Coherent algebras and noncommutative projective lines

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Abstract. A well-known conjecture says that every one-relator group is coherent. We state and partly prove an analogous statement for graded associative algebras. In particular, we show every that Gorenstein algebra A of global dimension 2 is graded coherent.

This allows us to define a noncommutative analogue of the projective line $\mathbb{P}^1$ as a noncommutative scheme based on the coherent noncommutative spectrum $\operatorname{qgr} A$ of such an algebra A, that is, the category of coherent A-modules modulo the torsion ones. This category is always abelian Ext-finite hereditary with Serre duality, like the category of coherent sheaves on $\mathbb{P}^1$. In this way, we obtain a sequence $\mathbb{P}^1_n$ $(n \geq 2)$ of pairwise non-isomorphic noncommutative schemes which generalize the scheme $\mathbb{P}^1 = \mathbb{P}^1_2$.

1. Introduction

We consider $\mathbb{N}$-graded algebras of the form $A = A_0 \oplus A_1 \oplus \dots$ over a fixed field k. All our algebras are assumed to be connected (that is, $A_0 = k$) and finitely generated. All vector spaces and modules are assumed to be $\mathbb{Z}$-graded, all their elements and maps of them are homogeneous.

Recall that an algebra (respectively, a group) is called coherent if every its finitely generated ideal (subgroup) is finitely presented, see Definition 2.1 below. A well-known conjecture says that every one-relator group is coherent [Ba]. An analogous statement for graded algebras seems to be true as well.

Conjecture 1.1. *Every graded algebra with a single defining relation is graded coherent.*

(For the definition of coherence, see subsection 2.1 below.) We prove this conjecture provided that the relation is quadratic.

Theorem 1.2 (Theorem 4.1)**.** *Every graded algebra defined by a single homogeneous quadratic relation is graded coherent.*

Note that there are non-coherent quadratic algebras with two relations, for example, the algebras $k\langle x, y, z, t | tz - zy, zx\rangle$ [Pi3, Prop. 10] or even $k\langle x, y, z | yz - zy, zx\rangle$ [Po, Example 2].

Recall [Z2] that a graded algebra A is called regular if it has finite global dimension (say, d) and satisfies the following Gorenstein property:

$$\operatorname{Ext}^i_A(k_A, k_A) \cong \begin{cases} 0, & i \neq d \\ k[l] \text{ for some } l \in \mathbb{Z}, & i = d. \end{cases}$$

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The most important class of regular algebras is the class of Artin-Shelter (AS) regular algebras, that is, the ones of polynomial growth. A well-known conjecture [AS] claims that all these algebras are Noetherian.

The following conjecture is due to A. Bondal (unpublished).

Conjecture 1.3. *Every regular algebra is graded coherent.*

Regular algebras of global dimension 2 have been described in [Z2]. All these algebras are one-relator. If such an algebra is generated in degree one, then it is quadratic, but in general such algebra is only 'generalized quadratic' — like, for example, the algebra $k\langle x, y | xy - yx = x^3 \rangle$.

Theorem 1.4 (Theorem 4.3)**.** *Every regular algebra of global dimension two is graded coherent.*

Two abelian categories may naturally be associated to any graded coherent algebra A, that is, the category $\mathrm{cmod}\, A$ of finitely presented (=graded coherent) right graded A-modules and its quotient category $\mathrm{qgr}\, A = \mathrm{cmod}\, A / \mathrm{tails}\, A$ by the category $\mathrm{tails}\, A$ of finite-dimensional modules. This category $\mathrm{qgr}\, A$ plays a role of projective spectrum for noncommutative coherent algebras [Po, BVdB], in generalization of the well-known construction (due to Artin and Zhang) of noncommutative schemes in the Noetherian case [AZ]. In this approach, a noncommutative projective scheme is a triple

$$(\mathrm{qgr}\, A, \mathcal{A}, s),$$

where A is a coherent algebra, noncommutative structural sheaf $\mathcal{A}$ is the the image of A in $\mathrm{qgr}\, A$, and s is the autoequivalence of $\mathrm{qgr}\, A$ induced by the shift of grading. Some details will be given in the subsection 2.2.

The noncommutative schemes of (Koszul) Noetherian (AS-)regular algebras of global dimension $n+1$ are usually considered as noncommutative generalizations of $\mathbb{P}^n$. However, in the case of the projective line $\mathbb{P}^1$, this Noetherian construction does not give any more than the standard commutative $\mathbb{P}^1$ again. On the other hand, there are other Noetherian abelian categories whose properties are close to the ones of the category of coherent sheaves on $\mathbb{P}^1$ (that is, they are hereditary Ext-finite with Serre duality) [RVdB], but the "coordinate rings" of the corresponding noncommutative schemes are far from being connected graded, in contrast to the coordinate ring $k[x_1, x_2]$ of $\mathbb{P}^1$.

