mathcal{H}$, with $b(g) = l(x)$. An object in this subcategory will be called a $(\mathcal{V}, \mathcal{H})$-*factorization* of $\mathcal{D}$.

Each diagram $(\mathcal{D}, j, i)$ has an associated double groupoid $\square(\mathcal{D}, j, i)$ defined as follows. Boxes in $\square(\mathcal{D}, j, i)$ are of the form

$$A = h \overset{x}{\underset{y}{\square}} g \in \square(\mathcal{V}, \mathcal{H}),$$

with $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that

$$i(x)j(g) = j(h)i(y) \quad \text{in } \mathcal{D}.$$

Notice that $\square(\mathcal{D}, j, i)$ is stable under vertical and horizontal products in $\square(\mathcal{V}, \mathcal{H})$; therefore it is itself a double groupoid. By its very definition, $\square(\mathcal{D}, j, i)$ is *thin*.

The assignment $(\mathcal{D}, j, i) \to \square(\mathcal{D}, j, i)$ just defined is clearly functorial.

R. Brown has kindly pointed out to us that the construction of the double groupoid $\square(\mathcal{D}, j, i)$ was found by him long time ago, and that it was taken up by Lu and Weinstein [LW89].

Example 3.1. Let G be a simply connected Poisson-Lie group, $\mathfrak{g}$ its Lie algebra. It is well known that $\mathfrak{g}$ is a Lie bialgebra; let $\mathfrak{g}^*$ be the dual Lie algebra and let $\mathfrak{d}$ be the correspondig Drinfeld double. Let G^* and D be simply connected Lie groups with Lie algebras $\mathfrak{g}^*$ and $\mathfrak{d}$, respectively. Then the maps $G \to D$ and $G^* \to D$ give rise to a double symplectic groupoid [LW89, Theorem 3].

Lemma 3.2. *The core groupoid of* $\square(\mathcal{D}, j, i)$ *is isomorphic to the groupoid* ${\mathcal{V}^{\text{op}}}_j \times_i \mathcal{H} := \{(g, x) \in {\mathcal{V}^{\text{op}}}_b \times_l \mathcal{H} : \, j(g) = i(x^{-1})\} \subseteq {\mathcal{V}^{\text{op}}}_b \times_l \mathcal{H}$.

Proof. An isomorphism is given by the map $\mathbf{E} \to {\mathcal{V}^{\text{op}}}_j \times_i \mathcal{H}$, defined by $E \mapsto (l(E), b(E))$. This map is surjective by construction of $\square(\mathcal{D}, j, i)$; it is injective as a consequence of the thin condition on $\square(\mathcal{D}, j, i)$. $\square$

Remark 3.3. If $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ is a factorization, then $\square(\mathcal{D}, j, i)$ satisfies the filling condition (2.2). Indeed, if $g \in \mathcal{V}$, $x \in \mathcal{H}$, are such that $r(x) = t(g)$, then the condition $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ implies that there exist $y \in \mathcal{H}$, $h \in \mathcal{V}$, such that $j(h)i(y) = i(x)j(g)$. Then, by construction, the box $h \overset{x}{\underset{y}{\square}} g$ is a filling in $\square(\mathcal{D}, j, i)$ for $\overset{x}{\urcorner} g$.

Example 3.4. Suppose $\mathcal{D} = \mathcal{P}^2$ is the coarse groupoid on $\mathcal{P}$. Let the maps $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$, be defined by $i(x) = (l(x), r(x))$, $x \in \mathcal{H}$, and $j(g) = (t(g), b(g))$, $g \in \mathcal{V}$. Let $\mathcal{B} = \square(\mathcal{P}^2, j, i)$ be the associated double groupoid. The relations $i(x)j(g) = j(h)i(y)$ are satisfied in $\mathcal{D}$, for all $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that $r(x) = t(g)$, $b(h) = l(y)$, $l(x) = t(h)$, $r(y) = b(g)$. Hence, for all such x, y, g, h there is a box $h \overset{x}{\underset{y}{\square}} g$ in $\mathcal{B}$. According to the composition rules in $\mathcal{B}$, it turns out that $\mathcal{B}$ is exactly the *coarse* double groupoid $\square(\mathcal{V}, \mathcal{H})$.

We shall show that double groupoids of the form $\square(\mathcal{D}, j, i)$ exhaust the class of thin double groupoids which satisfy the filling condition (2.2).

3.2. Diagonal groupoid of a thin double groupoid.

Let $\mathcal{B}$ be a double groupoid. In the free product $\mathcal{V} * \mathcal{H}$, we denote

$$[A] := xgy^{-1}h^{-1}, \quad \text{if} \quad A = h \overset{x}{\underset{y}{\square}} g \in \mathcal{B}.$$

Define the 'diagonal' groupoid $\mathcal{D}(\mathcal{B})$ to be the quotient of the free product $\mathcal{V} * \mathcal{H}$ modulo the relations $[A]$, $A \in \mathcal{B}$.