Here we introduce another noncommutative generalization of $\mathbb{P}^n$, that is, the noncommutative projective schemes corresponding to (degree-one generated) coherent regular algebras of dimension $n+1$. We show in Proposition 5.1 that the corresponding $\mathrm{qgr}\, A$ is an ext-finite category of cohomological dimension n, and the algebra A (its coordinate ring) may be recovered by this category via a suitable "representing functor". In the case of the projective line $\mathbb{P}^1$, we obtain an infinite sequence $\{\mathbb{P}^1_n\}_{n \geq 2}$ of pairwise non-isomorphic noncommutative schemes analogous to $\mathbb{P}^1 = \mathbb{P}^1_2$, where the coordinate ring of each $\mathbb{P}^1_n$ is a connected graded 2-dimensional algebra with n generators. The corresponding categories of coherent sheaves are Ext-finite hereditary and satisfy Serre duality and BGG-correspondence. This is shown in the following

Proposition 1.5. *Let A be a degree-one generated regular algebra of global dimension 2 with $n \geq 2$ generators.*

(a) The categories $\operatorname{cmod} A$ *and* $\operatorname{qgr} A$ *and the noncommutative scheme* $\mathbb{P}^1_n = \mathbb{P}^1_n(k)$ *constructed by* A *are defined (up to isomorphisms) by the ground field* k *and the number* n *only, and do not depend on the algebra* A *itself. All these noncommutative schemes* $\mathbb{P}^1_n$ *are pairwise non-isomorphic, with* $\mathbb{P}^1_2 \cong \mathbb{P}^1$.

(b) The category $\operatorname{qgr} A$ *is* Ext*-finite hereditary with Serre duality. If* $n \geq 3$*, then it is not Noetherian, hence it does not belong to the classification in* [RVdB].

(c) The category $\operatorname{qgr} A$ *is derived equivalent to the category of finite* B*-modules modulo projectives, where* B *is a commutative Artinian algebra*
$k[x_1, \dots, x_n]/(x_i x_j, x_i^2 - x_j^2 | i \neq j)$.

This paper is organized as follows. In section 2, we give a background on coherent algebras, regular algebras of global dimension 2, and (relative) noncommutative complete intersections. In section 3, we give the following criterion for coherence: if an algebra $B = A/I$ is a *relative noncommutative complete intersection* of A (that is, the ideal I is generated by a strongly free set), and B is right Noetherian, then A is graded coherent. In the next section 4, we apply the above criterion in order to prove Theorems 1.2 and 1.4. Finally, in section 5 we consider noncommutative schemes associated to coherent regular algebras. In particular, we prove Proposition 1.5.

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2. Backgroung

2.1. **Coherence.** A finitely generated (f. g.) right module M is called *coherent* if every its finitely generated submodule is finitely presented (that is, presented by a finite number of generators and relations). Analogously, a graded f. g. module is called graded coherent if every its graded f. g. submodule is finitely presented. In fact, this notion had been introduced by Serre [S] in a more general case of coherent sheaves.

Theorem–Definition 2.1. *A (graded) algebra* A *is called (graded) right coherent, if the following equivalent conditions hold:*

(i) every (homogeneous) finitely generated right-sided ideal in A *is finitely presented, that is,* A *is (graded) coherent as a right module over itself;*

(ii) every finitely presented (graded) right A*-module is (graded) coherent;*

(iii) all finitely presented (graded) right A*-modules form an abelian category.*

The proof of equivalence may be found in [C] (see also [F]). For example, every right Noetherian algebras are right coherent, as well as free associative algebras and path algebras.

Because all our algebras and modules are graded, by the word *coherent* we will mean *graded right coherent* algebras and modules. The idea of noncommutative geometry based on such algebras will be explained in the next subsection.

2.2. **Noncommutative schemes.** Let A be a graded algebra. By $\operatorname{Gr} A$ (respectively, $\operatorname{cmod} A$) we denote the abelian category of graded (resp., coherent) A-modules. Let $\operatorname{Tors} A$ (resp., $\operatorname{tors} A$) be the category of torsion A-modules (resp., finite dimensional modules), where a module M is called torsion if for every $x \in M$ there is $n > 0$ such that $x A_{\geq n} = 0$. Note that $\operatorname{Tors} A$ is a Serre subcategory of $\operatorname{Gr} A$; moreover, if A is coherent, then $\operatorname{tors} A$ is also a Serre subcategory of $\operatorname{cmod} A$.

The quotient abelian categories $\mathrm{Qgr}\, A = \mathrm{Gr}\, A/\, \mathrm{Tors}\, A$ and $\mathrm{qgr}\, A = \mathrm{cmod}\, A/\, \mathrm{tors}\, A$ (for coherent A) play roles of the categories of (quasi)coherent sheaves on the projective scheme associated to A. Due to classical Serre theorem [S], these categories of modules are indeed equivalent to the respective categories of sheaves provided that A is commutative.

The image $\mathcal{A}$ of A_A in $\mathrm{Qgr}\, A$ (or in $\mathrm{qgr}\, A$) plays the role of the structure sheaf, and the the degree shift $s : M \mapsto M[1]$ plays a role of polarization. Thus, a noncommutative scheme is a triple

$$X = (\mathcal{C}, \mathcal{A}, s),$$

where $\mathcal{C}$ is a suitable k-linear abelian category, $\mathcal{A}$ is an object, and s is an autoequivalence of $\mathcal{C}$. For $\mathcal{C} = \mathrm{Qgr}\, A$ with an arbitrary connected graded algebra A, this definition is due to Verevkin [V] (a general scheme). For $\mathcal{C} = \mathrm{qgr}\, A$ (coherent scheme), this definition is due to Artin and Zhang [AZ] in the case of noetherian A (noetherian scheme) and to Bondal and Van den Bergh [BVdB] and Polishchuk [Po] in a more general setting of coherent algebra A.

According to [AZ], a morphism $f : X \to X'$ of two schemes $X = (\mathcal{C}, \mathcal{A}, s)$ and $X' = (\mathcal{C}', \mathcal{A}', s')$ is a k-linear functor $f : \mathcal{C} \to \mathcal{C}'$ such that $f(\mathcal{A})$ is isomorphic to $\mathcal{A}'$ and there is an isomorphism of functors $fs \cong s'f$. A map of schemes is defined as an isomorphism class of morphisms. Such a morphism f (or a respective map) is called an isomorphism if it is an equivalence of categories $f : \mathcal{C} \cong \mathcal{C}'$.

Given such a triple $X = (\mathcal{C}, \mathcal{A}, s)$, we can apply an analogue of the Serre functor to define a connected graded algebra $A := \Gamma_{\geq 0}(X) = \bigoplus_{i \geq 0} \mathrm{Hom}(\mathcal{A}, s^i(\mathcal{A}))$ with the multiplication $a \cdot b := s^j(a) \circ b$ for $a : \mathcal{A} \to s^i(\mathcal{A}), b : \mathcal{A} \to s^j(\mathcal{A})$. In some cases, this algebra A is coherent and the scheme X itself is isomorphic to the scheme $(\mathrm{qgr}\, A, \mathcal{A}, s)$. This happens if the autoequivalence s is ample [AZ], that is, the shifts of $\mathcal{A}$ form an ample sequence in $\mathcal{C}$ [Po].

If two general schemes X and Y are isomorphic, then the algebra $\Gamma_{\geq 0}(Y)$ is isomorphic to a Zhang twist of $\Gamma_{\geq 0}(X)$; on the other hand, if a coherent algebra B is a Zhang twist of an algebra A, then the coherent (and general) schemes of these algebras are isomorphic [Z1, Th. 1.4]. Here an algebra B is called a Zhang twist of A if there are k-linear bijections $\tau_i : A_i \to B_i, i \geq 0$ such that $\tau_{m+n}(yz) = \tau_m(y)\tau_{m+n}(z)$ for homogeneous $y \in y \in A_n, z \in A$ [Z1, Prop. 2.8]. For example, the projective scheme of the quantum polynomial algebra $k\langle x, y | xy = qyx\rangle$ is isomorphic to $\mathbb{P}^1$ for every $q \neq 0$.

Let A be a graded algebra, let $M, N \in \mathrm{Gr}\, A$ be two modules, and let $\mathcal{M}$ and $\mathcal{N}$ be their images in $\mathrm{Qgr}\, A$. Let $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N}) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(\mathcal{M}, s^i\mathcal{N})$, and let Ext and $\underline{\mathrm{Ext}}$ be the derived functors of Hom and $\underline{\mathrm{Hom}}$. Because the obvious functor $\mathrm{qgr}\, A \to \mathrm{Qgr}\, A$ is fully faithful for a coherent algebra A, the functors Ext and $\underline{\mathrm{Ext}}$ on the category $\mathrm{qgr}\, A$ are restrictions of the respective functors on $\mathrm{Qgr}\, A$.

Following [V, AZ], we also define the cohomologies of objects of $\mathrm{Qgr}\, A$ as $\mathrm{H}^i(\mathcal{M}) = \mathrm{Ext}^i(\mathcal{A}, \mathcal{M})$ and $\underline{\mathrm{H}}^i(\mathcal{M}) = \underline{\mathrm{Ext}}^i(\mathcal{A}, \mathcal{M}) = \lim_{n\to\infty} \mathrm{Ext}^i_A(A_{\geq n}, M)$. According to [BVdB, Lemma 4.1.6], we have $\underline{\mathrm{H}}^i(\mathcal{M}) \cong \lim_{n\to\infty} \mathrm{Ext}^i_A(A_{\geq n}, M)$ and $\mathrm{H}^i(\mathcal{M}) \cong \lim_{n\to\infty} \mathrm{Ext}^i_A(A_{\geq n}, M)_0$.