In the rest of this section we suppose that $\mathcal{B}$ *is thin and satisfies the filling condition* (2.2).

Lemma 3.5. *The subgroupoid* J *generated by all relations* $[A]$, $A \in \mathcal{B}$, *is a normal subgroup bundle of the free product* $\mathcal{V} * \mathcal{H}$.

Hence, if $\mathcal{B}$ satisfies (2.2), then $\mathcal{D}(\mathcal{B}) = (\mathcal{V} * \mathcal{H})/J$.

Proof. It is clear that J is a subgroup bundle of $\mathcal{V} * \mathcal{H}$. Let $A = \begin{matrix} & x & \\ h & \square & g \\ & y & \end{matrix}$ be a box in $\mathcal{B}$. We shall show that the expressions $z(xgy^{-1}h^{-1})z^{-1}$ and $f(xgy^{-1}h^{-1})f^{-1}$ both belong to J, for all $z \in \mathcal{H}$, $f \in \mathcal{V}$, such that $r(z) = l(x) = b(f)$. This implies normality because $\mathcal{V}$ and $\mathcal{H}$ generate $\mathcal{V} * \mathcal{H}$. Let z, f as above. We have $r(z) = l(x) = t(h)$. Hence, since $\mathcal{B}$ satisfies (2.2), we may pick a box $\begin{matrix} & z & \\ r & \square & h \\ & s & \end{matrix}$ in $\mathcal{B}$. Then the horizontal composition

$$\begin{matrix} & z & \\ r & \square & h \\ & s & \end{matrix} \begin{matrix} & x & \\ h & \square & g \\ & y & \end{matrix} = \begin{matrix} & zx & \\ r & \square & g \\ & sy & \end{matrix}$$

is in $\mathcal{B}$. Therefore, the expressions $X = zhs^{-1}r^{-1}$ and $Y = zxgy^{-1}s^{-1}r^{-1}$ both belong to J. Hence so does the product $YX^{-1} = z(xgy^{-1}h^{-1})z^{-1}$. To show that $f(xgy^{-1}h^{-1})f^{-1}$ belongs to J, we argue as before, now picking a box $B = \begin{matrix} & u & \\ f & \square & v \\ & x & \end{matrix}$ in $\mathcal{B}$ and then taking the vertical composition $\begin{matrix} B \\ A \end{matrix}$ in $\mathcal{B}$. $\square$

Composing the inclusions $\mathcal{H}, \mathcal{V} \to \mathcal{V} * \mathcal{H}$ with the canonical projection $\mathcal{V} * \mathcal{H} \to \mathcal{D}$ we get canonical groupoid maps $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$.

The diagram $(\mathcal{D}(\mathcal{B}), j, i)$ is characterized by the following universal property: for every diagram $(\mathcal{G}, j_0, i_0)$ over $\mathcal{V}$ and $\mathcal{H}$, such that $i_0(x)j_0(g) = j_0(h)i_0(y)$, whenever the box $\begin{matrix} & x & \\ h & \square & g \\ & y & \end{matrix}$ is in $\mathcal{B}$, there is a unique morphism of diagrams $f : \mathcal{D}(\mathcal{B}) \to \mathcal{G}$.

It is clear that the assignment $\mathcal{B} \to (\mathcal{D}(\mathcal{B}), j, i)$ is functorial.

The filling condition on $\mathcal{B}$ corresponds to the factorizability condition on $\mathcal{D}(\mathcal{B})$, as we show next.

Lemma 3.6. $\mathcal{D}(\mathcal{B}) = j(\mathcal{V})i(\mathcal{H})$.

In particular, if $\mathcal{B}$ is finite then $\mathcal{D}(\mathcal{B})$ is a *finite* groupoid.

Proof. Every element in $\mathcal{V} * \mathcal{H}$ writes as a product $w_1 \dots w_m$, where each w_i is an element of $\mathcal{H}$ or $\mathcal{V}$. Hence every element in $\mathcal{D}$ factorizes as $\overline{w_1} \dots \overline{w_m}$, where $\overline{w_i}$ is the image of w_i under the canonical projection, which coincides with $i(w_i)$ or $j(w_i)$ according to whether w_i is an element of $\mathcal{H}$ or $\mathcal{V}$.

Therefore it is enough to see that every product $i(x)j(g)$, $x \in \mathcal{H}, g \in \mathcal{V}$, belongs to $j(\mathcal{V})i(\mathcal{H})$. Indeed, this implies that the elements in the factorization may be appropriately reordered to get an element in $j(\mathcal{V})i(\mathcal{H})$. To see this we use the assumption of condition (2.2) on $\mathcal{B}$: since the product $i(x)j(g)$, $x \in \mathcal{H}, g \in \mathcal{V}$, is defined, then there is a box $h \overset{x}{\underset{y}{\square}} g$ in $\mathcal{B}$. By construction of $\mathcal{D}$, $i(x)j(g) = j(h)i(y)$. The lemma follows. $\square$

3.3. **Main result.**

We can now prove the main result of this section. Lemma 3.8 encapsulates the most delicate part of the proof.

Theorem 3.7. *The assignments $\mathcal{B} \mapsto \mathcal{D}(\mathcal{B})$ and $\mathcal{D} \mapsto \square(\mathcal{D}, j, i)$ determine mutual category equivalences between*

(a) *The category of thin double groupoids* $\begin{matrix} \mathcal{B} & \rightrightarrows & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{matrix}$ *satisfying the filling condition* (2.2)*, with fixed $\mathcal{V}$ and $\mathcal{H}$, and*

(b) *The category of $(\mathcal{V}, \mathcal{H})$-factorizations of groupoids $\mathcal{D}$ on $\mathcal{P}$.*

Proof. It remains to show that the assignments are mutually inverse. Suppose first that $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$ is a factorization as in (b). Let $\mathcal{D}'$ be the diagonal groupoid associated to the double groupoid $\square(\mathcal{D}, j, i)$; so that there are groupoid maps $j' : \mathcal{V} \to \mathcal{D}'$, $i' : \mathcal{H} \to \mathcal{D}'$ such that $\mathcal{D}' = j'(\mathcal{V})i'(\mathcal{H})$, in view of Lemma 3.6.

The universal property of $\mathcal{D}'$ implies the existence of a unique groupoid map $f : \mathcal{D}' \to \mathcal{D}$ such that $fi' = i$ and $fj' = j$. Moreover, f is surjective because of the condition $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$. To prove injectivity of f, let $P \in \mathcal{P}$ and let $z \in \mathcal{D}'(P)$ such that $f(z) \in \mathcal{P}$. Write $z = j'(g)i'(x)$, $g \in \mathcal{V}$, $x \in \mathcal{H}$, such that $b(g) = l(x)$. Applying f to this identity, we get $\mathrm{id}_P = f(z) = j(g)i(x)$. In particular $t(g) = P = r(x)$, and $i(\mathrm{id}_P)j(\mathrm{id}_P) = j(g)i(x)$ in $\mathcal{D}$.

The definition of $\square(\mathcal{D}, j, i)$ implies that the box $g\,\underset{x}{\square}$ is in $\square(\mathcal{D}, j, i)$. Therefore, in view of the defining relations in $\mathcal{D}'$, we have $z = j'(g)i'(x) = \mathrm{id}_P$. This proves that f is injective and thus an isomorphism.

Let now $\mathcal{B}$ be a double groupoid as in (a), $\mathcal{D} = \mathcal{D}(\mathcal{B})$ the associated diagonal groupoid with the canonical maps $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$, and $\mathcal{B}' = \square(\mathcal{D}, j, i)$. Let $h\,\overset{x}{\underset{y}{\square}}\,g$ be a box in $\mathcal{B}$. Then $i(x)j(g) = j(h)i(y)$ in $\mathcal{D}$ and this relation determines a unique box $h\,\overset{x}{\underset{y}{\square}}\,g$ in $\mathcal{B}'$, since $\mathcal{B}'$ is thin. This defines a map $F : \mathcal{B} \to \mathcal{B}'$ that, because of the thin condition on $\mathcal{B}$, turns out to be an injective map of double groupoids.

We claim that F is also surjective, hence an isomorphism. To establish this claim, we shall need the presentation of the free product $\mathcal{V} * \mathcal{H}$ given in Subsection 1.2. Let $A = h\,\overset{x}{\underset{y}{\square}}\,g$ be a box in $\mathcal{B}'$, which means that

$$i(x)j(g) = j(h)i(y) \tag{3.1}$$

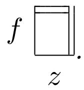
in $\mathcal{D}$. We shall prove that there is a box $h\,\overset{x}{\underset{y}{\square}}\,g$ in $\mathcal{B}$. First, using the filling condition (2.2) in $\mathcal{B}$, there is a box $A_0 = h_0\,\overset{x}{\underset{y_0}{\square}}\,g \in \mathcal{B}$. Then it is enough to show that the box $E = f\,\underset{z}{\square}$ is also in $\mathcal{B}$, where $f = h_0^{-1}h$ and $z = yy_0^{-1}$. In fact, if $E \in \mathcal{B}$, then $E \rightharpoondown A_0 \in \mathcal{B}$ is the desired box.