2.3. **Regular algebras of global dimension 2.** Let us recall some results of [Z2]. Let V be a vector space. A *rank* of an element $b \in T(V)$ is defined as the minimal number r of elements $l_1, \dots, l_r \in V$ such that $b = l_1 a_1 + \cdots + l_r a_r$ for some $a_1, \dots, a_r \in T(V)$.

Theorem 2.2 ([Z2]). *A graded algebra A is regular of global dimension 2 if and only if it is isomorphic to the algebra $k\langle x_1, \dots, x_n\rangle/(b)$, where $\operatorname{rank} b = n > 1$, or, equivalently, the following conditions hold:*

1. $n \geq 2$;

2. $1 \leq \deg x_1 \leq \cdots \leq \deg x_n$ with $\deg b = \deg x_i + \deg x_{n+1-i}$ for all i;

3. for some graded automorphism σ of the free algebra $k\langle x_1, \dots, x_n\rangle$ we have $b = \sum_{i=1}^{n} x_i \sigma(x_{n+1-i})$.

In this case, the algebra A is Noetherian if and only if $n = 2$.

In particular, ir follows that a regular two-dimensional algebra is Koszul if and only if it is degree-one generated.

2.4. **(Relative) noncommutative complete intersections.** In the next definition, we unite several statements from [A]. For discussions on strongly free sets as a noncommutative analogue of regular sequences and related topics, see also [Pi2, U]. Recall that a relative complete intersection, from an algebraic point of view, is a quotient of some graded or local commutative ring by an ideal generated by a regular sequence. Here we introduce relative noncommutative complete intersection (RNCI) as a quotient of a graded algebra by an ideal generated by a strongly free set. It is analogous to the term 'noncommutative complete intersection', that is, RNCI of a free algebra [A, G, EG].

Theorem–Definition 2.3 ([A]). *Suppose that a set X of homogeneous elements in a graded algebra A minimally generates a two-sided ideal I. Let $B = A/I$ be a quotient algebra. The set X is called strongly free, if the following equivalent conditions hold:*

(i) there are isomorphisms of graded vector spaces

$$\operatorname{Tor}_i^A(k,k) \simeq \operatorname{Tor}_i^B(k,k) \text{ for all } i \geq 3 \text{ and}$$

$$\operatorname{Tor}_1^A(k,k) \oplus \operatorname{Tor}_2^B(k,k) \simeq \operatorname{Tor}_1^B(k,k) \oplus \operatorname{Tor}_2^A(k,k) \oplus kX;$$

(ii) there is an isomorphism of graded vector spaces

$$B\langle X\rangle \simeq A,$$

where $B\langle X\rangle = B * k\langle X\rangle$ is a free product of B and a free algebra on X;

(iii) the Shafarevich complex $Sh(X, A)$ is acyclic in positive degrees;

In this situation, we refer to the algebra $B = A/I$ as relative noncommutative complete intersection (RNCI) of the algebra A.

In particular, it follows that if $B = A/I$ and there are isomorphisms $\operatorname{Tor}_i^A(k,k) \simeq \operatorname{Tor}_i^B(k,k)$ for all $i \geq 2$, then B is an RNCI of A.

3. A criterion for coherence

Lemma 3.1. *Let X be a strongly free set in a graded algebra A and let I be an ideal generated by X. Then I is free as right (and left) A-module.*

More precisely, let $B = A/I$, and let B' be any pre-image of B in A with the natural isomorphism of vector spaces $B' \cong B$. Then I as a free right A-module is minimally generated by the vector space $B'X$.

Proof. Obviously, $I = B'XA$. We have to show that the natural epimorphism $\gamma : B'X \otimes_k A \to B'XA$ is an isomorphism. Following [A], there is an isomorphism of graded vector spaces $\alpha : B\langle X\rangle \to A$ such that $\alpha(B) = B'$, $\alpha(BXB\langle X\rangle) = I$ (this follows also from the property (*ii*) of Theorem 2.3). The right $B\langle X\rangle$-module in the last equality is free, so, we get the desired isomorphisms of graded vector spaces $B'XA = I \cong BXB\langle X\rangle = B'X \otimes B\langle X\rangle \cong B'X \otimes_k A$. Therefore, the map γ is an isomorphism of graded vector spaces, hence it is also an isomorphism of modules. □

The next statement is similar to [Po, Prop. 3.3].