Since A_0 belongs to $\mathcal{B}$, $i(x)j(g) = j(h_0)i(y_0)$ in $\mathcal{D}$; combined with (3.1), this gives $j(f)i(z) \in \mathcal{P}$. The definition of $\mathcal{D}$ combined with Lemma 3.5 implies that the path fz belongs to the normal group bundle J. The proof of the Theorem will be achieved once the following Lemma is established. $\square$

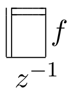
Lemma 3.8. *Let $f \in \mathcal{V}$ and $z \in \mathcal{H}$ such that*

- *$t(f) = r(z) =: P$ and $b(f) = l(z)$.*
- *There exist $A_1, \dots, A_n \in \mathcal{B}$, $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$ such that*

$$fz = [A_1]^{\epsilon_1} \dots [A_n]^{\epsilon_n}. \tag{3.2}$$

Then there exists $E \in \mathbf{E}$ such that $E =$ .

Proof. We proceed by induction on n. Note that the case $n = 0$ means that $fz = \mathrm{id}_P$ in $\mathcal{V} * \mathcal{H}$, so that $f = \mathrm{id}_P^{\mathcal{V}}$, $z = \mathrm{id}_P^{\mathcal{H}}$, since the word fz is not reduced; thus $E = \Theta_P$ is the desired element in $\mathcal{P}$.

Proof when $n = 1$. We need to work out this case by technical reasons, see Sublemma 3.10 below. We shall assume that $f \notin \mathcal{P}$, $z \notin \mathcal{P}$, so that the path fz is reduced; if either $f \in \mathcal{P}$ or $z \in \mathcal{P}$ the arguments are similar. We have $fz = [A_1]^{\epsilon_1}$. If $\epsilon_1 = 1$ this says that $fz = xgy^{-1}h^{-1}$ where we omit the subscript 1 for simplicity. Hence, the right-hand side is not reduced. Since the letters x, g, y, h belong alternatively to different groupoids, at least one of them should be in $\mathcal{P}$. If $x = \mathrm{id}_{\mathcal{H}} P$, then $fz = gy^{-1}h^{-1}$ hence necessarily $h = \mathrm{id}_{\mathcal{V}} P$, $f = g$, $z = y^{-1}$ and $A_1 =$ . Thus $E = (A_1)^h$ does the job. If $x \notin \mathcal{P}$ then it should be cancelled because the left-hand side begins by $f \in \mathcal{V}$; thus $g \in \mathcal{P}$, $y = x^{-1}$ and $fz = h$, a contradiction.

Assume now that $\epsilon_1 = -1$, that is, $fz = hyg^{-1}x^{-1}$. Again, the right-hand side is not reduced and one of the letters should be in $\mathcal{P}$. If $h = \mathrm{id}_{\mathcal{V}} P$ then $fz = yg^{-1}x^{-1}$ hence necessarily $y = \mathrm{id}_{\mathcal{H}} P$, $f = g^{-1}$, $z = x^{-1}$ and $E = (A_1)^{-1}$ does the job. If $h \notin \mathcal{P}$ then either

(a) $y \in \mathcal{P}$, $f = hg^{-1}$, $z = x^{-1}$ and $E = \begin{Bmatrix} \mathrm{id}\, h \\ (A_1)^{-1} \end{Bmatrix}$ does the job; or

(b) $y \notin \mathcal{P}$, $g \in \mathcal{P}$, $f = h$, $z = yx^{-1}$ and $E = A_1 \; \mathrm{id}\, x^{-1}$ does the job.

Assume now that the claim is true for $n - 1$. Assume that (3.2) holds for n. Our aim is to reduce the right-hand side of (3.2) to $n - 1$ factors, or to achieve a contradiction by comparison of the lengths in both sides.

Before we begin to analyze contiguous brackets, where 'bracket' means an element of the form $[A_i]^{\pm 1}$, let us set up some preliminaries. Let us say that $A_i \in \mathcal{B}$ is of *class ℓ* if exactly ℓ of its sides are not in $\mathcal{P}$. We shall assign a 'type' to any bracket $[A_i]^{\pm 1}$.

- If $\ell = 0$, all the sides of A_i are in $\mathcal{P}$; hence $[A_i]^{\epsilon_i}$ can be extirpated from the right-hand side of (3.2) and we are done by the inductive hypothesis. Hence, we can assume that $\ell > 0$ for all i.

- If $\ell = 1$, we distinguish two types.

(α): A_i has a non-trivial vertical side. We can assume that $A_i = h_i\square$.

For, if $A_i = \square\, g_i$ then $[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = \begin{Bmatrix} \operatorname{id} g_i^{-1} \\ A_i \end{Bmatrix} = g_i^{-1}\square$.

Moreover if $[A_i]$ is of type (α) then $[A_i]^{-1} = [A_i^v]$, again of type (α).

(β): A_i has a non-trivial horizontal side. We can assume that $A_i = \overset{x_i}{\square}$.

Moreover if $[A_i]$ is of type (β) then $[A_i]^{-1} = [A_i^h]$, again of type (β). In particular, if $\ell = 1$, then we can assume $\epsilon_i = 1$.

- If $\ell = 2$, we can assume that the two non-trivial sides live in different groupoids. For, if $A_i = h_i\square\, g_i$ then $[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = \begin{Bmatrix} \operatorname{id} g_i^{-1} \\ A_i \end{Bmatrix} = g_i^{-1}h_i\square$, whose bracket is of type (α). Similarly, if A_i has two non-trivial horizontal sides then $[A_i]^{\epsilon_i}$ can be replaced by a bracket of type (β).

We distinguish two types.

(γ): $[A_i] \in \mathcal{HV}$. We can assume that $A_i = h_i\overset{x_i}{\square}$. For, if $A_i = \overset{x_i}{\square}\, g_i$ then

$[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = \begin{Bmatrix} A_i \\ \operatorname{id} g_i^{-1} \end{Bmatrix} = g_i^{-1}\overset{x_i}{\square}$. Similarly, if $A_i = h_i\underset{y_i}{\square}$ then

$[A_i] = [\widetilde{A}_i]$ where $\widetilde{A}_i = A_i \operatorname{id} y_i^{-1} = h_i\overset{y_i^{-1}}{\square}$.

(δ): $[A_i] \in \mathcal{VH}$. Then $A_i = \underset{y_i}{\square}\, g_i$.

Moreover if $[A_i]$ is of type (γ) then $[A_i]^{-1} = [B]$, where $B = (A_i \operatorname{id} t(A_i)^{-1})^h$ is of type (δ). In particular, if $\ell = 2$ then we can assume $\epsilon_i = 1$.

- If $\ell = 3$, we can assume that the identity sides are either x_i or h_i; otherwise we replace the box by one with $\ell = 2$. There are two types.

$$(\zeta): \quad A_i = \begin{array}{c} x_i \\ \|\square\; g_i, \\ y_i \end{array} \qquad (\eta): \quad A_i = \begin{array}{c} h_i \overline{\square}\, g_i \\ y_i \end{array}.$$

Moreover if $[A_i]$ is of type (ζ) then $[A_i]^{-1} = [A_i^v]$ is again of type (ζ); if $[A_i]$ is of type (η) then $[A_i]^{-1} = [A_i^h]$ is again of type (η). In particular, if $\ell = 3$ then we can assume $\epsilon_i = 1$.

- If $\ell = 4$, we distinguish two types.

$$(\theta): \quad \epsilon = 1, \qquad (\kappa): \quad \epsilon = -1.$$

In conclusion the right-hand side of (3.2) *is a product of n brackets of types (α), ..., (κ) with the exponent $\epsilon_i = 1$ except for the type (κ).*

Next, we can assume several restrictions on the contiguity of these brackets, as summarized in the following statement. We prove below that, whenever these restrictions do not hold, then we can replace the pair of contiguous brackets by a single bracket and hence apply the inductive hypothesis.

Restrictions 3.9. *Let $1 \leq i \leq n$.*

(1). *If $[A_i]$ is of type (α), then $i > 1$ and $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (κ). Also, if $i < n$ then $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (θ).*

(2). *If $[A_i]$ is of type (β), then $i < n$ and $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (κ). Also, if $i > 1$ then $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (θ).*

(3). *If $[A_i]$ is of type (γ), then:*

(a): *If $i > 1$ then $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (η), (θ), or (κ) with $x_{i-1} \neq x_i$.*
(b): *If $i < n$ then $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (ζ), (θ), or (κ) with $h_{i+1} \neq h_i$.*

(4). *If $[A_i]$ is of type (δ), then:*

(a): *If $i > 1$ then $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (ζ), (θ), (κ), or (θ) with $h_{i-1} \neq g_i$.*
(b): *If $i < n$ then $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (η), (κ), or (θ) with $x_{i+1} \neq y_i$.*

(5). *If* $[A_i]$ *is of type* (ζ)*, then:*

- **(a):** *If* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (γ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i-1}]$ *is of type* (ζ) *then* $y_{i-1} \neq x_i$;
  (2) *if* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (κ) *then* $x_{i-1} \neq x_i$.
- **(b):** *If* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (δ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i+1}]$ *is of type* (ζ) *then* $y_i \neq x_{i+1}$;
  (2) *if* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (θ) *then* $y_i \neq x_{i+1}$.

(6). *If* $[A_i]$ *is of type* (η)*, then:*

- **(a):** *If* $i > 1$ *then* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (δ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i-1}]$ *is of type* (η) *then* $h_{i-1} \neq g_i$;
  (2) *if* $[A_{i-1}]^{\epsilon_{i-1}}$ *is of type* (θ) *then* $h_{i-1} \neq g_i$.
- **(b):** *If* $i < n$ *then* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (γ), (ζ), (η), (θ) *or* (κ). *Also,*
  (1) *if* $[A_{i+1}]$ *is of type* (η) *then* $g_{i+1} \neq h_i$;
  (2) *if* $[A_{i+1}]^{\epsilon_{i+1}}$ *is of type* (κ) *then* $h_{i+1} \neq h_i$.