Proposition 3.2. *Let $B = A/I$, where the algebra B is right Noetherian and the ideal I is free as a left A-module. Then the algebra A is right graded coherent.*

Proof. Let J be a proper finitely generated homogeneous right-sided ideal in A. We have to show that J is finitely presented, that is, $\dim_k \operatorname{Tor}_2^A(A/J, k) < \infty$.

Consider a standard spectral sequence $E^2_{p,q} = \operatorname{Tor}_p^B(\operatorname{Tor}_q^A(A/J, B), k) \Longrightarrow \operatorname{Tor}_*(A/J, k)$. Let $N_q = \operatorname{Tor}_q^A(A/J, B)$. Because the left module ${}_AB$ has projective dimension at most one (since it admits a free resolution $0 \to I \to A$), we have $N_q = 0$ for $q > 1$, hence $E^2_{p,q} = 0$ for $q > 1$. The right B-module $N_0 = A/J \otimes_A B$ is obviously finitely generated. Moreover, the short exact sequence

$$0 \to J \to A \to A/J \to 0$$

gives, after tensoring by B, an exact sequence

$$0 \to N_1 \to J \underset{A}{\otimes} B \to A \underset{A}{\otimes} B \to N_0 \to 0.$$

Since N_1 is a submodule of a finitely generated B-module $J \underset{A}{\otimes} B$, it is finitely generated as well. Therefore, we have $\dim_k E^2_{p,q} = \dim_k \operatorname{Tor}_p^B(N_q, k) < \infty$ for all p, q. Thus, $\dim_k \operatorname{Tor}_2^A(A/J, k) \le \dim_k E^2_{2,0} + \dim_k E^2_{1,1} < \infty$. □

Corollary 3.3. *Let B be an RNCI of a graded algebra A. If the algebra B is right Noetherian, then the algebra A is right graded coherent.*

4. One-relator quadratic algebras

Theorem 4.1. *Every algebra defined by a single homogeneous quadratic relation is graded coherent.*

Proof. Let $b \in V \otimes V$ be a quadratic element in the free algebra $T(V)$, where $\dim V = n$, and let A be a quotient algebra of $T(V)$ by an ideal $\operatorname{id}(b)$ generated by b. If $n = 1$, then the algebra A is finite-dimensional, hence Artinian, hence Noetherian, hence coherent. If $n = 2$, then either A is Noetherian or b has the form $b = xy$, where $x, y \in V$ [AS, p. 172]. In the last case, A is coherent by [Pi1, Th. 2].

Consider the case $n \ge 3$. If $\operatorname{rank} b = 1$, then $b = xy$ is a monomial on generators, and the algebra A is coherent, again by [Pi1, Th. 2]. So, we can assume that $\operatorname{rank} b \ge 2$.

Lemma 4.2. *Let V be an n-dimensional vector space with $n \ge 2$, and let $b \in V \otimes V$. Given an $(n-2)$-dimensional subspace $W \subset V$, let b' be the image of b in $(V/W) \otimes (V/W)$. Then either $b = xy$ for some $x, y \in V$ or there exists W such that rank $b' = 2$.*

Proof of Lemma 4.2. By the induction on n, we can assume that for every $x \in V$ such that the image b'' of b in $(V/kx) \otimes (V/kx)$ has rank ≤ 1. Let $\{x_1 = x, \dots, x_n\}$ be a basis of V.

If, for some x, we have $b'' = 0$, then $b = \alpha x^2 + x l_1 + l_2 x$ with $l_i \in k\{x_2, \dots, x_n\}, \alpha \in k$. If $\operatorname{rank} b \geq 2$, then $l_1 \neq 0$ and $l_2 \neq 0$. Now, if $l_2 \neq \beta l_1$ for $\beta \in k$, then the image $b = \alpha x^2 + x l_1 + l_1 x$ has rank 2 — a contradiction. Hence, $l_2 = \beta l_1$ for some $0 \neq \beta \in k$, hence $b = \alpha x^2 + x l_1 + \beta l_1 x$ has rank two.

So, we can assume that $\operatorname{rank} b'' = 1$ for every x. Then $b'' = uv$ for some nonzero $u, v \in k\{x_2, \dots, x_n\}$, hence $b = uv + \alpha x^2 + x l_1 + l_2 x$ with $l_i \in k\{x_2, \dots, x_n\}, \alpha \in k$. We can assume that either (1) $u = x_2, v = x_3$ or (2) $u = v = x_2$.

Suppose that $l_1 \neq 0$ and $l_2 \neq 0$. The image of b under the factorization by l_1 has unit rank, hence $l_2 = \beta u$ for some $\beta \in k$; analogously, $l_1 = \gamma v$ with $\gamma \in k$. In the case (1), let $W = k\{x_2 - x_3, x_4, \dots, x_n\}$; in the case (2), let us put $W = k\{x_3, \dots, x_n\}$. In both cases, the image b' of b in $(V/W) \otimes (V/W)$ has the same rank as b.

Now, it remains to consider the case $l_2 = 0$ (the case $l_1 = 0$ is analogous). Then $b = uv + x(\alpha x + l_1)$. If $\alpha = 0$ and $l_1 = 0$, then $\operatorname{rank} b = 1$, and there is nothing to prove. In the case (1), the image of b under the factorization by $(x_2 - x_3)$ must have rank one, hence $\alpha = 0$, $l_1 = \lambda v$ for some $\lambda \in k$, and $\operatorname{rank} b = 1$. In the case (2), because either $l_1 = 0$ or the the image of b under the factorization by l_1 has unit rank, we have $l_1 = \lambda x_2$ for some $\lambda \in k$. Then b depends on the variables x_1 and x_2 only, hence we may put $W = k\{x_3, \dots, x_n\}$. □

Recall that $\operatorname{rank} b \geq 2$. Let $x_1, \dots, x_n$ be a basis of V such that $W = k\{x_i | i = 3 \dots n\}$ be as in this Lemma. Then the image b' of b in $(V/W) \otimes (V/W) = k\{x_1, x_2\}^{\otimes 2}$ has rank 2. By Theorem 2.2, the algebra $B = A/\operatorname{id}(x_3, \dots, x_n) = k\langle x_1, x_2 | b' = 0\rangle$ is Noetherian. Now, the set $X = \{x_3, \dots, x_n\}$ is strongly free in the algebra A, because $A/\operatorname{id}(X) = B$, while $\operatorname{Tor}_1^B(k,k) \oplus kX \cong \operatorname{Tor}_1^A(k,k)$, $\operatorname{Tor}_2^B(k,k) \cong \operatorname{Tor}_2^A(k,k) \cong kb$, and $\operatorname{Tor}_i^B(k,k) = \operatorname{Tor}_i^A(k,k) = 0$ for all $i \geq 3$. Thus, it follows from Corollary 3.3 that the algebra A is coherent. □

Theorem 4.3. *Let A be a regular algebra of global dimension 2. Then A is graded coherent.*

Proof. According to Zhang's Theorem 2.2, the algebra A has the form $A = k\langle x_1, \dots, x_n\rangle/(b)$, where $n \geq 2$, $1 \leq \deg x_1 \leq \dots \leq \deg x_n$ with $\deg b = \deg x_i + \deg x_{n+1-i}$ for all i, and for some graded authomorphism σ of the free algebra $k\langle x_1, \dots, x_n\rangle$ we have $b = \sum_{i=1}^n x_i \sigma(x_{n-i})$. If $\deg x_1 = \dots = \deg x_n$, then $b \in k\{x_1, \dots, x_n\}^{\otimes 2}$, hence A is coherent by Theorem 4.1.

So, we can assume that $\deg x_1 = \dots = \deg x_p < \dots < \deg x_{n-p+1} = \dots = \deg x_n$. Because the definition of regular rings is left-right symmetric, it follows that there is another graded automorphism τ of the free algebra $k\langle x_1, \dots, x_n\rangle$ such that $b = \sum_{i=1}^n \tau(x_{n-i}) x_i$. Let $\tilde{b} = \sum_{i=n-p+1}^n (x_i \sigma(x_{n-i}) + \tau(x_{n-i}) x_i)$. Obviously, the element $b - \tilde{b}$ does not depend on the variables $x_{n-p+1}, \dots, x_n$, hence $\operatorname{rank}(b - \tilde{b}) \leq n - p$ (where rank is defined as the minimal number r of elements $l_1, \dots, l_r \in k\{x_1, \dots, x_n\}$ such that $b - \tilde{b} = l_1 a_1 + \dots l_r a_r$ for some $a_1, \dots, a_r \in k\langle x_1, \dots, x_n\rangle$). Now, we are interested in $\operatorname{rank} \tilde{b}$.

Consider the case $\operatorname{rank} \tilde{b} \leq 1$. Then $\operatorname{rank} b \leq \operatorname{rank}(b - \tilde{b}) + \operatorname{rank} \tilde{b} \leq n - p + 1$. Since $\operatorname{rank} b = n$ by Theorem 2.2, we have $p = 1$. Then $\tilde{b}b = x_n \sigma(x_1) + \tau(x_1) x_n = \alpha x_n x_1 + \beta x_1 x_n$ for some nonzero $\alpha, \beta \in k$. Thus, $\operatorname{rank} \tilde{b} = 2$ — a contradiction.