(7). *If* $[A_i]^{\epsilon_i}$ *and* $[A_{i+1}]^{\epsilon_{i+1}}$ *are both of types* (θ) *or* (κ) *for some* i, $1 \leq i \leq n-1$, *then either*

- $\epsilon_i = \epsilon_{i+1}$; *or*
- $\epsilon_i = 1 = -\epsilon_{i+1}$, *and* $h_i \neq h_{i+1}$; *or*
- $\epsilon_i = 1 = -\epsilon_{i+1}$, $h_i = h_{i+1}$ *and* $y_i \neq y_{i+1}$; *or*
- $\epsilon_i = -1 = -\epsilon_{i+1}$ *and* $x_i \neq x_{i+1}$; *or*
- $\epsilon_i = -1 = -\epsilon_{i+1}$, $x_i = x_{i+1}$ *and* $g_i \neq g_{i+1}$.

We deal with (1). If $[A_1]$ is of type (α) then $h_1 f z = [A_2]^{\epsilon_2} \dots [A_n]^{\epsilon_n}$; by the inductive hypothesis there exists $E_1 \in \mathbf{E}$ such that $E_1 = \begin{matrix} h_1 f & \square \\ & z \end{matrix}$. Then $E = \begin{Bmatrix} (A_1)^v \\ E_1 \end{Bmatrix}$ does the job. Thus we do not consider this possibility. Assume that $i > 1$ and $[A_{i-1}]^{\epsilon_{i-1}}$ is not of type (κ). We show that $[A_{i-1}][A_i] = [B]$; hence the right-hand side of (3.2) has really $n-1$ factors and the existence of E follows from the inductive hypothesis. Thus we do not consider this possibility. Explicitly, $B \in \mathcal{B}$ is given as follows.

- If $[A_{i-1}]$ is of type (α) or (η) then $B = \begin{Bmatrix} A_i \\ A_{i-1} \end{Bmatrix}$.
- If $[A_{i-1}]$ is of type (β) then $B = \begin{Bmatrix} A_{i-1} \\ A_i \end{Bmatrix}$.

- If $[A_{i-1}]$ is of type (γ) then $B = \left\{\begin{matrix} A_i \\ \operatorname{id} h_{i-1}^{-1} \\ A_{i-1} \end{matrix}\right\}$.
- If $[A_{i-1}]$ is of type (δ) or (ζ) then $B = A_i\, A_{i-1}$.
- If $[A_{i-1}]$ is of type (θ) then $B = \left\{\begin{matrix} A_i \\ A_{i-1}C \end{matrix}\right\} C^h$, where $C \in \mathcal{B}$ is a box filling the configuration $\begin{smallmatrix} & x_i^{-1} \\ g_i & \ulcorner \end{smallmatrix}$.

Now, if $i < n$ then write $[A_{i+1}]^{\epsilon_{i+1}} = [\widetilde{A}_{i+1}]^{-\epsilon_{i+1}}$. Then $[A_i][A_{i+1}]^{\epsilon_{i+1}} = \left([\widetilde{A}_{i+1}]^{-\epsilon_{i+1}}[A_i^v]\right)^{-1}$. We discard this possibility by the previous discussion unless $[\widetilde{A}_{i+1}]^{-\epsilon_{i+1}}$ is of type (κ), that is, unless $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (θ).

We deal with (2). If $[A_n]$ is of type (β) then $fzx_n^{-1} = [A_1]^{\epsilon_1}\dots[A_{n-1}]^{\epsilon_{n-1}}$; by the inductive hypothesis there exists $E_1 \in \mathbf{E}$ such that $E_1 = \begin{smallmatrix} f & \square \\ & zx_n^{-1} \end{smallmatrix}$. Then $E = E_1(A_n)^v$ does the job. Thus we do not consider this possibility. Assume that $i < n$ and $[A_{i+1}]^{\epsilon_{i+1}}$ is not of type (κ). We show that $[A_i][A_{i+1}] = [B]$; hence the right-hand side of (3.2) has really $n-1$ factors and the existence of E follows from the inductive hypothesis. Thus we do not consider this possibility. Explicitly, $B \in \mathcal{B}$ is given as follows.

- If $[A_{i+1}]$ is of type (α) or (η) then $B = \left\{\begin{matrix} A_i \\ A_{i+1} \end{matrix}\right\}$.
- If $[A_{i+1}]$ is of type (β) or (δ) or (ζ) then $B = A_i\, A_{i+1}$.
- If $[A_{i+1}]$ is of type (γ) then $B = A_i \left\{\begin{matrix} A_{i+1} \\ \operatorname{id} h_{i+1}^{-1} \end{matrix}\right\}$.
- If $[A_{i+1}]$ is of type (θ) then $B = \left\{\begin{matrix} A_i \left\{\begin{matrix} A_{i+1} \\ C \end{matrix}\right\} \\ C^v \end{matrix}\right\}$, where $C \in \mathcal{B}$ is a box filling the configuration $\begin{smallmatrix} & y_{i+1} \\ h_{i+1}^{-1} & \ulcorner \end{smallmatrix}$.