So, $\operatorname{rank} \tilde{b} \geq 2$. Note that $\tilde{b} \in V \otimes V$, where $V = k\{x_1, \dots, x_p, x_{n-p+1}, \dots, x_n\}$. According to Lemma 4.2, there is a $(2p-2)$-dimensional subset W in V (say, $W = k\{x_2, \dots, x_p, x_{n-p+1}, \dots, x_{n-1}\}$) such that the rank of the image b' of $\tilde{b}$ in $(V/W) \otimes (V/W)$ is 2. It follows from Theorem 2.2 that the algebra $B = k\langle x_1, x_n | b' \rangle = A/\operatorname{id}(x_2, \dots, x_{n-1})$ is Noetherian and has global dimension 2. By the same arguments as in the proof of Theorem 4.1, the set $X = \{x_2, \dots, x_{n-1}\}$ is strongly free in A. In the view of Corollary 3.3, we conclude that the algebra A is coherent. □

5. Non-noetherian $\mathbb{P}^1$

A module M over an algebra R is said to satisfy condition χ if $\dim_k \operatorname{Ext}^i(k, M) < \infty$ for all $i \geq 0$, see [AZ]. A coherent algebra A said to satisfy χ if every finitely presented A-module M satisfy χ.

The following proposition is analogous to [AZ, Th. 8.1]. The proof is more or less analogous too.

Proposition 5.1. *Let A be a graded coherent regular algebra of global dimension $d \geq 0$. Then*

(1) A satisfies the condition χ;

(2) the algebra A may be recovered from its noncommutative scheme $\operatorname{proj} A := (\operatorname{qgr} A, \mathcal{A}, s)$ *as*

$$A \cong \Gamma_{\geq 0}(\operatorname{proj} A);$$

(3) the category $\operatorname{qgr} A$ *is Ext-finite and has cohomological dimension $d - 1$.*

Notice that the condition (2) here means that s is ample [AZ], that is, that the shifts of $\mathcal{A}$ form an ample sequence in $\operatorname{qgr} A$ [Po].

Proof. Using the induction on the projective dimension p of a coherent module M, we will show that $\dim_k \operatorname{Ext}^i(k, M) < \infty$ for all $i \geq 0$. If $p = 0$, then M is a finitely generated free A-module, so, all $\operatorname{Ext}^i(k, M)$ are bounded by the Gorenstein condition. If $p > 0$, then there is a short exact sequence (presentation)

$$0 \to N \to F \to M \to 0 \tag{5.1}$$

with projective module P, where gl. $\dim N < p$. By the induction assumption, the condition χ holds for P and N; by the exact triangle of Ext s, it holds for M as well.

(2). Let $\mathcal{M}$ be an image of some $M \in \operatorname{cmod} A$ in $\operatorname{qgr} A$. For every $n > 0$, the short exact sequence $0 \to A_{\geq n} \to A \to A/A_{\geq n} \to 0$ gives an exact sequence

$$0 \to \operatorname{Hom}_A(A/A_{\geq n}, A) \to A \to \operatorname{Hom}_A(A_{\geq n}, A) \to \operatorname{Ext}^1_A(A/A_{\geq n}, A) \to 0.$$

Because A is regular, the left and right terms are zero. Hence $A \cong \operatorname{Hom}_A(A_{\geq n}, A) \cong \lim_{n \to \infty} \operatorname{Hom}_A(A_{\geq n}, A) = \underline{\mathrm{H}}^0(\mathcal{A})$.

(3). Notice that $\underline{\mathrm{H}}^i(\mathcal{A}) = \lim_{n \to \infty} \operatorname{Ext}^{i+1}_A(A_{\geq n}, A)$ for $i \geq 1$, hence $\underline{\mathrm{H}}^i(\mathcal{A}) = 0$ for $i \neq 0, d-1$ and $\underline{\mathrm{H}}^{d-1}(\mathcal{A}) = A^*[l]$ for some $l \in \mathbb{Z}$. It follows that $\operatorname{cd}(\operatorname{qgr} A) \geq d - 1$ and that the cohomologies $\mathrm{H}^i(\mathcal{A})$ are finite-dimensional.

Moreover, $\underline{\mathrm{H}}^i(\mathcal{M}) = \lim\limits_{n\to\infty} \mathrm{Ext}^{\,i}_A(A_{\geq n}, M) = 0$ for all $i \geq d$. If $\mathrm{pd}\, M = 0$, then $M = \bigoplus_i A[l_i]$ is a finitely generated free module, hence $\mathrm{Ext}^{\,i}(\mathcal{A}[l], \mathcal{M})$ is finite-dimensional for every $i \geq 0, l \in \mathbb{Z}$. By the induction on $\mathrm{pd}\, M$, it follows from the $\mathrm{Ext}\,(\mathcal{A}[l], -)$ triangle for the exact sequence (5.1) that the vector spaces $\mathrm{Ext}^{\,i}(\mathcal{A}[l], \mathcal{M})$ are finite-dimensional for all i, l, M.

Let $\mathcal{M}'$ be an image in $\mathrm{qgr}\, A$ of another coherent A-module M'. If $M' = \bigoplus_{i=0}^t A[l_i]$ is a free module, we can apply the functor $\mathrm{Ext}^{\,i}(-, \mathcal{M})$ to the short exact sequence $0 \to \bigoplus_{i=0}^{t-1} A[l_i] \to \mathcal{M}' \to \mathcal{A}[l] \to 0$. The derived exact triangle shows that the vector space $\mathrm{Ext}^{\,i}(\mathcal{M}', \mathcal{M})$ is finite-dimensional for every i and vanishes for $i \geq d$. For non-free modules M', we proceed by induction on $\mathrm{pd}\, M'$. Applying the same functor to the short exact sequence $0 \to N \to F \to M' \to 0$ analogous to (5.1), we deduce that the vector spaces $\mathrm{Ext}^{\,i}(\mathcal{M}', \mathcal{M})$ are finite-dimensional for all i and vanish for $i \geq d$ as well. It follows that the cohomological dimension of $\mathrm{qgr}\, A$ is $d-1$ and that the category $\mathrm{qgr}\, A$ is Ext-finite. □

Proof of Proposition 1.5. According to [Z1, Th. 1.4] (see also the subsection 2.2 above), the coherent scheme of A is independent (up to isomorphism) on the choice of the automorphism σ in Theorem 2.2. On the other hand, if the regular algebras A and A' of global dimension two have different numbers of generators (say, m and n), then they are not twists of each other because $\tau_1 : A_1 \to A'_1$ cannot be an isomorphism of vector space. This proves (a).

Let us give also a direct proof of the last statement. Let be $\mathcal{A}$ and $\mathcal{A}'$ the images of A and A' in respective qgr, and let s and s' be the shifts if grading in these qgr. Assume that the schemes $\mathbb{P}^1_n$ and $\mathbb{P}^1_m$ with the underlying algebras A and A' are isomorphic. By definition [AZ], this means that there is an equivalence of categories $F : \mathrm{qgr}\, A \to \mathrm{qgr}\, A'$ such that $F(\mathcal{A}) \cong \mathcal{A}'$ and $s'F \cong Fs$. Then F maps the exact sequence

$$0 \to s^2\mathcal{A} \to s\mathcal{A}^n \to \mathcal{A} \to 0$$

to an exact sequence

$$0 \to s'^2\mathcal{A}' \to s'\mathcal{A}'^n \to \mathcal{A}' \to 0.$$

Taking the Euler characteristics for the second exact sequence, we deduce that the following equality of formal power series holds for some polynomial $p(z) \in \mathbb{Z}[z]$ (because a pre-image of this sequence in $\mathrm{cmod}\, A'$ must be exact up to finite-dimensional modules):

$$A'(z)(1 - nz + z^2) = p(z),$$

where $A'(z) := \sum_{i\geq 0} (\dim A'_i) z^i = (1 - mz + z^2)^{-1}$. It follows that $m = n$.

(b) The Serre duality for $\mathrm{qgr}\, A$ follows from [MV].The hereditarity (that is, that $\mathrm{qgr}\, A$ has cohomological dimension ≤ 1) and Ext-finiteness follows from Proposition 5.1.

If $n \geq 3$, then A is not Noetherian by Theorem 2.2. Let us show that the image $\mathcal{A}$ of A in $\mathrm{qgr}\, A$ is not Noetherian as well. In the view of (a), we may assume that $b = x_1x_2 + x_2x_3 + \cdots + x_nx_1$. Then b forms a Groebner basis of the ideal $\mathrm{id}\,(b) \subset k\langle x_1, \ldots, x_n\rangle$ w. r. t. an arbitrary deg-lex order, therefore, there is a linear basis of A consisting of the monomials on the variables $x_1, \ldots, x_n$ which do not contain a subword x_1x_2. Now, it is easy to see that the monomials $x_1x_3, x_1^2x_3, \ldots, x_1^tx_3$ form a right Groebner basis of the right-sided ideal $I_t \subset A$ generated by them. It follows that every quotient module I_t/I_{t-1} is infinite-dimensional (because it contains a sequence of linearly independent monomials $x_1^t x_3^s, s \geq 1$), hence the image in $\mathcal{A}$ of the chain $I_1 \subset I_2 \subset \dots$ is strictly ascending. This proves (b).

The statement (a) allows us to choose any particular b of rank n; let us choose $b = x_1^2 + \cdots + x_n^2$. Then we have $B = A^!$ — a Koszul dual algebra. Now, the claim (c) follows from the Koszul duality and a noncommutative analogue of the Bernstein-Gelfand-Gelfand correspondence, see [MVS, Prop. 4.1 and Cor. 4.5]. $\square$

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