If $i > 1$, we may write $[A_{i-1}]^{\epsilon_{i-1}} = [\widetilde{A}_{i-1}]^{-\epsilon_{i-1}}$, hence $[A_{i-1}]^{\epsilon_{i-1}}[A_i] = \left([A_i^h][\widetilde{A}_{i-1}]^{-\epsilon_{i-1}}\right)^{-1}$. We discard this, unless $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (θ).

We deal with (3a). We may assume that $[A_{i-1}]^{\epsilon_{i-1}}$ is neither of type (α) nor (β) by (1) and (2). If $[A_{i-1}]^{\epsilon_{i-1}}$ is of type (γ), (δ), (ζ), or (κ) with $x_{i-1} = x_i$, then $[A_{i-1}]^{\epsilon_{i-1}}[A_i] = [B]$; hence the right-hand side of (3.2) has $n-1$ factors and, by the inductive hypothesis, we do not consider this possibility. Here $B \in \mathcal{B}$ is:

- If $[A_{i-1}]$ is of type (γ) then $B = \begin{Bmatrix} A_{i-1} \\ \mathrm{id}\, h_{i-1}^{-1} \\ \{A_i \,\mathrm{id}\, x_i^{-1}\} \end{Bmatrix}$.
- If $[A_{i-1}]$ is of type (δ) or (ζ) then $B = A_i \ \mathrm{id}\, x_i^{-1}\, A_{i-1}$.
- If $[A_{i-1}]^{-1}$ is of type (κ) with $x_{i-1} = x_i$, then $B = \begin{Bmatrix} A_i \,\mathrm{id}\, x_1^{-1} \\ A_{i-1} \end{Bmatrix}$.

We deal with (3b). We may assume that $[A_{i+1}]^{\epsilon_{i+1}}$ is neither of type (α) nor (β) nor (γ) by (1), (2) and (3a). If $[A_{i+1}]^{\epsilon_{i+1}}$ is of type (δ), (η), or (κ) with $h_i = h_{i+1}$ then $[A_i][A_{i+1}]^{\epsilon_{i+1}} = [B]$. By the inductive hypothesis, we do not consider this possibility. Here $B \in \mathcal{B}$ is:

- If $[A_{i+1}]$ is of type (δ) or (η) then $B = \begin{Bmatrix} A_i \\ \mathrm{id}\, h_i^{-1} \\ A_{i+1} \end{Bmatrix}$.
- If $[A_{i+1}]^{-1}$ is of type (κ) with $h_i = h_{i+1}$ then $B = \begin{Bmatrix} A_i \\ \mathrm{id}\, h_i^{-1} \end{Bmatrix} A_{i+1}^v$.

Now (4a) follows from (3b), and (4b) follows from (3a), by the inversion argument as for the second parts of (1) and (2).

We deal with (5a). If $[A_{i-1}]$ is of type (ζ) and $y_{i-1} = x_i$, then $[A_{i-1}][A_i] = [B]$ where $B = \begin{Bmatrix} A_{i-1} \\ A_i \end{Bmatrix}$. If $[A_{i-1}]^{-1}$ is of type (κ) and $x_{i-1} = x_i$, then $[A_{i-1}]^{-1}[A_i] = [B]^{-1}$ where $B = \begin{Bmatrix} A_i^v \\ A_{i-1} \end{Bmatrix}$. By the inductive hypothesis, we do not consider this possibility. Now (5b) follows from (5a) by the inversion argument.

We deal with (6a). If $[A_{i-1}]$ is of type (η) and $h_{i-1} = g_i$, then $[A_{i-1}][A_i] = [B]$ where $B = A_i A_{i-1}$. If $[A_{i-1}]$ is of type (θ) and $h_{i-1} = g_i$, then $[A_{i-1}][A_i] = [B]$ where $B = A_i A_{i-1}$. By the inductive hypothesis, we do not consider this possibility. Now (6b) follows from (6a) by the inversion argument.

We deal with (7). If $\epsilon_i = 1 = -\epsilon_{i+1}$, $h_i = h_{i+1}$ and $y_i = y_{i+1}$, then $[A_i][A_{i+1}]^{-1} = [B]$ where $B = \begin{Bmatrix} A_i \\ A_{i+1}^v \end{Bmatrix}$.

If $\epsilon_i = -1 = -\epsilon_{i+1}$, $x_i = x_{i+1}$ and $g_i = g_{i+1}$, then $[A_i]^{-1}[A_{i+1}] = [B]^{-1}$ where $B = A_i A_{i+1}^h$.

To conclude the proof of the lemma, and *a fortiori* of the theorem, we need to establish the following fact.

Sublemma 3.10. *Let $n \geq 2$. Consider an element $P_n \in \mathcal{V} * \mathcal{H}$, such that*

$$P_n = [A_1]^{\epsilon_1} \dots [A_n]^{\epsilon_n}, \tag{3.3}$$

where $A_1, \dots, A_n \in \mathcal{B}$, $\epsilon_1, \dots, \epsilon_n \in \{\pm 1\}$; the brackets $[A_i]^{\epsilon_i}$ are of type (α), ..., (κ), $1 \leq i \leq n$; and contiguous brackets satisfy Restrictions 3.9. Then

(i) the last letter of P_n is the last letter of $[A_n]^{\epsilon_n}$,

(ii) if $[A_n]^{\epsilon_n}$ has type (θ) or (κ) then the last two letters of P_n

equal the last two of $[A_n]^{\epsilon_n}$, and

(iii) the length of $P_n > 2$.

Proof. If $n = 2$ the claim follows by inspection of the possible cases; in fact, the length of P_2 turns out to be ≥ 5. Now assume that the claim is true for $n \geq 2$. Consider P_{n+1} as in (3.3) and write $P_{n+1} = P_n[A_{n+1}]^{\epsilon_{n+1}}$. By hypothesis, the length of P_2 is > 2 and its last letter is that of $[A_n]^{\epsilon_n}$. Now the pair $[A_n]^{\epsilon_n}[A_{n+1}]^{\epsilon_{n+1}}$ satisfies Restrictions 3.9. The sublemma now follows from Lemma 1.1. □

We can now finish the proof of the Lemma. If the right-hand side of (3.2) satisfies Restrictions 3.9, then the sublemma gives a contradiction, since the left-hand side has length ≤ 2. Thus at least one bracket can be eliminated in the right-hand side and the Lemma follows by induction. □

4. Fusion double groupoids

Let $\mathcal{B}$ be a finite double groupoid satisfying the filling condition (2.2). The following definition is motivated by [AN06, Proposition 3.16].

Definition 4.1. We say that $\mathcal{B}$ is *fusion* if and only if the following hold:

(F1) The vertical groupoid $\mathcal{V} \rightrightarrows \mathcal{P}$ is connected.

(F2) For any $x \in \mathcal{H}$, there exists at most one $E \in \mathbf{E}$ such that $b(E) = x$.

Observe that condition (F2) in the definition is equivalent to injectivity of the morphism $b : \mathbf{E} \to \mathcal{H}$ of groupoids over $\mathcal{P}$.

Let $\mathbf{E}$ be the core groupoid of $\mathcal{B}$ and let $\mathfrak{E}$ be the core groupoid of its frame $\mathcal{F}$. Then there is an exact sequence of groupoids

$$1 \longrightarrow \mathbf{K} \longrightarrow \mathbf{E} \longrightarrow \mathfrak{E} \longrightarrow 1.$$

In particular, for any $P \in \mathcal{P}$, we have

$$|\mathbf{E}(P)| = |\mathbf{K}(P)||\mathfrak{E}(P)|. \tag{4.1}$$

Proposition 4.2. *Suppose that $\mathcal{B}$ is fusion. Then $\mathcal{B}$ is thin.*

Proof. It follows immediately from condition (F2) in Definition 4.1. □

Then, in view of Theorem 3.7, a fusion double groupoid $\mathcal{B}$ is determined by a $(\mathcal{V}, \mathcal{H})$-factorization of its diagonal groupoid $\mathcal{D} = \mathcal{D}(\mathcal{B})$.

Proposition 4.3. *The following are equivalent:*

(i) *$\mathcal{B}$ is fusion.*

(ii) *$\mathcal{V} \rightrightarrows \mathcal{P}$ is connected and $j : \mathcal{V} \to \mathcal{D}(\mathcal{B})$ is injective.*

In particular, if $\mathcal{B}$ is fusion, then its diagonal groupoid $\mathcal{D} = \mathcal{D}(\mathcal{B})$ is connected.

Proof. It is enough to see that injectivity of the map $b : \mathbf{E} \to \mathcal{H}$ is equivalent to injectivity of the map j. Suppose first that $b : \mathbf{E} \to \mathcal{H}$ is injective. Let $g \in \mathcal{V}$ such that $j(g) \in \mathcal{P}$. By Theorem 3.7, $\mathcal{B} \simeq \mathcal{B}(\mathcal{D}, j, i)$. Then there is a box $g\boxed{} \in \mathcal{B}$. Since this box belongs to the core groupoid $\mathbf{E}$, the injectivity of b implies that $g \in \mathcal{P}$. Hence j is injective.

Conversely, suppose that j is injective. Let $E \in \mathbf{E}$ such that $b(E) \in \mathcal{P}$, so that $E = g\boxed{}$, for some $g \in \mathcal{V}$. The construction of $\mathcal{D}$ implies that $j(g) \in \mathcal{P}$ and therefore $g \in \mathcal{P}$. The thin condition on $\mathcal{B}$ now guarantees that $E \in \mathcal{P}$. Thus b is injective. □

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